Identification of the feasible set of initial conditions for the collision avoidance of two satellites

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Abstract

In this thesis the collision avoidance problem of two satellites is considered using the Hill-Clohessy-Wiltshire equations. A new approach to the computation of the viability kernel, based on the maximum principle, is applied to solve this problem. This method, which was never used in practice, is compared to a well-established approach based on dynamic programming.

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1 Introduction

Unmanned vehicles (UV) are widely used and their fields of application will further increase. However, the more UVs are in use, the greater is the risk of collisions. In space, the history of unmanned vehicles dates back to 1957, when the first satellite was launched. Today the number of operational satellites in space is estimated to be 1,900 and this number is likely to rise extremely fast due to the recent growing interest in mega-constellations, consisting of hundreds or thousands of satellites working together to fulfill a certain task. With the increasing number of satellites there is a great need of reliable systems for collision avoidance.

The classical approach to collision avoidance starts with the intuitive idea of a safety volume $C$ which shall not be penetrated by any other spacecraft. The mathematical formulation of the safety and collision regions is related to the viability analysis for a dynamical system associated to the safety volume. This analysis seeks to find all the initial points that allow to always remain outside of the safety volume for a certain available control.

More precisely, let us consider a certain number of spacecrafts, each of which has its proper propulsion system. The state at a certain time consists of the current position and velocity of the spacecraft. Given all the initial states, the goal is to find controls such that all the spacecrafts stay in a given domain and also respect a certain minimum distance to each other. The controls are bounded by a given value $\epsilon$. Let

$$\dot{\xi} = f(x, u), \|u\| \leq \epsilon$$

be a ‘sufficiently regular’ controlled system. The collision zone $\mathcal{C}$ is defined by

$$\mathcal{C} = \{\xi = (x_i, v_i)_{i=1,...,n} \in (\mathbb{R}^{2d})^n : \forall i \neq j \|x_i - x_j\| \leq \epsilon, \forall i \|x_i\| \leq \Delta\},$$

for a given $\Delta > 0$, where $n$ is the number of spacecrafts and $d$ is the dimension of the space in position and velocity. We consider an initial point $\xi_0 \in (\mathbb{R}^{2d})^n \setminus \mathcal{C}$. We search an admissible control $u(\cdot)$ such that the associated trajectory $\xi(\cdot)$ remains in $(\mathbb{R}^{2d})^n \setminus \mathcal{C}$ until a given time $T$:

$$\begin{align*}
(i) \quad &\xi(0) = \xi_0, \quad \dot{\xi}(t) = f(\xi(t), u(t)) \text{ p.p.} \\
(ii) \quad &u(t) \in U \quad \forall t \in [0, T] \\
(iii) \quad &\xi(t) \in (\mathbb{R}^{2d})^n \setminus \mathcal{C} \quad \forall t \in [0, T]
\end{align*}$$

Since the control is bounded, there are clearly initial states (initial velocity sufficiently huge) where the collision is impossible to avoid.

Figure 1: initial state that leads to a collision
Problem definition

All of the definitions in this chapter are taken out of [2].

**Definition 1.** A target or smooth target of $\mathbb{R}^n$ is an open set, whose boundary is a smooth submanifold of $\mathbb{R}^n$ of dimension $n - 1$.

**Definition 2.** Let $u$ be a measurable function of $\mathbb{R}$ in $\mathbb{R}^m$. We call $u$ an admissible control if it verifies
\[ \|u(t)\| \leq \epsilon \] for every $t \geq 0$. We denote the set of admissible controls by $\mathcal{U}_\epsilon(\mathbb{R}^m)$.

Let us consider a target $\Omega \subset \mathbb{R}^n$ and the linear controlled system
\[ \dot{x}(t) = Ax(t) + Bu(t), \] where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$ and $u \in \mathcal{U}_\epsilon(\mathbb{R}^m)$. We denote by $x(t)$ the absolutely continuous solution of (D) associated to the control $u$ and a starting point $x_0$ which verifies $x_0 = x_0, u(0) = x_0$. Note that the existence and uniqueness of this solution is a well-known result.

**Definition 3.** The reachable set $\mathcal{R}(x_0,t)$ at time $t$ starting from a point $x_0$ at time 0 for the system (D) is defined by
\[ \mathcal{R}(x_0,t) = \{x_0, u(t) \in \mathbb{R}^n : u \in \mathcal{U}_\epsilon(\mathbb{R}^m)\}. \]

**Definition 4.** Let $\Omega$ be a subset of $\mathbb{R}^n$. We define the viability kernel associated to the dynamics (D) as
\[ \text{Viab}(\Omega) = \{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U}_\epsilon(\mathbb{R}^m) : x_0, u(\mathbb{R}^+) \in \Omega\}. \] Equivalently the viability kernel until a given time $T > 0$ is defined as
\[ \text{Viab}_T(\Omega) = \{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U}_\epsilon(\mathbb{R}^m) : x_0, u([0,T]) \in \Omega\}. \]

Space mechanics

3.1 Relative orbit description

The following chapter is based on [3]. Our task is to avoid a collision between two satellites. Therefore we are interested in the trajectory of one satellite relative to the other one, rather than the actual trajectories of the two satellites. To achieve this relative orbit description we have to introduce a new set of coordinates.

We start by choosing one satellite about which the motion of the other satellite is referenced and call it the chief satellite. The other is referred to as the deputy satellite. The position of the chief satellite at time $t$ is denoted by $r_c(t)$, while the deputy satellite position is given by $r_d(t)$. We introduce the so called Hill coordinate frame which expresses how the deputy satellite orbit is seen by the chief.
origin of this coordinate system is the chief satellite position. The orientation of the axes is given by the vectors $\hat{o}_r$, $\hat{o}_\theta$ and $\hat{o}_h$ as shown in figure 2.

Figure 2: relative orbit description (from [3])

$\hat{o}_r$ goes in the same direction as $r_c$. $\hat{o}_h$ points in the direction of the normal vector on the orbital plane and $\hat{o}_\theta$ is then simply given as the cross product of these two vectors. All of these frame orientation vectors are normalized at 1.

\[
\begin{align*}
\hat{o}_r &= \frac{r_c}{\|r_c\|} \\
\hat{o}_\theta &= \hat{o}_r \times \hat{o}_h \\
\hat{o}_h &= \frac{r_c \times \dot{r}_c}{\|r_c \times \dot{r}_c\|}
\end{align*}
\]

The relative orbit position vector of the deputy satellite is expressed in hill frame coordinates as

\[
\rho = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

Hereby the $(x, y)$ coordinates describe the relative position of the deputy in the chief orbit plane. The $z$ coordinate describes any displacement out of the chief orbit plane.

### 3.2 Hill-Clohessy-Wiltshire equations

We assume that all orbits in this chapter are keplerian orbits which means that they can be parametrized into six orbital elements, as explained in [3]. We are in
need of a description of the relative orbit in terms of the cartesian coordinate vector \( \rho = (x, y, z)^T \) in the rotating Hill frame. The derivation of the equations describing the relative orbit requires some more background in orbital mechanics and won't be detailed in this report. The exact nonlinear relative equations of motion are given by

\[
\ddot{x} - 2 \dot{f} \left( \dot{y} - y \frac{\dot{r}_c}{||r_c||} \right) - x \dot{f}^2 - \frac{\mu}{r_c^3} (r_c + x) = 0 \tag{10a}
\]

\[
\ddot{y} + 2 \dot{f} \left( \dot{x} - x \frac{\dot{r}_c}{||r_c||} \right) - y \dot{f}^2 = -\frac{\mu}{r_c^3} y \tag{10b}
\]

\[
\ddot{z} = -\frac{\mu}{r_c^3} z, \tag{10c}
\]

where \( \mu \) is the gravitational parameter and \( f \) is the angular velocity. We could use these equations to solve our problem, but simplifications can be made by making some additional assumptions. In the following we assume that the relative orbit coordinates \((x, y, z)\) are small compared to the chief orbit radius \(||r_c||\). This assumption is legitimate in our case, since a collision analysis only makes sense if the two satellites are close to each other. Therefore assuming that their distance to each other is small compared to the distance of the chief satellite to the earth is not a huge concession for us. Furthermore we assume that the chief satellite orbit is circular, meaning that the eccentricity \( e \) equals zero and the chief orbit radius \(||r_c||\) is constant. For a circular orbit the time derivative of the true anomaly \( \dot{f} \) is equal to the mean orbital rate \( n \). Taking in account all these assumptions the relative orbital equations reduce to the simple form

\[
\ddot{x} - 2n \dot{y} - 3n^2 x = 0 \tag{11a}
\]

\[
\ddot{y} + 2n \dot{x} = 0 \tag{11b}
\]

\[
\ddot{z} + n^2 z = 0 \tag{11c}
\]

These equations are also known as the Hill-Clohessy-Wiltshire (HCW) equations. In the following we want to use them as the dynamics for our control problem. From now on we will suppose that \( n = 1 \). Changing the mean orbital rate normally should not lead to a change of complexity of the problem, at least numerically. Furthermore note that equation 11c is independent of the other two. In addition this differential equation is very easy to solve. We will suppose \( z = 0 \) to reduce the dimension of our problem. Therefore we define a state variable

\[
\xi = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}. \tag{12}
\]

The dynamics can then be written as

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}. \tag{13}
\]
Defining the matrix as
\[ A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \] (14)
we can write the dynamics in the short form
\[ \dot{\xi} = A\xi. \] (15)
If we add a control \( u = (u_1, u_2) \) to this dynamics it becomes
\[ \dot{\xi} = A\xi + Bu, \quad \|u\| \leq \epsilon, \] (16)
where
\[ B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (17)

4 Uncontrolled Case

We consider the uncontrolled Hill-Clohessy-Wiltshire equations, given by
\[ \frac{d}{df} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}. \] (18)
Suppose that \( \rho > 0 \) is given and \( C_\rho = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | \sqrt{x^2 + y^2} \leq \rho\} \).

Studying the case without control is an interesting consideration itself and can also deliver insight into the controlled case. In the following we will mainly use polar coordinates to represent the state.

\[ (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 \rightarrow (r, \theta, v, \gamma) \in (\mathbb{R} \times [0, 2\pi])^2, \]
where \((r, \theta)\) represents the position and \((v, \gamma)\) represents the velocity. In the uncontrolled case we only need to know the starting point of a trajectory in order to deduct its position and velocity at any time \( t > 0 \). Since there exists an explicit solution of the state transition matrix for the HCW-equations it is not even an integration needed to compute the trajectory. See [4] for more information on this explicit solution. Let us denote the starting point by
\[ \xi_0 := (r_0, \theta_0, v_0, \gamma_0). \] (19)
The state at any time \( t > 0 \) is then simply given by
\[ \xi(t) = M(t)\xi_0, \] (20)
where $M$ is the state transition matrix. Our practical idea starts by projecting the obstacle $C_\rho$ onto its position variables. We then consider a circle in dimension 2 around the origin with radius $R > \rho$ and choose a certain number $N$ of points along this circle. So $r_0 := R$ is fixed and $\theta_0$ takes different values in $[0, 2\pi]$. Furthermore we consider an upper bound $v_{\text{max}}$ for the initial velocity. We let $\gamma_0$ vary over $[\theta_0 + \frac{\pi}{2}, \theta_0 + \frac{3\pi}{2}]$ and $v_0$ over $[0, v_{\text{max}}]$. Then we compute for each initial point $\xi_0 = (r_0, \theta_0, v_0, \gamma_0)$ the corresponding trajectory using the state transition matrix.

In figure 3 a velocity is colored in blue if the corresponding trajectory leaves the interior of the circle in red with radius $R$ without crossing the obstacle beforehand. Such a velocity is regarded as being a feasible velocity. Of course it can happen that a trajectory leaves the circle and then reenters it to cross the collision zone later. In this model such a velocity is considered to be feasible anyway. However, the state where the trajectory reenters the circle is then regarded as infeasible, since from this point on, the trajectory touches the collision zone before quitting the circle again. Of course, in this model there is no point in considering initial velocities with $\gamma_0 \notin [\theta_0 + \frac{\pi}{2}, \theta_0 + \frac{3\pi}{2}]$, because in this case the corresponding trajectory immediately leaves the circle. So all of these velocities are feasible.

There are a few things to point out concerning the obtained results. We can observe an interesting symmetry in the above figure. The half circles containing the feasible and infeasible velocities corresponding to opposite starting positions are the same, rotated by 180 degrees. Also it is interesting to note that all the infeasible velocities
are velocities where \( \gamma_0 \in [\theta_0 + \pi, \theta_0 + \frac{3\pi}{2}] \). Roughly speaking, facing the obstacle from any considered starting position it is 'more dangerous' to go to the left than to go to the right.

Figure 4: feasible and unfeasible initial velocities for \( r_0 = 4 \) and \( \theta_0 = \frac{9\pi}{5} \)

Figure 4 is just a zoom in of the previous figure, but it draws the attention on an interesting phenomenon. Looking at the previous figure it can be seen that collisions are more likely for bigger initial velocities. However, there are certain angles where a big initial velocity avoids a collision while a smaller one leads to a collision. The above figure highlights such a case. In figure 5 we can see the trajectories corresponding to such a case. While the trajectories in the 'extreme' cases for \( v_0 = 6.0 \) and \( v_0 = 8.0 \) are clearly feasible, many of the intermediate values lead to a collision.
Figure 5: trajectories for different norms of the initial velocity corresponding to the highlighted direction in figure 4.

Finally we want to investigate what happens when we change the initial distance $r_0$ to the collision zone. In figure 6 the same computations are repeated for two smaller circles of radius $r_0 = 2$ and $r_0 = 3$. It can be observed that the number of infeasible directions increases when the initial position approaches the obstacle. Imagine the extreme case where $r_0 = 1$, which means that the starting position belongs to the boundary of the collision zone. In this case any angle of the velocity $\gamma_0 \in (\theta_0 + \frac{\pi}{2}, \theta_0 + \frac{3\pi}{2})$ would immediately lead to a collision. So it is well conform with our intuition that for decreasing values of $r_0$ the set of infeasible directions approaches this extreme case.
5 New approach based on the maximum principle

5.1 Theory

The following theory is based on [2]. The goal of this approach is to establish a necessary condition for a point being in the viability kernel. The idea is to define a set $F$ of weakly viable initial points:

$$F = \{ \xi_0 \in \Omega : \forall t \geq 0 \; A(\xi_0, t) \cap \Omega \neq \emptyset \}$$

Comparing the definitions of both sets it is clear that $F$ contains the viability kernel. The strategy is to characterize the boundary $\partial F$ of this set through the supplementary condition

$$P(\xi_0) : \exists T > 0 \text{ s.t. } \text{Int}(A(\xi_0, T) \cap \Omega) = \emptyset$$

Note that $F$ is a closed set, so the elements of $\partial F$ belong to $F$.

**Definition 5 ([2]).** A controlled system $(D)$ is called $\mathcal{K}$-controllable if it verifies the two following conditions:

(i) The reachable sets $A(\xi_0, t)$ have a non-empty interior for every time $t > 0$.

(ii) For every initial point $\xi_0$ and every compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists a neighborhood $O$ of $\xi_0$ and a time $t^* \geq 0$ such that:

$$\forall \xi \in O \; \forall t \geq t^* \; \mathcal{K} \subset A(\xi, t).$$
The following theorem is the main result of this approach. Its proof is quite long and relies on some other lemmata. The interested reader can find the complete proof in [2].

**Theorem 1** ([2]). Let \((\mathcal{D})\) be a \(K\)-controllable system and \(\xi_0 \in \partial \mathcal{F} \cap \text{Int}(\Omega)\), where \(\Omega\) is a target of \(\mathbb{R}^n\). Then there exist \(T > 0\), \(u \in \mathcal{U}(\mathbb{R}^m)\), \(\xi : \mathbb{R}^+ \to \mathbb{R}^n\) and \(p : \mathbb{R}^+ \to \mathbb{R}^n\) absolutely continuous, such that

(i) \(\xi, u\) and \(p\) are solutions of the following equations:

\[
\begin{align*}
\dot{\xi}(t) &= A\xi(t) + Bu(t) \quad \text{(24)} \\
\xi(0) &= \xi_0 \quad \text{(25)} \\
\dot{p}(t) &= -p(t)A \quad \text{(26)} \\
p(t)Bu(t) &= \max_{\|v\| \leq \epsilon} p(t)Bv \quad \text{for a.e. } t \quad \text{(27)}
\end{align*}
\]

(ii) The trajectory \(\xi\) arrives at the boundary of \(\Omega\) at time \(T\):

\[\xi(T) \in \partial(\Omega) \quad \text{(28)}\]

(iii) The adjoint vector \(p(T)\) is tangent to the tangent plane \(T_{\xi(T)}\Omega\):

\[\langle p(T) \mid v \rangle = 0 \quad \forall v \in T_{\xi(T)}\Omega \quad \text{(29)}\]

(iv) The trajectory \(\xi\) arrives at \(\partial \Omega\) with a velocity tangent to \(\partial \Omega\):

\[\langle p(T) \mid A\xi(T) + Bu(T) \rangle = 0 \quad \text{(30)}\]

Note that (i), (ii) and (iii) of the previous theorem are just the maximum principle. Point (iv) is the main novelty and will be very useful. See for example [5] for more information on the maximum principle.

It is important to mention that theorem 1 provides us a necessary condition for an initial point belonging to the viability kernel, since the four properties are necessary for weak viability which is again necessary for viability.

The following figure tries to explain the geometry behind this theorem and especially the property (iv).

![Figure 7: viable and non-viable initial conditions](image-url)

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Think of \( C \) as a collision zone being a subset of \( \mathbb{R}^n \). Its the set that we try to avoid and if at one point a trajectory crosses it, this is regarded as a collision. Therefore its complementary set \( \mathbb{R}^n \setminus C \) corresponds to the set \( \Omega \) in theorem 1.

Suppose that all of the three trajectories above satisfy property (i) of Theorem 1. Note that these trajectories are artificially constructed and do not correspond to real dynamics. The initial point \( \xi_1 \) clearly belongs to the viability kernel \( Viab(\Omega) \), since its corresponding trajectory never crosses the collision zone. However, according to theorem 1 it does not belong to the boundary, because property (ii) is not satisfied. The initial point \( \xi_2 \) does not belong to the viability kernel, since it crosses the collision zone. This time, the corresponding trajectory satisfies property (ii). There is a time \( T > 0 \) when the trajectory touches the boundary of the collision zone. However, property (iv) is not satisfied, the velocity vector at time \( T \) is not tangent to \( \partial C \). Finally \( \xi_0 \) satisfies all four properties of Theorem 1. Therefore it is a candidate for being a viable initial point.

The following figure demonstrates why the four properties of Theorem 1 are not sufficient for showing that an initial point belongs to the boundary of the viability kernel.

![Figure 8: non-viable trajectory satisfying the properties of Theorem 1](image)

There is a time \( T > 0 \) where the trajectory touches the boundary of \( C \) and the velocity vector is tangent to it. Therefore all four properties are satisfied. However the trajectory crosses the collision zone afterwards, so we can not conclude that \( \xi_0 \) is viable.

### 5.2 Numerical Continuation

Numerical continuation is a method that computes solutions of a system of parametrized nonlinear equations,

\[
F(u, \lambda) = 0, \tag{31}
\]

where \( u \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \). Numerical Continuation in more than one parameter is also possible, but not of importance in our case. An initial solution \((u_0, \lambda_0)\) has to be known to start the continuation at this point. Then the algorithm returns a set of points \((u, \lambda)\) which satisfy \( F(u, \lambda) = 0 \) and are connected to the initial solution \((u_0, \lambda_0)\) by a path of solutions \((u(s), \lambda(s))\) which satisfies

\[
F(u(s), \lambda(s)) = 0, \quad \forall s \in [0, 1] \tag{32}
\]

\[
(u(0), \lambda(0)) = (u_0, \lambda_0) \tag{33}
\]

Our strategy to find initial points on the boundary of the viability kernel consists in performing several different continuations. In other words, given a trajectory
that satisfies all four properties of Theorem 1, we want to perform a continuation, such that all the trajectories corresponding to the returned solution path still satisfy these properties. There exist many softwares that can solve numerical continuation problems. We will use the software 'hampath'.

5.3 Application to the collision avoidance problem

We need to take all the properties of theorem 1 into account. In the following we define the matrices $A$ and $B$ according to our dynamics as

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

Also we define the collision zone $C$ as

$$C = \{ \xi \in (\mathbb{R} \times [0, 2\pi])^2 \mid r \leq 1 \}. \tag{35}$$

The costate $p = (p_x, p_y, p_x, p_y)$ is given by the equation

$$\dot{p} = -pA. \tag{36}$$

Note that we only need to know the value of $p(t)$ at one point in time to obtain the solution of the previous equation for an arbitrary time, thanks to the state transition matrix available for the HCW-equations. Remember that we search a time $T > 0$, at which the trajectory touches the boundary of the collision zone at a velocity tangent to it. So the state at time $T$ can be written as

$$\xi(T) = (1, \theta(T), v(T), \theta(T) + \pi). \tag{37}$$

According to point (iii) of theorem 1 the costate $p(T)$ has to be tangent to the tangent plane $T_{\xi(T)}\Omega$, so

$$p_{rad}(T) = (0, 0, 1, \theta(T)), \tag{38}$$

or in cartesian coordinates

$$p(T) = (0, 0, \cos(\theta(T)), \sin(\theta(T))). \tag{39}$$

We see that, given $T$ and $\theta(T)$, we obtain the costate at time $T$ which we can then use to compute the costate for every time $t > 0$ using the state transition matrix. Let us write $p_e = (p_x, p_y)$. Knowing $p_e$, equation (27) is easily solved in our case, since

$$p(t)Bv = \langle p_e, v \rangle \tag{40}$$

and therefore

$$u = \arg\max_{\|v\| \leq u_{max}} \langle p_e, v \rangle = \frac{u_{max}}{\|p_e\|} p_e. \tag{41}$$

The linear differential equation

$$\dot{\xi}(t) = A\xi(t) + Bu(t) \tag{42}$$
can be solved, given the initial condition
\[ \xi(0) = \xi_0. \] (43)

We limit the norm of the control \( u \) to be smaller than a certain \( u_{\text{max}} \). Furthermore let us suppose that an initial position \((r_0, \theta_0)\) is given and also an initial angle of the velocity \( \gamma_0 \). Only the initial norm of the velocity \( v_0 \) is assumed to be unknown. We would like the solution of the differential equation (42) to satisfy the equations
\[ x(T) - \cos(\theta_T) = 0 \] (44a)
\[ y(T) - \sin(\theta_T) = 0 \] (44b)
\[ x(T)\dot{x}(T) + y(T)\dot{y}(T) = 0. \] (44c)

The first two equations (44a) and (44b) make sure that the trajectory \( \xi \) touches the boundary of the collision zone at time \( T \). The third equation (44c) ensures that the trajectory arrives at this point with velocity tangent to \( \partial \mathcal{C} \). This corresponds to the properties (ii) and (iv) of theorem 1.

Obviously (44) depends on \( T \) and \( \theta_T \). Note that it also depends on \( v_0 \), since equation (1) can not be solved without it. So we have three nonlinear equations depending on three variables. In other words we have a nonlinear function \( F : \mathbb{R}_+ \times [0, 2\pi] \times [0, v_{\text{max}}] \to \mathbb{R}^3 \), defined by
\[ F(T, \theta_T, v_0) = \begin{pmatrix} x(T) - \cos(\theta_T) \\ y(T) - \sin(\theta_T) \\ x(T)\dot{x}(T) + y(T)\dot{y}(T) \end{pmatrix}, \] (45)

whose zeros yield us an initial norm of the velocity \( v_0 \), such that together with the fixed values \( r_0, \theta_0 \) and \( \gamma_0 \) this forms an initial state \( \xi_0 = (r_0, \theta_0, v_0, \gamma_0) \) which fulfills the necessary condition for belonging to the boundary of the viability kernel.

Adding a continuation parameter to the function \( F \), we can compute a path of zeros of \( F \) along this parameter utilizing the software 'hampath'. As continuation parameter we could take any of the parameters that we fixed for the time being, like \( r_0, \theta_0, \gamma_0 \) or \( u_{\text{max}} \). Let us take \( \gamma_0 \) as the continuation parameter. Then we have to specify a range \([\gamma_{\text{init}}, \gamma_{\text{end}}]\) for \( \gamma_0 \) and compute the solution of
\[ F(\cdot, \gamma_{\text{init}}) = 0 \] (46)

to start the continuation from this point. However, keep in mind that it is never guaranteed that for a given \( \gamma_{\text{init}} \) there exists a solution of the above equation. So the question of where to search for an initial solution is legitimate. Luckily, in section 5.3 we already delivered a detailed analysis of the problem without control. This can assist our search for an initial solution. So suppose that \( u_{\text{max}} = 0 \) and let us again take a look at figure 6. As a reminder, the blue region corresponds to 'safe' initial directions. The white region corresponds to directions that lead to a collision. Therefore on the border between these two regions we should find initial positions being on the boundary of the viability kernel. It seems like for each fixed initial position \((r_0, \theta_0)\) and initial velocity \( v_0 \) there exist two angles \( \gamma_0^1 \) and \( \gamma_0^2 \),
corresponding to different solution paths.
The plan is to find these two angles for \( v_0 = v_{\text{max}} \) and to perform a continuation for each of the found initial solutions. In 'hampath' there is a method called 'ssolve' which finds a solution of the shooting problem, given a point very close to it.

Figure 9: initial trajectories for the two continuations at \((r_0, \theta_0) = (4, 0)\)

In figure 9 we see the trajectories corresponding to the two angles \( \gamma_0^1 \) and \( \gamma_0^2 \) that were found by 'ssolve'. Clearly they are both tangent to the collision zone at one point and therefore solutions of the shooting problem.
In figure 10 we see again the feasible directions in blue, simply obtained using the state transition matrix for the HCW-equations like in chapter . In black we see the two solution paths, obtained by the continuation through 'hampath'. As expected the two paths closely follow the boundary of the blue set, showing that the results of the two methods conform with each other.

However, both solution paths continue approaching each other until the lower bound of the half circle, meaning that all the initial directions between these two paths lead to a collision. On the other side for the 'extreme' angles even the points between the paths are considered to be feasible by the 'simple' approach.
In figure 11 the points in red correspond to trajectories that leave the sphere but then reenter the sphere and enter the collision zone. We can see that this occurs quite often. Especially the points between the two continuation paths are all such directions. Like explained in section 5.3 we want these points to be marked as feasible. So how can we do that in the case of the continuation approach? Note that similarly to the boundary between the initial feasible and non-feasible directions, there should be a boundary between directions that make the trajectory leaving the sphere and those that don’t. Again for the directions belonging to this boundary there should exist a point of tangency, but this time on the sphere and not on the collision zone. This is actually almost the same problem as before, only the radius of the sphere where the tangency occurs is different. We can even generalize the function $F$ to

$$F(T, \theta_T, v_0) = \begin{pmatrix} x(T) - R \cos(\theta_T) \\ y(T) - R \sin(\theta_T) \\ x(T) \dot{x}(T) + y(T) \dot{y}(T) \end{pmatrix},$$

(47)

where $R$ is a parameter indicating the radius of the sphere where the tangency should occur. For $R = 1$ we receive the same problem as before. With $R = 4$ we hope to detect the trajectories tangent to the sphere.
Figure 12: results for \((r_0, \theta_0) = (4, 0)\) including continuation of the no-exit boundary

Figure 13: zoom in on the results for \((r_0, \theta_0) = (4, 0)\) including continuation of the no-exit boundary

In figure 12 and 13 we can see that the solution path of the continuation for the no-exit boundary cuts off the directions where the collision occurs after leaving the
sphere. This is exactly what we wanted to achieve. Summing up we have to perform three different continuations to recover the set of feasible directions found by the simple approach.

Figure 14: double tangent trajectory corresponding to the highlighted point in figure 13

The above trajectory corresponds to the intersection point of the no-exit boundary and one of the feasibility boundary parts (see figure 13). As expected this trajectory is tangent to both the collision zone and the outer sphere.

Note that until now we still fixed $u_{\text{max}}$ to be zero, so all of the previous results are only the solution for the case without control. Of course we also want to investigate the case with control, so let us now consider the case where $u_{\text{max}} = 1$. In theory, the set of feasible initial directions should increase, since the collision can now be actively avoided.

Again we want to perform a continuation along the angle of the initial velocity $\gamma_0$. Of course, we need a starting point for the continuation for $u_{\text{max}} = 1$. Currently we don’t know about a solution in the controlled case. However, we can perform another continuation to find such an initial point. We would like to start again on the boundary of the halfcircle, so we fix $v_0 = v_{\text{max}}$ and take $u_{\text{max}}$ as the continuation parameter. We start at $u_{\text{max}} = 0$, which means that the starting point for
this continuation is exactly the same as for the continuation along $\gamma_0$ in the uncontrolled case. We perform the continuation until $u_{\text{max}} = 1$, so the endpoint of this continuation will be the starting point for the continuation along $\gamma_0$ where we fix $u_{\text{max}} = 1$.

Figure 15: results for $(r_0, \theta_0) = (4, 0)$ in the controlled case with $u_{\text{max}} = 1$

In figure 15 we see in green the solution paths of the first continuations to find the appropriate starting points. In red we have the continuations for $u_{\text{max}} = 1$ along $\gamma_0$. Both of these solution curves are pushed towards the interior of the white region which means that for $u_{\text{max}} = 1$ all the directions between the black and the red curves are also feasible. As expected the set of feasible directions clearly increases when a control is allowed.

6 Established approach based on dynamic programming

6.1 Theory

The following chapter is based on [1]. The ‘level set’ approach is used in order to characterize the viability kernel. In particular an optimal control problem is considered whose objective function indicates for each initial point whether it belongs to the viability kernel or not. The dynamics are supposed to be of the form (24) and satisfy the following properties:
(a) There exists $L_f > 0$, such that for every $\xi_1, \xi_2 \in \mathbb{R}^n$ and $u \in U$:

$$|f(\xi_1, u) - f(\xi_2, u)| \leq L_f |\xi_1 - \xi_2|,$$

where $L_f > 0$ is the Lipschitz constant.

(b) For every $x \in \mathbb{R}^n$, $f(x, U)$ is a convex set of $\mathbb{R}^n$.

Let us now introduce the idea for the characterization of the viability kernel. We consider a given parameter $\lambda > 0$ and a lipschitz continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$g(\xi) \leq 0 \Leftrightarrow \xi \in \mathbb{R}^n \setminus C. \quad (49)$$

Note that the signed distance function

$$d_C(\xi) = \begin{cases} d(x, \partial C), & \text{if } \xi \in C \\ -d(x, \partial C), & \text{if } \xi \notin C \end{cases}$$

always satisfies the previous relation. However it can be practically beneficial to choose another function if this is possible. We choose $\lambda$ such that

$$\lambda > L_f$$

and consider the optimal control problem

$$v(\xi) := \inf_{u \in U} \max_{\theta \in [0, \infty)} [e^{-\lambda \theta} g(\xi_0^u(\theta))]. \quad (50)$$

Note that the objective function $\max_{\theta \in [0, \infty)} [e^{-\lambda \theta} g(\xi_0^u(\theta))]$ of the previous infinite horizon control problem penalizes a trajectory $\xi_0^u$ which crosses the collision zone.

The following two theorems are the main results and explain the utility of problem (50).

**Theorem 2** (Characterization of the viability kernel). Let $g$ be a Lipschitz continuous function defined by (49). Let $v$ be the value function defined by (50). Then the viability kernel $\text{Viab}(\mathbb{R}^n \setminus C)$ is given by

$$\text{Viab}(\mathbb{R}^n \setminus C) = \{\xi \in \mathbb{R}^n, v(\xi) \leq 0\}. \quad (51)$$

**Theorem 3.** Under the property (a), for any $\lambda > L_f$ and for any Lipschitz continuous function $g$ satisfying (49), the value function $v$ is the unique continuous viscosity solution of the variational inequality

$$\min(\lambda v + H(\xi, \nabla v), v - g(\xi)) = 0, \quad \xi \in \mathbb{R}^n. \quad (52)$$
Theorem 3 provides us a method to compute the value function of the infinite horizon problem (50) in practice. Theorem 2 then tells us how to deduct the viability kernel from the computed value function. Therefore we have all the tools at hand to compute the viability kernel. Note that Theorem 3 is valid for any choice of function \( g \) satisfying (49) and for any \( \lambda > L_f \). While the value function \( v \) depends on \( g \) and \( \lambda \), the set \( \{ \xi \in \mathbb{R}^n, v(\xi) \leq 0 \} \) does not depend on either of them.

One solver capable of solving problems of the form (52) is the solver ROC-HJ. ROC-HJ stands for 'Reachability, Optimal Control and Hamilton-Jacobi equations' and is a c++ MPI/OpenMP library for solving \( d \)-dimensional Hamilton-Jacobi-Bellman equations by finite difference methods, or semi-lagrangian methods. First order and second order HJ equations can be solved, where the equation can be time-dependent or steady. Note that the equation that we want to solve is steady, since it does not depend on time.

In the general case, ROC-HJ solves a problem of the following form: Find \( u = u(x) \) solution of a steady Hamilton-Jacobi-Bellman equation

\[
\lambda(x)u(x) + H(x, u(x), \nabla u(x)) = 0, \quad x \in \Omega \tag{53}
\]

\[
u(x) = g_{\text{border}}(x), \quad x \in \partial \Omega. \tag{54}
\]

However, it is also possible to solve an obstacle problem, given by

\[
\min \left[ \lambda u(x) + H(x, \nabla u), u(x) - g(x) \right] = 0, \quad x \in \Omega \tag{55}
\]

\[
u(x) = g_{\text{border}}(x), \quad x \in \partial \Omega \tag{56}
\]

This is exactly the type of equation that we want to solve. It can be solved by a 'value iteration algorithm'. For steady equations (as well as time dependent equations), the basic finite difference (FD) scheme is based on the following iteration: for \( n \geq 0 \) and for a fixed \( \Delta t > 0 \),

\[
\frac{u^{n+1}(x) - u^n(x)}{\Delta t} + \lambda u^{n+1}(x) + h(x, u^n(x), Du^n(x)) = 0, \quad x \in \Omega, \tag{57}
\]

where \( h(x, u^n(x), Du^n(x)) \) is a numerical approximation of \( H(x, u^n(x), \nabla u^n(x)) \).

The iterations are stopped as soon as \( \|u^{n+1} - u^n\|_\infty \) is smaller than a given threshold.

### 6.2 Results

In figure we see the value function 50 plotted for \((\dot{x}, \dot{y}) = (1, 0)\) in the case without control. According to relation 49 all the points \( \xi = (x, y, 1, 0) \) with \( g(\xi) \leq 0 \) belong to the viability kernel. In figure 17 we see in blue all the pair \((x, y)\), such that \( \xi \) belongs to the viability kernel. Furthermore in red the boundary of the viability kernel is given. In gray we also see the collision zone which obviously does not belong to the viability kernel.

We can see that the starting positions directly to the left of the collision zone all lead to a collision which is expected since the vector of the initial velocity points to the right. In figures 18 to 22 we see the same results in the controlled case.
Each figure corresponds to a different $u_{\text{max}}$. We notice that the bigger $u_{\text{max}}$, the bigger is the viability kernel of the problem. This is of course the result what we hoped to obtain, since a greater $u_{\text{max}}$ should increase the capability of avoiding the collision. In particular for $u_{\text{max}} = 5$ it seems that all the points outside of the collision zone belong to the viability kernel. Note that in general for a bounded control and $\|v_0\| > 0$ there is always a starting position close enough to the collision zone where the collision can not be avoided. However, a control of norm 5 is very big in comparison to an initial velocity of norm 1, so probably the discretization is not fine enough to consider a point sufficiently close to the collision zone.
Figure 17: cut of the viability kernel at $(\dot{x}, \dot{y}) = (1, 0)$ without control

Figure 18: cut of the viability kernel at $(\dot{x}, \dot{y}) = (1, 0)$ with $u_{max} = 1$
Figure 19: cut of the viability kernel at $(\dot{x}, \dot{y}) = (1, 0)$ with $u_{max} = 2$

Figure 20: cut of the viability kernel at $(\dot{x}, \dot{y}) = (1, 0)$ with $u_{max} = 3$
Figure 21: cut of the viability kernel at \((\dot{x}, \dot{y}) = (1, 0)\) with \(u_{max} = 4\)

Figure 22: cut of the viability kernel at \((\dot{x}, \dot{y}) = (1, 0)\) with \(u_{max} = 5\)
7 Comparison between the two approaches

7.1 Results

Figure 23: results for \((r_0, \theta_0) = (4, 0)\) in the case without control

Figure 24: results for \((r_0, \theta_0) = (4, 0)\) in the case where \(u_{\text{max}} = 1\)
Figure 25: results for \((r_0, \theta_0) = (4, 60^\circ)\) in the case without control

Figure 26: results for \((r_0, \theta_0) = (4, 60^\circ)\) in the case where \(u_{\text{max}} = 1\)
Figure 27: results for \((r_0, \theta_0) = (4, 150^\circ)\) in the case where \(u_{\text{max}} = 1\)

Figure 28: results for \((r_0, \theta_0) = (4, 150^\circ)\) in the case where \(u_{\text{max}} = 3\)
7.2 Discussion and Conclusion

In figures 23 to 30 we fix different initial positions and an upper bound for the control norm and show the obtained 'cut' of the viability kernel, obtained by the different approaches. The blue points correspond to initial directions in the viability kernel, according to ROC-HJ. The red curves show the directions that lay on the boundary of the viability kernel, according to the PMP approach. We can see that in all the plots the results obtained by the two approaches are very similar. However in most figure there are small inconsistencies. Note that in the cases without control (figures 23 and 25) there tend to feasible directions of ROC-HJ inside the 'danger zone' obtained by the PMP approach. On the other side, in the controlled case, especially for big controls (figures 28 and 30), there are more often directions that
are regarded as infeasible by ROC-HJ inside of the viability kernel, computed by the PMP approach. A possible explanation for this is that the control is discretized very roughly. For the norm of the control only 0 and $u_{\text{max}}$ are allowed. For the angle of the control 10 values between 0 and $2\pi$ are allowed. The bigger $u_{\text{max}}$ gets, the more important is the role of the control. Therefore it can happen that there are initial positions that require a control of a very specific angle to avoid a collision which is not taken into account because of the rough discretization.

In these figures (except for 25 and 26) there also many points on the continuation curves which clearly belong to the feasible region, according to ROC-HJ. This shows in practice that the points on the continuation curves only consist of points satisfying all the necessary conditions of theorem 1, but don’t necessarily belong to the boundary of the viability kernel.

In figures 26 a special case occurs where the two curves join each other and form one curve. In this case all the points of the curve should belong to the boundary of the viability kernel. In figure 30 there seems to be a problem with the ROC-HJ approach, since there are some feasible points separated from all the other ones.

Summing up it is verified in practice that the four conditions in theorem 1, combined with the practical method of numerical integration, provide a legitimate approach for finding points on the boundary of the viability kernel. However, note that we always fixed the initial position $(x_0, y_0)$ and searched for directions $(\dot{x}_0, \dot{y}_0)$ such that the initial state $\xi_0 = (x_0, y_0, \dot{x}_0, \dot{y}_0)$ belongs to the boundary of the viability kernel. At this time we are not capable of computing the whole viability kernel using the PMP approach and therefore it can not deliver a complete answer to the problem. In particular cases though, when we are especially interested in the solution at a certain initial position, the PMP approach can deliver a faster response to the problem. In one sense the solution provided by the PMP approach is also more accurate, since it does not rely on a discretization of the state space. On the other hand it can not be forgotten that the computed points only satisfy a necessary condition and are never guaranteed to be part of the boundary of the viability kernel which is a big disadvantage compared to the HJB approach. All in all the PMP approach should be seen as a supplement, rather than an alternative, to the more standard HJB method that can provide deeper insight into particular cases of the problem.

References


