From double brackets to integrable systems

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UNIVERSITY OF LEEDS

Plan for the talk

1. **Double brackets and associated structures**
2. Jordan quiver : Poisson case
3. Jordan quiver : quasi-Poisson case

Double brackets

We follow (Van den Bergh,'08), *double Poisson algebras*

A denotes an arbitrary associative $\mathbb{C}$-algebra. $\otimes = \otimes_{\mathbb{C}}$.

For $d \in A^{\otimes 2}$ we write $d = d' \otimes d''$ and $\tau_{(12)} d = d'' \otimes d'$.

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Definition

A *double bracket* on A is a $\mathbb{C}$-bilinear map $\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}$ which satisfies

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = b\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} c$ (outer derivation)

$$\{\!\{a,bc\}\!\} = (b\{\!\{a,c\}\!\}') \otimes \{\!\{a,c\}\!\}'' + \{\!\{a,b\}\!\}' \otimes (\{\!\{a,b\}\!\}'' c)$$

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$$\begin{aligned}\{\!\{bc,a\}\!\} &= \{\!\{c,a\}\!\}' \otimes (b\{\!\{c,a\}\!\}'') + (\{\!\{b,a\}\!\}' c) \otimes \{\!\{b,a\}\!\}'' \\ &= b * \{\!\{c,a\}\!\} + \{\!\{b,a\}\!\} * c \qquad \text{(inner derivation)}\end{aligned}$$

Associated bracket

We write $\{-,-\} : A^{\otimes 2} \xrightarrow{\{\!\{-,-\}\!\}} A^{\otimes 2} \xrightarrow{m} A$

This is the concatenation of $\{\!\{-,-\}\!\}$, as $\{a,b\} = \{\!\{a,b\}\!\}' \{\!\{a,b\}\!\}''$

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Get $\{-,-\} : A \to \mathrm{Der}(A)$, $a \mapsto \{a,-\}$

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How to get symmetries? i.e. when do such derivations commute?

Associated bracket : a second issue

$\{-,-\} : A^{\otimes 2} \to A$, $\{a,b\} = \{\!\{a,b\}\!\}' \{\!\{a,b\}\!\}''$

For vector space $A/[A,A]$, consider map $A \to A/[A,A] : a \mapsto \bar{a}$
($H_0(A) := A/[A,A]$ is first Hochschild homology of A)

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$\{[A,A],-\} = 0$, so factors through $A/[A,A] \times A \to A$
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This gives an antisymmetric bilinear map on $A/[A,A]$. Is it a Lie bracket?

A useful result

Lemma (Van den Bergh,'08)

For any $a, b, c \in A$,

$$\{a, \{b, c\}\} - \{\{a, b\}, c\} - \{b, \{a, c\}\} = \{a, b, c\}_3 - \{b, a, c\}_3$$

where $\{-, -, -\}_3$ *is a map completely determined by* $\{\!\{-, -\}\!\}$.

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In particular, assuming $\{-,-,-\}_3 = 0$,

- the derivations $\{a,-\}$, $\{b,-\}$ commute when $\{a,b\} \in [A,A]$
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There are two interesting cases :
double *Poisson* brackets, and double *quasi-Poisson* brackets

The different brackets

$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$ is a double (quasi-)Poisson bracket

$$\downarrow$$

$\{-,-\} = m \circ \{\!\{-,-\}\!\} : A \times A \to A$ is the associated bracket

$\{[A,A],-\} = 0$ and $\{a,-\}$ is a derivation

$$\downarrow$$

$\{-,-\}$ descends to $A/[A,A]$ where it is a Lie bracket

Link to representation

Fix $n \in \mathbb{N}^{\times}$. Write $\operatorname{Rep}(A, n)$ for the moduli space whose points are representations of A on $\mathbb{C}^n$.
Define $\mathcal{X}(a)(\rho) := \rho(a)$, $\forall a \in A$, $\rho \in \operatorname{Rep}(A, n)$

Proposition (Crawley-Boevey,'05,'11)

If $A/[A, A]$ *is endowed with a Lie bracket* $\{-, -\}$ *such that the map* $\{\bar{a}, -\}$ *is induced by a derivation* $d_a : A \to A$ *for each* $a \in A$*, then there is a Poisson structure* $\{-, -\}_P$ *on* $\operatorname{Rep}(A, n)/\!/ \operatorname{GL}_n$ *satisfying*

$$\{\operatorname{tr} \mathcal{X}(a), \operatorname{tr} \mathcal{X}(b)\}_P = \operatorname{tr} \mathcal{X}(\{\bar{a}, \bar{b}\}) \tag{1}$$

We say that $\{-, -\}$ is an H_0-Poisson structure

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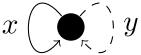
double Jordan quiver $\bar{Q}_0$



$$A = \mathbb{C}\bar{Q}_0 = \mathbb{C}\langle x, y\rangle$$

$$\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\},$$
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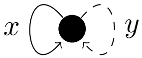

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Derivation properties : $\{\!\{y^k, y^l\}\!\} = 0$, so $\{y^k, y^l\} = 0 \in [A, A]$

For $\frac{d}{dt_k} = \frac{1}{k}\{y^k, -\}$, $\frac{dx}{dt_k} = y^{k-1}$, $\frac{dy}{dt_k} = 0$

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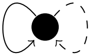

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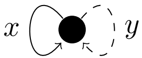
On $\mathfrak{gl}_n \times \mathfrak{gl}_n \ni (X, Y)$: $X(t_k) = X_0 + t_k Y_0^{k-1}$, $Y(t_k) = Y_0$

Jordan quiver 2

double Jordan quiver $\bar{Q}_0$



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We have $\{[x, y] - \lambda, A\} = 0$ and $\{A, [x, y] - \lambda\} = 0$ for all $\lambda \in \mathbb{C}$

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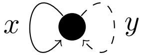

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The bracket $\{-, -\}$ descends to $A_\lambda = A/([x, y] - \lambda)$
It defines H_0-Poisson structure on $A_\lambda/[A_\lambda, A_\lambda]$

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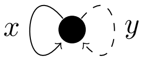

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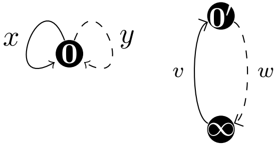
But $\mathrm{Rep}(A_\lambda, n) = \emptyset$ for $\lambda \neq 0$...

Framing

$A = \mathbb{C}\bar{Q}^{dis}$ is a B-algebra, $B = \mathbb{C}e_0 \oplus \mathbb{C}e_{0'} \oplus \mathbb{C}e_\infty$

$\bar{Q}^{dis}$



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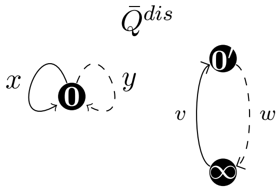

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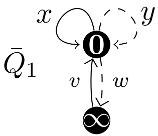
We form $\bar{A}$ where "$e_0 = e_{0'}$" (fusion)

Framed Jordan quiver

$\bar{A} = \mathbb{C}\bar{Q}_1$ is a B-algebra, $B = \mathbb{C}e_0 \oplus \mathbb{C}e_\infty$



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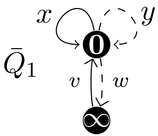

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Now $\mu = [x, y] + [v, w]$ is moment map.
Get H_0-Poisson structure on
$\bar{A}_\lambda = \bar{A}/([x, y] - wv = \lambda_0 e_0, vw = \lambda_\infty e_\infty)$, for $\lambda_0, \lambda_\infty \in \mathbb{C}$

Calogero-Moser space

$\bar{A}_\lambda = \bar{A}/([x,y] - wv = \lambda_0 e_0, vw = \lambda_\infty e_\infty)$

We attach $\mathbb{C}^n$ at 0 and $\mathbb{C}$ at ∞.

$$\mathcal{M} := \{X, Y \in \mathfrak{gl}_n, V \in \mathrm{Mat}_{1\times n}, W \in \mathrm{Mat}_{n\times 1}\}$$
$$\mathcal{M}_\lambda := \{[X,Y] - WV = \lambda_0 \operatorname{Id}_n\} \subset \mathcal{M}$$

For $g \cdot (X,Y,V,W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$, $g \in \mathrm{GL}_n$,

$$\mathrm{Spec}(\mathbb{C}[\mathrm{Rep}(\bar{A}_\lambda,(1,n))]^{\mathrm{GL}_n}) = \mathcal{M}_\lambda /\!/ \mathrm{GL}_n$$

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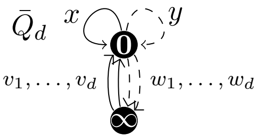
This is n-th Calogero-Moser space (Wilson, 98)
$(\mathrm{tr}\, Y^k)$ Poisson commute by Crawley-Boevey's Theorem
(because $\{\!\{y^k, y^l\}\!\} = 0$, thus $\{\overline{y^k}, \overline{y^l}\} = 0$ in $\bar{A}_\lambda/[\bar{A}_\lambda, \bar{A}_\lambda]$)

Spin Calogero-Moser space

(Bielawski-Pidstrygach,'10; Tacchella,'15; Chalykh-Silantyev,'17)



$d \geq 2$. $\bar{A} = \mathbb{C}\bar{Q}_d$

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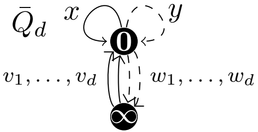

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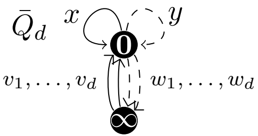

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This is n-th Calogero-Moser space with d spins/intermal degrees of freedom
$(\mathrm{tr}\, Y^k)$ Poisson commute but only n functionally independent...

Spin Calogero-Moser space (bis)

Using double bracket, we get on $\mathcal{M}_\lambda // \mathrm{GL}_n$ for $t^l_{\alpha\beta} = \mathrm{tr}\, W_\beta V_\alpha Y^l$

$$\{\mathrm{tr}\, Y^k, \mathrm{tr}\, Y^l\}_{\mathrm{P}} = 0 = \{\mathrm{tr}\, Y^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

$$\{t^k_{\alpha\beta}, t^l_{\gamma\epsilon}\}_{\mathrm{P}} = \delta_{\beta\gamma} t^{k+l}_{\alpha\epsilon} - \delta_{\alpha\epsilon} t^{k+l}_{\gamma\beta}$$

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The elements $(\operatorname{tr} Y^k, t^l_{\alpha\alpha})$ *are all in involution.*

Hence, we get Liouville integrability for any trY^k
But we can remark...

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Proposition

The subalgebra generated by $(\operatorname{tr} Y^k, t^l_{\alpha\beta})$ *is Poisson, of dimension* $2nd - n$*. Its centre has dimension* n*.*

Hence we get degenerate integrability for any trY^k

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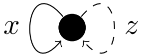

$A = (\mathbb{C}\bar{Q}_0)_{x,z} = \mathbb{C}\langle x^{\pm 1}, z^{\pm 1}\rangle$

$\{\!\{z, z\}\!\} = \frac{1}{2}(1 \otimes z^2 - z^2 \otimes 1)$
$+$ complicated bracket...

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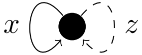
Van den Bergh endowed any (double) quiver with a double *quasi*-Poisson bracket

double Jordan quiver $\bar{Q}_0$



$A = (\mathbb{C}\bar{Q}_0)_{x,z} = \mathbb{C}\langle x^{\pm 1}, z^{\pm 1}\rangle$

$\{\!\{z, z\}\!\} = \frac{1}{2}(1 \otimes z^2 - z^2 \otimes 1)$
$+$ complicated bracket...

Again, $\{z^k, z^l\} = 0$, so could give interesting dynamics :

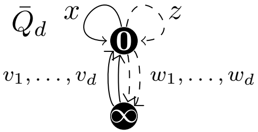
- get $\{\overline{z^k}, \overline{z^l}\} = 0$ in $A/[A, A]$
- $\{\operatorname{tr} Z^k, \operatorname{tr} Z^l\}_{\mathrm{P}} = 0$ on rep. spaces by Crawley-Boevey.

Remark

Now, the bracket factors through $A/(xzx^{-1}z^{-1} - q)$, $q \in \mathbb{C}^\times$.

Spin Ruijsenaars-Schneider space

This is joint work with O. Chalykh. Case $d = 1$: (Chalykh-F.,'17)



$A = \mathbb{C}\bar{Q}_d$ localised at
$x, z, (e_0 + v_\alpha w_\alpha), (e_\infty + w_\alpha v_\alpha)$

Fix q not root of unity

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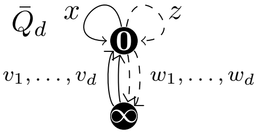

$A = \mathbb{C}\bar{Q}_d$ localised at
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Fix q not root of unity

$$\mathcal{M} := \{X, Z \in \mathrm{GL}_n, V_\alpha \in \mathrm{Mat}_{1\times n}, W_\alpha \in \mathrm{Mat}_{n\times 1}\}$$

$$\mathcal{M}_q /\!/ \mathrm{GL}_n := \{XZX^{-1}Z^{-1}\prod_\alpha (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n\} /\!/ \mathrm{GL}_n$$

We can rearrange to write $XZX^{-1} - qZ = q\mathcal{A}\mathcal{C}$,
for $\mathcal{A}_{i\alpha} = \mathbf{a}_i^\alpha$, $\mathcal{C}_{\alpha j} = \mathbf{c}_\alpha^j$, then Z is spin trigo RS Lax matrix

Spin Ruijsenaars-Schneider space (bis)

We have

$$\{\operatorname{tr} Z^k, \operatorname{tr} Z^l\}_{\mathrm{P}} = 0 = \{\operatorname{tr} Z^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

As in double Poisson case,

Proposition (Chalykh,F.)

Any function $\operatorname{tr} Z^k$ *is degenerately integrable.*

Spin Ruijsenaars-Schneider space (bis)

We have

$$\{\operatorname{tr} Z^k, \operatorname{tr} Z^l\}_{\mathrm{P}} = 0 = \{\operatorname{tr} Z^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

As in double Poisson case,

Proposition (Chalykh,F.)

Any function $\operatorname{tr} Z^k$ *is degenerately integrable.*

Can we get Liouville integrability using $t^l_{\alpha\beta} = \operatorname{tr} W_\beta V_\alpha Z^l$?

Spin Ruijsenaars-Schneider space (ter)

(recall $t^l_{\alpha\beta} = \operatorname{tr} W_\beta V_\alpha Z^l$)

$$
\begin{aligned}
\{\operatorname{tr} Z^k, \operatorname{tr} Z^l\}_{\mathrm{P}} =& 0 = \{\operatorname{tr} Z^k, t^l_{\alpha\beta}\}_{\mathrm{P}} \\
\{t^k_{\epsilon\gamma}, t^l_{\beta\alpha}\}_{\mathrm{P}} =& \frac{1}{2}\left[o(\gamma,\beta) + o(\epsilon,\alpha) - o(\epsilon,\beta) - o(\gamma,\alpha)\right] t^k_{\epsilon\gamma} t^l_{\beta\alpha} \\
&+ \frac{1}{2} o(\gamma,\beta)\; t^{k+l}_{\epsilon\alpha} t^0_{\beta\gamma} + \frac{1}{2} o(\epsilon,\alpha)\; t^0_{\epsilon\alpha} t^{k+l}_{\beta\gamma} \\
&- \frac{1}{2} o(\epsilon,\beta)\; t^l_{\epsilon\alpha} t^k_{\beta\gamma} - \frac{1}{2} o(\gamma,\alpha)\; t^k_{\epsilon\alpha} t^l_{\beta\gamma} \\
&- \delta_{\gamma\beta}\left[t^{k+l}_{\epsilon\alpha} + \frac{1}{2} t^{k+l}_{\epsilon\alpha} t^0_{\beta\gamma} + \frac{1}{2} t^k_{\epsilon\gamma} t^l_{\beta\alpha}\right] \\
&+ \delta_{\alpha\epsilon}\left[t^{k+l}_{\beta\gamma} + \frac{1}{2} t^0_{\epsilon\alpha} t^{k+l}_{\beta\gamma} + \frac{1}{2} t^k_{\epsilon\gamma} t^l_{\beta\alpha}\right] \\
&+ \frac{1}{2}\epsilon(u)\left[\sum_{\tau=1}^{k-1} t^{k+l-\tau}_{\epsilon\alpha} t^\tau_{\beta\gamma} + \sum_{\sigma=1}^{l} t^{k+\sigma}_{\epsilon\alpha} t^{l-\sigma}_{\beta\gamma}\right] \\
&- \frac{1}{2}\epsilon(u)\left[\sum_{\sigma=1}^{l-1} t^\sigma_{\epsilon\alpha} t^{k+l-\sigma}_{\beta\gamma} + \sum_{\tau=1}^{k} t^{k-\tau}_{\epsilon\alpha} t^{l+\tau}_{\beta\gamma}\right]
\end{aligned}
$$

Let me know if you find nd Poisson commuting functions !
(we know it for $d = 1, 2$, or for $n = 1$)

Thank you for your attention

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