Around Van den Bergh's double brackets

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Motivation: Kontsevich-Rosenberg principle

Fix (unital) associative algebra A over field $\Bbbk$ (char($\Bbbk$)$= 0$)

For $n \geq 1$, n-th representation space $\mathrm{Rep}_n(A)$ is scheme with B-points

$$\mathrm{Rep}_n(A)(B) := \mathrm{Hom}_{Alg.}(A, \mathrm{Mat}(n \times n, B))$$

$\Bbbk[\mathrm{Rep}_n(A)]$ is generated by 'matrix' symbols a_{ij} for $a \in A$, $1 \leq i, j \leq n$

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Motto: [Kontsevich-Rosenberg,'00]

"A noncommutative structure of some kind on A should give an analogous commutative structure on all schemes $\mathrm{Rep}_n(A)$, $n \geq 1$."

structure $\mathcal{S}_{nc}$ in $Alg.$ (*e.g.* formally smooth) $\longrightarrow$ structure $\mathcal{S}$ in $Com.Alg.$ (*e.g.* smooth)

Van den Bergh's double brackets (1)

$A^{\otimes 2} := A \otimes_{\Bbbk} A$, mult. $(a \otimes b)(c \otimes d) = ac \otimes bd$, swap $\tau_{(12)} a \otimes b = b \otimes a$.

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1) \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a) \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} (d \otimes 1)$ (inner derivation)

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Proposition ([Van den Bergh,'08])

If A is endowed with a double bracket $\{\!\{-,-\}\!\}$, then $\mathrm{Rep}_n(A)$ admits a unique $\mathrm{GL}_n(\Bbbk)$-invariant antisymmetric biderivation $\{-,-\}$ satisfying

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj} \; \{\!\{a,b\}\!\}''_{il} \,. \tag{1}$$

(We write $\{\!\{a,b\}\!\} =: \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}'' \in A^{\otimes 2}$)

Van den Bergh's double brackets (2)

double (quasi-)Poisson bracket $\{\!\{-,-\}\!\}$ on A $\overset{\text{Eq. (1)}}{\dashrightarrow}$ (quasi-)Poisson bracket $\{-,-\}$ on $\mathrm{Rep}_n(A)$

$\downarrow$ $(\Bbbk = \bar{\Bbbk})$

Poisson bracket on $\mathrm{Rep}_n(A)/\!\!/ \mathrm{GL}_n(\Bbbk)$

Van den Bergh's double brackets (2)

double (quasi-)Poisson bracket
$\{\!\{-,-\}\!\}$ on A
moment map $\mu \in A$ ($\in A^\times$)

Eq. (1) ⇢

(quasi-)Poisson bracket
$\{-,-\}$ on $\mathrm{Rep}_n(A)$
mom. map $\hat{\mu}$ to $\mathfrak{gl}_n(\Bbbk)$ (to $\mathrm{GL}_n(\Bbbk)$)

↓ (left)

↓ ($\Bbbk = \bar{\Bbbk}$) (right)

Lie bracket on
$H_0(A) := A/[A,A]$
$H_0(A/\langle \mu - \lambda \mathrm{id}_A \rangle)$
fixed $\lambda \in \Bbbk$ ($\in \Bbbk^\times$)

⇢

Poisson bracket on
$\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n(\Bbbk)$
$\hat{\mu}^{-1}(\lambda \, \mathrm{Id}_n) /\!\!/ \mathrm{GL}_n(\Bbbk)$

Examples: affine (multiplicative) quiver varieties

Why should we care?

Double brackets are ...

- a starting point for noncommutative Poisson geometry
- equivalent to other important algebraic structures
- useful in the study of integrable systems
- ...

NC Poisson geometry (1)

double (quasi-)Poisson bracket $\{\!\{-,-\}\!\}$ on A
(+ moment map $\mu \in A$)

$\xrightarrow{\text{— } \operatorname{Rep}_n(A) \text{ —}}$ ($\Bbbk = \bar{\Bbbk}$)

Poisson bracket on $\operatorname{Rep}_n(A) /\!\!/ \operatorname{GL}_n(\Bbbk)$
$\hat{\mu}^{-1}(\lambda \operatorname{Id}_n) /\!\!/ \operatorname{GL}_n(\Bbbk)$

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1. $A = \Bbbk[x]$, $\mu = x \longrightarrow$ closed orbits in $\mathfrak{gl}_n(\Bbbk)$ [VdB,'08]
2. $A = \Bbbk\overline{Q}$, $\mu = \sum_{a\in\overline{Q}} \epsilon(a)\, aa^* \longrightarrow$ quiver varieties [VdB,'08]
3. $A = \Bbbk\overline{Q}_{\texttt{loc}}$, $\mu = \prod_{a\in\overline{Q}} (1 + aa^*)^{\epsilon(a)} \xrightarrow{\text{quasi}}$ mult. quiver varieties [VdB,'08]
4. $A = \Bbbk\pi_1$, $\mu = \prod_{i=1}^{g} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1} \prod_k \gamma_k \xrightarrow{\text{quasi}}$ character varieties [Massuyeau-Turaev,'14]
5. $A =$'Boalch algebra' —??$\longrightarrow$ wild character varieties [F.-Fernández, *ongoing*]

NC Poisson geometry (2)

Applications of the Kontsevich-Rosenberg principle

structure $\mathcal{S}_{nc}$ on A - - - - - - - - → structure $\mathcal{S}$ on $\mathrm{Rep}_n(A)$

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(0) A associative algebra

[VdB,'08]

double (quasi-)Poisson algebra $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ - - - - - - - - → (quasi-)Poisson algebra $\{-,-\} : \Bbbk[\mathrm{Rep}_n(A)]^2 \to \Bbbk[\mathrm{Rep}_n(A)]$

(1) A associative algebra with derivation ∂

[DeSole-Kac-Valeri,'15]

double Poisson vertex algebra $\{\!\{-_\lambda-\}\!\} : A \times A \to A^{\otimes 2}[\lambda]$ - - - - - - - - → Poisson vertex algebra $\{-_\lambda-\} : \Bbbk[\mathrm{Rep}_n(A)]^2 \to \Bbbk[\mathrm{Rep}_n(A)][\lambda]$

(2) A associative algebra with automorphism S

[F.-Valeri,'22]

double mult. Poisson vertex algebra $\{\!\{-_\lambda-\}\!\} : A \times A \to A^{\otimes 2}[\lambda]$ - - - - → mult. Poisson vertex algebra $\{-_\lambda-\} : \Bbbk[\mathrm{Rep}_n(A)]^2 \to \Bbbk[\mathrm{Rep}_n(A)][\lambda]$

Can get (1) in terms of alternative def. of PVA using maps $\Bbbk[\mathrm{Rep}_n(A)] \to \mathrm{End}(\Bbbk[\mathrm{Rep}_n(A)])$ [Bozec-F.-Moreau,*ongoing*]

Algebraic structures (1)

Explicit correspondences using double brackets:

1. double Lie algebra (= dPA without Leibniz rules) on $V \simeq \Bbbk^n$
   $\Leftrightarrow$ skew-symm. solution to AYBE on $\mathrm{Mat}(n \times n, \Bbbk)$
   $\Leftrightarrow$ skew-symm. Rota-Baxter operator on $\mathrm{Mat}(n \times n, \Bbbk)$
   [Schedler,'09] [Odesskii-Rubtsov-Sokolov,'13] [Goncharov-Kolesnikov,'18]
2. double Poisson algebras
   $\Leftrightarrow$ bi-symplectic $\mathbb{N}Q$-algebras [Alvarez-Consul–Fernández,'15]
3. double Poisson vertex algebras gen. in degrees $0, 1$
   $\Leftrightarrow$ double Courant-Dorfman algebras [Alvarez-Consul–Fernández–Heluani,'22]
4. local lattice double Poisson algebras
   $\Leftrightarrow$ double multiplicative Poisson vertex algebras [F.-Valeri,'22]

Algebraic structures (2)

Similarities with double brackets:

- Modified double Poisson brackets [Arthamonov,'17]
- Parameter-dependent double brackets [Odesskii-Rubtsov-Sokolov,'14]
- Change in the Leibniz rules [F.-McCulloch,'22]
- HUGE interest in structures inducing Poisson bracket on $\mathrm{Rep}_n(A)/\!\!/\,\mathrm{GL}_n(\Bbbk)$ + their derived versions:

  pre-Calabi-Yau algebras, Calabi-Yau or shifted symplectic structures, ...

  e.g. [Berest,Chen + collab.],[Iyudu-Kontsevich-Vlassopoulos,'21],[Leray-Vallette,'22],...
- Quantization? Seems very hard...

  [Avan-Ragoucy-Rubtsov,'16],[Olshanesky,'22], (and I am thinking about that...)

Integrable systems

- Calogero-Moser/Ruijsenaars-Schneider systems live on some (multiplicative) quiver varieties "=reduction from $\operatorname{Rep}_n(\Bbbk\overline{Q})$"
  $\rightsquigarrow$ understood from double bracket on $A = \Bbbk\overline{Q}$ [F. + Chalykh or Görbe]
- Kontsevich system from modified double Poisson bracket [Arthamonov,'17]
- (multiplicative) Poisson vertex algebras are useful for non-abelian integrable PDEs [DeSole-Kac-Valeri'15] or DΔEs [Casati-Wang,'22],[F.-Valeri,'22]

Currently: generalize double Poisson cohomology [Pichereau-Van de Weyer,'08] to seek more integrable systems [F.-Valeri,*ongoing*]

Thank you for your attention

This was a personal view, not a survey! *More on:*
www.imo.universite-paris-saclay.fr/en/perso/maxime-fairon/double-brackets/

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