Plateau problem(s)

Camille Labourie

Université Paris-Sud

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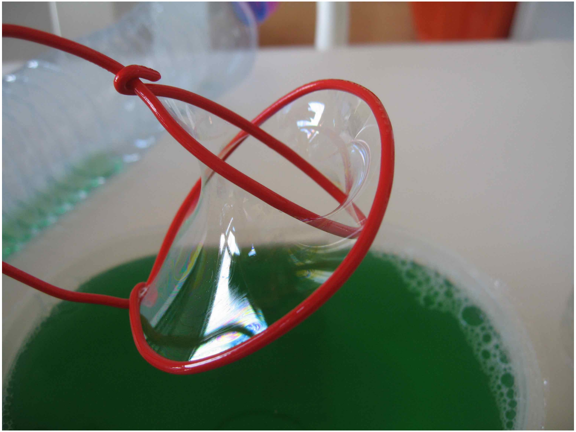
Introduction

Plateau problem consists in minimizing the area of a surface spanning a boundary. It is inspired by soap films.



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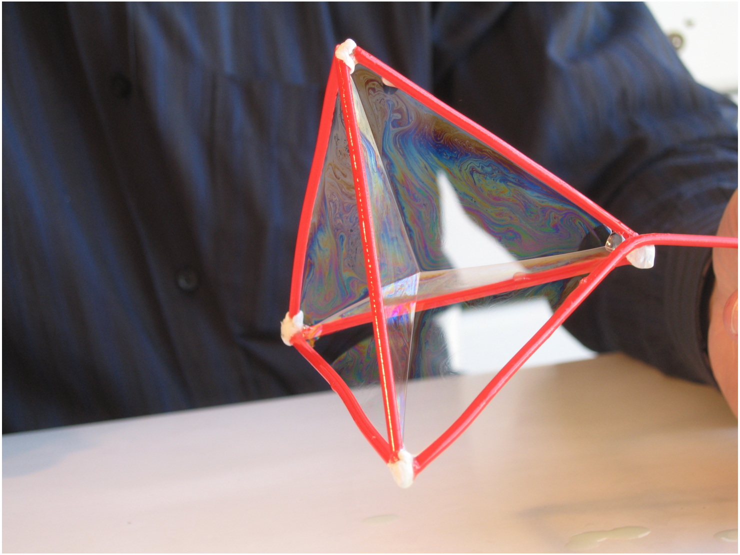

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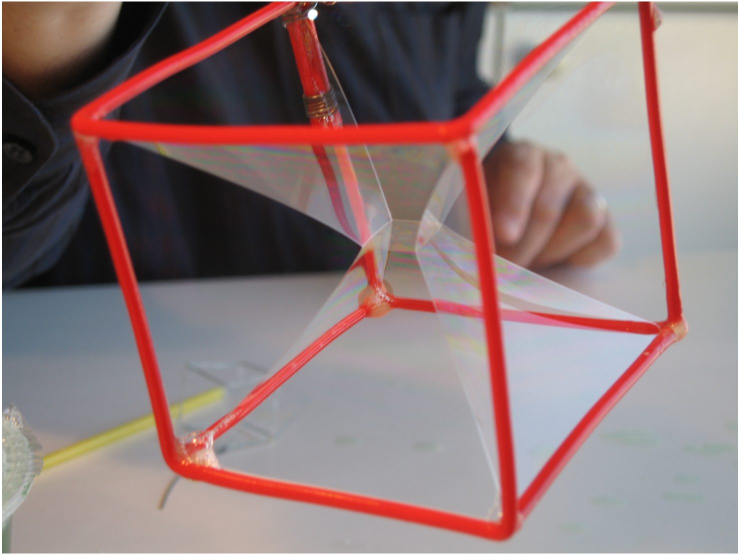

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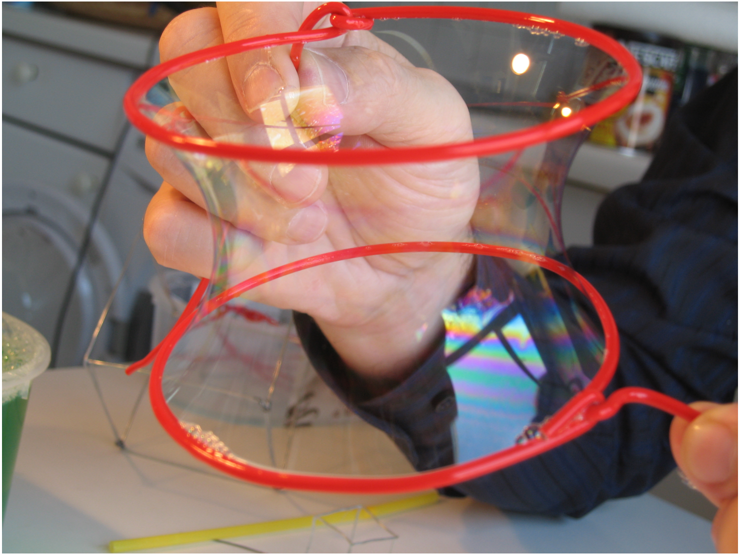

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1. ..define the class of "surfaces spanning a given boundary" (also called *competitors*) and their "area";
2. ..lend itself to the direct method of the calculus of variation;
3. ..stay close to Plateau's original motivations: describing soap films.

The formulation of Rado and Douglas (1930s)

Let

$$D^2 = \{\, (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \,\}$$
$$S^1 = \{\, (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \,\}$$

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1. A surface spanning Γ is defined as a continuous map $f\colon D^2 \to \mathbf{R}^3$ such that f sends S^1 homemorphically onto Γ.
2. The area of such a surface is defined as the total variation of f (also called the area-integral or area functional).

The formulation of Federer and Fleming (1960s)

Federer and Fleming work with *integral currents* and minimize their *mass* (an area computed with multiplicity). They have developped the *flat convergence* for which integral currents enjoy a compactness principle and the mass is lower semicontinuous.

The formulation of Reifenberg (1960s)

Reifenberg works with *sets* of the Euclidean space which span a boundary in the sense of algebraic topology and minimizes their (spherical) *Hausdorff measure*. A set E spans the boundary Γ if E contains Γ and cancels its generators.

Reifenberg competitors

Fix Γ a closed subset of $\mathbf{R}^n$ and let L be a subgroup of the homology group $H_{d-1}(\Gamma)$.

Definition (Reifenberg competitors)

A Reifenberg competitor is a compact subset $E \subset \mathbf{R}^n$ such that E contains Γ and the morphism induced by inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E \cup \Gamma),$$

is zero on L.

Hausdorff measures

Definition (Hausdorff measure H^d)

$$H^d(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_k \mathrm{diam}(A_k)^d \;\middle|\; E = \cup_k A_k, \mathrm{diam}(A_k) \le \delta \right\}$$

where (A_k) is a sequence of balls.

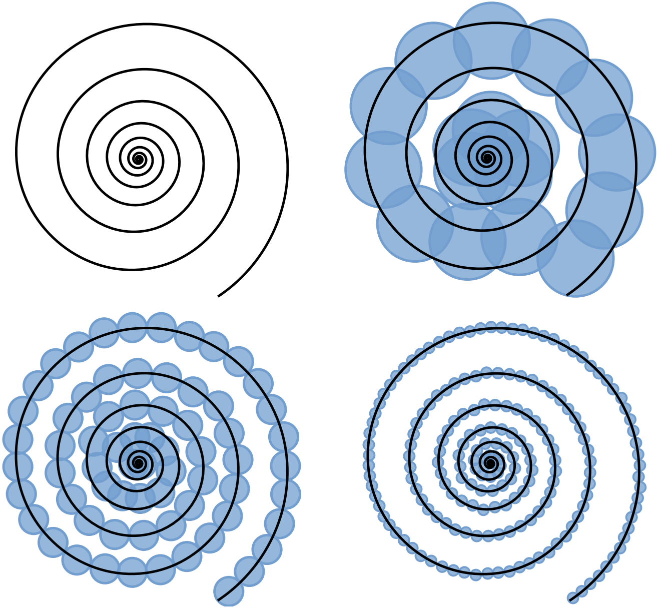

Figure: Computing the H^1 measure of a spiral

Sliding deformations

We fix Γ a closed subset of $\mathbf{R}^n$.

Definition (Sliding deformation along a boundary)

Let E be a closed, H^d-locally finite subset of $\mathbf{R}^n$. A sliding deformation of E is a Lipschitz map $f\colon E \to \mathbf{R}^n$ such that there exists a continuous homotopy $F\colon I \times E \to \mathbf{R}^n$ satisfying the following conditions:

$$\begin{aligned} &F_0 = \mathrm{id} \quad \& \quad F_1 = f \\ &\forall t \in [0,1],\ F_t(E \cap \Gamma) \subset \Gamma \\ &\forall t \in [0,1],\ F_t = \mathrm{id} \text{ in } E \setminus K \end{aligned}$$

where K is some compact subset of E.

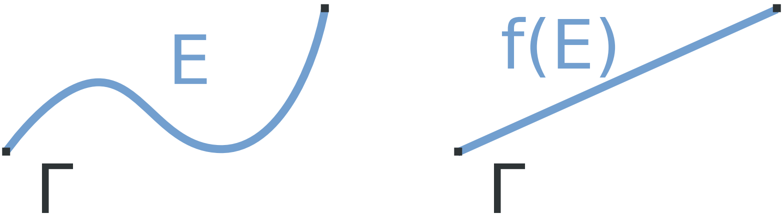

Figure: Fixed boundary; $f = \mathrm{id}$ on Γ.

Figure: Free boundary; $f(E \cap \Gamma) \subset \Gamma$.

Sliding competitors

Definition (Sliding competitors)

We fix E_0 a compact, H^d finite subset of $\mathbf{R}^n$. The sliding competitors induced by E_0 are the images of E_0 under sliding deformations.

Unknown existence!

Minimal sets

A (sliding) minimal set is a closed, H^d-locally finite sets $E \subset \mathbf{R}^n$ such that for every sliding deformation f of E,

$$H^d(E \cap W) \leq H^d(f(E \cap W)).$$

where H^d is the d-dimensional Hausdorff measure and W is the set

$$W = \{\, x \in \mathbf{R}^n \mid f(x) \neq x \,\}.$$

A few results are known about their regularities.

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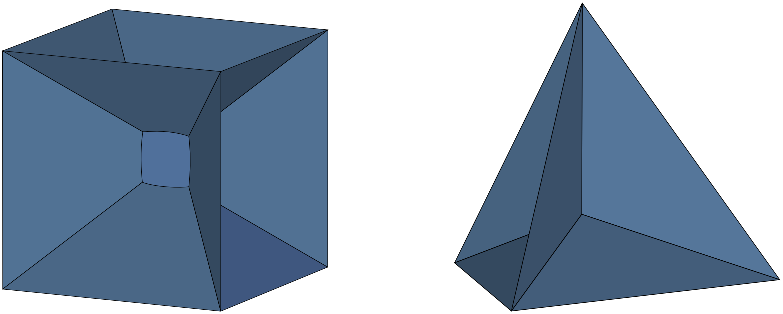

Figure: Soap films spanning the skeleton of a tetrahedron (left) and the skeleton of a cube (right).

Minimal sets in small dimensions

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Minimal cones (without boundaries)

1. $\mathbf{d} = \mathbf{1}$: a line or three half-lines making an angle of $\frac{2\pi}{3}$ in a plane.
2. $\mathbf{d} = \mathbf{2}, \mathbf{n} = \mathbf{3}$: planes, three half-planes making an angle of $\frac{2\pi}{3}$, the cone passing through the edges of a regular tetrahedron centered at 0.
3. $\mathbf{n} > \mathbf{3}$: No complete list..

Minimal sets in small dimensions

Theorem (Jean Taylor)

We work in $\mathbf{R}^3$ *with* $d = 2$. *We define*

$$E^* = \left\{ x \in E \;\middle|\; \forall r > 0,\ H^d(E \cap B(x,r)) > 0 \right\}$$

Every point of $E^* \setminus \Gamma$ *admits a neighborhood in which* E *is* C^1*-diffeomorph to a minimal cone.*

Alhfors-regularity and rectifiability

Proposition

Let

$$E^* = \left\{ x \in E \,\middle|\, \forall r > 0,\ H^d(E \cap B(x,r)) > 0 \right\}.$$

There exist constants $C > 1$ *(depending on* n*,* Γ*) and* $\delta > 0$ *(depending on* n*,* Γ*) such that for all* $x \in E^*$*, for all* $0 < r \leq \delta$*,*

$$C^{-1} r^d \leq H^d(E \cap B(x,r)) \leq C r^d.$$

Moreover, E *is* H^d *rectifiable.*

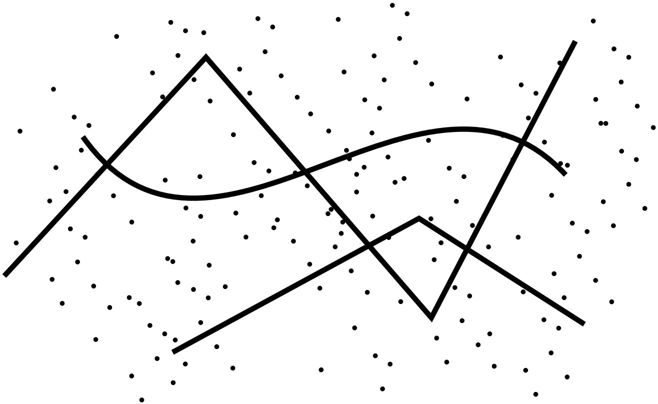

Figure: A H^1 rectifiable set.

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Direct method

Strategy initiated by De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi.

Proposition

Let $\mathcal{C}$ be a nonempty class of closed, H^d-finite subsets $E \subset \mathbf{R}^n$. We assume that for all $E_0 \in \mathcal{C}$, for all sliding deformation f of E_0 in $\mathbf{R}^n$,

$$\inf \{ H^d(E) \mid E \in \mathcal{C} \} \leq H^d(f(E_0)).$$

If (E_k) is a minimizing sequence of the Hausdorff measure H^d in $\mathcal{C}$, then, up to a subsequence, there exists a sliding minimal set E such that

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E.$$

In particular $H^d(E) \leq \inf \{ H^d(E) \mid E \in \mathcal{C} \}$.

Solution of the Reifenberg problem...

Theorem (...away from the boundary)

We assume that

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg} \} < \infty \tag{2}$$

and that there exists a compact set $C \subset \mathbf{R}^n$ such that

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg}, \; E \subset C \} . \tag{3}$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E \setminus \Gamma) = m$.

Solution of the Reifenberg problem...

Theorem (...with the boundary)

We assume that

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg} \} < \infty \tag{4}$$

and that there exists a compact set $C \subset \mathbf{R}^n$ *such that*

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg}, \ E \subset C \} . \tag{5}$$

Then there exists a Reifenberg competitor $E \subset C$ *such that* $H^d(E) = m$.

Fin