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On completeness and frame properties of certain exponential families

(The talk is based on joint work with A. Kulikov and A. Ulanovskii)

We study completeness and frame properties of the system

$$E(\Lambda,\Gamma) := \{t^k e^{2\pi i\lambda t} : \lambda\in\Lambda, k\in\Gamma\},\quad \Gamma\subset\mathbb{N}_0=\{0,1,2,\dots\},\ \Lambda\subset\mathbb{R}.$$

Let $X(I)$ be a space of functions supported on I, e.g. $X=C(I)$ or $X=L^p(I)$, where $I=[-\sigma,\sigma]$. The radius of completeness of the family $E(\mathbb{Z},\Gamma)$ in the space X is denoted by

$$r_X(E(\mathbb{Z},\Gamma)) = \sup\{a\geq 0 : E(\mathbb{Z},\Gamma) \text{ is complete in } X(-a,a)\}.$$

It is well-known that

- $$r_{L^2}(E(\mathbb{Z},\{0\})) = r_{L^2}(\{e^{2\pi int}\}_{n\in\mathbb{Z}}) = r_C(\{e^{2\pi int}\}_{n\in\mathbb{Z}}) = \frac{1}{2};$$
- if $\Gamma=\{0,1,2,\dots,N\}$ then
$$r_C(E(\mathbb{Z},\Gamma)) = r_{L^2}(E(\mathbb{Z},\Gamma)) = \frac{\#\Gamma}{2} = \frac{N+1}{2}.$$

One may ask the following

Question Is it true that for any $\Gamma\subset\mathbb{N}_0$ we have

$$r_C(E(\mathbb{Z},\Gamma)) = r_{L^2}(E(\mathbb{Z},\Gamma))?$$

It turns out that in general this is false. More precisely, if Γ has "gaps" then the answer depends on $\#\Gamma_{odd}$ and $\#\Gamma_{even}$, where

$$\Gamma_{odd} = \Gamma\cap(2\mathbb{Z}+1) \quad\text{and}\quad \Gamma_{even}=\Gamma\cap 2\mathbb{Z}.$$

We proved the following

Theorem 1 (A. Kulikov, A. Ulanovskii, I. Z., 2022)**.** *Given a finite set $\Gamma\subset\mathbb{N}_0$ satisfying $0\in\Gamma$. Then*

$$r_{L^2}(E(\mathbb{Z},\Gamma)) = \frac{\#\Gamma}{2} \qquad r_C(E(\mathbb{Z},\Gamma)) = \begin{cases} \#\Gamma_{odd}+\frac{1}{2}, & \text{if } \#\Gamma_{odd}<\#\Gamma_{even},\\ \#\Gamma_{even}, & \text{if } \#\Gamma_{odd}\geq\#\Gamma_{even}.\end{cases}$$

Our argument is based on a description of certain uniqueness sets for lacunary polynomials.

References

[1] Aleksei Kulikov, Alexander Ulanovskii, Ilya Zlotnikov, *Completeness of Certain Exponential Systems and Zeros of Lacunary Polynomials,*(2022) arxiv.org/abs/2210.00504