UNIVERSITÉ PARIS-SACLAY

INTERNSHIP REPORT

Game Theory in Competitive Viral Marketing and Effectiveness of Mitigation Measures against the Covid-19 Epidemic

Author: Siying Lin

Supervisor: Samson LASAULCE
Constantin Morărescu

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Part I

A Survey on Game Theory for Competitive Viral Marketing
1 Introduction

Today, social networks such as Facebook and Instagram have rapidly gained popularity, providing great opportunities for viral marketing campaigns. A consumer’s purchasing decisions are heavily influenced by suggestions and recommendations from his family, friends, and colleagues. One of the best known examples of very simply engineered viral marketing comes from Hotmail. Hotmail offer free email addresses and services to users and each email sent by a Hotmail user added a short phrase with the URL of the service. When the initial users send emails to their personal networks, more users would sign up as they discover Hotmail from them. As a result, the subscribers of Hotmail grew from zero to 12 million users in just eighteen months, on an advertising budget of only $50,000 [23].

In [24], the influence maximization is first formulated as a discrete optimization problem as follow: given a social graph $G$ and a positive integer $k$, selects a set $S \in V$, $|S| \leq k$ as the seed set to maximize the influence spread $\sigma(S)$. Influence is propagated in the network according to stochastic diffusion models, including the independent cascade model and the linear threshold model. These stochastic diffusion models define how influence starts from the initial nodes and spreads through the network. Under both IC and LT models, this problem is NP-hard [24]. However, they show that the influence function $\sigma(S)$ is monotone and submodular. With those two desired properties, a greedy seed selection algorithm that guarantees the $(1 - 1/e - \varepsilon)$-approximation of the optimal solution is proposed. Mathematical formalizations of influence maximization have been proposed and widely studied by many researchers [10, 30, 44, 45].

Most studies in this field assume that only one company is promoting one product and there is no competition [10, 32, 46]. However, in the real world, it is prevalent that two or more companies are competing in the same market, which extends the classical influence maximization problem to a competitive environment. The goal of each company is to obtain the maximum number of adopters at the end of the diffusion process and defeat the opponent. There are various methods to model diffusion processes. One of the most efficient approaches to study the diffusion problem in competitive environment is game theory. These studies deal with the diffusion process as a strategic game where companies are players and compete to maximize the influence of their options (such as information or products) in the social network. They seek to find the most influential initial nodes so as to maximize the total number of infected nodes. The present studies on the competitive environment can be divided into two sub-categories: seed selection and budget allocation. The game-theoretic approaches that have been proposed along these directions mainly split into two types. The first is to view the process as a Stackelberg game, where the competitors of a product first choose their strategy, and then a last mover choose the strategy with complete information. The second is to capture the competition as a simultaneous game, where firms pick their initial set of nodes or decide how many resources to allocate to their potential customers at the same time, and then the diffusion process follows.
This survey aims to provide a better understanding of the current research issues in this field with overviews of the main ideas and the basic game types of various approaches. This will enable researchers to better understand the game-theoretic solutions of viral marketing and identify further research trends. For this part, we begin by introducing and analyzing some classical diffusion models and their extensions to network models. We then discuss various game modeling used in the competitive environment and several extensions.

2 Preliminaries

We start with some definitions that are common to all of the models we will study.

2.1 Social Networks

Formally, a social network is represented as a graph $G = (V, E)$, which can be either directed or undirected depending on the particular model and application. Each vertex $v \in V$ represents an individual and edges represent social relationship between nodes in the network. In a directed graph, an edge $(u, v) \in E$ means that $u$ has direct influence on the decisions made by $v$. Similarly, in an undirected graph, the edge $(u, v) = (v, u)$ means mutual influence between $u$ and $v$. In a weighted graph, each edge $(u, v) \in E$ is associated with an influence weight $ω_{u,v}$, capturing the extent to which $v$ will be influenced by $u$ once $u$ adopts a new technology or product. We denote by $d_v$ the degree of node $v$, i.e. $d_v = |N(v)|$. Finally, we denote by $N(v)$ the neighbors of $v$. This set contains all individuals who have direct influence on $v$. More precisely, $N(v) = \{u \in V : (u, v) \in E\}$. When $(u, v) \notin E$, we fix $A_{u,v} = 0$. The matrix $A$ with components $A_{u,v}$ is called the weighted adjacency matrix.

Node $u$ is said to be connected with node $v$ if the graph $(V, E)$ contains a directed path from $u$ to $v$ i.e., if there exists at least one sequence $(u = u_1, u_2, ..., u_{k+1} = v)$ such that

$$(u_h, u_{h+1}) \in E \Leftrightarrow A_{u_h,u_{h+1}} > 0, \ \forall h \in \{1, 2, ..., k\}$$

The graph $G = (V, E)$ is strongly connected if for any two distinct nodes $(u, v) \in V^2$, $u$ is connected to $v$.

2.2 Submodular Functions

Definition 1 Let $V$ be a finite set. A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if, for any $S \subseteq T \subseteq V$ and any $v \in V \setminus T$,

$$f(S + v) - f(S) \geq f(T + v) - f(T)$$
3 Models of Diffusion

The influence of any seed set is defined based on the process of information diffusion among the individuals. In this section, we survey various mathematical models of network diffusion that have emerged in recent years from economics and sociology. Each of which makes different assumptions on the ways in which a node is influenced by his neighbors. The models we present next are perhaps the two most studied diffusion models in the literature and is covered in a large number of different papers.

3.1 Linear Threshold Model

Linear Threshold (LT) is a diffusion model, which was studied by Granovetter [17] and Schelling [36]. The basic idea of LT is that a user can switch its status from inactive to active if a “sufficient” number of its incoming neighbors are active.

In this model, each node \( v \in V \) has a nonnegative weight \( \omega_{u,v} \) for every \( u \in N(v) \), where \( \sum_{u \in N(v)} \omega_{u,v} \leq 1 \). Moreover, each user \( v \) is also associated with a threshold \( \theta_v \). Considering an instance of the diffusion process, the LT model first samples the value of \( \theta_v \) of each user \( v \) uniformly at random from \([0,1]\). Then, it proceeds in discrete steps. Given these thresholds and an initial set \( S \) of active nodes, the process unfolds deterministically in a sequence of steps. In step \( t \), all nodes that were active in step \( t - 1 \) remain active, and any node \( v \) that was inactive in step \( t - 1 \) switches to active if the total weight of its active neighbors in \( N(v) \) is at least \( \theta_v \):

\[
\sum_{u \in N(v)} \omega_{u,v} \geq \theta_v.
\]

The diffusion instance terminates when no more user is to be activated. Intuitively then, the weight \( \omega_{v,u} \) represents the extent to which \( v \) is influenced by \( u \), and the threshold \( \theta_v \) represents the personal tendency of \( v \) to adopt a new technology when his neighbors do. The expected spread of influence of a set \( S \), denoted by \( \sigma(S) \), is defined as the expected number of active nodes by the end of diffusion.

3.2 Independent Cascade Model

Independent Cascade (IC) is a classic and well-studied diffusion model [15]. Each individual \( v \) is activated by each of its incoming neighbors independently by introducing an influence probability \( p_{u,v} \) to each edge \( e = (u,v) \). starting with an initial active set \( S \), the process unfolds in discrete steps. Each active node \( u \) in step \( t \) will activate each of its outgoing neighbor \( v \) that is inactive in step \( t - 1 \) with probability \( p_{u,v} \). Note that \( u \) has only one chance to activate its outgoing neighbors. Whether or not \( v \) becomes active, \( u \) stays active and stops the activation. The diffusion instance terminates when no more nodes can be activated. The influence spread of seed set \( S \) under the IC model is the expected
number of activated nodes when $S$ is the initial active node set and the above stochastic activation process is applied.

3.3 Extension

The model for Dynamic Influence in Competitive Environments (DICE) is proposed in [9]. The existing models of influence propagation, including linear threshold and independent cascade models, can be derived as special cases of DICE. Unlike existing approaches, DICE allows nodes to switch between adopted products over time. This allows modeling of the case where a new product is able to overtake or replace an existing product. Based on DICE, game-theoretic models of competition between ideas are developed.

The work in [3] proposes a competitive independent cascade model to multiple competing influences and gives a $(1 - 1/e)$ approximation algorithm for computing the best response to an already known opponent’s strategy.

The authors in [8] present the Wave Propagation model and the Distance Based model, which were based on the Independent Cascade model for the spread of influence of competing technologies from the follower’s perspective. Under their models, the authors calculate the best response of the company to a competitor’s move in a Stackelberg game.

The work in [6] proposes extensions to the LT model to deal with competing products. The Separated-Threshold Model model is monotone. The vital characteristic of this model is that the probability that technology $B$ will propagate cannot increase as a result of $A$-activating additional nodes. This stems from the definition of the model, in which each set of technology specific neighbours relate to a separate threshold value.
4 Seed selection

In this section, we survey the game-theoretic models for seed selection problem that have emerged in recent years. Each firm needs to find the most influential nodes to diffuse the influence of its products. The goal of each firm is to get the maximum number of nodes, that is, the largest number customers adopting its products.

The first is to view the process as a Stackelberg game, where the seed selection happens in turn. A firm first choose his strategy, and then the opponent needs to make a decision on the set of nodes to target [3,8,25]. Another method is to capture the competition as a simultaneous game, that is, the firms select their initial set of nodes at the same time, and then the diffusion process follows. This was first proposed in [1], and has also been studied later by [9,16,40].

4.1 Stackelberg Game

We define the general stackelberg game with two players in competing environment.

The Stackelberg competing game consists of two players $p_k$, $k = 1, 2$ on a graph $G(V,E)$. Each player owns a competing product $I_k$. One of the players (without loss of generality, assume it is $p_1$) selects a set of nodes $V_1$ to adopt $I_1$ at time 0. The second player $p_2$, observes $V_1$ and then chooses a set of nodes $V_2 \subseteq V \setminus V_1$ in which to adopt $I_2$. The influences then propagate through the graph as specified by the propagation model. The payoff for player $p_k$ is the number of nodes that adopt product $I_k$ when the diffusion ends.

Intuitively, one would think that the first player has an advantage over the second player because it has more choices and can always persuade more nodes than the second player, but this is not true.

The work [8] finds the best response to the first player’s strategy in Stackelberg game. The influence of products is supposed to be propagated by the independent cascade model in their work. The authors prove that the second player faces an NP-hard problem if aiming at selecting an optimal strategy. Using well known results on submodular functions [31], it is possible to give a $(1 - \frac{1}{e})$-approximation algorithm for finding the best response to the first player’s move. In fact, by using knowledge of the social network and the set of consumers targeted by the competitor, the follower can obtain a majority of the market by targeting a relatively small set of the appropriate consumers.

Around the same time, the authors in [3] introduce roughly the same model for competing rumors and they also gave a $(1 - \frac{1}{e})$ approximation algorithm for computing the best response to an opponent’s strategy, and prove that the “price of competition” of the game is at most 2. Moreover they present an FPTAS for the problem of maximizing the influence of a single player when the underlying graph is a tree.

The work [25] model the selection of starting nodes for the rumors as a stackelberg game. They show that computing the optimal strategy for both the first and the second player is NP-complete. Moreover determining an approximate
solution for the first player is NP-complete as well. Similarly, the authors show that the first to decide is not always an advantage, namely there exist networks where the second player can convince more nodes than the first, regardless of the first player’s decision.

The authors in [9] provide the algorithms that can be used to approximate the solution for both players. They consider the graph $G$ is strongly connected and the topology is probabilistic, then the follower’s strategy can be found by using submodular optimization techniques.

### 4.2 Competitive diffusion game

Competitive diffusion games have been introduced by authors in [1] as a game-theoretic approach towards modeling the process of diffusion of influence in social networks. We give the formal definitions of the game.

A game $\Gamma = (G, N)$ is defined by an undirected and unweighted graph $G = (V, E)$, representing the underlying social network. Denote by $N = \{1, ..., n\}$ the set of players(agents). Each player $i$ has its distinct color $i \in N$. The strategy space of each player is the set of vertices $V$ in the graph, that is, each player $i$ selects a single vertex(user) that is colored in color $i$ at time 1. If two players choose the same vertex $v$, then this vertex is removed from the graph.

A strategy profile of $n$ players is a vector $x = (x_1, ..., x_n) \in V^n$, where $x_i \in V$ is the initial vertex selected by player $i$. We also denote the strategy profile of the other $n-1$ players by $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

Given a strategy profile $x \in V^n$, the utility of player $i \in N$, denoted $U_i(x)$, is the number of vertices that are colored in color $i$ after the following propagation process. At time $t+1$, any so far uncolored vertex that has only uncolored neighbors and neighbors colored in $i$ (and no neighbors with other colors $j \in N \setminus \{i\}$) is colored in $i$. Any uncolored vertex with more than two different colors among its neighbors is removed from the graph. The process terminates when the coloring of the vertices does not change between consecutive steps (see Figure 1).

A pure strategy profile $x$ is a Nash equilibrium of the game $\Gamma$ if an agent cannot benefit by unilaterally changing their strategy, i.e., for every $i \in N$ and $x'_i \in V$ it holds that $U_i(x'_i, x_{-i}) \leq U_i(x)$. We define Competitive Diffusion as the problem of deciding whether $(G, N)$ has a Nash equilibrium.

Note that the diffusion process as defined above is a special case of progressive diffusion processes, which means the state of the nodes can not switch from being active to being inactive.

#### 4.2.1 Existence of Nash Equilibrium

If we can find the Nash equilibrium, then we can predict the company’s behavior and the outcome of the competitive diffusion process. We will show that existence of Nash Equilibrium depends on the topology of the network, and the number of the players.
The work [1] studies the relation between the diameter of a given network and existence of pure Nash equilibria in the game. In this work, the authors prove that if the diameter of the graph is at most two then an equilibrium exists with any number of players and can be found in polynomial time, whereas if the diameter is greater than two then an equilibrium is not guaranteed to exist even with two players. The diameter of the graph, denoted $D(G)$, is the maximum distance between a pair of vertices, that is, $D(G) = \max_{u, v \in V} d(u, v)$, where $d(u, v)$ denotes the length of the shortest path between $u$ and $v$ (in terms of the number of edges). However, the authors in [39] give a counterexample which does not admit a pure Nash equilibrium even if the diameter is two and correct the argument as follows. Let $N_v$ be the neighborhood of vertex $v$ including $v$:

**Theorem 4.1** When $N_u \cap N_v = V$ holds for any pair of vertices $u$ and $v$, the diffusion game with any number of players admits a pure Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.

For a couple of pairs of vertices $(v, u)$ and $(z, w)$, let $D_{u,v}(z, w)$ be the distance between $z$ and $w$ in the graph deleting $(N_v \cap N_u) \setminus \{u, v\}$.

**Theorem 4.2** For any pair of vertices $(v, u)$ and any vertex $z \in V \setminus (N_v \cup N_u)$, $D_{u,v}(u, z) = D_{u,v}(v, z)$ holds, the diffusion game with two players admits a pure Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.

Rather than restricting the diameter of the network, many works study the relationship between the type of graphs and the existence of Nash Equilibrium in the two-players competitive diffusion game. It has been shown in [38] that if the underlying graph $G$ has a tree structure, there always exists an pure-strategy Nash Equilibrium. In [34], the authors consider game on several classes of graphs such as paths, cycles, and grid graphs. The author proved that on sufficiently large grids, there always exists a Nash equilibrium for two players. Two special but well-studied classes of networks, namely the lattice and the hypercube are considered in [12] for which the authors determine the set of pure-strategy Nash equilibrium of the two networks.

In general network, the decision process on the existence of pure strategy Nash equilibrium is an NP-hard problem. In [12], the following necessary con-
dition for existence of pure-strategy Nash equilibrium for the 2-player diffusion game with a single seed placement is presented.

**Theorem 4.3** Suppose that \((a^*, b^*) \in V \times V\) is an equilibrium profile for the diffusion game. Then,

\[
\left\lceil \frac{n-1}{d(a^*)} \right\rceil \leq U_B(a^*, b^*), \quad \left\lceil \frac{n-1}{d(b^*)} \right\rceil \leq U_A(a^*, b^*),
\]

where \(U_A(a^*, b^*)\) and \(U_B(a^*, b^*)\) denote the utilities of players A and B given the initial seed placement at \((a^*, b^*)\), and \(d(a^*)\) and \(d(b^*)\) denote, respectively, the degrees of nodes \(a^*\) and \(b^*\) in an arbitrary network \(G\).

It has been shown in [1], [37] and [12] that competitive diffusion game may or may not admit pure-strategy Nash equilibria depending on the topology of the network \(G\), and the number of the players. In fact, for the case of three or more players may not have a pure Nash equilibrium. The game has Nash Equilibrium with 2 players on any tree of any diameter, but this conclusion does not hold for three or more players [38]. The work in [7] extends the results of [34] for two players to three or more players on paths, cycles, and grid graphs. In particular, there is no Nash equilibrium for three players on \(m \times n\) grids with \(\min\{m, n\} \geq 5\).

We generalize the competitive diffusion game to \(n\) players and weighted graphs, where a weight on a vertex represents a level of importance of an individual; negative weights are admitted to express very demanding customers. Competitive diffusion problem is basically a computational hard problem. However, it can be solved for some particular graph classes [21]. The authors provided an algorithm to find a Nash equilibrium of a chain graph.

### 4.3 Switching-selection game

The authors in [16] propose switching-selection game model.

We consider a 2-player game of competitive adoption on a (possibly directed) graph \(G\) over \(n\) vertices. \(G\) is known to the two players, \(R(\text{ed})\) and \(B(\text{lu})\). We use \(R, B\) and \(U(\text{uninfected})\) to denote the state of a vertex in \(G\), where \(U\) represents Uninfected, \(R\) represents infection by Red, and \(B\) represents infection by Blue. The two players simultaneously choose some number of vertices to initially seed. After this seeding, the stochastic dynamics of local adoption determine how each player’s seeds spread throughout \(G\) to generate new adoptions.

More precisely, we assume that player \(p \in \{R, B\}\) has budget \(K_p \in \mathbb{N}\). Each player \(p\) then chooses an allocation \(a_p = (a_{p1}, a_{p2}, ..., a_{pn})\) of budget across the \(n\) vertices, where \(a_{pj} \in \mathbb{N}\) and \(\sum_{j=1}^{n} a_{pj} = K_p\). The pure strategy space \(A_p\) of player \(p\) is the set of allocations. A mixed strategy for player \(p\) is a probability distribution \(\sigma_p\) on \(A_p\). Let \(A_p\) denote the set of probability distributions for player \(p\). The two players simultaneously choose their strategies \((\sigma_R, \sigma_B)\). Consider any realized initial allocation \((a_R, a_B)\) for the two players. Let \(V(a_R) = \{v| a_{Rv} > 0\}\), \(V(a_B) = \{v| a_{Bv} > 0\}\) and let
\( V(a_R, a_B) = V(a_R) \cup V(a_B) \). A vertex \( v \) becomes initially infected if one or more players assigns a seed to infect \( v \). If both players assign seeds to the same vertex, then the probability of initial infection by a player is proportional to the seeds allocated by the player (relative to the other player).

Following the allocation of seeds, the stochastic contagion dynamics on \( G \) determines how these Red and Blue infections create adoptions by new nodes in the network. Time is considered to be discrete for this process, and the state of a vertex \( v \) at time \( t \) is represented by \( s_{vt} \in \{U, R, B\} \). We assume once a vertex is infected, the state of the vertex cannot be changed. We apply the switching-selection model for the stochastic update of an uninfected vertex \( v \), which decomposes the local adoption decisions into two stages: switching and selection. In this model, updating is determined by the application of two functions to \( v \)'s local neighborhood: \( f(x) \) (the switching function), and \( g(y) \) (the selection function). Let \( \alpha_R \) and \( \alpha_B \) be the fraction of \( v \)'s neighbors infected by \( R \) and \( B \), respectively, at the time of the update, and let \( \alpha = \alpha_R + \alpha_B \) be the total fraction of infected neighbors. The function \( f \) maps \( \alpha \) to the interval \([0,1]\) and \( g \) maps \( \alpha_R/(\alpha_R + \alpha_B) \) (the relative fraction of infections that are \( R \)) to \([0,1]\). These two functions determine the stochastic update in the following way:

- \( v \) becomes infected by either \( R \) or \( B \) with probability \( f(\alpha) \); \( v \) remains in state \( U \) (uninfected) with probability \( 1 - f(\alpha) \), and the update ends.
- If \( v \) is infected, it is infected by \( R \) with probability
  \[
  g(\alpha_R/(\alpha_R + \alpha_B)),
  \]
  or infected by \( B \) with probability
  \[
  g(\alpha_B/(\alpha_R + \alpha_B)).
  \]

We assume \( f(0) = 0 \) (infection requires exposure), \( f(1) = 1 \) (full neighborhood infection forces infection), and \( f \) is increasing (more exposure yields more infection); and \( g(0) = 0 \) (players need some local market share to win an infection), \( g(1) = 1 \). Note that since the selection step above requires that an infection takes place, we also have \( g(y) + g(1-y) = 1 \), which implies \( g(1/2) = 1/2 \).

The payoff of player \( p \in \{R, B\} \) is a function of his final adopters, represented by \( \Pi_p(\sigma_R, \sigma_B) \). Let \( \mathcal{X}_p \) denote the number of infections at the termination of the dynamics for the strategy profile \((\sigma_R, \sigma_B)\). Given strategy profile \((\sigma_R, \sigma_B)\), the payoff to player \( p \in \{R, B\} \) is \( \Pi_p(\sigma_R, \sigma_B) = \mathbb{E}[\mathcal{X}_p(\sigma_R, \sigma_B)] \), where the expectation is over any randomization in the player strategies, in the choice of initial allocations, and the randomization in the stochastic updating dynamics.

Given any payoff function, each player seeks to maximize her own expected payoff, and this results in competition among the players. In the resulting game a Pure Nash Equilibrium is a profile of pure strategies \((\sigma_R, \sigma_B)\) such that \( \sigma_p \) maximizes player \( p \)'s payoff given the strategy \( \sigma_{-p} \) of the other player.
4.3.1 Analysis of Equilibrium

Here we focus on understanding two general features of equilibrium: Price of Anarchy and Budget Multiplier.

The Price of Anarchy (PoA) is a measure of the inefficient use of resources due to the non-cooperative behavior by the two players. For a fixed graph and fixed local dynamics (given by $f$ and $g$), the maximum social welfare allocation is the (deterministic) allocation $(a^*_R, a^*_B)$ (obeying the budget constraints if any exists) that maximizes $\Pi_R(a_R, a_B) + \Pi_B(a_R, a_B)$ subject to $|S_R| = K_R$ and $|S_B| = K_B$, and let $\sigma_R$ and $\sigma_B$ be Nash equilibrium strategies obeying the budget constraints that minimize the joint payoff $\Pi_R(\sigma_R, \sigma_B) + \Pi_B(\sigma_R, \sigma_B)$ across all Nash equilibria. The Price of Anarchy is then defined as:

$$\frac{\Pi_R(a^*_R, a^*_B) + \Pi_B(a^*_R, a^*_B)}{\Pi_R(\sigma_R, \sigma_B) + \Pi_B(\sigma_R, \sigma_B)}$$

The Budget Multiplier, which measures the extent to which network structure and dynamics can amplify initial resource inequality across the players. For example, when we have external budget constraints $K_R, K_B$, with $K_R \geq K_B$, we define the Budget Multiplier as follows: for any fixed graph $G$ and stochastic update dynamics, let $(\sigma_R, \sigma_B)$ be the equilibrium that maximizes the ratio $\frac{\Pi_R(\sigma_R, \sigma_B)}{\Pi_B(\sigma_R, \sigma_B)} \times \frac{K_B}{K_R}$ among all equilibria. The resulting maximized ratio is the Budget Multiplier, and it measures the extent to which the larger budget player can obtain a final market share that exceeds her share of the initial budgets.

The work in [16] provides bounds on the efficiency of equilibria (i.e. the price of anarchy) in this game. The authors identify broad conditions on the adoption dynamics—namely, decreasing returns to local adoption—under which the PoA is uniformly bounded above, across all networks. They also find sufficient conditions on the adoption dynamics, that is, proportional local adoption between competitors, which guarantees that the budget multiplier remains bounded for bounded pure strategy Budget Multiplier. $f$ is any concave switching function (satisfying $f(0) = 0$, $f(1) = 1$ and $f$ increasing) and $g$ is linear voter function functions, then upper-bounded of the Price of Anarchy is 4 for any graph $G$. Moreover, if the switching function is concave and the selection function is linear, then the (pure strategy) Budget Multiplier is bounded above by 2, uniformly across all networks. In fact, the concavity of $f$ is very important for both the PoA and Budget Multiplier. Even a slight convexity of $f$ will lead to unbounded PoA and Budget Multiplier. In addition, If $f$ is fixed to be linear (which is concave), slight deviations from linearity of $g$ towards polarizing $g$ can lead to unbounded Budget Multiplier, for similar reasons as in the PoA case: graph structure can amplify a slightly polarizing $g$ towards arbitrarily high punishment of the minority player. It can be shown that the this model of [16], the equilibrium is where both players put their only seed on $v$ and each get an expected payoff equal to $(4N + 1)/2$. 

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Building on the switching-selecting framework, the authors in [11] introduce the connectivity model in which the payoff to firms comprises not only the number of vertices who adopt their (competing) technologies, but also the number of edges among those adopters. More precisely, consider a pure strategy profile \((a_R, a_B)\) where \(a_p\) denotes the strategy of player \(p \in \{R, B\}\). We define the random variable \(\gamma_p\) to be the eventual number of edges among adopters of product \(p\). Player \(p\) then seeks to maximize her payoff which is equal to

\[ E[\gamma_p + \mathcal{X}_p | (a_R, a_B)] \].

The upper bounds on both the pure strategy PoA and the Budget Multiplier which depend on the firm budgets and the maximum degree of the network, but no other structural properties.

The budgets available to firms to seed initial infections in the network are fixed and exogenously determined in [16]. Budgeting decisions are endogenous in the endogenous budgets model. In this setting, the firms are allowed to choose the number of seeds to initially infect (at a fixed cost per seed), as well as which nodes to select as seeds. Table 1 shows a summary of results in [11] compared to [16].

<table>
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<td>(\Theta(1))</td>
<td></td>
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<td>(\Theta(K_{max}^2, d_{max}))</td>
<td>(\Theta(K_{max}))</td>
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</tbody>
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Table 1: Summary of upper bound results for the case where the adoption dynamics exhibit decreasing returns to local adoption. Here \(K_{max}\) denotes the maximum number of “seeds” firms can spend to initialize the adoption of their product in the network, and \(d_{max}\) is the maximum degree of a vertex in the network.

The switching-selection model can be considered an instance of a general threshold model in time-expanded graph. From this observation, it can be deduced that the game is a valid utility game, which in turn implies an upper bound of 2 on the coarse PoA [4,20,35,43]. The authors in [20] show a stronger bound under a generalized version of the framework used in [16].

**Theorem 4.4** The coarse PoA is upper-bounded by 2 under the switching-selection model with concave switching functions and linear selection functions.

There are some difference between two frameworks. First, and most importantly, the model in [20] allows for an arbitrary number of players. Second, multiple players can target the same individual and each player can put multiple units of budget on the same individual. This generalization larges the strategy space from sets to multisets and some what complicates the analysis of this model. Third, individual in the network has a weight measuring the
importance of the node. Fourth, the authors generalize the adoption functions defined on the fraction of already adopting neighbors to arbitrary set functions defined on the individuals who have previously adopted the product.

4.3.2 Extension

The model proposed in [14] considers a similar setting to [16] with the difference that agents determine which technology to adopt based on the outcome of a local coordination. In [14], two firms initially decide on the production cost of their products (which results in the quality of their products and incentive for people to consume them) and the number of consumers they initially free offer their products to. Then neighboring consumers play the following local coordination game which determines the dynamics of the spreading.

There are \( n \) consumers \( V = \{1, ..., n\} \) in a social network. The relationship among consumers is represented by an undirected graph \( G = (V, E) \). Consumers \( i, j \in V \) are neighbors if \((i, j) \in E\). The adjacency matrix of the graph \( G \) is denoted by \( A \) where \( a_{ij} = 1 \) if \((i, j) \in E\) and \( a_{ij} = 0 \) otherwise. We assume \( a_{ii} = 0 \) meaning there is no self loop. We denote the degree of node \( i \) by \( d_i \) and the diagonal matrix of degrees of the graph \( G \) by \( D = \text{diag}(d) \). The \( i \)-th largest eigenvalue of the row stochastic matrix \( D^{-1}A \) is represented by \( \lambda_i \). We also assume that there are two competing firms \( a \) and \( b \) producing products \( a \) and \( b \). These two firms initially offer their products to a set of consumers in the network. Let the binary variable \( x_i(t) \) denotes the choice of consumer \( i \) at time \( t \). We assume \( x_i(t) = 0 \) if consumer \( i \) chooses the product \( a \) and \( x_i(t) = 1 \) if consumer \( i \) chooses the product \( b \). Therefore, the state of consumers at time \( t \) is represented by a vector \( \vec{x}(t) \). Denote by \( S_a \) the vector of initial seeds of firm \( a \) for which the \( i \)-th element is equal to 1 if product \( a \) is initially offered to consumer \( i \). The vector \( S_b \) is defined similarly. Denote the norm of these vectors by \( S_a = \|S_a\|_1 \) and \( S_b = \|S_b\|_1 \). Initially all consumers are seeded either by firm \( a \) or firm \( b \), therefore, \( S_a + S_b = 1 \). The two products have some payoffs for neighboring consumers depending on their states. If two neighbors in the graph choose the product \( a \) they receive a payoff of \( p_a \), if they both choose the product \( b \) they receive a payoff of \( p_b \) and they receive zero if they choose different products. Therefore, the payoff of the interaction between consumer \( i \) and consumer \( j \) can be displayed as the following local coordination game:

<table>
<thead>
<tr>
<th></th>
<th>( x_j = 0 )</th>
<th>( x_j = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i = 0 )</td>
<td>( p_a )</td>
<td>0</td>
</tr>
<tr>
<td>( x_i = 1 )</td>
<td>0</td>
<td>( p_b )</td>
</tr>
</tbody>
</table>

Thus, the total payoff of a consumer is simply the sum of her payoffs obtained from her interactions with her neighbors

\[
u_i(x_i) = \sum_{j \in N_i} u_i(x_i, x_j),\]

13
where $\mathcal{N}_i$ is the set of neighbors of consumer $i$. We assume consumers repeatedly apply myopic best response. This means that each consumer considering her neighbors, chooses a product that gives her the most payoff. For example, consumer $i$ already adopted to the product $a$ switches to the product $b$ if enough of her neighbors have already adopted to the product $b$. For consumer $i$ already adopted to the product $a$, the payoff of choosing the product $a$ and $b$ can be written as

$$u_i(x_i = 0) = p_a \sum_{j \in \mathcal{N}_i} (1 - x_j)u_i(x_i = 1) = p_b \sum_{j \in \mathcal{N}_i} x_j.$$ 

Consumer $i$ will switch to the product $b$ if we have $u_i(x_i = 0) < u_i(x_i = 1)$ that is

$$\frac{\sum_{j \in \mathcal{N}_i} x_j}{d_i} > \frac{p_a}{p_a + p_b}, \tag{1}$$

similarly, consumer $i$ already adopted to the product $b$ will switch to the product $a$ if we have $u_i(x_i = 1) < u_i(x_i = 0)$ that is

$$\frac{\sum_{j \in \mathcal{N}_i} (1 - x_j)}{d_i} > \frac{p_b}{p_a + p_b}. \tag{2}$$

Define the right hand side of equations (1) and (2) as

$$r_a := \frac{p_b}{p_a + p_b}, \quad r_b := \frac{p_a}{p_a + p_b}. \tag{3}$$

Note that $r_a$ and $r_b$ in (3) are the degree of risk dominance of actions $a$ and $b$ respectively. This means that if for consumer $i$ already adopted to the product $a$, the fraction of her neighbors adopting to the product $b$ is greater than $r_b$, then consumer $i$’s best response is to switch to the product $b$. We can explain $r_a$ similarly.

By analyzing this local coordination game and utilizing earlier results in the literature, the lower and upper bounds of the process $\vec{x}(t)$ with myopic best response dynamics can be shown to be

$$1 - \sqrt{\frac{S_a}{n} \exp\left(\frac{t}{r_a}\right)} \leq \sum_{i=1}^n \mathbb{E}(x_i(t)) \leq \sqrt{\frac{S_b}{n} \exp\left(\frac{t}{r_b}\right)}.$$ 

These bounds depend on the payoff of products for consumers and the number of initial seeds of the network.

However, it is desirable for firms to have targeted marketing strategies in which they decide on not only how many people, but also who they offer the products to. Therefore, the authors in [14] derive new bounds for the spread of products which depend on the location of initial seeds. These bounds are tighter than bounds in [13] in which only the size of the initial adoptions matters:

$$1 - \frac{T^T \exp((D^{-1}A)t)\vec{S}_a}{n} \leq \sum_{i=1}^n \mathbb{E}(x_i(t)) \leq \frac{T^T \exp((D^{-1}A)t)\vec{S}_b}{n}.$$
4.4 Other approaches

Previous games supposes that each advertiser has detailed knowledge of the social network topology, the ability to compute or converge to equilibrium strategies, and the power to target arbitrary individuals in the network. Differing from a line of prior work that studies equilibria of the game in which each advertiser selects their seed set directly. The authors in [5] assumed that the targeted advertising goes through an intermediary (the social network), which selects seed sets on the players’ behalf. The authors designed the 2-approximate strategyproof mechanism for use when there are two competing advertisers, under a very general model of influence spread. This mechanism uses a novel technique for monotonizing the expected utilities of the agents using geometric properties of the problem in the two-player case.

The authors in [26] consider the scenario where player knows the existence of its rivals but does not know the strategies adopted by their rivals. They propose a novel algorithm called GetReal to find whether there exists a Nash Equilibrium. Experimental results show that no matter which strategy the competitor adopts, it can find the optimal strategy for each group.

Contrary to previous research, in the game proposed by [22], players not only find the most influential initial node, but also the best information content. The proposed novel game is implemented on a real data set, in which, due to the influence of neighbors and the content of the information received, the individuals’ tendencies toward the players’ options that change over time. This means that the diffusion process can be remarkably affected by information content, the topology of the graph and the initial tendency of the individual.
5 Resource allocation

Most existing studies focus on the seed selecting problem. However, only a few studies focus on the resource allocation scenario. We mainly study the budget allocation scenario in [27, 42]. In this scenario, companies can allocate different values of their budget to a node in the network, and nodes will prefer the product of the company that has offered a higher value. The authors in [27] analyze the case of two competitors and use the voter model with discrete states for their diffusion dynamics. Combining game theory, especially the Colonel Blotto game, to calculate the Nash equilibrium. The authors in [42] propose a game-theoretical formulation of the problem and conduct the complete corresponding equilibrium analysis. In addition, the work in [2] proposes a two-stage budget allocation scenario, which combines seed selection and budget allocation. They design a new payoff estimation method to calculate the Nash equilibrium.

5.1 Voter Model

We consider that the opinion of consumers is not only combined influenced by their neighbors, but also by competing marketers. We study the case where two marketing campaigns competing to simultaneously decide how many resources to allocate to potential customers to advertise their products. The process and dynamics by which influence is spread is given by the voter model. Before introducing the voter model, we focus on the Colonel Blotto game.

5.1.1 Colonel Blotto games

In the simplest version of the Colonel Blotto game, two generals simultaneously allocate one divisible unit of military resources across three equally valued battlefields to capture three battlefields. If a general allocates more resources than his opponent across a battlefield then he can obtain the battlefield. The goal of each general is to maximize the number of captured battlefields. We extend the simple Colonel Blotto game to $n$ unequally valued battlefields. We assume there are two players, denoted $X$ and $Y$, and $n$ objects. Player $X$ has budget $B_X$ and he can allocate the fraction of his budget $x_i$ to object $i$ for $1 \leq i \leq n$. Similarly, player $Y$ has budget $B_Y$ and he can allocate the fraction of his budget $y_i$ to object $i$ for $1 \leq i \leq n$. We assume that both players have the same total divisible budget of 1, i.e., $B_X = B_Y = 1$.

A pure strategy for player $X$ can be written as an $n$-dimensional allocation vector $x = (x_1, x_2, ..., x_n)$ with

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \in [0, 1],$$

where $x_i$ represents a proportion of budget allocated to $i$. Thus, the set of pure
strategies is the \((n-1)\)-dimensional simplex
\[
\Delta^{(n-1)} = \{(x_1, \ldots, x_n) : x_i \geq 0, \ 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} x_i = 1\}.
\]

A mixed strategy is an \(n\)-variate distribution function \(F : \Delta^{(n-1)} \rightarrow [0, 1]\). Let \(F_i\) represent the \(i\)th one-dimensional marginal of \(F\), i.e., the unconditional distribution of \(x_i\).

The object \(i\) has an associated non-negative payoff \(A_i\) for \(1 \leq i \leq n\). Let \(A\) denotes the sum of the payoffs of all objects, i.e.,
\[
A = \sum_{i=1}^{n} A_i.
\]

For all \(i\), we define
\[
a_i = \frac{A_i}{A},
\]
which represents the relative value of object \(i\) and note that
\[
\sum_{k=1}^{n} a_i = 1.
\]

We assume that a battlefield is captured by a player if he allocates more resources there than his opponent. Ties are resolved by flipping a coin.

For any pair \((x, y)\) of pure strategies, the excess aggregate value for player \(X\), denoted by \(g(x, y)\), of objects captured by player \(X\) if he plays the pure strategy \(x\) while player \(Y\) plays the pure strategy \(y\) is given by
\[
g(x, y) = \sum_{i=1}^{n} a_i \text{sgn}(x_i - y_i),
\]
where \(\text{sgn}(\cdot)\) is the sign function defined as
\[
\text{sgn}(u) = \begin{cases} 
1 & \text{if } u > 0, \\
2 & \text{if } u = 0, \\
-1 & \text{if } u < 0.
\end{cases}
\]

The excess aggregate value \(g(x, y)\) is the gain for pure strategy \(x\) against pure strategy \(y\).

**Definition 2** We define the Colonel Blotto game as the two-player, zero-sum game defined by the payoff function \(g\).

If \(F\) and \(G\) are two mixed strategies the payoff to mixed strategy \(F\) against mixed strategy \(G\) is:
\[
K(F, G) = \int_{x \in \Delta^{(n-1)}} \int_{y \in \Delta^{(n-1)}} g(x, y) dF(x) dG(y).
\]
The expected payoff for mixed strategy $F$ against pure strategy $y$ is given by

$$K(y) = \int_{x \in \Delta^{(n-1)}} g(x, y) dF(x). \quad (5)$$

The game is symmetric so to prove that a strategy is optimal we only need to show that $K(y) \geq 0$ for every $y$.

From eq.(4) and eq.(5), we have that

$$K(y) = \sum_{i=1}^{n} a_i (\mathbb{P}(x_i > y_i) - \mathbb{P}(y_i > x_i))$$

We assume $n \geq 2$ otherwise the game always ends in a tie.

There is no pure strategy Nash equilibrium in the general case for $n > 2$ [27]. For the case of three battlefields, Gross and Wagner [18] proved the existence of a mixed strategy solution given as follows.

**Theorem 5.1** (Gross and Wagner [18]). For $n = 3$, the Colonel Blotto game with heterogeneous battlefield values has a mixed strategy equilibrium in which the marginal distribution over front $i$ is uniform on $[0, 2a_i]$ for $i \in \{1, 2, 3\}$.

**Theorem 5.2** (Gross [19]). Consider the Colonel Blotto Game with heterogeneous battlefield values. Let $F^*$ be a probability distribution of $x \in \Delta^n$ such that each vector coordinate $x_i$ is uniformly distributed on $[0, 2a_i]$ for $i \in \{1, ..., n\}$. Then $(F^*, F^*)$ constitutes a symmetric Nash equilibrium.

In the following we will use Theorem 5.2 to the strategic resource allocation on the voter model of social networks.

### 5.1.2 Voter model

Each potential customer changes its opinion by randomly selecting one of his neighbor’s opinion and adopting his neighbor’s opinion. As we proposed before, in the threshold models once a node is activated, it never change its state. However, the voter model allows to change preferences.

Let $G = (V, E)$ be an undirected graph with selfloops where $G$ has $n$ nodes, i.e. $|V| = n$. The players of the game are the competing marketing campaigns $X$ and $Y$, and the nodes of the graph correspond to the potential customers. We use function $f_0$ to label a node $v \in V$ by its initial preference between two players. We denote by $f_0(v) = 1$ when node $v \in V$ prefers the product promoted by marketing campaign $X$, $f_0(v) = 1$ when node $v$ prefers the product promoted by marketing campaign $Y$, and $f_0(v) = 0$ when node $v$ is indifferent between both products.

We assume that the initial opinion assigned to node $i$ is determined by the amount of marketing budget, i.e.,

$$f_0(i) = \begin{cases} 
1 & \text{if } x_i > y_i, \\
0 & \text{if } x_i = y_i, \\
-1 & \text{if } x_i < y_i.
\end{cases}$$
We notice that \( f_0(i) \) corresponds to \( \text{sgn}(x_i - y_i) \).

The evolution of the system will be described by the voter model. Starting from any arbitrary initial preference assignment to the vertices of \( G \), at each time \( t \geq 1 \), each node selects uniformly at random one of its neighbors and adopts its opinion. More formally, starting from any assignment \( f_0 : V \to \{1, 1\} \), we inductively define

\[
f_{t+1}(v) = \begin{cases} 
1 & \text{with prob. } \frac{|\{u \in N(v) : f_t(u) = 1\}|}{|N(v)|} \\
-1 & \text{with prob. } \frac{|\{u \in N(v) : f_t(u) = -1\}|}{|N(v)|}.
\end{cases}
\]

Now we can define the strategic resource allocation problem.

Let \( G = (V, E) \) be a graph representing a social network with \( n \) nodes. There are two players, denoted \( X \) and \( Y \), and \( n \) potential consumers. The players of the game are the competing marketing campaigns. Both players have the same total budget \( B \) which need to be simultaneously allocated to potential customers. If \( x_i > y_i \) (or \( x_i < y_i \)), a node \( i \in V \) will choose \( X \) (or \( Y \)). Otherwise, if \( x_i = y_i \), it will flip a coin to decide between both marketing campaigns. Then the strategic resource allocation problem is the problem of finding an initial assignment of resources \( x = (x_1, x_2, ..., x_n) \) for player \( X \) that will maximize at a target time \( \tau \) the expected number of nodes:

\[
\mathbb{E}[\sum_{v \in V} f_{\tau}(v)],
\]

subject to the budget constraint

\[
\sum_{v \in V} x_v \leq B.
\]

5.1.3 Application

The authors in [27] characterize the optimal strategic allocation of resources for the voter model of social networks by using Colonel Blotto games. In case where the intrinsic value of customers is the only consideration, the optimal long-term strategy is to choose from marginal distributions uniform \( x_i^{(1)} \sim U(0, 2/n) \). If we consider the network value of the customers, the optimal strategy is to choose from marginal distributions uniform \( x_i^{(2)} \sim U(0, d_i/|E|) \).

Later, the work in [28] calculates the best response function for player \( i \) considering the network value of each customer at a target time \( \tau \), given that the strategy of the opponent \( i \) is \( x_{-i} \),

\[
x_{i,j}^* = -x_{-i,j} + (B_i + B_{-i}) \frac{\sqrt{v_{i,j} d_{-i,j}}}{\sum_{k=1}^n \sqrt{v_{i,k} d_{-i,k}}},
\]

They extended the problem to the case of more than two players and presented the voter model with ranking scores in [29], finding a closed form expression of the symmetric equilibrium offer strategy for the marketing campaigns from which no campaign has any incentive to deviate.
In both works, the action space considered is continuous, which means that any fraction of the budget can be allocated to a node. In addition, all players have an equal amount of budget.

5.2 Static Game Model

In [27] the authors assume an undirected graph and a voter model for opinion dynamics resulting in strategies that are independent of the node centralities, i.e., the agent influence power. A marketing resource allocation based on the influence power that each individual has over the social network is reported in [42].

5.2.1 Game Definition

There are \( N \) agents \( V = \{1, ..., n\} \) in a social network. Let \( G = (V, E, \Omega) \) be a fixed weighted directed graph, with \( \Omega \) representing the matrix of corresponding weights. We assume that there are two competing Firms 1 and 2 producing their products. Let the normalized scalar opinion \( x_n(t) \in (0, 1) \) denotes the opinion of agent \( n \) in favor of the product of Firm 1 at time \( t \). Therefore, for agent \( n \) at time \( t \), the revenue of Firm 1 is proportional to \( x_n(t) \) and for Firm 2 the revenue is proportional to \( 1 - x_n(t) \). Denote by \( x(t) = (x_1(t), x_2(t), ..., x_N(t))^{\top} \) the state of the network at any time \( t \), where \( x(t) \in \mathcal{X}_0 \) and \( \mathcal{X}_0 = (0, 1) \). We use \( x_{n;\text{i}}(t) \) to denote the opinion of agent \( n \) in favor of Firm \( \text{i} \), with \( x_{n;\text{i}}(t) = x_n(t) \) and \( x_{n;\text{2}}(t) = 1 - x_n(t) \).

A game follows \( G = ([1, 2], \{A_1, A_2\}, \{u_1, u_2\}) \). The action space for Firm \( \text{i} \) is

\[
A_i := \left\{ a_i \in [0, b_i]^{\mathbb{N}} \mid \sum_{n=1}^{N} a_{i,n} \leq B_i \right\}
\]

where \( b_i \leq B_i \in \mathbb{R} \geq 0 \) for \( i \in \{1, 2\} \) denote the maximum influence/discount for one specific agent and the total budget, respectively. The vector \( a_i, i \in \{1, 2\} \) is the action of Firm \( i \), with \( a_{i,n} \) being the marketing expenditure targeted at agent \( n \). The campaign update the opinions of agents according to a function \( \Phi(x_0, a_1, a_{-1}) : \mathcal{X}_0 \times A_1 \times A_{-1} \rightarrow \mathcal{X}_0 \) where \( x_0 = (x_{0,1}, ..., x_{0,N})^{\top} \in \mathcal{X}_0 \) is the vector collecting the initial opinions of the agents before the campaign in favour of Firm \( i \), i.e. \( x_0 = x(0) \). Without any loss of generality, we assume that the campaign occurs at \( t = 0 \). In the sequel we consider the function

\[
\phi(x_{0,1;i}, a_{i,1}, a_{-i,1}) = \frac{x_{0,n;i} + a_{i,n}}{1 + a_{i,n} + a_{-i,n}}, \quad \forall n \in \{1, ..., N\}.
\]

After the marketing campaign, the opinions of consumers are only influenced by the other consumers of the networks, characterized the following model:the

\[
\begin{align*}
\dot{x}(t) &= -Lx(t) \\
x_n(0^+) &= \phi(x_{0,n}, a_{1,n}, a_{2,n}) \quad \forall t \in \mathbb{R}_{\geq 0} \setminus \{0\} \quad \forall n \in V
\end{align*}
\]

20
where $L \in (R)^{N \times N}$ is the Laplacian matrix over the graph.

We assume that the profit is based only on the opinion the agents have after some time $T > 0$. The agent influence power (AIP) of Agent $n$ is given by $\rho_n > 0$

$$\rho = 1_N^T \exp(-LT)$$

where $1_N$ is a column vector of ones. Thus, the total payoff of Firm $i$ is the difference between the profit and the marketing expenses resulting in

$$u_1(x_0, a_1, a_2) := \gamma_1 \rho x(0^+) - \lambda_1 1_N a_1,$$

$$u_2(x_0, a_1, a_2) := \gamma_2 \rho [1_N - x(0^+)] - \lambda_2 1_N^T a_2.$$

where $x(0^+) = \phi(x_{0, n}, a_{1, n}, a_{2, n}), \gamma_i \geq 0$ is the revenue generated per consumer for Firm $i$, and $\lambda_i \geq 0$ is the advertising efficiency or pricing factor for Firm $i$.

### 5.2.2 Game-theoretic Analysis

**Definition 3** A strategy profile $(a^*_1, a^*_2) \in A_1 \times A_2$ is a pure NE for $G$ for a given $x_0$ if $\forall i \in \{1, 2\}$,

$$\forall a_i \in A_i, u_i(x_0, a^*_i, a_{-i}) \geq u_i(x_0, a_i, a^*_{-i})$$

where $i$ stands for the other firm than $i$.

In [42] the authors combine the results in [33] to prove that the game $G$ has a pure and unique NE.

The top plot of Figure 2 shows the AIP $\rho_n$ and initial opinion $x_n(0)$ for agent $n$. The amount of budget allocated to agent $n$ at NE is proportional to $\rho_n$. If the initial opinion of agent $n$ already favours Firm $i$, the amount of budget allocated to agent $n$ at NE will decrease. Intuitively, if $x_n(0)$ is closer to 1, that is, initially biased towards Firm 1, then Firm 2 will invest more to bring it closer to 0 while a starting opinion close to 0 makes Firm 1 invest more. Both firms invest more on agents with a larger AIP $\rho_n$.

The authors [41] consider long-term utilities which result from averaging stage utilities over the $K$ stages to conduct a long-term analysis. For the one-shot game, it is possible to show the best response. Therefore, by intersection, the one-shot Nash equilibrium actions can be obtained. This is not possible to do so when it comes to strategies, which are (possibly infinite) sequences of functions. They analyze the long-term performance of the repeated NE strategy and propose a coopetition marketing strategy which outperforms the one-shot NE strategy. The proposed coopetition plan consists of two phases, the first phase is composed of all the stages $k \in \{1, 2, ..., K_1\}$ and the second phase lasts for the remaining duration, i.e. $k \in \{K_1 + 1, ..., K\}$. During the first phase, both marketers repeatedly play the one-shot NE. Then, in phase two, the players switch to a non-aggressive operating point such that no marketing is performed.

In Figure 3, it shows that for any $1 \leq K_1 < 5$, both marketers obtain a better long-term utility by stopping their marketing after $K_1$ campaigns. The proposed coopetition strategy profile Pareto-dominates the solution of [42]. (For theoretical details see [41].)
Figure 2: The allocation of budget by the two players at the NE.

Figure 3: The long-term utilities for the two marketers.
6 Conclusion

We have reviewed and analyzed some basic game models in competitive viral marketing. The areas of competitive viral marketing have attracted a great deal of attention, since many companies take advantage of viral marketing to maximize the influence of their products or information. The game modelings, results, and analysis surveyed in this report provide a framework for reasoning about the diffusion of products or information on social networks. The social network is represented as a graph with nodes corresponding to potential consumers. The players of the game are competing marketers who try to maximize the influence of their products or information over the social network. The problem can be divided into two scenario: seed selection and budget allocation. In the first case, the competitors need to chose the initial nodes which will make maximum influence. In the other case, the competitors decide how many resources to allocate to a node. Most existing studies focus on the seed selection problem, however, only a few studies analyze the budget allocation scenario.

In reality, there are more than two marketers. However most of the frameworks proposed only apply to two players rather than $n$ players. There is a lot of work left to be done to really utilize the power of these results and make a real social impact.
Part II
Effectiveness of Mitigation Measures against the Covid-19 Epidemic
1 Introduce

In order to mitigate the epidemic, the government has adopted a series of measures. We measure the quality of the lockdown measures taken by a government to mitigate the macroeconomical and health impact of the Covid-19 epidemics. The quality is measured in terms of trading off the Gross Domestic Product (GDP) loss, the total number of infected people over a given period of time, and the maximum number of people requiring Intensive Care Units (ICU). We use the standard SEIR (susceptible-exposed-infected-removed) model for an entire country and propose a model for capturing the tradeoff between health and economic aspect. Unlike existing works, we consider 3-phase epidemics control policies that is, the reproduction number $R(t)$ is assumed to be a piecewise constant function (see Fig. 4).

Figure 4: One of the goals of this work is to determine numerically, for a given tradeoff between health and economic aspects, the best 3-phase lockdown policy that is, the best values for $\tau_0$, $\tau_1$, $R_1$, and $R_2$ (the epidemic time horizon $T$ being given and the natural reproduction number $R_0$ being fixed).
2 Problem statement

2.1 Epidemic model

We apply the SEIR epidemiological model to the recent SARS-CoV-2 outbreak in the world, with a focus on France. The total population is partitioned into the following compartments:

- $s$ is the fraction of susceptible individuals,
- $e$ is the fraction of exposed individuals (those who have been infected but are not yet infectious),
- $i$ is the fraction of infectious individuals,
- $r$ is the fraction of recovered individuals.

Note that the variables give the fraction of individuals, that is, we have normalized them so that

$$s(t) + e(t) + i(t) + r(t) = 1$$

The epidemics is assumed to obey the following time-continuous dynamics:

$$\frac{ds}{dt}(t) = -\beta(t)i(t)s(t)$$
$$\frac{de}{dt}(t) = \beta(t)i(t)s(t) - \gamma e(t)$$
$$\frac{di}{dt}(t) = \gamma e(t) - \delta i(t)$$
$$\frac{dr}{dt}(t) = \delta i(t)$$

(1)

$$s(t) + e(t) + i(t) + r(t) = 1$$

where:

- $\beta(t)$, $t \in \mathbb{R}$, denotes the time-varying virus transmission rate,
- $\gamma$ denotes the rate constant at which the exposed individuals become infected. The period $\frac{1}{\gamma}$ is called an incubation period,
- $\delta$ denotes the recovery rate and $\frac{1}{\delta}$ is called the average recovery period.

We assume that the decision-maker (the government or possibly a more local decision-maker) can control the transmission virus rate according to the following linear relation:

$$\beta(t) = R_0\delta - u(t)$$

(2)

where

- $R_0$ is the basic reproduction number,
• \( u(t) \in [0, U] \) is the control action or severity level. \( U \) is the most drastic or severe control action, it could theoretically reach the value \( R_0 \delta \) and make the reproduction number vanishing. \( u(t) \) is assumed to be a piecewise-constant function.

We define the time-varying reproduction number:

\[
R(t) = \frac{\beta(t)}{\delta} = R_0 - \frac{u(t)}{\delta}.
\]

### 2.2 Decision-maker behavior model

The behavior model we proposed for the decision-maker is to implement a given tradeoff between economic and health aspects. We assume that economic cost is quadratic in the control action and health cost is derived from the number of infected people over the given period of time. Therefore, the proposed overall cost consists of a convex combination of these two costs. By minimizing the combined cost, one realizes the desired tradeoff. The proposed behavior model for the decision-maker consists in assuming that it wants to find a solution to the following optimization problem:

\[
\begin{align*}
\min_u & \quad \alpha K_e \left\{ \int_0^{\tau_0 + \tau_1} u^2(t)dt + \frac{1}{\mu^2} \int_{\tau_0 + \tau_1}^T u^2(t)dt \right\} + (1 - \alpha)K_h [s(0) - s(T)] \\
\text{s.t.} & \quad \forall t \in [0, T], \sigma N_i(t) \leq N_{\text{ICU}}^{\text{max}} \\
& \quad \text{Equations (1) and (2)}
\end{align*}
\]

where:

• \( 0 \leq \alpha \leq 1 \) is the weight assigned to the macroeconomical impact of the epidemics;

• \( K_e > 0 \) and \( K_h > 0 \) are constants that weight the economic and health cost functions (they act as conversion factors allowing one to obtain appropriate units and orders of magnitude);

• \( \tau_0 \) and \( \tau_1 \) respectively represent the lockdown starting time and duration;

• the parameter \( \mu \geq 1 \) accounts for possible differences in terms of economic impact between the lockdown and after-lockdown phases;

• \( s(0) \) and \( s(T) \) are respectively the fractions of the population infected at the beginning and the end of the analysis;

• \( N \) is the population size, \( N_{\text{ICU}}^{\text{max}} \) represents the maximum number of ICU patients, and \( 0 \leq \delta \leq 1 \) is the percentage of infected people requiring intensive care.
We use \( \Delta \text{GDP} \) to denote the GDP loss over the lockdown period for a given country. We have that:

\[
K_e = \frac{\Delta \text{GDP}}{\tau_1 \delta^2 (R_0 - R_1)^2}
\]

We focus on specific categories of control or lockdown policies, which can be solved through exhaustive search. By using the relation \( u(t) = \delta [R_0 - R(t)] \), the OP (4) can be rewritten under a more convenient form for numerical purposes:

\[
\begin{align*}
\min_{(\tau_0, \tau_1, R_1, R_2)} & \quad \alpha K_e \delta^2 (R_0 - R_1)^2 \tau_1 + \frac{\alpha K_e \delta^2 (R_0 - R_1)^2 [T - (\tau_0 + \tau_1)]}{\mu^2} + (1 - \alpha) K_h [s(0) - s(T)] \\
\text{s.t.} & \quad \forall t \in [0, T], \sigma N_i(t) \leq N_{\text{ICU}}^{\text{max}} \\
& \quad \text{Equations (1) and (2)}
\end{align*}
\]

\( 3 \) Numerical analysis

3.1 General simulation setup

We need to quantify the time and amplitude to implement exhaustive search for the quadruple of variables \((\tau_0, \tau_1, R_1, R_2)\). Time is discretized with a step of 2.4 hours, that is, with 10 samples for each day. A time horizon is \( T = 180 \) days from March 1 to August 31, which is approximately equivalent to six months. The possible lockdown starting day, the lockdown duration, and the reproduction numbers are set as follows: \( \tau_0 \in \{1, 2, ..., 30\} \), \( \tau_1 \in \{1, 2, ..., 120\} \), \( R_1 \in \{0.1, 0.2, ..., 3.5\} \), \( R_2 \in \{0.1, 0.2, ..., 3.5\} \). Denote \( \delta = 0.1875 \) such that \( \frac{1}{\delta} = 5.4 \) days. Denote \( \gamma = 0.16 \) and \( \frac{1}{\gamma} = 6.25 \) days. The set for population size is \( N = 66.10^6 \) and the maximum number of ICU patients is \( N_{\text{ICU}}^{\text{max}} = 15.10^3 \) and \( \sigma = 1.5\% \). Notice that this number is only reached for very small values of \( \alpha \) (for which the total number of people infected over the analysis duration would be around 9 millions). The set for initial number of infected people on March 1 2020 is \( N_i(0) = 1.33.10^5 \). For France, OFCE estimates the GDP loss over the lockdown period at 120 billions \( \text{€} \) and we have that \( \tau_{\text{France}} = 55 \) days with \( R_0^{\text{France}} = 3.5 \) and \( R_1^{\text{France}} = 0.6 \). From (6), we have \( K = 7.379.10^6 \) \( \text{€} \).

3.2 Optimal tradeoff between economic and health impacts

Giving a value to the parameter \( \alpha \) is equivalent to implementing a given tradeoff between economic costs and health costs. Economic costs and health costs are measured by GDP loss and total number of infected people respectively.

For \( \mu = 1.41 \), Figure 5 shows for various values of \( \alpha \) in the interval \([10^7, 10^4]\) the total GDP loss and number of infected people that is obtained after choosing the quadruplet \((\tau_0, \tau_1, R_1, R_2)\) that minimizes the combined cost given by Equation (5). When \( \alpha \) is relatively large and equal to \( 10^4 \), it is clear that the best lockdown strategy leads to a GDP loss over the entire study period [March
1, August 31] of 125 billions € with about 9.5 millions infected, and 15 000 patients requiring intensive care. When \( \alpha \) is relatively small (\( \alpha = 10^7 \)), the GDP loss reaches values as high as 260 billions € with a total number of newly infected people over the period [March 1, August 31] as low as 48 180.

With our model, the French government policy leads to the GDP loss over the entire period of time of interest is 215 billions € and the total number of infected people is about 3.7 million. However, we can see that the GDP loss and the number of infected people can be reduced by adopting the optimal tradeoff shown in the blue curve. We show a point where the number of infected people could be less than 1 million people with a total GDP loss of 157 billions €. The number of infected people is only a quarter of the expected value from the current policy and the GDP loss is about 30% less than the current policy.

### 3.3 Optimal lockdown starting time

We study the features of the optimal lockdown policy which implements a given tradeoff between economic and health costs. We consider two values of \( \mu \):

1. \( \mu = 1.41 \) means that the economic cost associated with a given intensity or severity level is lower after lockdown than during it;
2. \( \mu = 1 \) means that the economic losses are considered to be uniform over time.

Figure 6 indicates the optimal day to start the lockdown, for one hundred values of \( \alpha \) ranging from
10^{-4} to 10^{-6}. For \( \mu = 1.41 \), the lockdown should be implemented immediately, which means starting from March 1 (versus March 17 in France). For \( \mu = 1 \), there is some economic incentive to delay the lockdown, but for the actual range of \( \alpha \), this delay is considered to be at most 3 days. By March 17, there were official figures about the epidemic which were sufficiently critical to make the population accept the measures whereas, starting on March 4 (the optimal starting date for \( \mu = 1 \)) the situation might have not been critical enough to create full adhesion to government measures.

### 3.4 Optimal lockdown duration

For \( \mu = 1 \), \( \mu = 1.22 \), and \( \mu = 1.41 \), Figure 7 shows the best lockdown duration (in days) for values of \( \alpha \) ranging from 10^{-4} to 10^{-6}. For \( \mu = 1 \), the optimal duration ranges from 67 days to 85 days for the considered interval of \( \alpha \). For \( \mu = 1.22 \) and \( \mu = 1.41 \), i.e., the economic losses is smaller after lockdown, the optimal lockdown duration should be shorter. Actually, our results shows that the lockdown phase should be absent if the values for the tradeoff parameter \( \alpha \) exceed a critical value \( \alpha \). This means that the optimal control consists in having only one phase instead of three: such a phase should start as soon as possible and would typically last 5-6 months. Here, we assume that the epidemic would disappear naturally, e.g., because of seasonal effects. All of these observations hold under an optimal choice for the lockdown starting time. If the lockdown
phase starts too late, this approach can no longer be adopted.

3.5 Optimal severity levels

Figure 8 shows the number of infected people over a period of six months and the associated 3-phase reproduction number for the French government policy. The reproduction number values of the first two phases come from past and quite accurate evaluations, the value $R_0 = 0.9$ corresponds to the assumption that the people will try their best to avoid a second wave.

Figure 9 shows the number of infected people and the reproduction number under optimal lockdown policy, i.e., $\mu = 1$ and $\alpha = 5.10^{-5}$. It is clear that the optimal lockdown starting time delays 3 days. Then the severity level of the lockdown phase should be as high as possible, here $R_1 = 0.1$. In France, such a value can be achievable by wearing face masks, maintaining a safe distance from people, digital contact tracing. At the end of the second phase it naturally reaches the lowest number of infected people 48. This would then be followed by a third phase, in which the number of reproduction should be approximately 2.1 and can be obtained through natural intervals and appropriate equipment. This means that a second wave may occur, but this wave may be damped by using appropriate detection, quarantine measures, and seasonal effects.

Figure 7: The optimal lockdown duration days.
Figure 8: The number of infected people over a period of 6 months and the associated 3-phase reproduction number profile for the French government policy.

Figure 9: Optimal lockdown policy and corresponding dynamics for the Covid-19 epidemic.
4 The multiple decision-maker scenario

We consider two coupled subsystems because it corresponds to the situation in which two regions take different control/lookdown measures. In this case the spreading of the virus is different too. We are interested in:

- the effects of interaction between decision-makers controlling each a geographical area (region or country);
- the effects of the presence of multiple performance criteria (that is, accounting for tradeoffs);
- the effects of uncertainty (asymptomatics, seasonablity) on the decision-making.

We assume an SEIAR model with no randomness:

- the fraction of asymptomatic cases is assumed to be zero;
- the influence of weather conditions of the infection rate is assumed to be completely negligible;
- no reinfection is allowed.

For Region 1, the epidemics is assumed to obey the following time-continuous dynamics:

\[
\begin{align*}
\dot{s}_1 &= - (\beta_{11} i_1 + \beta_{12} i_2 + a_1) s_1 \\
\dot{e}_1 &= (\beta_{11} i_1 + \beta_{12} i_2 + a_1) s_1 - \gamma_1 e_1 \\
\dot{i}_1 &= p_1 \gamma_1 e_1 - \delta_1 i_1 \\
\dot{a}_1 &= (1 - p_1) \gamma_1 e_1 - \epsilon_1 a_1 \\
\dot{r}_1 &= \delta_1 i_1 + \epsilon_1 a_1
\end{align*}
\]

where:

- \( \beta_{11} \) represents the transmission rate within Region 1;
- \( \beta_{12} \) represents the equivalent transmission rate regarding Region 2;
- \( \gamma_1 \) denotes the probability rate at which the exposed subject develops clinically relevant symptom. The period \( \frac{1}{\gamma_1} \) is called an incubation period;
- \( p_1 \) represents the probability of developing symptoms;
• $\delta_1$ and $\epsilon_1$ respectively, denotes the remove rate, so $\frac{1}{\delta_1}$ is the average period to isolation for infected symptomatic subjects, and $\frac{1}{\epsilon_1}$ is the average recovery period for asymptomatic subjects. We assume that $\frac{1}{\epsilon_1} > \frac{1}{\delta_1}$ since the Covid-19 patients are treated or isolated very quickly after developing symptoms.

We assume a local decision-maker can control the transmission rate according to the following linear relation:

$$\beta_{ii}(t) = b_{ii} - u_{ii}(t)$$

where $b_{ii}$ is the transmission rate without control and $u_{ii}(t) \in [0, U]$ is the control of Decision-Maker $i$, $U > 0$ corresponding to the most drastic lockdown measures the DM could take. We assume he control is assumed to be piecewise constant:

$$u_{ii}(t) = a_{i,k} \text{ for } t \in [t_k, t_{k+1})$$

We assume that $t_{k+1} = t_k + \Delta t$ where $\Delta t$ could e.g., represent the duration of a day, a week, or a month.

The cost function of decision-maker $i$ is assumed to implement a tradeoff between sanitary and socio-economical aspects:

$$J_i(a_1, a_2) = \alpha_i \int_0^T u_{i_1}^2(t) dt + \int_0^T i_i(t) dt$$

where $a_i = (a_{i,1}, ..., a_{i,K})$, $K$ being the total number of actions taken over the epidemics span (that is $T = K \Delta t$); $\alpha_i \geq 0$ is just there to capture the tradeoff. Then, we might add constraints on the number of infected people or so.

The goal is to study the Nash equilibrium of this game. We consider the influence of the following parameters: the time at which the epidemics starts in a given region $\tau_i$ and of course the intrinsic/natural/spontaneous propagation speed within Region $i$ which is given by $b_{ii}$. Then one interesting result would be to derive the best interconnection policy given by:

$$(\beta_{12}(t)^*, \beta_{21}(t)^*) \in \arg \min_{\beta_{12}(t), \beta_{21}(t)} \int_0^T i_1^{NE}(t) + i_2^{NE}(t) dt$$

where $i_1^{NE}(t)$ corresponds to the/a Nash equilibrium of the game $(J_1, J_2)$.

Then, we should introduce the effect of randomness (e.g., asymptomatics) on the results. Note that the effect of weather conditions can be seen as a time-varying multiplicative function that is not controllable but might be assumed to be estimated:

$$\beta_{ii} = b_{ii}(t) - w_i(t)$$

where $w_i$ is a random process which models the weather effects. For example, it could be a noisy weather forecast $w_i(t) = f_i(t) + z_i(t)$ ($f_i$ increasing linearly with the average day ambient temperature).
5 Conclusion

We propose a behavior model for decision-maker that implements a given tradeoff between economic and health aspects. By modeling the Covid-19 epidemic with a standard SEIR model, we derive the optimal parameters of the 3-phase lockdown policy. Our analysis shows that it would be possible for France to have 4 times less than expected number of infected people, while saving about 30% Gross Domestic Product loss with the current policy. When economic losses are uniform over the entire period (6 months), the lockdown can be delayed by up to 3 days. The optimal lockdown duration ranges from 60 days to 80 for the possible interval tradeoff, which is close to the lockdown duration in France (55 days). When economic cost is lower after lockdown, it is optimal to start locking down as soon as possible. The optimal lockdown duration should be shorter and even absent if the economic aspect becomes more important.

Most existing studies focus on single decision maker scenario. For future work, we plan to generalize the case to multiple decision-maker scenario and try to characterize Nash equilibrium in corresponding game. Furthermore, we will consider the effect of weather on Covid-19 epidemic.
References


