CHAPTER 7

Perelman's entropy and (some of) its consequences

1. Definition and properties

Let us start with a formal definition. Given a closed Riemannian manifold (M^n, g), a smooth function f on M and a positive constant τ, Perelman's entropy is defined by:

$$\mathcal{W}(g, f, \tau) := \int_M \left(\tau\left(\mathrm{R}_g + 2\Delta_g f - |\nabla^g f|_g^2\right) + (f - n)\right) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu_g.$$

Since $\int_M \left(\Delta_g f - |\nabla^g f|_g^2\right) e^{-f} d\mu_g = 0$ by observing that the integrand can be written as $\mathrm{div}_g\left(e^{-f}\nabla^g f\right)$, Perelman's entropy can be reformulated as:

$$\mathcal{W}(g, f, \tau) = \int_M \left(\tau\left(\mathrm{R}_g + |\nabla^g f|_g^2\right) + (f - n)\right) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu_g. \tag{1.1}$$

This formula will be more suited to show that $\mathcal{W}$ is coercive in a sense to be defined later.

Observe that if ψ is a diffeomorphism of M, then the change of variable theorem shows that:

$$\mathcal{W}(\phi^* g, \phi^* f, \tau) = \mathcal{W}(g, f, \tau).$$

Moreover, if $\lambda > 0$,

$$\mathcal{W}(\lambda g, f, \lambda\tau) = \mathcal{W}(g, f, \tau).$$

One motivation for considering such a quantity comes from shrinking gradient Ricci soliton:

Lemma 7.1. *Let (M^n, g) be a Riemannian manifold endowed with a smooth function f. Let*

$$T := \mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_{\nabla^g f}(g) - \frac{g}{2}.$$

Then,

$$2\left(\mathrm{div}_g T - T(\nabla^g f)\right) = \nabla^g\left(\mathrm{R}_g + 2\Delta_g f - |\nabla^g f|_g^2 + f\right). \tag{1.2}$$

Proof. The proof is related to the proof of the Bianchi identity as in the proof of Lemma 2.5. Indeed,

$$\begin{aligned} 2(\mathrm{div}_g T - T(\nabla^g f)) &= 2\,\mathrm{div}_g \mathrm{Ric}(g) + \mathrm{div}_g \mathcal{L}_{\nabla^g f}(g) - 2\,\mathrm{Ric}(g)(\nabla^g f) - \nabla^g|\nabla^g f|_g^2 + \nabla^g f \\ &= \nabla^g\left(\mathrm{R}_g - |\nabla^g f|_g^2 + f\right) + \frac{1}{2}\nabla^g \mathrm{tr}_g \mathcal{L}_{\nabla^g f}(g) + \Delta_g \nabla^g f - \mathrm{Ric}(g)(\nabla^g f) \\ &= \nabla^g\left(\mathrm{R}_g + 2\Delta_g f - |\nabla^g f|_g^2 + f\right), \end{aligned}$$

where we have used the Bochner formula for vector fields and functions in the second line and the last lien respectively. □

Corollary 7.2. *Let (M^n, g, X) be a closed shrinking Ricci soliton. Then there exists a smooth function f on M such that $(M^n, g, \nabla^g f)$ is a shrinking **gradient** Ricci soliton if and only if there exists a smooth function f such that*

$$\mathrm{R}_g + 2\Delta_g f - |\nabla^g f|_g^2 + f = cst. \tag{1.3}$$

Notice that the quantity in (1.3) is exactly the integrand in the definition of Perelman's entropy up to an additive constant n. This constant is designed for the integrand to be 0 on a shrinking gradient Ricci soliton thanks to Lemma 2.5.

PROOF. Let $T := \mathrm{Ric}(g)+\frac{1}{2}\mathcal{L}_{\nabla^g f}(g)-\frac{g}{2}$ as in Lemma 7.1. The tensor T measures the obstruction to be a shrinking **gradient** Ricci soliton.

Notice that since (M^n, g, X) is a shrinking Ricci soliton, $T = \frac{1}{2}\mathcal{L}_{\nabla^g f - X}(g)$.

Now, observe the following integration by parts:

$$2\int_M |T|_g^2\, e^{-f} d\mu_g = \int_M \langle \mathcal{L}_{\nabla^g f - X}(g), T\rangle_g\, e^{-f} d\mu_g = -2\int_M \left\langle \nabla^g f - X, \mathrm{div}_g\left(e^{-f}T\right)\right\rangle_g d\mu_g.$$

Since $\mathrm{div}_g\left(e^{-f}T\right) = \mathrm{div}_g T - T(\nabla^g f)$, the corollary follows. □

As suggested by Corollary 7.2, let us see now if Perelman's entropy can be minimized. The first step is to check whether $\mathcal{W}$ is bounded from below on functions f such that $\int_M e^{-f}\, d\mu_g = (4\pi\tau)^{n/2}$. For doing so, remark that if $\varphi := (4\pi\tau)^{-n/4}e^{-f/2}$, formula (1.1) can be reformulated as:

$$\overline{\mathcal{W}}_\tau(g,\varphi) := \mathcal{W}(g,f,\tau) = \int_M \tau\left(4|\nabla^g\varphi|_g^2 + \mathrm{R}_g\,\varphi^2\right) - \varphi^2\log\varphi^2\, d\mu_g - \left(n + \frac{n}{2}\log(4\pi\tau)\right)\int_M \varphi^2\, d\mu_g.$$

Let $H^1(M)$ denotes the Sobolev space of functions φ in L^2 such that their distributional gradient lies in L^2 as well.

Lemma 7.3. *Let (M^n, g) be a closed Riemannian manifold. Then for any $a > 0$, there exists a constant $C = C(a, g)$ such that if $\varphi \in H^1(M)$ with $\|\varphi\|_{L^2} = 1$,*

$$\int_M \varphi^2\log\varphi^2 \le a\int_M |\nabla^g\varphi|_g^2\, d\mu_g + C.$$

In particular, for $\tau > 0$,

$$\inf\left\{\overline{\mathcal{W}}_\tau(g,\varphi)\,|\,\varphi\in H^1(M),\ \|\varphi\|_{L^2} = 1\right\} > -\infty.$$

PROOF. Let $c(n) > 0$ such that $c(n)\varphi^2\log\varphi^2 \le \varphi^{2+2/n}$ for all $\varphi\in\mathbb{R}$. Then, if $\|\varphi\|_{L^2} = 1$,

$$\begin{aligned}\int_M \log\varphi^2\varphi^2\, d\mu_g &\le c(n)^{-1}\int_M \varphi^{2+2/n}\, d\mu_g \le \varepsilon\int_M \varphi^{2+4/n}\, d\mu_g + c(n,\varepsilon)\int_M \varphi^2\, d\mu_g\\ &\le \varepsilon\int_M \varphi^2\cdot\varphi^{4/n}\, d\mu_g + c(n,\varepsilon)\\ &\le \varepsilon\|\varphi\|^2_{L^{\frac{2n}{n-2}}}\|\varphi\|^{\frac{4}{n}}_{L^2} + c(n,\varepsilon) = \varepsilon\|\varphi\|^2_{L^{\frac{2n}{n-2}}} + c(n,\varepsilon).\end{aligned}$$

Now, the Sobolev inequality $\|\varphi\|^2_{L^{\frac{2n}{n-2}}} \le C_S\left(\|\nabla^g\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2}\right)$ for all $\varphi\in H^1(M)$ implies:

$$\int_M \log\varphi^2\varphi^2\, d\mu_g \le \varepsilon C_S\|\nabla^g\varphi\|^2_{L^2} + \underbrace{c(n,\varepsilon)C_S}_{=c(n,\varepsilon,g)},$$

if $\|\varphi\|_{L^2} = 1$ as expected. □

Corollary 7.4. *Let (M^n, g) be a closed Riemannian manifold. Then for each $\tau > 0$, there exists a smooth minimizer $f = f(\tau, g)$ of Perelman's entropy, i.e. there exists a smooth function f_τ such that:*

$$\mathcal{W}(g,f,\tau) = \inf\left\{\overline{\mathcal{W}}_\tau(g,\varphi)\,|\,\varphi\in H^1(M),\ \|\varphi\|_{L^2} = 1,\ \varphi\ge 0\right\}.$$

Thanks to this corollary, we can define the following invariant for a metric g and a number $\tau > 0$:

$$\mu(g,\tau) := \inf\left\{\mathcal{W}(g,f,\tau)\,|\,f\in C^\infty(M),\ \int_M e^{-f}\, d\mu_g = (4\pi\tau)^{\frac{n}{2}}\right\}. \tag{1.4}$$

Due to the aforementioned scaling properties: if ψ is a diffeomorphism of M and if $\lambda > 0$,

$$\mu(\lambda\psi^* g, \lambda\tau) = \mu(g,\tau).$$

PROOF.

Claim 7.5. There exists a nonnegative minimizer φ in H^1 with unit L^2 norm.

Let $(\varphi_i)_i$ be a minimizing sequence of $\overline{\mathcal{W}}_\tau(g,\cdot)$ such that $\|\varphi_i\|_{L^2}=1$. Then Lemma 7.3 ensures that $(\varphi_i)_i$ is bounded in H^1. By Sobolev embeddings (Kondrakov's Theorem [**Aub98**, Chapter 2]): $H^1(M)\hookrightarrow L^q(M)$, $q<2n/n-2$, is compact so that there exists a subsequence converging strongly to some function φ in the L^q, $q<2n/n-2$, topology and such that it converges weakly in the H^1 norm. In particular, we get that φ has unit L^2 norm. Since $\liminf_{i\to+\infty}\|\nabla^g\varphi_i\|_{L^2}\geq\|\nabla^g\varphi\|_{L^2}$, we get that $\mu(g,\tau)=\lim_{i\to+\infty}\overline{\mathcal{W}}_\tau(g,\varphi_i)\geq\overline{\mathcal{W}}_\tau(g,\varphi)$, i.e. φ is a minimizer which can be assumed to be nonnegative since $\overline{\mathcal{W}}_\tau(g,\psi)\geq\overline{\mathcal{W}}_\tau(g,|\psi|)$ for all $\psi\in H^1$.

Let us stick to $\tau=1$ for the sake of clarity from now on.

Moreover, φ satisfies the PDE

$$-4\Delta_g\varphi+\underbrace{\underbrace{\left(\mathrm{R}_g-\left(n+\frac{n}{2}\log(4\pi)\right)-\mu(g,1)\right)}_{=:V}\varphi-\varphi\log\varphi^2}_{=:F}=0,\tag{1.5}$$

in the weak sense.

Claim 7.6. $\Delta_g\varphi$ belongs to L^p for some $p>n/2$.

Indeed, since φ is in $L^{2n/n-2}$ and V is in $L^{n/(2-\varepsilon)}$ for some small $\varepsilon>0$ since it is bounded, the product $V\cdot\varphi$ lies in $L^{2n/(n+2-2\varepsilon)}$ for $\varepsilon>0$ small enough. Now, $\varphi\in L^{2n/n-2}$ implies that for any $\delta>0$ small enough, $\varphi\log\varphi^2$ lies in $L^{(2n/n-2)-\delta}$. Therefore, F lies in $L^{2n/(n+2-2\varepsilon)}$ for $\varepsilon>0$ small enough. Calderon-Zygmund elliptic estimates show that φ lies in $L^{2n/(n-2-2\varepsilon)}$ ($\Delta_g u=f\in L^p\Rightarrow u\in L^r$ where $r^{-1}:=p^{-1}-2/n$). This improvement can be iterated a finite number of times to achieve the desired claim.

Claim 7.7. φ is $C^{2,\alpha}_{loc}$ for every $\alpha\in(0,1)$.

Since $\Delta_g\varphi$ belongs to L^p for some $p>n/2$, De Giorgi-Nash-Moser theory (see [**HL11**, Chapter 4] for instance) ensures that φ is locally bounded. Therefore, $\Delta_g\varphi$ is locally bounded which implies by Morrey's elliptic estimate that φ is $C^{1,\alpha}_{loc}$ for every $\alpha\in(0,1)$. The function $x\log x$ being locally Hölder, $\Delta_g\varphi$ is locally Hölder too which by Schauder estimates imply that φ is $C^{2,\alpha}_{loc}$. Therefore, φ satisfies (1.5) in the pointwise sense.

Claim 7.8. φ is positive.

Assume by contradiction that there is some point p in M such that $\varphi(p)=0$. Define for a smooth radial cut-off function $\tilde{\psi}(x)=\psi(r_p(x))$ where ψ is a smooth cut-off function on $\mathbb{R}$. Then, (1.5) is equivalent to:

$$4\int_0^R\psi'(r)\underbrace{\int_{S_g(p,r)}g(\nabla^g\varphi,\nabla^g r_p)\,d\sigma_g}_{=:G(r)A(r)}\,dr+\int_0^R\psi(r)\left(\underbrace{\int_{S_g(p,r)}V\varphi\,d\sigma_g}_{=:V(r)A(r)}-\underbrace{\int_{S_g(p,r)}\varphi\log\varphi^2\,d\sigma_g}_{=:L(r)A(r)}\right)dr=0,$$

where $A(r):=\int_{S_g(p,r)}d\sigma_g$. If R is small enough compared to the injectivity radius of g at p, $A(r)$ is equivalent to r^{n-1}. Now define $\Phi(r):=A(r)^{-1}\int_{S_g(p,r)}\varphi\,d\sigma_g$ and observe that since φ is $C^{2,\alpha}_{loc}$ by the previous claim, $\Phi(r)$ and $G(r)$ are differentiable on $(0,R)$ and:

$$|\Phi'(r)A(r)-G(r)A(r)|\leq\int_{S_g(p,r)}\left|H_r-\frac{A'(r)}{A(r)}\right|\varphi\,d\sigma_g\leq Cr\Phi(r),\quad r\in(0,R),\tag{1.6}$$

where H_r denotes the mean curvature of the geodesic sphere $S_g(p,r)$.

Moreover, $V(r)\leq v_0\Phi(r)$ by smoothness of V and by concavity of the function $x\log x$: $L(r)\geq\Phi(r)\log\Phi(r)^2$.

Therefore, for $r \in (0, R)$:

$$4\frac{d}{dr}(G(r)A(r)) = V(r)A(r) - L(r)A(r) \leq v_0\Phi(r)A(r) - 2\Phi(r)\log\Phi(r)A(r), \quad r \in (0, R).$$

By integrating between $r = 0$ and $r \in (0, R)$, on account that $\lim_{r\to 0^+}\Phi(r) = 0$ by assumption on φ at p:

$$G(r)A(r) \leq \frac{v_0}{4}\int_0^r \Phi(s)A(s)\,ds - \frac{1}{2}\int_0^r \Phi(s)\log\Phi(s)A(s)\,ds, \quad r \in (0, R).$$

Invoking (1.6), a further integration shows that:

$$\Phi(r) \leq C\int_0^r s\Phi(s)\,ds + \frac{v_0}{4}\int_0^r\int_0^s \Phi(t)A(t)\,dt\frac{ds}{A(s)} - \frac{1}{2}\int_0^r\int_0^s \Phi(t)\log\Phi(t)A(t)\,dt\frac{ds}{A(s)}, \quad r \in (0, R).$$

If $\delta \in (0, 1]$, let us take R sufficiently small so that $\Phi(r) \in [0, e^{-1}]$ and $-\Phi(r)\log\Phi(r) \leq C\delta^{-1}\Phi(r)^{1-\delta}$ for $r \in [0, R]$, C being a universal constant. Then, inserting this bound back to the previous estimate together with the fact that $A(r)$ is equivalent to r^{n-1} leads to:

$$\Phi(r) \leq C\left(\int_0^r s\Phi(s)\,ds + \delta^{-1}\int_0^r\int_0^s \Phi(t)^{1-\delta}t^{n-1}\,dt\frac{ds}{s^{n-1}} + \int_0^r\int_0^s \Phi(t)t^{n-1}\,dt\frac{ds}{s^{n-1}}\right), \quad r \in (0, R),$$

for some uniform positive constant C. Assume that for some $k \geq 1$, $\Phi(r) \leq r^k$ for $r \in [0, R]$ then, if $\delta := k^{-1}$, the previous estimate gives the improvement:

$$\Phi(r) \leq C\left(\frac{r^{k+2}}{k+2} + \frac{kr^{k+1}}{(k+1)(n+k-1)} + \frac{r^{k+2}}{(k+2)(k+n)}\right) \leq r^{k+1/2},$$

for $r \in [0, R]$, R being independent of k. Iterating this reasoning, we end up by proving that $\Phi(r) = 0$ for $r \in [0, R]$, i.e. $\varphi \equiv 0$ on $B_g(p, R)$ which leads by an open-closed argument that φ vanishes on M identically, contradicting the fact that its L^2-norm is 1. □

2. Rigidity of shrinking gradient Ricci solitons

The first goal of this section is to prove that shrinking Ricci solitons are *gradient* on a closed manifold. This is one of the first breakthrough due to Perelman.

Theorem 7.9. *Let (M^n, g, X) be a shrinking Ricci soliton on a closed manifold. Then there exists a smooth function f on M such that $(M^n, g, \nabla^g f)$ is a shrinking gradient Ricci soliton.*

Proof. Let $\tau = 1$ and let f be a smooth minimizer of Perelman's entropy $\mathcal{W}(g, \cdot, 1)$ ensured by Corollary 7.4. If $\varphi := e^{-f/2}/(4\pi)^{n/2}$ then the Euler-Lagrange equation satisfied by φ is:

$$\int_M \left(-4\Delta_g\varphi + \mathrm{R}_g\,\varphi - \left(\mu(g, 1) + n + \frac{n}{2}\log 4\pi + \log\varphi^2\right)\varphi\right)\psi\,d\mu_g = 0,$$

for all $\psi \in C^\infty(M)$. This is equivalent to:

$$2\Delta_g f - |\nabla^g f|_g^2 + \mathrm{R}_g + f = cst.$$

Corollary 7.2 lets us conclude the proof of this theorem. □

The second goal of this section we are concerned with is the existence of periodic solutions to the Ricci flow, also called **breathers**. Recall that the Ricci flow is infinite dimensional dynamical system on the space of metrics of a given manifold modulo scalings and diffeomorphisms. A solution to the Ricci flow $(M^n, g(t))_{t\in[0,T]}$ is a **Ricci-breather** if there exists $0 \leq t_1 < t_2 \leq T$ satisfying $g(t_2) = \alpha\phi^*g(t_1)$ for some $\alpha > 0$ and some diffeomorphism ϕ of M. Observe that a shrinking Ricci soliton is a Ricci-breather with $\alpha < 1$, a steady Ricci soliton is a Ricci-breather with $\alpha = 1$ and an expanding Ricci soliton is a Ricci-breather with $\alpha > 1$. The question whether there exists non-trivial Ricci-breathers with $\alpha < 1$ on a closed manifold was answered by Perelman:

Theorem 7.10. *Let $(M^n, g(t))_{t\in[0,T]}$ be a Ricci-breather with $\alpha < 1$. Then there exists a smooth function f_0 on M such that $(M^n, g(0), \nabla^{g(0)}f_0)$ is a shrinking gradient Ricci soliton.*

Before proving Theorem 7.10, we need to recall the fundamental monotonicity of Perelman's entropy along the Ricci flow:

Proposition 7.11. *Let $(M^n, g(t))_{t\in(0,T)}$ be a solution to the Ricci flow on a closed manifold. Let $(f(t), \tau(t))$ evolve as follows:*

$$\frac{\partial f}{\partial t} = -\Delta_{g(t)} f - \mathrm{R}_{g(t)} + |\nabla^{g(t)} f|^2_{g(t)} + \frac{n}{2\tau}, \quad \frac{d\tau}{dt} = -1. \tag{2.1}$$

Then:

$$\left[\frac{\partial}{\partial t} + \Delta_{g(t)} - \mathrm{R}_{g(t)}\right]\left[\tau(t)\left(\mathrm{R}_{g(t)} + 2\Delta_{g(t)} f(t) - |\nabla^{g(t)} f(t)|^2_{g(t)}\right) + f(t) - n\right)(4\pi\tau(t))^{-\frac{n}{2}} e^{-f(t)}\right] =$$

$$2\tau(t)\left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2} f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)} (4\pi\tau(t))^{-\frac{n}{2}} e^{-f(t)}.$$

In particular,

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau(t)\int_M \left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2} f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)} \frac{e^{-f(t)}}{(4\pi\tau(t))^{\frac{n}{2}}} d\mu_{g(t)}, \quad t \in (0, T).$$

In particular, $t \to \mu(g(t), \tau(t))$ is increasing unless $(M^n, g(t))_{t\in(0,T)}$ is a shrinking gradient Ricci soliton.

The system (2.1) can appear slightly ad-hoc at first sight, it can be reinterpreted in terms of prescribing a density $\rho(t) := (4\pi\tau(t))^{-\frac{n}{2}} e^{-f(t)}$ as follows:

$$\frac{\partial}{\partial t}\rho(t) = -\Delta_{g(t)}\rho(t) + \mathrm{R}_{g(t)}\,\rho(t), \quad \frac{d\tau}{dt} = -1. \tag{2.2}$$

The density $\rho(t)$ must satisfy a backward heat equation with a potential given by the scalar curvature along the Ricci flow. System (2.2) is not well-posed, it can however be solved backward in time, i.e. by prescribing an "initial" condition later in time and by solving the corresponding heat-type equation obtained by reversing the time variable.

The operator $\partial_t + \Delta_{g(t)} - \mathrm{R}_{g(t)}$ is the formal adjoint of the heat operator $\partial_t - \Delta_{g(t)} =: \Box_{g(t)}$ along the Ricci flow as it can be checked by a straightforward integration by parts:

$$\int_{M\times[a,b]} (\Box_{g(t)}\varphi)\psi \, d\mu_{g(t)} dt = \int_{M\times[a,b]} \varphi(\Box^*_{g(t)}\psi)\, d\mu_{g(t)} dt,$$

for any smooth function φ, ψ with compact support in $M \times (a, b)$. Here $\Box^*_{g(t)}$ denotes the operator $-\partial_t - \Delta_{g(t)} + \mathrm{R}_{g(t)}$.

In particular, the system (2.2) already shows thanks to a similar reasoning applied to $\psi = \rho$ and $\varphi = 1$ that the density $\rho(t)$ is a probability density along the Ricci flow:

$$\frac{d}{dt}\int_M \rho(t)\, d\mu_{g(t)} = 0, \quad t \in (0, T).$$

Proof of Proposition 7.11. It is a brutal force computation. The difficulty is not to lose track of the expected result!

Observe first that if $u(t) := 2\Delta_{g(t)} f(t) + \mathrm{R}_{g(t)} - |\nabla^{g(t)} f(t)|^2_{g(t)}$,

$$\begin{aligned}\frac{\partial}{\partial t}u(t) = {} & \Delta_{g(t)}\mathrm{R}_{g(t)} + 2|\mathrm{Ric}(g(t))|^2_{g(t)} \\ & + 2\left(\Delta_{g(t)}(\partial_t f(t)) + 2\langle \mathrm{Ric}(g(t)), \nabla^{g(t),2} f(t)\rangle_{g(t)}\right) \\ & - 2\,\mathrm{Ric}(g(t))(\nabla^{g(t)} f(t), \nabla^{g(t)} f(t)) - 2g(t)(\nabla^{g(t)}\partial_t f(t), \nabla^{g(t)} f(t)).\end{aligned}$$

Now,

$$\begin{aligned}\Delta_{g(t)}(\partial_t f(t)) &= -\Delta_{g(t)}\Delta_{g(t)}f(t) - \Delta_{g(t)}\,\mathrm{R}_{g(t)} + \Delta_{g(t)}|\nabla^{g(t)}f(t)|^2_{g(t)}\\ &= -\Delta_{g(t)}(\Delta_{g(t)}f(t) + \mathrm{R}_{g(t)}) + 2|\nabla^{g(t),2}f(t)|^2_{g(t)}\\ &\quad + 2\,\mathrm{Ric}(g(t))(\nabla^{g(t)}f(t), \nabla^{g(t)}f(t)) + 2g(t)(\nabla^{g(t)}\Delta_{g(t)}f(t), \nabla^{g(t)}f(t)),\end{aligned}$$

and,

$$-2g(t)(\nabla^{g(t)}\partial_t f(t), \nabla^{g(t)}f(t)) = 2g(t)(\nabla^{g(t)}\left(\Delta_{g(t)}f(t) + \mathrm{R}_{g(t)} - |\nabla^{g(t)}f(t)|^2_{g(t)}\right), \nabla^{g(t)}f(t)),$$

so that,

$$\frac{\partial}{\partial t}u(t) = -\Delta_{g(t)}u(t) + 2g(t)(\nabla^{g(t)}u(t), \nabla^{g(t)}f(t)) + 2|\,\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t)|^2_{g(t)}.$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\tau(t)u(t) + f(t) - n\right] =\\ &\tau(t)\left(-\Delta_{g(t)}u(t) + 2g(t)(\nabla^{g(t)}u(t), \nabla^{g(t)}f(t))\right) + 2\tau(t)|\,\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t)|^2_{g(t)}\\ &- \left(\mathrm{R}_{g(t)} + 2\Delta_{g(t)}f(t) - |\nabla^{g(t)}f(t)|^2_{g(t)}\right) - \Delta_{g(t)}f(t) - \mathrm{R}_{g(t)} + |\nabla^{g(t)}f(t)|^2_{g(t)} + \frac{n}{2\tau(t)}\\ &= \tau(t)\left(-\Delta_{g(t)}u(t) + 2g(t)(\nabla^{g(t)}u(t), \nabla^{g(t)}f(t))\right) + 2\tau(t)\left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)}\\ &\quad + 2|\nabla^{g(t)}f(t)|^2_{g(t)} - \Delta_{g(t)}f(t)\\ &= -\Delta_{g(t)}v(t) + 2g(t)(\nabla^{g(t)}v(t), \nabla^{g(t)}f(t)) + 2\tau(t)\left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)}.\end{aligned}$$

As an intermediate conclusion, if $v(t) := \tau(t)u(t) + f(t) - n$,

$$\begin{aligned}-\Box^*_{g(t)}\left(v(t)\rho(t)\right) &= \left(\partial_t v(t) + \Delta_{g(t)}v(t)\right)\rho(t) + 2g(t)(\nabla^{g(t)}v(t), \nabla^{g(t)}\rho(t)) - v(t)\left(\Box^*_{g(t)}\rho(t)\right)\\ &= \left(\partial_t v(t) + \Delta_{g(t)}v(t)\right)\rho(t) + 2g(t)(\nabla^{g(t)}v(t), \nabla^{g(t)}\rho(t))\\ &= \left(\partial_t v(t) + \Delta_{g(t)}v(t) - 2g(t)(\nabla^{g(t)}v(t), \nabla^{g(t)}f(t))\right)\rho(t)\\ &= 2\tau(t)\left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)}\rho(t),\end{aligned}$$

as expected. Therefore,

$$\begin{aligned}\frac{d}{dt}W(g(t), f(t), \tau(t)) &= \int_M -\Box^*_{g(t)}\left(v(t)\rho(t)\right)d\mu_{g(t)}\\ &= 2\tau(t)\int_M\left|\mathrm{Ric}(g(t)) + \nabla^{g(t),2}f(t) - \frac{g(t)}{2\tau(t)}\right|^2_{g(t)}\rho(t)d\mu_{g(t)},\end{aligned}$$

by integration by parts. □

We are in a good position to prove Theorem 7.10:

Proof of Theorem 7.10. Let $\tau(t) := \tau - t > 0$ where $\tau > t_2$ remains to be defined. Then by the monotonicity result from proposition 7.11,

$$\mu(g(t_1), \tau(t_1)) \leq \mu(g(t_2), \tau(t_2)),$$

with equality if and only if $(M^n, g(t))_{t\in(0,T)}$ is a shrinking gradient Ricci soliton. Now, by assumption, $g(t_2) = \alpha\phi^* g(t_1)$. One can check that: $\mu(g(t_2), \tau(t_2)) = \mu(g(t_1), \alpha^{-1}\tau(t_2))$ by the scaling properties of the geometric quantities involved in the definition of $\mathcal{W}$. Therefore, all we need to ensure is a number τ satisfying the conditions $\tau > t_2$ and $\alpha^{-1}\tau(t_2) = \tau(t_1)$. The number $\tau := (\alpha^{-1}t_2 - t_1)/(\alpha^{-1} - 1)$ does the job since $\alpha < 1$ and we are done.

$\square$