CHAPTER 5

Uniformization of surfaces of high genus

The goal of this chapter is to prove the following theorem:

Theorem 5.1. *Let (Σ^2, g_0) be a closed Riemannian surface with negative Euler characteristic. Then the unique solution to the normalized Ricci flow starting from g_0 converges exponentially fast in any C^k norm to a smooth constant-curvature metric g_∞ with negative curvature as t tends to $+\infty$.*

By Gauss-Bonnet theorem, it is equivalent to require that the mean value of the scalar curvature of g_0 is negative, i.e. $\int_\Sigma \mathrm{R}_{g_0}\, d\mu_{g_0} < 0$.

Also, since we are dealing with the normalized Ricci flow, by definition, we get:

$$\frac{\partial}{\partial t} g(t) = -\mathrm{R}_{g(t)}\, g(t) + r_{g(t)} g(t), \quad r_{g(t)} := (\mathrm{vol}_{g(t)}\, \Sigma)^{-1} \int_\Sigma \mathrm{R}_{g(t)}\, d\mu_{g(t)}, \quad t \in (0, T).$$

Invoking Gauss-Bonnet theorem once more, we see that $r_{g(t)}$ is constant in time and therefore, we will denote this constant by r so that we will work with the following flow:

$$\frac{\partial}{\partial t} g(t) = -\mathrm{R}_{g(t)}\, g(t) + r g(t), \quad t \in (0, T).$$

The strategy for proving Theorem 5.1 is as follows:

(i) Prove that the normalized Ricci flow exists for all time.

(ii) Prove that the scalar curvature converges to a negative constant as t tends to $+\infty$ exponentially fast.

(iii) Prove that all higher covariant derivatives of the scalar curvature decays exponentially fast as t tends to $+\infty$.

(iv) Invoke Proposition 3.10 to end the proof of Theorem 5.1.

1. Proof of Theorem 5.1

Since we will mainly work with the normalized Ricci flow (NRF), we need to adapt our previous computations from previous chapters.

Let us derive the evolution of the scalar curvature along the NRF first:

Lemma 5.2. *Under the NRF on a surface, one has:*

$$\frac{\partial}{\partial t} \mathrm{R}_{g(t)} = \Delta_{g(t)} \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)}(\mathrm{R}_{g(t)} - r), \quad t \in (0, T).$$

Proof. Recall that if $g = e^u g_0$ for some smooth function f on Σ then:

$$\mathrm{R}_g = e^{-u}(-\Delta_{g_0} u + \mathrm{R}_{g_0}).$$

Now, if we apply this formula to our favorite NRF solution $(g(t) = e^{u(t)} g_0)_{t \in (0,T)}$ and if we differentiate it, it gives:

$$\begin{aligned}
\frac{\partial}{\partial t} \mathrm{R}_{g(t)} &= -\frac{\partial u}{\partial t} \mathrm{R}_{g(t)} - e^{-u(t)} \Delta_{g_0} \left(\frac{\partial u}{\partial t} \right) \\
&= -\frac{\partial u}{\partial t} \mathrm{R}_{g(t)} - \Delta_{g(t)} \left(\frac{\partial u}{\partial t} \right),
\end{aligned}$$

where we have used the conformal invariance of the Laplacian in dimension 2. Since $\partial_t g(t) = -(\mathrm{R}_{g(t)} - r) g(t)$, $\partial_t u(t) = r - \mathrm{R}_{g(t)}$. This fact implies the expected result.

□

From now on, let us consider the solution $(g(t))_{t\in(0,T)}$ to the normalized Ricci flow coming out of g_0 with $r<0$.

The minimum principle from Chapter 3 applied to the evolution equation satisfied by the scalar curvature obtained in Lemma 5.2 lets us bound the scalar curvature from below by comparing with the solution to the ODE $\frac{d}{dt}s(t)=s(t)(s(t)-r)$, $s(0)=\min_\Sigma \mathrm{R}_{g_0}$:

Proposition 5.3. *Along the NRF, if $r<0$, then*

$$\mathrm{R}_{g(t)}-r\geq \frac{r}{1-\left(1-\frac{r}{\min_\Sigma \mathrm{R}_{g_0}}\right)e^{rt}}-r\geq (\min_\Sigma \mathrm{R}_{g_0}-r)e^{rt}.$$

Exercise 5.4. *Derive an analogous statement of Proposition 5.3 for $r=0$ and $r>0$.*

All we would need to complete the first step of the proof of Theorem 5.1 as explained at the beginning of this chapter is an upper bound on the scalar curvature analogous to the one obtained for the minimum of scalar curvature in the previous proposition. Using the maximum principle is not useful here since the upper bound blows up whenever $\max_\Sigma \mathrm{R}_{g_0}<0$ in finite time.

To circumvent this issue, we mimic gradient Ricci solitons $(\Sigma,g,\nabla^g f)$ which here take the form:

$$(r-\mathrm{R}_g)g=2\nabla^{g,2}f.$$

By tracing this equation, we get a Poisson equation:

$$\Delta_g f=r-\mathrm{R}_g,\quad \text{on } \Sigma.$$

This equation is solvable since the righthand side has zero mean value. Therefore, we define a function $f_0(t)$ for each time $t\in(0,T)$ such that

$$\Delta_{g(t)}f_0(t)=r-\mathrm{R}_{g(t)},\quad \text{on } \Sigma. \tag{1.1}$$

Such a solution $f_0(t)$ is unique if for instance its mean value is zero which makes it differentiable with respect to the parameter t. The next lemma ensures that there exists another normalization that is more suitable for our purpose here:

Lemma 5.5. *There exists a function $c(t)$ of time only such that $f(t):=f_0(t)+c(t)$ satisfies:*

$$\frac{\partial}{\partial t}f(t)=\Delta_{g(t)}f(t)+rf(t),\quad \textit{on } \Sigma.$$

Proof. Let us differentiate (1.1) with respect to time with the help of Proposition 1.15 from Chapter 1:

$$\begin{aligned}\frac{\partial}{\partial t}\Delta_{g(t)}f_0(t)&=(\mathrm{R}_{g(t)}-r)\Delta_{g(t)}f_0(t)+\Delta_{g(t)}\left(\frac{\partial}{\partial t}f_0\right)\\&=-(\mathrm{R}_{g(t)}-r)^2+\Delta_{g(t)}\left(\frac{\partial}{\partial t}f_0\right)\\&=-(\Delta_{g(t)}f_0)^2+\Delta_{g(t)}\left(\frac{\partial}{\partial t}f_0\right).\end{aligned}$$

On the other hand, Lemma 5.2 gives:

$$\begin{aligned}\frac{\partial}{\partial t}\Delta_{g(t)}f_0(t)&=-\frac{\partial}{\partial t}(\mathrm{R}_{g(t)}-r)\\&=-\Delta_{g(t)}\mathrm{R}_{g(t)}-\mathrm{R}_{g(t)}(\mathrm{R}_{g(t)}-r)\\&=\Delta_{g(t)}(\Delta_{g(t)}f_0)+\mathrm{R}_{g(t)}\Delta_{g(t)}f_0\\&=\Delta_{g(t)}(\Delta_{g(t)}f_0)-(\Delta_{g(t)}f_0)^2+r\Delta_{g(t)}f_0.\end{aligned}$$

Therefore, combining the two previous computations leads to:

$$\Delta_{g(t)}\left(\frac{\partial}{\partial t}f_0-\Delta_{g(t)}f_0-rf_0\right)=0.$$

As a harmonic function on a closed Riemannian manifold, it must be constant: there exists $d(t)$ such that $\frac{\partial}{\partial t}f_0-\Delta_{g(t)}f_0-rf_0=d(t)$ for each $t\in(0,T)$. Let $c(t)$ be the solution to the ODE $\frac{d}{dt}c-rc=-d$, $c(0)=0$. This choice implies the desired statement. □

As a straightforward consequence of the maximum and the minimum principle, we get:

Corollary 5.6. *Along the NRF, there exists a positive constant C such that for all $t\in(0,T)$:*

$$|f(t)|\leq Ce^{rt}.$$

Heuristically, if $T=+\infty$, we already see that the potential goes to 0 exponentially fast as t tends to $+\infty$ in case $r<0$.

We can already show that along the NRF, the metrics are uniformly equivalent:

Proposition 5.7. *Along the NRF, there exists a uniform constant $C\geq 1$ depending on g_0 only such that:*

$$C^{-1}g_0\leq g(t)\leq Cg_0.$$

Proof. It is just a matter of combining all the bounds we got so far:

$$\partial_t g(t)=(r-\mathrm{R}_{g(t)})g(t)=\Delta_{g(t)}f(t)g(t)=(\partial_t f(t)-rf(t))g(t),\quad t\in(0,T).$$

Here we have used the definition of the NRF together with the normalisation of the potential $f(t)$ given by Lemma 5.5.

If $p\in\Sigma$ and $v\in T_p\Sigma$ is a time-independent tangent vector, the function $g(t)(v,v)=:y(t)$ satisfies the ODE $\partial_t\log y=\partial_t f(t)-rf(t)$, $y(0)=g_0(v,v)$. Integrating this ODE leads to:

$$\log\left(\frac{g(t)(v,v)}{g_0(v,v)}\right)=f(t)-f(0)-r\int_0^t f(s)\,ds.$$

Now, thanks to Corollary 5.6, the righthand side of the previous estimate is uniformly bounded in time if $r<0$. This implies the desired bounds once the previous estimate is exponentialized.

□

In view of completing the existence of the NRF for all time, the previous bounds are not strong enough. All we need in order to apply the criteria from Proposition 3.10 is to bound the scalar curvature from above uniformly in time. For this purpose, define the first order quantity:

$$\mathrm{R}_{g(t)}-r+|\nabla^{g(t)}f(t)|^2_{g(t)}=:H(t).$$

This quantity naturally appeared in the context of Ricci solitons: see Chapter 2. More precisely, the soliton identities from Lemma 2.5 show that $H(t)-rf(t)$ is a constant on a gradient Ricci soliton with soliton vector field $\nabla^{g(t)}f(t)$. The next crucial result computes the evolution equation satisfied by the auxiliary function $H(t)$:

Proposition 5.8. *Along the NRF,*

$$\frac{\partial}{\partial t}H(t)=\Delta_{g(t)}H(t)-2|M(t)|^2_{g(t)}+rH(t),$$
$$M(t):=\nabla^{g(t),2}f(t)-\frac{\Delta_{g(t)}f(t)}{2}g(t).$$

The tensor $M(t)$ is trace free, i.e. $\mathrm{tr}_{g(t)}M(t)=0$ and measures how far the solution is from being a Ricci soliton since if $M(t)=0$ then $(r-\mathrm{R}_{g(t)})g(t)=\Delta_{g(t)}f(t)g(t)=2\nabla^{g(t),2}f(t)$.

Proof. We have already computed the evolution equation satisfied by $\mathrm{R}_{g(t)}$ in Lemma 5.2:

$$\partial_t(\mathrm{R}_{g(t)}-r)=\Delta_{g(t)}(\mathrm{R}_{g(t)}-r)+\mathrm{R}_{g(t)}(\mathrm{R}_{g(t)}-r)=\Delta_{g(t)}(\mathrm{R}_{g(t)}-r)+(\Delta_{g(t)}f(t))^2+r(\mathrm{R}_{g(t)}-r).$$

Now,

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^{g(t)} f(t)|^2_{g(t)} &= -(r - \mathrm{R}_{g(t)})|\nabla^{g(t)} f(t)|^2_{g(t)} + 2g(t)\left(\nabla^{g(t)}\left(\frac{\partial}{\partial t} f(t)\right), \nabla^{g(t)} f(t)\right) \\ &= (\mathrm{R}_{g(t)} + r)|\nabla^{g(t)} f(t)|^2_{g(t)} + 2g(t)\left(\nabla^{g(t)}\Delta_{g(t)} f(t), \nabla^{g(t)} f(t)\right) \\ &= r|\nabla^{g(t)} f(t)|^2_{g(t)} + 2g(t)\left(\Delta_{g(t)}\nabla^{g(t)} f(t), \nabla^{g(t)} f(t)\right) \\ &= r|\nabla^{g(t)} f(t)|^2_{g(t)} - 2|\nabla^{g(t),2} f(t)|^2_{g(t)} + \Delta_{g(t)}|\nabla^{g(t)} f(t)|^2_{g(t)}.\end{aligned}$$

Here we have used the Bochner formula in the third equality. Combining the two previous computations gives:

$$\frac{\partial}{\partial t} H(t) = \Delta_{g(t)} H(t) + rH(t) - 2|\nabla^{g(t),2} f(t)|^2_{g(t)} + (\Delta_{g(t)} f(t))^2,$$

which leads to the expected reaction diffusion equation by definition of $M(t)$. □

As an immediate consequence of the maximum principle applied to the equation satisfied by H:

Corollary 5.9. *Along the NRF, there exists a uniform positive constant C such that:*

$$\mathrm{R}_{g(t)} - r \leq H(t) \leq Ce^{rt}, \quad t \in (0, T).$$

The combination of Proposition 5.3 and Corollary 5.9 gives not only a uniform bound on the scalar curvature (which implies the existence of the solution for all time by Theorem 3.14) but also an exponential convergence rate for the difference $\mathrm{R}_{g(t)} - r$: there exists a positive constant C such that for all $t \geq 0$,

$$|\,\mathrm{R}_{g(t)} - r| \leq Ce^{rt}.$$

This estimate alone proves that the NRF converges in the C^0 norm to a metric g_∞. *If* we knew that the convergence was happening in any C^k norm for $k \geq 2$ then the same estimate shows that the curvature of the metric g_∞ is constant. We are then left with establishing an exponential decay rate on higher covariant derivatives.

Notice that Bernstein-Shi estimates for the Ricci flow (not the normalized Ricci flow) proved in [Proposition 3.8, Chapter 3], are not enough to conclude. (Check this: the associated Ricci flow is $\tilde{g}(\tau) := (-r)\tau g(\log \tau/(-r))$, $\tau \geq 1$.)

One way to circumvent this issue is to prove Bernstein-Shi estimates for the normalized Ricci flow in the same way we proved Proposition 3.8. We admit them here and refer to [**CK04**, Chapter 5] for a proof.

Then we are in a good position to invoke Proposition 3.10 to ensure the smooth convergence of the NRF to a constant curvature metric.

2. The torus case

We present this section as a sequence of a problem to solve by yourself, the main goal being:

Theorem 5.10. *Let (Σ^2, g_0) be a closed Riemannian surface with vanishing Euler characteristic. Then the unique solution to the normalized Ricci flow starting from g_0 converges polynomially in any C^k norm to a smooth flat metric g_∞ as t tends to $+\infty$.*

Proof. Theorem 5.10 is proved through a series of claims.

- Prove the analogous statement of Proposition 5.3: $\mathrm{R}_{g(t)} \geq -\frac{1}{t}$ for $t \in (0, T)$.
- Prove that the potential $f(t)$ is uniformly bounded.
- Deduce from this that the metrics are uniformly equivalent: this echoes Proposition 5.7.
- Prove that $H(t)$ is uniformly bounded from above: see Corollary 5.9.
- Conclude that the NRF exists for all time.
- In order to show curvature decay:
  - (i) Derive a diffusion equation for the function
  $$t|\nabla^{g(t)} f(t)|^2_{g(t)} + f(t)^2,$$
  and show that
  $$|\nabla^{g(t)} f(t)|^2_{g(t)} \leq \frac{C}{1+t},$$
  for $t \geq 0$ and for some uniform positive constant C.
  - (ii) Show that the function $\mathrm{R}_{g(t)} + 2|\nabla^{g(t)} f(t)|^2_{g(t)}$ is a subsolution to the heat equation. More precisely,
  $$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left(\mathrm{R}_{g(t)} + 2|\nabla^{g(t)} f(t)|^2_{g(t)}\right) \leq -\mathrm{R}^2_{g(t)}.$$
  - (iii) Show that there exists a uniform positive constant C such that:
  $$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left[t\left(\mathrm{R}_{g(t)} + 2|\nabla^{g(t)} f(t)|^2_{g(t)}\right)\right] \leq -\frac{t}{2}\left(\mathrm{R}_{g(t)} + 2|\nabla^{g(t)} f(t)|^2_{g(t)}\right)^2 + \frac{C}{t}.$$
  - (iv) Conclude that $|\mathrm{R}_{g(t)}| \leq \frac{C}{1+t}$, $t \geq 0$.
  - (v) Invoke Shi's estimates to prove that
  $$|\nabla^{g(t),k} \mathrm{R}_{g(t)}| \leq \frac{C_k}{(1+t)^{\frac{k}{2}+1}}, \quad t \geq 0,$$

  and conclude from Lemma 3.11 that the covariant derivatives of $g(t)$ with respect to g_0 are uniformly bounded on Σ.
- Is this enough to conclude the convergence to a flat metric ? If yes, stop here. If not, have a look below.

(i) Define $\lambda(g(t))$ to be the first positive eigenvalue of $-\Delta_{g(t)}$ acting on functions which can be characterized by:

$$\lambda(g(t)) = \inf_{\varphi\in H^1(\Sigma)\setminus\{0\}\,|\,\int_M \varphi\, d\mu_{g(t)}=0} \frac{\|\nabla^{g(t)}\varphi\|^2_{L^2(g(t))}}{\|\varphi\|^2_{L^2(g(t))}}.$$

Show that $\inf_{t\geq 0}\lambda(g(t)) =: \lambda_0 > 0$.

Hint: use the minimax principle that states that

$$\lambda(g(t)) = \min\left\{\max\left\{\frac{\|\nabla^{g(t)}\varphi\|^2_{L^2(g(t))}}{\|\varphi\|^2_{L^2(g(t))}},\ \varphi\in E\right\}\,|\,\mathrm{dim} E = 2\right\},$$

in order to compare $\lambda(g(t))$ and $\lambda(g(0))$.

(ii) Show that the L^2 norm of $\mathrm{R}_{g(t)}$ decays exponentially fast as t tends to $+\infty$.

(iii) Conclude that the curvature $\mathrm{R}_{g(t)}$ decays pointwise exponentially fast together with its covariant derivatives with the help of suitable interpolation inequalities whose proofs can be found for instance in [**Aub98**]: for $1\leq q, r\leq\infty$, and integers $0\leq j<m$, for all smooth functions ϕ on a compact manifold (M^n, g) such that $\int_M \phi\, d\mu_g = 0$,

$$\|\nabla^{g,j}\phi\|_{L^p(g)} \leq C\|\nabla^{g,m}\phi\|^{1-a}_{L^r(g)}\|\phi\|^a_{L^q(g)}, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r}-\frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all $a\in[j/m, 1]$ and $p\geq 0$ and where C is a positive constant depending on the geometry of g and the above parameters.

Hint: first apply those to the background metric $g(0)$ and $p=+\infty$, $j=0$ and suitable other parameters to show that $\mathrm{R}_{g(t)}$ decays pointwise exponentially fast.

(iv) Conclude the proof of Theorem 5.10

$\square$