

# Around Van den Bergh's double brackets

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## Motivation: Kontsevich-Rosenberg principle

Fix (unital) associative algebra  $A$  over field  $\mathbb{k}$  ( $\text{char}(\mathbb{k})=0$ )

For  $n \geq 1$ ,  $n$ -th representation space  $\text{Rep}_n(A)$  is scheme with  $B$ -points

$$\text{Rep}_n(A)(B) := \text{Hom}_{\text{Alg.}}(A, \text{Mat}(n \times n, B))$$

$\mathbb{k}[\text{Rep}_n(A)]$  is generated by 'matrix' symbols  $a_{ij}$  for  $a \in A$ ,  $1 \leq i, j \leq n$

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Motto: [Kontsevich-Rosenberg, '00]

*"A noncommutative structure of some kind on  $A$  should give an analogous commutative structure on all schemes  $\text{Rep}_n(A)$ ,  $n \geq 1$ ."*

structure  $\mathcal{S}_{nc}$  in *Alg.*  $\longrightarrow$  structure  $\mathcal{S}$  in *Com.Alg.*  
(e.g. formally smooth) (e.g. smooth)

# Van den Bergh's double brackets (1)

$A^{\otimes 2} := A \otimes_{\mathbb{k}} A$ , mult.  $(a \otimes b)(c \otimes d) = ac \otimes bd$ , swap  $\tau_{(12)}a \otimes b = b \otimes a$ .

Definition ([Van den Bergh, *double Poisson algebras*, '08])

A **double bracket** on  $A$  is a  $\mathbb{k}$ -bilinear map  $\{\{-, -\}\} : A \times A \rightarrow A^{\otimes 2}$  with

- 1  $\{\{a, b\}\} = -\tau_{(12)} \{\{b, a\}\}$  (cyclic antisymmetry)
- 2  $\{\{a, bc\}\} = (b \otimes 1) \{\{a, c\}\} + \{\{a, b\}\} (1 \otimes c)$  (outer derivation)
- 3  $\{\{ad, b\}\} = (1 \otimes a) \{\{d, b\}\} + \{\{a, b\}\} (d \otimes 1)$  (inner derivation)

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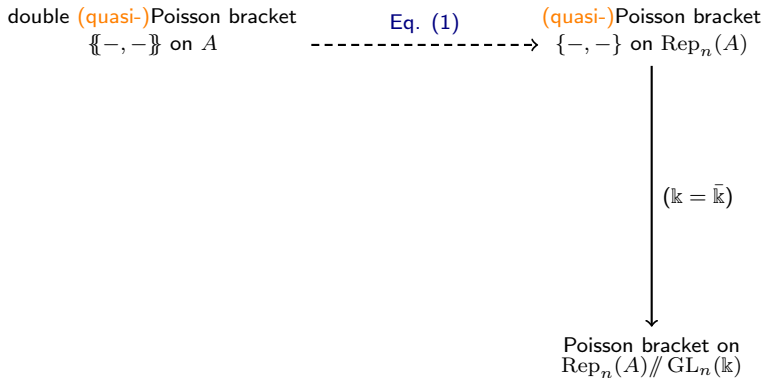
**Proposition** ([Van den Bergh, '08])

If  $A$  is endowed with a double bracket  $\{\{-, -\}\}$ , then  $\text{Rep}_n(A)$  admits a unique  $\text{GL}_n(\mathbb{k})$ -invariant antisymmetric biderivation  $\{-, -\}$  satisfying

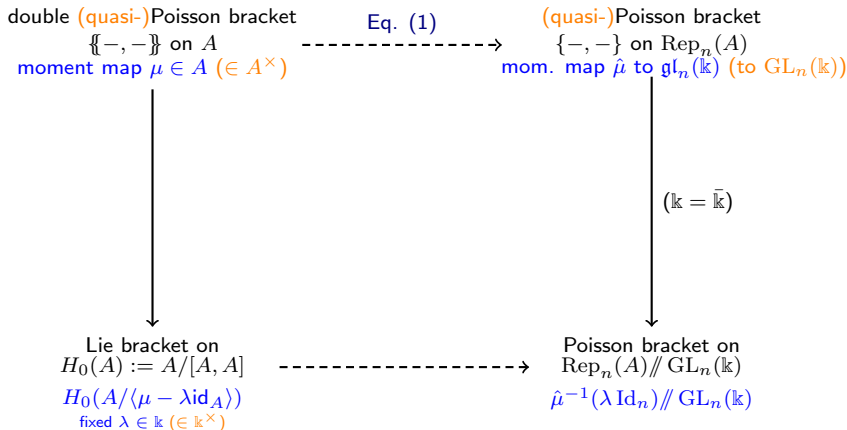
$$\{a_{ij}, b_{kl}\} = \{\{a, b\}\}'_{kj} \{\{a, b\}\}''_{il}. \quad (1)$$

(We write  $\{\{a, b\}\} =: \{\{a, b\}\}' \otimes \{\{a, b\}\}'' \in A^{\otimes 2}$ )

## Van den Bergh's double brackets (2)



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Examples: affine (multiplicative) quiver varieties

# Why should we care?

Double brackets are ...

- a starting point for noncommutative Poisson geometry
- equivalent to other important algebraic structures
- useful in the study of integrable systems
- ...



# NC Poisson geometry (1)

double (quasi-)Poisson bracket  
 $\{\{-, -\}$  on  $A$   
 (+ moment map  $\mu \in A$ )

$\xrightarrow[\text{(\mathbb{k} = \bar{\mathbb{k}})}]{\text{--- Rep}_n(A) \text{ ---}}$

Poisson bracket on  
 $\text{Rep}_n(A) // \text{GL}_n(\mathbb{k})$   
 $\hat{\mu}^{-1}(\lambda \text{Id}_n) // \text{GL}_n(\mathbb{k})$

- 1  $A = \mathbb{k}[x]$ ,  $\mu = x \longrightarrow$  closed orbits in  $\mathfrak{gl}_n(\mathbb{k})$  [VdB,'08]
- 2  $A = \mathbb{k}\bar{Q}$ ,  $\mu = \sum_{a \in \bar{Q}} \epsilon(a) aa^* \longrightarrow$  quiver varieties [VdB,'08]
- 3  $A = \mathbb{k}\bar{Q}_{\text{loc}}$ ,  $\mu = \prod_{a \in \bar{Q}} (1 + aa^*)^{\epsilon(a)} \xrightarrow{\text{quasi}}$  mult. quiver varieties [VdB,'08]
- 4  $A = \mathbb{k}\pi_1$ ,  $\mu = \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_k \gamma_k \xrightarrow{\text{quasi}}$  character varieties  
 [Massuyeau-Turaev,'14]
- 5  $A = \text{'Boalch algebra'}$   $\text{---??--}$  wild character varieties [F.-Fernández, *ongoing*]

# NC Poisson geometry (2)

## Applications of the Kontsevich-Rosenberg principle

structure  $S_{nc}$  on  $A$  -----> structure  $S$  on  $\text{Rep}_n(A)$

### (0) $A$ associative algebra

double (quasi-)Poisson algebra  $\xrightarrow{[\text{VdB}, '08]}$  (quasi-)Poisson algebra  
 $\{\{-, -\}\} : A \times A \rightarrow A^{\otimes 2}$   $\{-, -\} : \mathbb{k}[\text{Rep}_n(A)]^2 \rightarrow \mathbb{k}[\text{Rep}_n(A)]$

### (1) $A$ associative algebra with derivation $\partial$

double Poisson vertex algebra  $\xrightarrow{[\text{DeSole-Kac-Valeri}, '15]}$  Poisson vertex algebra  
 $\{\{-_\lambda -\}\} : A \times A \rightarrow A^{\otimes 2}[\lambda]$   $\{-_\lambda -\} : \mathbb{k}[\text{Rep}_n(A)]^2 \rightarrow \mathbb{k}[\text{Rep}_n(A)][\lambda]$

### (2) $A$ associative algebra with automorphism $S$

double mult. Poisson vertex algebra  $\xrightarrow{[\text{F.-Valeri}, '22]}$  mult. Poisson vertex algebra  
 $\{\{-_\lambda -\}\} : A \times A \rightarrow A^{\otimes 2}[\lambda]$   $\{-_\lambda -\} : \mathbb{k}[\text{Rep}_n(A)]^2 \rightarrow \mathbb{k}[\text{Rep}_n(A)][\lambda]$

Can get (1) in terms of alternative def. of PVA using maps  $\mathbb{k}[\text{Rep}_n(A)] \rightarrow \text{End}(\mathbb{k}[\text{Rep}_n(A)])$  [Bozec-F.-Moreau, ongoing]

# Algebraic structures (1)

Explicit correspondences using double brackets:

- 1 double Lie algebra (= dPA without Leibniz rules) on  $V \simeq \mathbb{k}^n$   
 $\Leftrightarrow$  skew-symm. solution to AYBE on  $\text{Mat}(n \times n, \mathbb{k})$   
 $\Leftrightarrow$  skew-symm. Rota-Baxter operator on  $\text{Mat}(n \times n, \mathbb{k})$   
[Schedler,'09] [Odesskii-Rubtsov-Sokolov,'13] [Goncharov-Kolesnikov,'18]
- 2 double Poisson algebras  
 $\Leftrightarrow$  bi-symplectic  $\mathbb{N}Q$ -algebras [Alvarez-Consul-Fernández,'15]
- 3 double Poisson vertex algebras gen. in degrees 0, 1  
 $\Leftrightarrow$  double Courant-Dorfman algebras [Alvarez-Consul-Fernández-Heluani,'22]
- 4 local lattice double Poisson algebras  
 $\Leftrightarrow$  double multiplicative Poisson vertex algebras [F.-Valeri,'22]

## Algebraic structures (2)

Similarities with double brackets:

- Modified double Poisson brackets [Arthamonov,'17]
- Parameter-dependent double brackets [Odesskii-Rubtsov-Sokolov,'14]
- Change in the Leibniz rules [F.-McCulloch,'22]
- HUGE interest in structures inducing Poisson bracket on  $\text{Rep}_n(A) // \text{GL}_n(\mathbb{K})$   
+ their derived versions:  
pre-Calabi-Yau algebras, Calabi-Yau or shifted symplectic structures, ...  
e.g. [Berest,Chen + collab.],[Iyudu-Kontsevich-Vlassopoulos,'21],[Leray-Vallette,'22],...
- Quantization? Seems very hard...  
[Avan-Ragoucy-Rubtsov,'16],[Olshanesky,'22], (and I am thinking about that...)

# Integrable systems

- Calogero-Moser/Ruijsenaars-Schneider systems live on some (multiplicative) quiver varieties “=reduction from  $\text{Rep}_n(\mathbb{k}\overline{Q})$ ”  
 $\rightsquigarrow$  understood from double bracket on  $A = \mathbb{k}\overline{Q}$  [F. + Chalykh or Görbe]
- Kontsevich system from modified double Poisson bracket [Arthamonov,'17]
- (multiplicative) Poisson vertex algebras are useful for non-abelian integrable PDEs [DeSole-Kac-Valeri'15] or DΔEs [Casati-Wang,'22],[F.-Valeri,'22]

Currently: generalize double Poisson cohomology [Pichereau-Van de Weyer,'08] to seek more integrable systems [F.-Valeri,*ongoing*]

# Thank you for your attention

*This was a personal view, not a survey! More on:*

[www.imo.universite-paris-saclay.fr/en/perso/maxime-fairon/double-brackets/](http://www.imo.universite-paris-saclay.fr/en/perso/maxime-fairon/double-brackets/)

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