Well-behaved averages, Random walk on $\mathbf{R}$ with "linear exponential law" and rooted-oriented trees.

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Abstract

Well-behaved averages were introduced by J. Ecalle to give a positive answer to the problem of real resummation : how to assign a real sum to a real divergent series of "natural origin" (stemming for example from a differential equation).

Each average can be described as a collection of weights, that is coefficients indexed by words of plus or minus signs. Among the well-behaved averages, some are strongly linked to sets of probabilities associated to random walks on the real axis : For a given word $(\varepsilon_1, \ldots, \varepsilon_n)$ $(\varepsilon_i = \pm)$, the weight of the average is simply the probability to be on $\mathbf{R}^{\varepsilon_1}$ at time 1, on $\mathbf{R}^{\varepsilon_2}$ at time 2, ..., on $\mathbf{R}^{\varepsilon_n}$ at time n.

When the probability density associated to the random walk is a linear combination of elementary exponential laws, it induces an average with explicit weights. These ones can be decomposed in terms of elementary coefficients that are indexed by finite rooted oriented trees.

This tree-decomposition yields a first partial answer to the following question : When does a random walk induce the same average as the random walk with exponential law.

Keywords : Rooted oriented trees, random walk, real resummation.

MSC classification : 05C05, 60C05, 60J15, 40H05.

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1 Introduction

Well-behaved uniformizing averages (WBA) were introduced by J. Ecalle to answer the problem of real resummation : how to assign a real sum to a real divergent series of "natural origin", for example a formal solution of a differential equation. The need for "uniformizing" some analytic ramified functions appears naturally in this kind of problem. For example, let φ be an analytic function with singularities over $\mathbf{N}^*$, analytically continuable on the universal covering of $\mathbf{C}/\mathbf{N}^*$. For a given non-negative integer n we can label the 2^n determinations of φ over the interval $]n, n+1[$ that are obtained by analytic continuation of φ along the 2^n paths dodging the singularities $\{1, \ldots, n\}$ to the left or to the right. If the sign $+$ (resp. $-$) is assigned when dodging to the right (resp. to the left), these 2^n determinations of φ over $]n, n+1[$ are labeled by the addresses $(\varepsilon_1, \ldots, \varepsilon_n)$ ($\varepsilon_i = \pm$ and $1 \leq i \leq n$). Such functions appear naturally in the real-resummation theory and it's as much natural to try to associate a uniform function to φ, that is to "uniformize". The most simple way to do so is , over the interval $]n, n+1[$, to do an "average" of the 2^n determinations of φ, pondering them by 2^n "weights" $\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n}$, of sum (for a given n) equal to 1.

Thus, a *uniformizing average* $\mathbf{m}$ is just a collection of weights :

$$\mathbf{m} = \{\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n}\,;\, n \geq 0\,;\, \varepsilon_i = \pm\,;\, 1 \leq i \leq n\} \tag{1.1}$$

As it will be reminded in section 2 some supplementary, analytic and algebraic, conditions are imposed to such averages, so that they become a very powerful tool in real resummation. In these conditions, the average $\mathbf{m}$ is called a *well-behaved average* (WBA).For details see section 2 and [2, 3, 6, 7]. The study of particular averages, which appeared to be WBA, proved the existence of such objects.

For example, J. Ecalle found a great class of WBA, which were named *averages induced by diffusion*. There is a way to generate an average (i.e. to *to induce*), from a *random walk* (abusively called *diffusion*) : Let $(X_n)_{n\geq 1}$ be a sequence of independent, identically distributed, real random variables, having an integrable even function as probability density (we will often write "the diffusion f"). We can define the random variables $(S_n)_{n\geq 1}$:

$$\forall n \geq 1\,; \quad S_n = X_1 + \cdots + X_n$$

and this *induces* an average $\mathbf{m}$:

$$\forall n \geq 1\,; \forall(\varepsilon_1, \ldots, \varepsilon_n) \in \{+, -\}\,; \quad \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n} = \mathbf{P}(\varepsilon_1 S_1 > 0, \ldots, \varepsilon_n S_n > 0) \tag{1.2}$$

J. Ecalle proved that such averages are WBA. Moreover, for the purpose of real resummation, we can omit that f is positive. A complete expository of these results can be found in [6].

Let us give the example of the "Catalan" average, that is, the average induced by the diffusion with the exponential law :

$$f(x) = \frac{1}{2} e^{-|x|}$$

This average **man** is called the "Catalan average" because it was first defined by its weights, in which the Catalan numbers appear. Moreover, it was proved then, independently of the results on averages induced by diffusion, that this is a WBA (see [2, 6]).

This average is also the point of origin of this paper. In his work on WBA (see [4]), C. Even conjectured that the average **man** was also induced by the diffusion of "law" :

$$f(x) = 3e^{-|x|} - 15e^{-2|x|} + 15e^{-3|x|}$$

Although this function is no more positive, we call it a "law". We prove in this paper that this result is true and we give a sufficient condition for "diffusion" (i.e. "diffusion laws") to induce the Catalan average. As it is suggested by the above example, on one hand, we will restrict our study to "linear exponential diffusions" (or "exponential diffusions"), that is to say laws that are linear combinations of exponentials :

$$f(x) = \sum_{i=1}^{d} a_i e^{-\lambda_i |x|} \quad (d \geq 1\,;\ \lambda_i > 0)$$

on the other hand, we omit the fact that such functions should be positive.

We prove that, if f is an "exponential diffusion" such that d is odd, and :

$$\forall 1 \leq i \leq d\,; \quad a_i = \frac{1}{4} \frac{\prod_{j=1}^{d} (\lambda_i + \lambda_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{d} (\lambda_i - \lambda_j)}$$

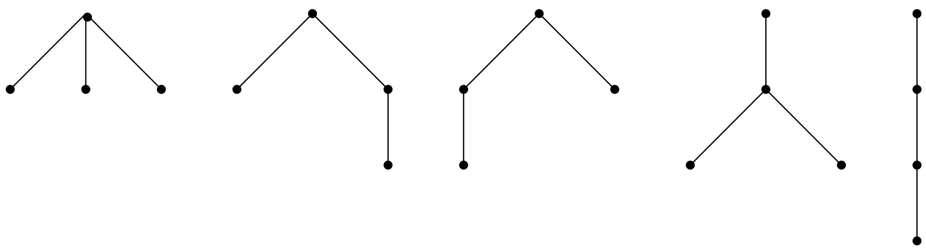
then f induces the average **man** by diffusion.

This result is based on computations of the weights of the averages induced by exponential diffusions. More precisely, we give decompositions for the average (and for some other objects associated to) in terms of simple coefficients that are indexed by *rooted-oriented trees.*

Without any details, let us just give an example. If **m** is induced by an exponential diffusion, for example :

$$\mathbf{m}^{+++-} = \sum_{T \in Ca_3} \mathbf{m}^T$$

where Ca_3 is the set of rooted-oriented trees with three edges :



The set Ca_3.

and the coefficients $\mathbf{m}^T$ are described in theorem 4.

This *tree-decomposition* should lead, in a forthcoming paper, to give a complete answer to the question : when do two "exponential diffusions" induce the same average ?

The section 2 is a general introduction to the real resummation that motivates the definition of WBA. In sections 3 and 4 we define several objects, and among them the averages, that are "induced" by diffusion, and more specifically by exponential diffusions. The section 5 contains our main results about "tree-decomposition". This will be helpful to give, in section 6, a sufficient criteria for an exponential diffusion to induce the Catalan average. The proofs of our main results are illustrated in section 7 and we shall then conclude by giving some perspectives on this work and some results that shall be available soon.

Contents

2 Reminder on real resummation theory.

2.1 Some heuristics on real resummation. The need for well-behaved averages.

The resummation scheme

All the series and functions in z introduced here are considered at infinity.

Let $\tilde{\varphi}(z)$ be a real (with real coefficients) divergent series of "natural origin": for instance the formal solution of a local analytic equation or system:

$$E(\tilde{\varphi}) = 0 \tag{2.1}$$

The most simple resummation scheme for resumming $\tilde{\varphi}(z)$ goes like this:

$$\begin{array}{ccccc} \widetilde{\varphi}(z) & - & - & \rightarrow & \varphi(z) \\ & \searrow & & \nearrow & \\ & & \hat{\varphi}(\zeta) & & \end{array} \tag{2.2}$$

We begin by subjecting $\widetilde{\varphi}(z)$ to the formal Borel transform (to obtain $\hat{\varphi}(\zeta)$) which, for instance, turns each monomial $z^{-\sigma}$ into $\zeta^{\sigma-1}/\Gamma(\sigma)$ $(\sigma > 0)$.

Then we carry out a Laplace transform:

$$\hat{\varphi}(\zeta) \longrightarrow \varphi(z) = \int_0^{+\infty} e^{-z\zeta}\hat{\varphi}(\zeta)d\zeta \tag{2.3}$$

This procedure for turning the formal object $\widetilde{\varphi}(z)$ into a geometric one $\varphi(z)$ is the most simple one, but it is already representative of the difficulties arising from the need for a *real* resummation.

Although the move $\hat{\varphi}(\zeta) \mapsto \varphi(z)$ seems to be one single step, it actually involves three distinct sub-steps.

Three steps in one.

(i) First sub-step : calculating a germ. The function $\hat{\varphi}(\zeta)$ is obtained, by the Borel transform, as a germ near $\zeta = +0$ and, generally speaking, it converges only for small enough values of ζ.

(ii) Second sub-step : getting a global function. We must continue this germ from $+0$ to $+\infty$ so as to get a global function, which could be Laplace transformed. This will be generally possible by analytic continuation, owing to the "natural origin" of $\tilde{\varphi}(z)$.

(iii) Third sub-step : uniformizing the global function. Although there are no obstacles to analytic continuation, there may be analytic singularities. Indeed, the existence of singularities in the ζ-plane is precisely what causes the divergence of $\tilde{\varphi}$. There are often compelling reasons for them to be located on $\mathbf{R}^+$. Then $\hat{\varphi}$ is *multi-valued* (many-branched) over $\mathbf{R}^+$ (its determination depends on the choice of a path of analytic continuation and on the way this one dodges the singularities.). If so, we must turn $\hat{\varphi}(\zeta)$, "in some suitable way", into a *univalued* (uniform) function $(\mathbf{m}\hat{\varphi})(\zeta)$, so as to be able to carry out the Laplace transform.

What does "suitable way" mean ?

If we assume that $\hat{\varphi}(\zeta)$ is multivalued, we will turn it into a uniform function by making an average of the different analytic continuations of $\hat{\varphi}$ and then, the difficulty relates to the choice of a "suitable" (or well-behaved) uniformizing average $\mathbf{m}$. Here, suitable means three things:

P1: $\mathbf{m}$ must respect convolution: it is indispensable in all non-linear situations that $\mathbf{m}$ turns convolution into convolution. The Borel and Laplace transforms are algebra homomorphisms. Thus our average $\mathbf{m}$ must be an algebra homomorphism (for the convolution of "ramified" functions and the convolution of "uniform" functions) so as to assign to $\tilde{\varphi}(z)$ a sum $\varphi(z)$ which is also a solution of the original equation.

P2: $\mathbf{m}$ must respect realness: this is rather necessary if $\tilde{\varphi}(z)$ has real coefficients and if we want to assign a real sum for some compelling reason: for example, if it represents a physical or real-geometric object.

P3: $\mathbf{m}$ must respect the lateral growth: for a series $\tilde{\varphi}$ of natural origin, $\hat{\varphi}$ displays, generally speaking, the good growth rate (that is to say exponential growth) which allows to carry out the Laplace transform. But that statement must be restricted. In fact, this exponential growth is obtained only:
- on singularity-free axes Γ_θ (from 0 to infinity in the direction θ).
- on both sides (right and left) of a singularity-carrying axis.

If $\hat{\varphi}(\zeta)$ has singularities on $\mathbf{R}^+$, this exponential growth is, generally speaking, also ensured on paths Γ which are close to the positive axis and cross it only a finite number of times. But, if $\mathbf{R}^+$ carries infinitely many singularities, on paths Γ following $\mathbf{R}^+$, but with infinitely many crossings, the function $\hat{\varphi}(\zeta)$ often has *faster-than-exponential* growth:

$$| \hat{\varphi}(\zeta) | \leq c_0 exp(c_1(| \zeta | + | \zeta | \, log \, | \zeta |)) \tag{2.4}$$

Unfortunately, the uniformizing averages which are **P1** and **P2** will involve the analytic continuations of $\hat{\varphi}(\zeta)$ on such "often-crossing" paths. Thus we must carefully choose the average $\mathbf{m}$ such that $(\mathbf{m}\hat{\varphi})(\zeta)$ has a no-faster-than-exponential growth ($| (\mathbf{m}\hat{\varphi})(\zeta) | \leq c_0 exp(c_1 | \zeta |)$).

Now an average $\mathbf{m}$ will be called a "well-behaved" uniformizing average if the three properties **P1**, **P2**, **P3** hold.

In order to define and present such well-behaved averages, we need to introduce the convolution algebras of resurgent functions. But, for the sake of simplicity, we will restrict ourselves to the definition of the convolution algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ of resurgent functions, with singularities over $\mathbf{N}^*$ and which are locally integrable. Nonetheless, the following statements can be extended to more general convolution algebras (with some different set of singularities and without the condition of local integrability)

2.2 The algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$.

Definition: *The algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ is defined as follows. Let $\hat{\varphi}(\zeta)$ be an element of this algebra, then:*

- *$\hat{\varphi}(\zeta)$ is defined and holomorphic at the root of $\mathbf{R}^+$ (on $]0, \epsilon[$).*
- *$\hat{\varphi}(\zeta)$ is analytically continuable along any path that follows $\mathbf{R}^+$ and dodges each point of $\mathbf{N}^*$ to the left or to the right, but without ever going back.*
- *All the determinations of $\hat{\varphi}(\zeta)$ are locally integrable on $\mathbf{R}^+$.*

Moreover, the convolution is defined by:

$$\hat{\varphi}_3(\zeta) = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) = \int_0^\zeta \hat{\varphi}_1(\zeta_1)\, \hat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1 \quad (0 < \zeta \ll 1) \tag{2.5}$$

$$(\hat{\varphi}_1, \hat{\varphi}_2 \in RESUR(\mathbf{R}^+//\mathbf{N}, int.))$$

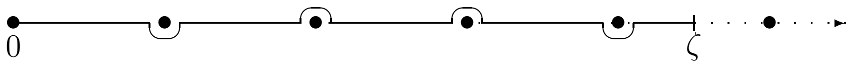
This expression is purely local (at $\zeta = 0$) and the germ $\hat{\varphi}_3(\zeta)$ must then be extended, by analytic continuation, to a global function.

For details see [2].

Now, for a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$, we can give the following notation:

Let $\hat{\varphi}(\zeta)$ be a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ be a sequence of n plus or minus signs, then, for ζ in $]n, n+1[$, we will note $\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta)$ the analytic continuation of $\hat{\varphi}$ from 0 to ζ on the path that follows $\mathbf{R}^+$ and dodges each singularity k $(1 \leq k \leq n)$ to the *right* (resp. to the *left*) if $\varepsilon_k = +$ (resp. $\varepsilon_k = -$).

Example: If $\zeta \in]4, 5[$, then $\hat{\varphi}^{+,-,-,+}(\zeta)$ is the analytic continuation of $\hat{\varphi}$ along the following path:



Of course, $\hat{\varphi}^{\emptyset}(\zeta)$ $(O < \zeta < 1)$ is the unique determination of $\hat{\varphi}$ on $]0, 1[$.

Once this notation is given, for any fixed integer n, a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ has 2^n possibly different determinations $\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta)$ *over* the interval $]n, n+1[$ and a uniformizing average $\mathbf{m}$ will return an actual average of these 2^n determinations.

2.3 The uniformizing averages.

A uniformizing average $\mathbf{m}$ is a uniformizing projection of the space $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ into the space $UNIF(\mathbf{R}^+, int.)$ of uniform, locally integrable functions on $\mathbf{R}^+$.

It can be define as a collection of "weights":

$$\mathbf{m} = \{\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n};\ n \in \mathbf{N}\ ;\ \varepsilon_i = \pm\ ;\ \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n} \in \mathbf{C}\} \tag{2.6}$$

subject to the self-consistency relations:

$$\sum_{\varepsilon_n = \pm} \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n} = \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_{n-1}}\ (resp.\ \mathbf{m}^{\emptyset} = 1)\ if\ n > 1\ (resp.\ n = 1) \tag{2.7}$$

and the action of the average $\mathbf{m}$ on a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ is defined as follows:

$$\forall n \in \mathbf{N}\ ;\ \forall \zeta \in]n, n+1[\quad (\mathbf{m}\hat{\varphi})(\zeta) = \sum_{\varepsilon_1 = \pm \ldots \varepsilon_n = \pm} \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n} \hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta) \tag{2.8}$$

Thus an average $\mathbf{m}$ turns a multivalued function into a uniform one by averaging its different determinations and it is important to point out that the self-consistency relations are a necessity: for instance, whenever a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ has only *fictive* singularities, that is to say $\hat{\varphi}$ is uniform, then we would like to obtain $\mathbf{m}\hat{\varphi} = \hat{\varphi}$, which is ensured by the self-consistency relations.

Once these definitions are given, there exists more precise statements for the properties P1, P2, P3:

P1: An average $\mathbf{m}$ respects convolution if and only if, for any two functions $\hat{\varphi}$ and $\hat{\psi}$ in $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$:

$$\mathbf{m}(\hat{\varphi} * \hat{\psi}) = (\mathbf{m}\hat{\varphi}) * (\mathbf{m}\hat{\psi}) \tag{2.9}$$

where the first star $*$ (resp. the second) denotes the convolution on $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ (resp. on $UNIF(\mathbf{R}^+, int.)$).

This condition is ensured if and only if the weights of $\mathbf{m}$ verify a universal multiplication table which reads, for example:

$$\begin{cases} \mathbf{m}^+\mathbf{m}^+ &= \mathbf{m}^{+,+} - \mathbf{m}^{-,+} \\ \mathbf{m}^+\mathbf{m}^- &= \mathbf{m}^{+,-} + \mathbf{m}^{-,+} \\ \mathbf{m}^-\mathbf{m}^- &= \mathbf{m}^{-,-} - \mathbf{m}^{+,-} \\ \mathbf{m}^+\mathbf{m}^{+,+} &= \mathbf{m}^{+,+,+} - \mathbf{m}^{+,-,+} - \mathbf{m}^{-,+,+} \\ & \vdots \end{cases} \tag{2.10}$$

For proofs and complements, see [2, 3].

P2: The fact that an average $\mathbf{m}$ respects realness can easily be read on its weights. Let $\hat{\varphi}$ be a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ and let us assume that $\hat{\varphi}$ is the formal Borel transform of a real divergent series. Then $\hat{\varphi}(\zeta)$ is *real* for small enough real values of ζ and assumes complex conjugate values on complex conjugate paths of analytic continuation:

$$\forall n \in \mathbf{N} \ ; \ \forall \zeta \in]n, n+1[\ ; \ \forall \varepsilon_i \in \{+,-\} \quad \hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta) = \overline{\hat{\varphi}^{\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_n}(\zeta)} \tag{2.11}$$

where $\bar{\varepsilon}_i$ is the opposite sign to ε.

Therefore, a uniformizing average $\mathbf{m}$ respects realness ($(\mathbf{m}\hat{\varphi})$ is real on $\mathbf{R}^+$) if and only if:

$$\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_n} = \overline{\mathbf{m}}^{\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_n} \quad (\forall n \geq 0 \ ; \ \forall \varepsilon_i \in \{+,-\}) \tag{2.12}$$

P3: Although this appears to be the main demand, we won't go into details in this introductory paper (see [2]). This condition does not reduce to growth conditions on the weights. It actually involves some compensation phenomena. This faster-than-exponential growth on "often-crossing" paths is a precise mechanism, which has to do with the nature of the "acting alien algebra". But, whenever a function $\hat{\varphi}$ is of "natural origin", the analysis of this nuisance (with the "Bridge equation") shows how to construct, independently of $\hat{\varphi}$ itself, "well-behaved" averages $\mathbf{m}$ which produce mean values $\mathbf{m}\hat{\varphi}$ with the requisite exponential growth.

These three properties tend to be mutually exclusive but the main fact is that such well-behaved averages exist. Among them, we shall give the example of the averages induced by diffusion.

3 Averages induced by a diffusion.

3.1 Definition.

We fix an integrable function f on $\mathbf{R}$ such that:

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \tag{3.1}$$

The function f may be viewed as representing the probability distribution at the time $t = 1$, on the vertical axis $1 + i\mathbf{R}$, of a particle starting from the origin at $t = 0$, moving along $\mathbf{R}^+$ with unit speed, and diffusing randomly in the vertical direction. To any such "diffusion", we may associate a uniformizing average $\mathbf{m}$ with weights defined as follows :

Definition 1 *(J. Ecalle) $\mathbf{m}^{\varepsilon_1,\dots,\varepsilon_n}$ is the probability for the particle to hit the half-axis $n + i\varepsilon_n\mathbf{R}^+$ at the time n after successively crossing each half-axis $j + i\varepsilon_j\mathbf{R}^+$ $(1 \leq j < n)$ at the time j.*

Analytically, this translates into the following formula :

$$\mathbf{m}^{\varepsilon_1,\dots,\varepsilon_n} = \int f(x_1)\dots f(x_n)\sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\dots\sigma_{\varepsilon_n}(x_1+\dots+x_n)\,dx_1\dots dx_n \tag{3.2}$$

with integration over $\mathbf{R}^n$ and with the classical step functions σ_+ and σ_- :

$$\sigma_\pm(x) \equiv 1 \ (resp.\ 0) \ \ if \quad \pm x > 0 \ \ (resp. \pm x \leq 0) \tag{3.3}$$

J. Ecalle proved that any average induced by a diffusion respects both convolution and lateral growth. Moreover, as soon as the function f is even ($f(x) = f(-x)$), the average also respects realness and thus is a well-behaved uniformizing average (for details see [2]).

The construction and the study of the averages induced by a diffusion are due to J. Ecalle. Moreover, this method for building well-behaved averages allows to *generalize* them, so that they can act on resurgent functions having their singularities in any discrete additive semi-group of $\mathbf{R}^+$. J. Ecalle also proved that the condition of local integrability, for the resurgent functions, is not a necessity.

3.2 Some probability theory : Random walk on $\mathbf{R}$.

We are aware that the above definition would perhaps be shocking for probabilists. First, we didn't make any assumption on the sign of f (note that we will always consider even functions). Second, it seems that the terminology is unusual. Thus let us give here a short dictionary to link averages induced by diffusion to random walks on $\mathbf{R}$.

Suppose that f is a positive function on $\mathbf{R}$. It represents a probability density. Consider now a sequence of independent random variables $(X_n)_{n\in \mathbf{N}^*}$ having f as probability density. Then we can consider the sums :

$$\forall n \in \mathbf{N}^*, \quad S_n = X_1 + \cdots + X_n \tag{3.4}$$

The relation between averages and random walk is now clear because, for any sequence $(\varepsilon_1, \ldots, \varepsilon_r)$:

$$\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_r} = P(\varepsilon_1 S_1 > 0, \ldots, \varepsilon_r S_r > 0) \tag{3.5}$$

We shall end this section by an example that has been deeply studied as well in resummation theory as in probability theory and that motivates the study of "exponential diffusions".

3.3 The Catalan average and the exponential law.

The Catalan average was studied independently from averages induced by diffusion. Nevertheless, this average, well-behaved, is induced by the exponential law :

$$f(x) = \frac{1}{2} e^{-|x|} \tag{3.6}$$

and provides us a first example of average induced by "exponential diffusion". Moreover, some questions, raised by the study of the Catalan average, were the start point of the present paper.

Let us resume first the elementary properties of the Catalan average.

The Catalan average man.

The weights of **man** assume rational values, and can be obtained by the following formula :

$$\mathbf{man}^{\varepsilon_1,\ldots,\varepsilon_n} \equiv 4^{-n} ca_{n_1} ca_{n_2} \ldots ca_{n_s} (1 + n_s) \tag{3.7}$$

with the classical Catalan numbers :

$$ca_n \overset{def}{=} \frac{(2n)!}{n!\,(n+1)!} \quad (ca_n \in \mathbf{N}) \tag{3.8}$$

which in this case are indexed by the integers $n_1, n_2, \ldots, n_s$ which denote the numbers of identical consecutive signs within the address $(\varepsilon_1, \ldots, \varepsilon_n)$:

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\pm)^{n_1} (\mp)^{n_2} \ldots (\varepsilon_n)^{n_s} \quad (\textit{of course } n_1 + \cdots + n_s = n) \tag{3.9}$$

The Catalan average **man** is interesting because it is a first example of *well-behaved uniformizing average* (**P1**, **P2**, **P3**). As it was mentioned, this average motivated the study of averages induced by diffusion.

Some questions about exponential diffusions.

First of all, the weights of this average, induced by the exponential law, have explicit expressions, in term of some combinatorial numbers. As we shall see, this property remains true for any exponential diffusion :

$$f(x) = \sum_{i=1}^{d} a_i e^{-\lambda_i |x|} \tag{3.10}$$

We will thus obtain an exhaustive description of the averages induced by exponential diffusion. Moreover C. Even, in his work (see [4]), suggested that the Catalan average is also induced by the following diffusion :

$$f(x) = 3e^{-|x|} - 15e^{-2|x|} + 15e^{-3|x|} \tag{3.11}$$

It shall be proved in theorem 5 that this statement is true and we will give a wider family of exponential diffusion inducing **man**.

We should first deal with some general algebraic considerations on diffusions, exponential diffusions and some underlying objects (weighted functions, operators ...) associated to an average induced by diffusion.

4 Diffusions and exponential diffusions.

We first give here some definitions of objects associated to an average induced by "diffusion".

4.1 Weighted functions and operators

The weighted functions associated to a diffusion.

Let us consider a diffusion f :

$$f(-x) = f(x) \quad and \quad \int_{\mathbf{R}} f(x)dx = 1$$

The definition 1 of the average induced by f suggests to consider the following weighted functions :

Definition 2 *let $(\varepsilon_1, \ldots, \varepsilon_r)$ a sequence of r signs. The weighted function $f^{\varepsilon_1,\ldots,\varepsilon_r}$ is defined by the following induction :*

$$\left\{ \begin{array}{rcl} f^{\varepsilon_1}(x) & = & f(x)\sigma_{\varepsilon_1}(x) \\ \\ f^{\varepsilon_1,\ldots,\varepsilon_r}(x) & = & (f * f^{\varepsilon_1,\ldots,\varepsilon_{r-1}})(x)\sigma_{\varepsilon_r}(x) \quad if\, r \geq 2 \end{array} \right. \tag{4.1}$$

where $\sigma_- = 1 - \sigma_+$ and $\sigma_+(x) = 1$ (resp. 0) if $x \geq 0$ (resp. $x < 0$).

Using this definition, for a sequence $(\varepsilon_1, \ldots, \varepsilon_r)$, the weight associated to the average $\mathbf{m}$ induced by f is :

$$\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_r} = \int_{\mathbf{R}} f^{\varepsilon_1,\ldots,\varepsilon_r}(x)dx \tag{4.2}$$

These weighted functions inherits some of the property **P2** (see section 2.3) of the weights of $\mathbf{m}$. Thus, we will focus in the next section on lightly modified version of these weighted functions.

The even weighted functions.

As we start with an even diffusion, we can define the following even weighted functions :

Definition 3 *To any given sequence $(\varepsilon_1, \ldots, \varepsilon_r)$ we associate the even function $F^{\varepsilon_1,\ldots,\varepsilon_r}$ which coincides with the function $f^{\varepsilon_1,\ldots,\varepsilon_r}$ on the axis $\boldsymbol{R}^{\varepsilon_r}$.*

As f is even, it's not difficult to check that :

$$F^{\varepsilon_1,\ldots,\varepsilon_r} = F^{\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_r} \quad (\bar{\varepsilon}_i \textit{ opposite sign to } \varepsilon_i) \tag{4.3}$$

and these functions are defined by the induction :

$$\begin{cases} F^{\varepsilon_1}(x) &= f(x) \\ F^{\varepsilon_1,\ldots,\varepsilon_r}(x) &= (f * F^{\varepsilon_1,\ldots,\varepsilon_{r-1}}\sigma_{\varepsilon_{r-1}})(x)\sigma_{\varepsilon_r}(x) = T_f^{\varepsilon_{r-1},\varepsilon_r} F^{\varepsilon_1,\ldots,\varepsilon_{r-1}} \quad \text{if } r \geq 2,\ \varepsilon_r x \geq 0 \end{cases} \tag{4.4}$$

As f is even :

$$\begin{aligned} T_f^{+,+} &= T_f^{-,-} = T_f^1 \\ T_f^{+,-} &= T_f^{-,+} = T_f^2 \end{aligned} \tag{4.5}$$

For any sequence $(\varepsilon_1, \ldots, \varepsilon_r)$, which can be written $(\pm)^{n_1}, (\mp)^{n_2} \ldots (\varepsilon_r)^{n_t}$, that is in stacks of identical signs,

$$F^{(\pm)^{n_1},(\mp)^{n_2}\ldots(\varepsilon_r)^{n_t}} = (T_f^1)^{n_t-1} T_f^2 (T_f^1)^{n_{t-1}-1} T_f^2 \ldots (T_f^1)^{n_2-1} T_f^2 (T_f^1)^{n_1-1} f \tag{4.6}$$

and if we have a complete understanding of the operators $(T_f^1)^{n-1}$ and $T_f^2(T_f^1)^{n-1}$ $(n \geq 1)$, we easily obtain all the even weighted functions. Moreover we should see now that we can restrict our study to the second kind of operators, namely $T_n^f = T_f^2(T_f^1)^{n-1}$ $(n \geq 1)$.

We have the following property :

Proposition 1 *If the action of the operators $T_n^f = T_f^2(T_f^1)^{n-1}$ is known, then we can compute all the weights of the associated average $\boldsymbol{m}$.*

Proof : Because of the equation (4.2), for any sequence $(\varepsilon_1, \ldots, \varepsilon_r)$:

$$\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_r} = \int_{\mathbf{R}^+} F^{\varepsilon_1,\ldots,\varepsilon_r}(x)dx$$

Now, $\mathbf{m}^+ = \mathbf{m}^- = 1/2$, and we suppose that, for positive integers $n_1, \ldots, n_t$ we know :

$$\mathbf{m}^{n_1,\ldots,n_t} = \mathbf{m}^{(\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_t)^{n_t},\bar{\varepsilon}_t} = \int_{\mathbf{R}^+} T^f_{n_t} \ldots T^f_{n_2} T^f_{n_1} F(x)dx$$

where the signs $\varepsilon_1, \ldots, \varepsilon_t, \bar{\varepsilon}_t$ are alternate.

To prove the above property, we have to prove that we can now compute the other weights of $\mathbf{m}$, that is to say the weights of $\mathbf{m}$ for the sequences $(\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_t)^{n_t}$ with $t \geq 1$, $n_t \geq 2$ and alternate signs $\varepsilon_1, \ldots, \varepsilon_t$. But because of the general equation :

$$\forall(\varepsilon_1, \ldots, \varepsilon_{s-1}) \quad \sum_{\varepsilon_s = \pm} \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_{s-1},\varepsilon_s} = \mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_{s-1}}$$

We easily obtain that, if $n_t \geq 2$:

$$\mathbf{m}^{(\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_t)^{n_t}} = \mathbf{m}^{(\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_{t-1})^{n_{t-1}},\varepsilon_t} - \sum_{k=1}^{n_t-1} \mathbf{m}^{(\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_t)^{n_t-k},\bar{\varepsilon}_t} \tag{4.7}$$

and then the above property is proven.

We can note that we have also obtained the following result : If two averages $\mathbf{m}_1$, $\mathbf{m}_2$ coincides for the sequences $((\varepsilon_1)^{n_1}(\varepsilon_2)^{n_2}\ldots(\varepsilon_t)^{n_t}, \bar{\varepsilon}_t)$, then $\mathbf{m}_1 = \mathbf{m}_2$.

To resume the above section, the definition of the weights of an average induced by diffusion gives rise to the definition of some other objects : two different but very similar kind of weighted functions and some fundamental operators T^f_n. We proved (see property above) that a deep understanding of these operators would lead us to a full knowledge of the average induced by f. We shall now restrict ourself to the case of "exponential" diffusions. In this case, we'll be able to give some simple matricial representations of the operators T^f_n, which will lead us to the main results on averages induced by exponential diffusion.

4.2 The exponential diffusions.

In the following sections, we consider diffusions f :

$$f(x) := \sum_{i=1}^{d} a_i e^{-\lambda_i |x|} \tag{4.8}$$

with

$$\begin{cases} \forall\ i,\ 1 \leq i \leq d,\ a_i \in\ \mathbf{R} \\ \displaystyle\sum_{i=1}^{d} a_i/\lambda_i = 1/2\ \left(\Leftrightarrow \int_{\mathbf{R}} f(x)dx = 1\right) \\ \forall\ i,j \in \{1,d\}\ ,\ \lambda_i > 0\ , and\ \lambda_i \neq \lambda_j\ if\ i \neq j \end{cases}$$

Let us first study the operators T^1_f and T^2_f with some help of linear algebra.

Some elementary algebra.

Let us consider the following vector space :

$$E = \bigoplus_{i=1}^{d} E_i = \bigoplus_{i=1}^{d} \mathbf{R}[|\, x \,|] e^{-\lambda_i |x|} \tag{4.9}$$

and its basis :

$$\{e_{i,k}(x)\}_{\substack{1\leq i\leq d \\ k\in \mathbf{N}}} \quad , \quad e_{i,k}(x) = \frac{|\, x \,|^k}{k!} e^{-\lambda_i |x|} \tag{4.10}$$

Note that

$$T_f^1 = \sum_{i=1}^{d} a_i T_{e_{i,0}}^1 \qquad T_f^2 = \sum_{i=1}^{d} a_i T_{e_{i,0}}^2 \tag{4.11}$$

and using the definition of such operators (see (4.4)) :

Proposition 2 *T_f^1, $T_{e_{i,0}}^1$, T_f^2 and $T_{e_{i,0}}^2$ are endomorphisms of E and :*

$$T_{e_{i,0}}^1 e_{j,k} = \begin{cases} e_{i,k+1} + \displaystyle\sum_{l=0}^{k} \frac{1}{(\lambda_i+\lambda_j)^{k+1-l}} e_{j,l} & if \quad j=i \\ \displaystyle\frac{1}{(\lambda_j-\lambda_i)^{k+1}} e_{i,0} + \sum_{l=0}^{k} \left(\frac{1}{(\lambda_i+\lambda_j)^{k+1-l}} - \frac{1}{(\lambda_j-\lambda_i)^{k+1-l}} \right) e_{j,l} & if \quad i \neq j \end{cases} \tag{4.12}$$

$$T_{e_{i,0}}^2 e_{j,k} = \frac{1}{(\lambda_i+\lambda_j)^{k+1}} e_{i,0} \tag{4.13}$$

Moreover, using the equations (4.11), we obtain the same kind of relations for T_f^1 and T_f^2.

Our goal remains to study the operators $T_f^2 (T_f^1)^{n-1} = T_n^f$ $(n \geq 1)$. Instead of using the above relations, we can express T_f^1 and T_f^2 as infinite matrices in the basis $\{e_{i,k}(x)\}_{\substack{1\leq i\leq s \\ k\in \mathbf{N}}}$. To do so, we use the following notations.

Some notations on infinite matrices.

Let $\alpha(x) = \sum_{n=0}^{+\infty} \alpha_n x^n$ a formal series. Then :

Definition 4 *We define the following infinite matrices :*

- *$A_{\alpha(x)}$ is the infinite matrix such that $A_{\alpha(x),(i,j)} = \alpha_{j-1}$ (resp. 0) if $i=1$ (resp. $i>1$).*
- *$B_{\alpha(x)}$ is the infinite matrix such that $B_{\alpha(x),(i,j)} = \alpha_{j-i+1}$ (resp. 0) if $j-i+1 \geq 0$ (resp. otherwise).*

These matrices are useful to represent the operators T_f^1 and T_f^2 and,

Proposition 3 *For any given formal series $\alpha_1(x)$ and $\alpha_2(x)$:*

$$\begin{aligned} A_{\alpha_1(x)}A_{\alpha_2(x)} &= A_{\alpha_1(0)\alpha_2(x)} && (4.14)\\ A_{\alpha_1(x)}B_{\alpha_2(x)} &= A_{\Delta(\alpha_1(x)\alpha_2(x))} && (4.15)\end{aligned}$$

where Δ is the shift operator :

$$if\ \alpha(x) = \sum_{n=0}^{+\infty} \alpha_n x^n \ then\ \Delta(\alpha(x)) = \frac{\alpha(x)-\alpha(0)}{x} = \sum_{n=0}^{+\infty} \alpha_{n+1}x^n \tag{4.16}$$

We have now an easy way to represent the operators T_f^1 and T_f^2.

The representative matrices of T_f^1 and T_f^2.

We can decompose the operators T_f^1 and T_f^2 with respect to the basis :

$$e = \{e_{i,k}(x)\}_{\substack{1\le i\le d\\ k\in \mathbf{N}}} = \cup_{i=1}^d \{e_{i,k}(x)\}_{k\in \mathbf{N}} \tag{4.17}$$

and using this ordering for the basis and proposition 2,

Proposition 4

- *The infinite representative matrix M of T_f^1 can be decomposed in a $d\times d$ matrix of infinite matrices $M = ((M_{i,j}))_{\substack{1\le i\le d\\ 1\le j\le d}}$ with*

$$M_{i,j} = A_{\alpha_{i,j}(x)}\ with\ \alpha_{i,j}(x) = \frac{a_i}{\lambda_j - \lambda_i - x}\ for\ i\neq j \tag{4.18}$$

$$M_{i,i} = B_{\alpha_{i,i}(x)}\ with\ \alpha_{i,i}(x) = \sum_{k=1}^{d} \frac{2\lambda_k a_i x}{\lambda_k^2 - (\lambda_i - x)^2} \tag{4.19}$$

- *The infinite representative matrix N of T_f^2 can be decomposed in a $d\times d$ matrix of infinite matrices $N = ((N_{i,j}))_{\substack{1\le i\le d\\ 1\le j\le d}}$ with*

$$N_{i,j} = A_{\beta_{i,j}(x)}\ with\ \beta_{i,j}(x) = \frac{a_i}{\lambda_i + \lambda_j - x} \tag{4.20}$$

Using these matricial representations, we obtain our first theorem on the operators $T_n^f = T_f^2(T_f^1)^{n-1}$ $(n\ge 1)$.

4.3 A first theorem on averages induced by exponential diffusions.

We give here a theorem on the representative matrices of the operators $T_n^f = T_f^2 (T_f^1)^{n-1}$. It is obvious that the representative matrix of T_n^f is :

$$P_n^f = N \,.\, M^{n-1}$$

Theorem 1 *For $n \geq 1$, the infinite representative matrix P_n^f of T_n^f can be decomposed in a $d \times d$ matrix of infinite matrices $P_n^f = ((P_{n,(i,j)}^f))_{\substack{1\leq i\leq d \\ 1\leq j\leq d}}$ with*

$$P_{n,(i,j)}^f = A_{\beta_{i,j}^n(x)} \tag{4.21}$$

where the series $\beta_{i,j}^n(x)$ are defined by the following induction :

$$\begin{aligned} \beta_{i,j}^1(x) &= \beta_{i,j}(x) && (4.22)\\ \beta_{i,j}^{n+1}(x) &= \sum_{\substack{k=1\\k\neq j}}^{d} \beta_{i,k}^n(0)\alpha_{k,j}(x) \;+\; \Delta(\beta_{i,j}^n(x)\alpha_{j,j}(x)) && (4.23)\end{aligned}$$

Proof : This result is a very simple consequence propositions 3 and 4. It can easily be proved by induction on n.

As $P_1^f = N$, the theorem is true for $n = 1$. Suppose it's true for a fixed n, then :

$$P_{n+1}^f = N.M^n = N.M^{n-1}.M = P_{n-1}^f.M$$

Thus

$$\begin{aligned} P_{n+1,(i,j)}^f &= \sum_{k=1}^{d} P_{n,(i,k)} M_{k,j} \\ &= \sum_{\substack{k=1\\k\neq j}}^{d} A_{\beta_{i,k}^n(x)} A_{\alpha_{k,j}(x)} \;+ A_{\beta_{i,j}^n(x)} B_{\alpha_{j,j}(x)} \\ &= \sum_{\substack{k=1\\k\neq j}}^{d} A_{\beta_{i,k}^n(0)\alpha_{k,j}(x)} \;+ A_{\Delta(\beta_{i,j}^n(x)\alpha_{j,j}(x))} \\ &= A_{\left(\sum_{\substack{k=1\\k\neq j}}^{d} \beta_{i,k}^n(0)\alpha_{k,j}(x) \;+\Delta(\beta_{i,j}^n(x)\alpha_{j,j}(x))\right)} \end{aligned}$$

This clearly ends the proof of this theorem.

With this theorem we obtained a first description of the operators T_n^f and it also yields a finite-dimensional description :

Proposition 5 *The vector space $E_0 = Vect\{e^{-\lambda_1|x|}, \dots, e^{-\lambda_d|x|}\}$ is stable under the operators T_n^f. To any given element*

$$g(x) = \sum_{i=1}^{d} b_i e^{-\lambda_i|x|}$$

we can associate the vector $\vec{b} = (b_1, \dots, b_d)$. Now, for all $n \geq 1$,

$$T_n^f g = \sum_{i=1}^{d} b_i^n e^{-\lambda_i|x|}$$

and the vector $\vec{b^n}$ can be deduced from the vector $\vec{b}$:

$$\vec{b^n} = \widetilde{P}_n^f \vec{b}$$

where $\widetilde{P}_n^f$ is a $d \times d$ matrix with entries $\widetilde{P}_{n,(i,j)}^f = \beta_{i,j}^n(0)$.

This proposition is a consequence of theorem 1 and of the definition of matrices A (see definition 4).

We can also note that, for a function g of E_0, represented by the vector $\vec{b}$, integration over $\mathbf{R}^+$ is :

$$\int_{\mathbf{R}^+} g(x)\, dx = < \vec{\Lambda} \,|\, \vec{b} > \tag{4.24}$$

where $< . \,|\, . >$ is the standard scalar product on $\mathbf{R}^d$ and $\vec{\Lambda} = (1/\lambda_1, \dots, 1/\lambda_d)$.

We can consider theorem 1 as the fundamental theorem of our paper and it already points out what is easier, that is more explicit, when we restrict ourself to exponential diffusions. Nevertheless, in order to have a deeper understanding of averages induced by exponential diffusions, we must have some sharper evaluation of the operators T_n^f, that is of the functions $\beta_{i,j}^n$. This will be done in the following section, by using combinatorial objects, strongly linked to the Catalan numbers : *rooted-oriented trees.*

5 Tree-decompositions for exponential diffusions.

Most of our results are based on theorem 1 and on the fact that any object (weights, weighted functions, operators) associated to an exponential diffusion can be "decomposed" with the help of elementary coefficients indexed by rooted-oriented trees. Before we give these results, let us introduce some definitions and notations on trees.

5.1 Trees.

Some reminder about trees.

Let us define the sets Ca_n of rooted-oriented trees with n edges. These sets are drawn for the first values of n in the figure 1 below. The figure is quite sufficient to understand what rooted-oriented trees are and the reader could refer to [1, 5] for details. We called these sets Ca_n because their cardinal is the Catalan number ca_n. We can also introduce some terminology.

Let T be a tree of Ca_n. This tree has n edges and $n+1$ vertices : 1 root and n "children". The length $l(T)$ of the tree is n. The definition of the first children of a vertex of T $(fc(T))$ and of the "father" of a vertex are obvious.

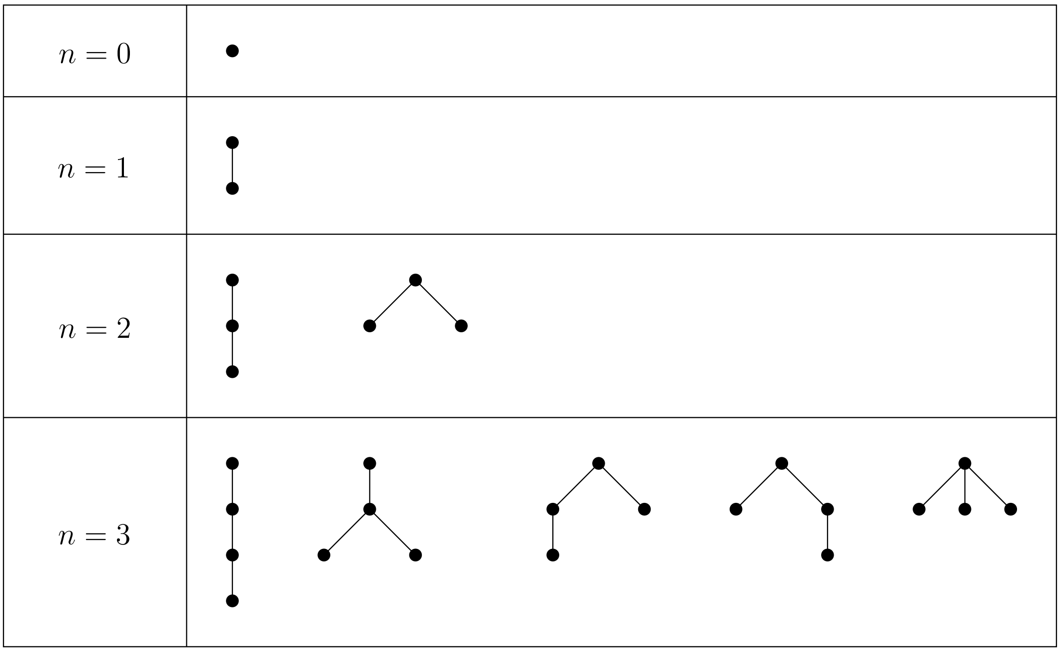


Figure 1: First sets Ca_n of rooted-oriented trees

Before we introduce some coefficients indexed by trees let us define a canonical way to "label" a tree, that is to give a name to each vertex of the tree. Then we should introduce the notion of "indexing" a tree with respect to a sequence.

Labeling of a tree.

Let us define the "labeling" of a tree with respect to a sequence. Let T be a tree of length n and $\boldsymbol{l}$ a sequence of index of length n ($\boldsymbol{l} = (l_0, \ldots, l_{n-1})$). Let us also give another index j. The labeled tree $T_{j:\boldsymbol{l}}$ is obtained by labeling the tree T with respect to the sequence $j : \boldsymbol{l}$: we name the root j and then name the other vertices from left to right and from the root to the children. For example :

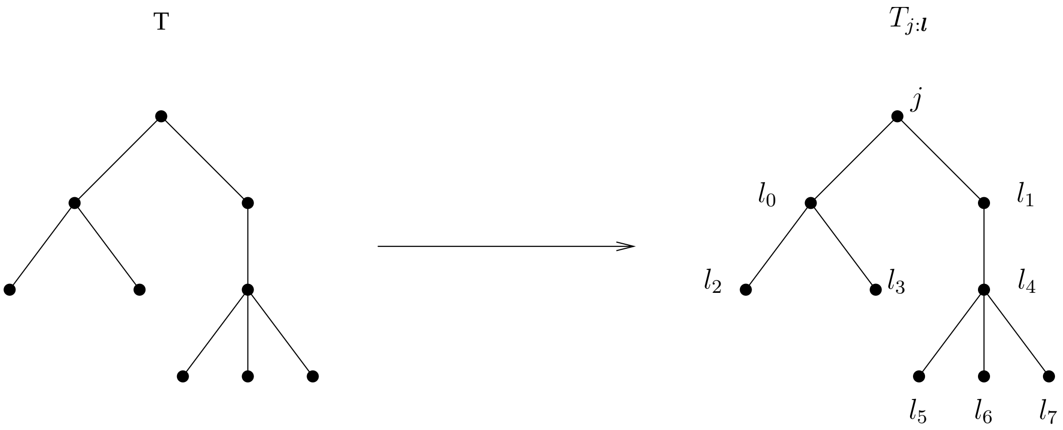


Figure 2: Labeling the tree T with respect to the sequence $j : \boldsymbol{l} = (j : l_0, \ldots, l_7)$.

Note that labeling a tree T with respect to a sequence $j : \boldsymbol{l}$ induces a natural partial ordering on this sequence and we can define, for each index in $\boldsymbol{l}$, an antecedent. For instance, in figure 2 :

$$\begin{cases} l_0^- = l_1^- = j \\ l_2^- = l_3^- = l_0 \\ l_3^- = l_1 \\ l_5^- = l_6^- = l_7^- = l_4 \end{cases} \tag{5.1}$$

To end with this section, we shall give two more definitions about some special labeled trees and about the decomposition of a labeled tree.

Rakes

For $n \geq 0$, in each set Ca_n, we shall give the name $\mathcal{T}^n$ to the "rake", that is the tree with one root and n first children :

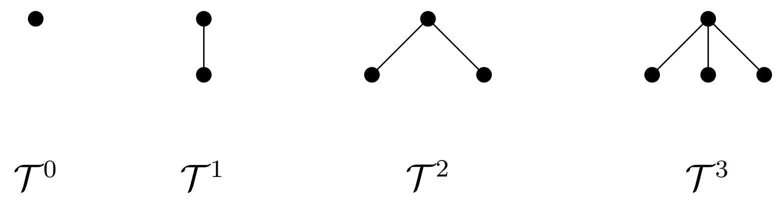


Figure 3: Rakes for $n = 0, 1, 2, 3$.

Decomposition of a labeled tree.

For $n \geq 0$, let us consider a tree T of Ca_n labeled by the sequence $j : \boldsymbol{l} = j : l_0, \ldots, l_{n-1}$. If k ($k \geq n-1$) is the number of first children of j, then we first consider the rake $\mathcal{T}^k_{j:l_0,\ldots,l_k}$.

Then we define, for $0 \leq p \leq k$, the subtree $T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}}$ as the subtree having l_p as root. The labeling sequence $\boldsymbol{l}^{<l_p}$ is naturally inherited from the initial tree. We have "decomposed" the tree :

$$T_{j:\boldsymbol{l}} = \mathcal{T}^k_{j:l_0,\ldots,l_{k-1}} \;\cup_{p=0}^{k-1} T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}} \tag{5.2}$$

For example :

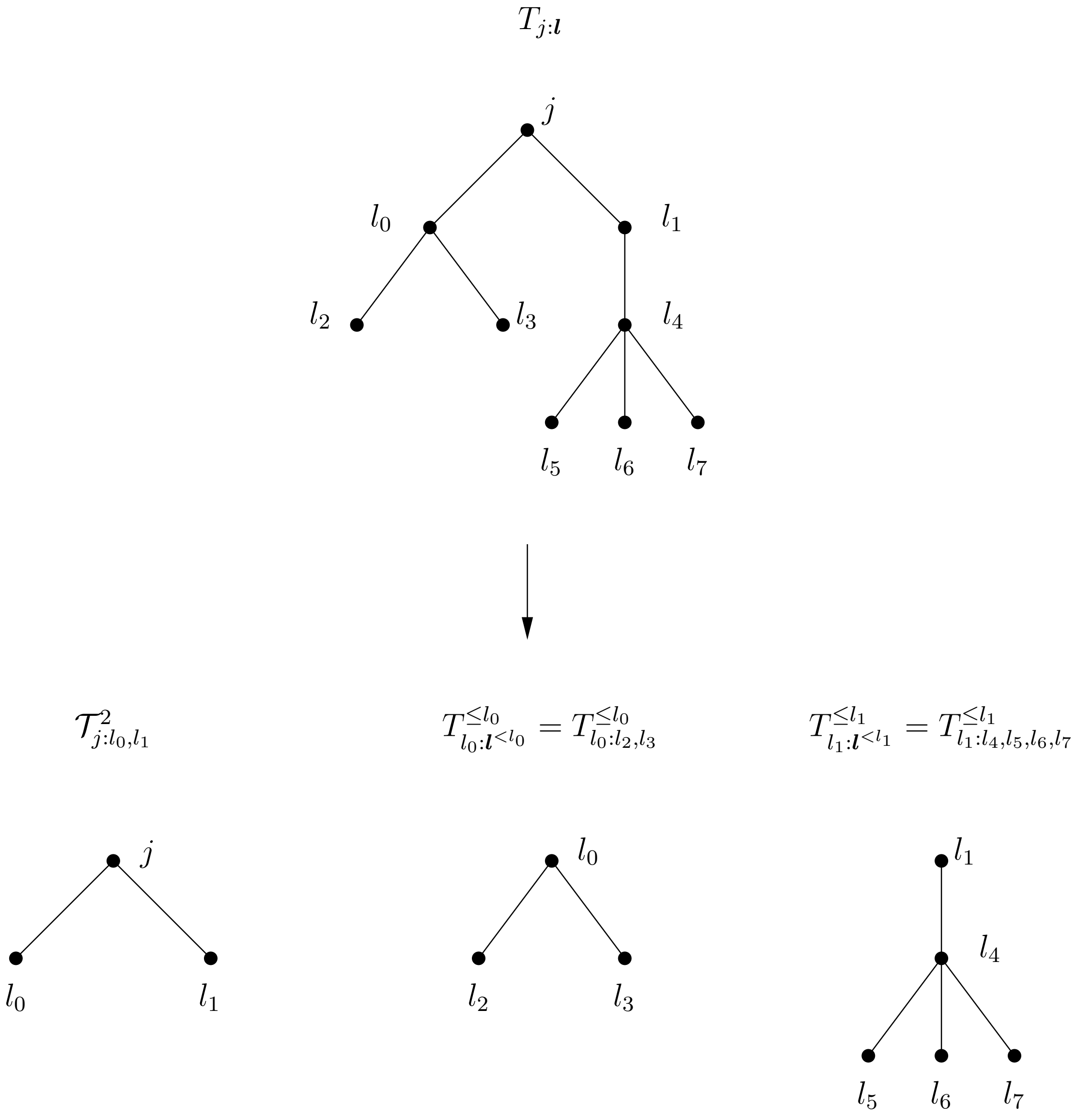


Figure 4: Decomposition of the tree $T_{j:l_0,\ldots,l_7}$.

Note that :

- The new sequences $\boldsymbol{l}^{<l_p}$ are graphically obvious. Nonetheless, for a given tree, it is not really easy to give these sequences explicitly.
- We can iterate this decomposition on each of the subtrees $T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}}$ that are not rakes so that we easily get a rake-decomposition of trees.
- If a labeled rake with k edges and k labeled trees are given, then, assuming that the indices are compatible, then we can re-compose the unique labeled tree having this decomposition. The compatibility of indices is quite obvious : it simply means that each index associated to a first child of the rake is equal to the root of one and only one of the k labeled trees.

We can now give the definition of miscellaneous coefficients indexed by trees that will be useful to study the "tree-decomposition" of the objects associated to an exponential diffusion.

5.2 Notations.

Let us remind that we considered an exponential diffusion :

$$f(x) = \sum_{i=1}^{d} a_i e^{-\lambda_i |x|}$$

We can associate to such a diffusion the following elementary coefficients :

$$1 \leq i, j \leq d, \quad R^i_j(x) = \frac{a_i}{\lambda_i + \lambda_j - x}, \quad R^i_j = R^i_j(0) = \frac{a_i}{\lambda_i + \lambda_j} \tag{5.3}$$

For a given tree of Ca_n, we can give a first family of coefficients indexed by labeled trees. Let us consider a tree T labeled with respect to the sequence $j : \boldsymbol{l} = j : l_0, \ldots, l_{n-1}$ ($1 \leq l_k \leq d$). Among the children $c(T)$ in the tree, we distinguish the first children of the root $fc(T)$ and to any child labeled by l_k, we can associate its father whose labeled is noted l_k^-. Now let us define the following coefficients :

$$R^{T_{j:l_0,\ldots,l_{n-1}}}(x) = \prod_{l_k \in fc(T)} R^{l_k}_j(x) \prod_{l_k \in c(T)/fc(T)} R^{l_k}_{l_k^-} \tag{5.4}$$

$$R^{T_{j:l_0,\ldots,l_{n-1}}} = R^{T_{j:l_0,\ldots,l_{n-1}}}(0) = \prod_{l_k \in c(T)} R^{l_k}_{l_k^-} \tag{5.5}$$

For example, if we consider the tree T pictured in figure 5, then, for $j : \boldsymbol{l} = j : l_0, \ldots, l_7$:

$$R^{T_{j:\boldsymbol{l}}}(x) = R^{l_0}_j(x) R^{l_1}_j(x) R^{l_2}_j(x) R^{l_3}_{l_1} R^{l_4}_{l_1} R^{l_5}_{l_3} R^{l_6}_{l_3} R^{l_7}_{l_3} \tag{5.6}$$

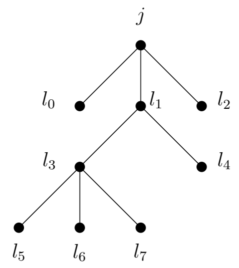


Figure 5: Example of tree.

$$R^{T_{j:\boldsymbol{l}}} \quad = \quad R_j^{l_0} R_j^{l_1} R_j^{l_2} R_{l_1}^{l_3} R_{l_1}^{l_4} R_{l_3}^{l_5} R_{l_3}^{l_6} R_{l_3}^{l_7} \tag{5.7}$$

Note that we could have define these coefficients by induction, using the previously defined decomposition of labelled tree :

For example, on figure 5, we have

$$T_{j:\boldsymbol{l}} = \mathcal{T}^3_{j:l_0,l_1,l_2} \ \cup \ T^{\leq l_0}_{l_0:\emptyset} \ \cup \ T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6,l_7} \ \cup \ T^{\leq l_3}_{l_3:\emptyset}$$

Thus

$$R^{T_{j:\boldsymbol{l}}}(x) = R^{\mathcal{T}^3_{j:l_0,l_1,l_2}}(x) \ R^{T^{\leq l_0}_{l_0:\emptyset}} R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6,l_7}} R^{T^{\leq l_3}_{l_3:\emptyset}}$$

and, by induction, we simply need the following definitions

$$R^{T^{\leq l_0}_{l_0:\emptyset}}(x) = R^{\mathcal{T}^0_{l_0:\emptyset}}(x) = 1$$

$$R^{\mathcal{T}^k_{j:l_0,\dots,l_{k-1}}} = \prod_{p=0}^{k-1} R_j^{l_p}$$

We can also give a notation that will be useful : If T is a tree of Ca_n ($n \geq 1$), then we can label this tree by the sequence $(j, l_0, l_1, \dots, l_{n-1})$ and then define :

$$R^T_{l_0,j}(x) \quad = \quad \sum_{(l_1,\dots,l_{n-1}) \in \{1,\dots,d\}^{n-1}} R^{T(j:l_0,l_1,\dots,l_{n-1})}(x) \tag{5.8}$$

$$R^T_{l_0,j} \quad = \quad \sum_{(l_1,\dots,l_{n-1}) \in \{1,\dots,d\}^{n-1}} R^{T(j:l_0,l_1,\dots,l_{n-1})} \tag{5.9}$$

With the use of these notations, we give now many theorems on Tree-decomposition of the objects associated to an exponential diffusion.

5.3 Tree-decomposition for operators.

Let us first remind that, because of theorem 1, the operators T^f_n are strongly related to some infinite matrices :

For $n \geq 1$, the infinite representative matrix P_n^f of T_n^f can be decomposed in a $d \times d$ matrix of infinite matrices $P_n^f = ((P^f_{n,(i,j)}))_{\substack{1\leq i\leq d\\ 1\leq j\leq d}}$ with

$$P^f_{n,(i,j)} = A_{\beta^n_{i,j}(x)} \tag{5.10}$$

where the series $\beta^n_{i,j}(x)$ are defined by the following induction :

$$\begin{aligned} \beta^1_{i,j}(x) &= \beta_{i,j}(x) && (5.11)\\ \beta^{n+1}_{i,j}(x) &= \sum_{\substack{k=1\\ k\neq j}}^{d} \beta^n_{i,k}(0)\alpha_{k,j}(x) \;+\; \Delta(\beta^n_{i,j}(x)\alpha_{j,j}(x)) && (5.12)\end{aligned}$$

with

$$\begin{aligned} \alpha_{i,j}(x) &= \frac{a_i}{\lambda_j - \lambda_i - x} \; for\ i \neq j && (5.13)\\ \alpha_{i,i}(x) &= \sum_{k=1}^{d} \frac{2\lambda_k a_k x}{\lambda_k^2 - (\lambda_i - x)^2} && (5.14)\\ \beta_{i,j}(x) &= \frac{a_i}{\lambda_i + \lambda_j - x} && (5.15)\end{aligned}$$

We shall see now that the coefficients $\beta^n_{i,j}(x)$ can be expressed in term of tree-indexed coefficients.

Theorem 2 *For $n \geq 1$ and $1 \leq i, j \leq d$:*

$$\beta^n_{i,j}(x) = \sum_{T \in Ca_n} R^T_{i,j}(x) \tag{5.16}$$

Scheme of the proof : Let us consider the matrices $\beta^n(x) = ((\beta^n_{i,j}(x)))_{1\leq i,j\leq d}$ then theorem 1 defines a transformation $\mathbf{U}\ :\ \beta^n(x) \mapsto \beta^{n+1}(x)$ (see above) that can be trivially extended to a linear operator on $d\times d$ matrices with coefficients in $\mathbf{C}[[x]]$. To prove theorem 2, by induction on n, we shall study the action of $\mathbf{U}$ on the matrices :

$$R^T_{\bullet,\bullet}(x) = \big(\big(R^T_{i,j}(x)\big)\big)_{1\leq i,j\leq d} \qquad (T \in Ca_n\ ;\ n \geq 1) \tag{5.17}$$

The proof will be easily obtained, using the two following lemmas.

Lemma 1 *Let $1 \leq n_0 \leq n$ and T a tree of Ca_n with n_0 first children. We have :*

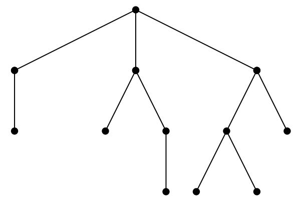
$$\boldsymbol{U}.R^T_{\bullet,\bullet}(x) = \sum_{k=0}^{n_0} R^{kT}_{\bullet,\bullet}(x) \tag{5.18}$$

where the trees of Ca_{n+1} ${}^0T, \ldots, {}^{n_0}T$ stem from well-defined transformations of the tree T :

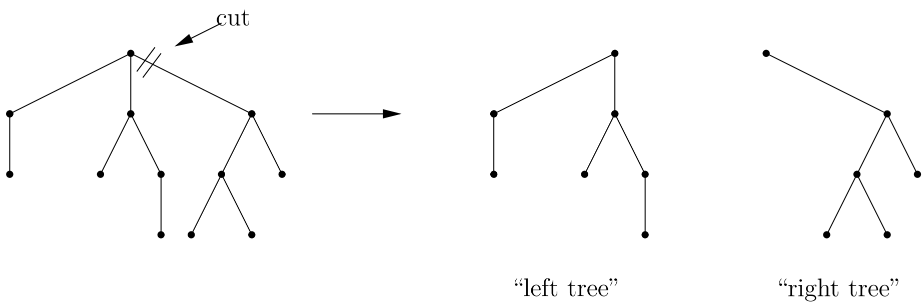
- *If $0 \leq k \leq n_0 - 1$, then "cut" the tree T at the root, just after the edge of the $(k+1)^{th}$ first child (first children counted from left to right). We have now a "left" tree and a "right" one. On the left tree (with $k+1$ first children), add an edge to the $(k+1)^{th}$ first child, to the right of its other edges, and then identify ("glue") the lower extremity of this edge to the root of the right tree. This new tree is ${}^kT$.*

- *If $k = n_0$ then simply add an edge to the root, to the right of its other edges*

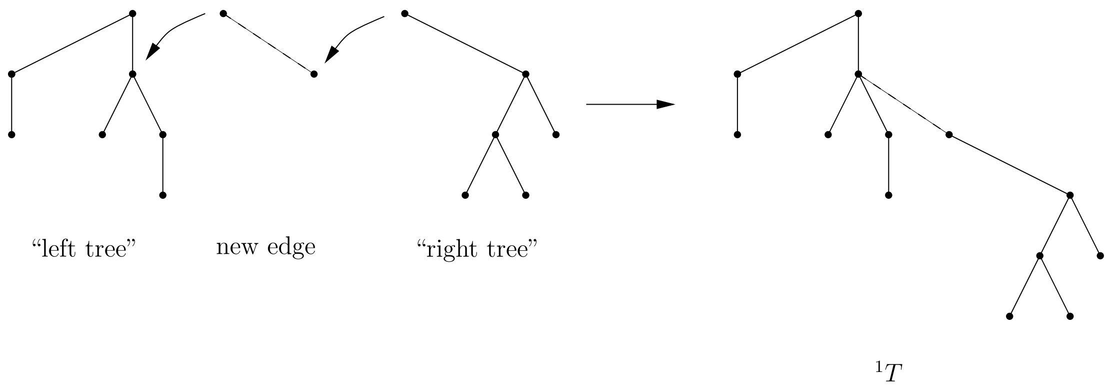
The graphical interpretation of these operations is really simple. For example, let us consider the following tree T with $n_0 = 3$ and $n = 11$:



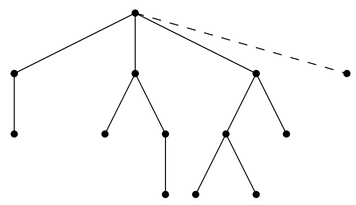
To build the tree ${}^1T$ ($k = 1$) we cut after the second first edge to get the left and right trees :



then we add a new edge to the right first child of the left tree and glue the right tree to the lower extremity of this edge :



For the specific case $k = n_0$, it is clear that, for this example we get :



${}^1T$

Using lemma 1, for a given integer n, the transformation $\mathbf{U}$ induces an application $\mathbf{U}^*$ from the set :

$$\widetilde{Ca}_n \overset{def}{=} \{(k,T)\ ;\ T \in Ca_n\ ;\ 0 \leq k \leq \mathrm{Card}(fc(T))\} \tag{5.19}$$

into Ca_{n+1}, defined by :

$$\begin{array}{rrcl} \mathbf{U}^*\ : & \widetilde{Ca}_n & \to & Ca_{n+1} \\ & (k,T) & \mapsto & {}^kT \quad \text{(see Lemma 1)} \end{array} \tag{5.20}$$

Then :

Lemma 2 ***U**** *is a bijection.*

Because of these lemmas, the proof of theorem 2 is straightforward. The result is true for $n = 1$ (see the definitions). Suppose that the result is true for $n \geq 1$:

$$\beta^n(x) = \sum_{T \in Ca_n} R^T_{\bullet,\bullet}(x)$$

then

$$\begin{array}{rcl} \beta^{n+1}(x) & \underset{\text{def}}{=} & \mathbf{U}.\beta^n(x) \\ & = & \displaystyle\sum_{T \in Ca_n} \mathbf{U}.R^T_{\bullet,\bullet}(x) \\ & \underset{\text{Lemma 1}}{=} & \displaystyle\sum_{T \in Ca_n} \sum_{k=0}^{\mathrm{Card}(fc(T))} R^{{}^kT}_{\bullet,\bullet}(x) \\ & \underset{\text{Eq. (5.20)}}{=} & \displaystyle\sum_{(k,T) \in \widetilde{Ca}_n} R^{\mathbf{U}^*(k,T)}_{\bullet,\bullet}(x) \\ & \underset{\text{Lemma 2}}{=} & \displaystyle\sum_{T \in Ca_{n+1}} R^T_{\bullet,\bullet}(x) \end{array}$$

This ends the proof of theorem 2.

We shall "illustrate" the proofs for these two lemmas at the end of this part. But right now, we will deduce some similar theorems for the weighted functions associated to a diffusion and also for the weights of an average induced by an "exponential diffusion".

5.4 Tree-decomposition for weighted functions.

Let us define the following weighted functions associated to a diffusion f (see definition 3) :

$$\begin{aligned} F^{n_1,\dots,n_t} &= T^f_{n_t}\dots T^f_{n_1}F \\ &= F^{(\pm)^{n_1}(\mp)^{n_2}\dots(\varepsilon_t)^{n_t}\bar{\varepsilon}_t} \end{aligned} \tag{5.21}$$

for $t\geq 1$ and $n_i\geq 1$ $(1\leq i\leq t)$.

Because of theorem 1 and proposition 5,

$$F^{n_1,\dots,n_t}(x)=\sum_{i=1}^{d} b_i^{n_1,\dots,n_t}e^{-\lambda_i|x|}$$

with the vector of dimension d :

$$\vec{b}^{n_1,\dots,n_t}=\beta^{n_t}(0)\dots\beta^{n_1}(0)\,.\,\vec{a}$$

and of course $f(x)=\displaystyle\sum_{i=1}^{d}a_ie^{-\lambda_i|x|}$.

Using the results of the previous section, and once again manipulating some trees, we get the following result :

Theorem 3 *For a given sequence of positive integers $(n_1,\dots,n_t)$, we remind that*

$$F^{n_1,\dots,n_t}(x)=\sum_{i=1}^{d} b_i^{n_1,\dots,n_t}e^{-\lambda_i|x|} \tag{5.22}$$

For this sequence we can find, in a constructive way, a subset $Ca_{n_1,\dots,n_t}$ of $Ca_{n_1+\dots+n_t}$ containing exactly $ca_{n_1}ca_{n_2}\dots ca_{n_t}$ elements such that :

$$\forall i\ ,\ 1\leq i\leq d\ ,\qquad b_i^{n_1,\dots,n_t}=a_i\sum_{T\in Ca_{n_1,\dots,n_t}}R_i^T \tag{5.23}$$

with

$$R_i^T=\sum_{l_0=1}^{d}R^T_{l_0,i}(0) \tag{5.24}$$

Once again we shall simply illustrate the proof of this theorem. Nonetheless we can easily give the explicit and inductive construction of $Ca_{n_1,\dots,n_t}$:

Step 0 : For each integer s $(1\leq s\leq t)$ chose a tree T^s in Ca_{n_s}.

Step 1 : Pull up the left first child of T^1 so that it becomes the root of a new tree $\widetilde{T}^1$. If $t = 1$ the procedure is over.

Step 2 : Glue $\widetilde{T}^1$ and T^2 using the following rules : the roots of both trees are identified and $\widetilde{T}^1$ is located to the right of the first edges of T^2. Repeat Step 1 to this tree to obtain a tree $\widetilde{T}^2$ of $Ca_{n_1+n_2}$.

Step s : $(s \geq 2)$ Repeat Step 2 for the trees $\widetilde{T}^{s-1}$ and T^s to get a tree $\widetilde{T}^s$ of $Ca_{n_1+\cdots+n_s}$.

Step t : Repeat Step s for $s = t$ to finally get a tree $\widetilde{T}^t$ of $Ca_{n_1+\cdots+n_t}$.

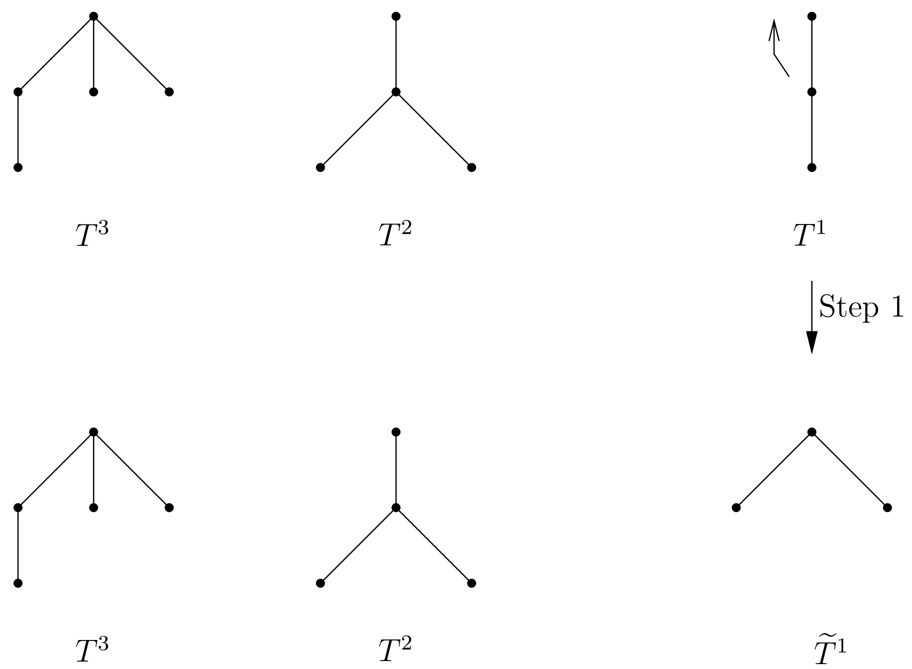
Thus to each t-uple $(T^1, \ldots, T^t)$ of $Ca_{n_1} \times \cdots \times Ca_{n_t}$ we can associate a tree $\widetilde{T}^t = H(T^1, \ldots, T^t)$ of $Ca_{n_1+\cdots+n_t}$ and

$$Ca_{n_1,\ldots,n_t} = \{H(T^1, \ldots, T^t) \ ; \ (T^1, \ldots, T^t) \in Ca_{n_1} \times \cdots \times Ca_{n_t}\} \tag{5.25}$$

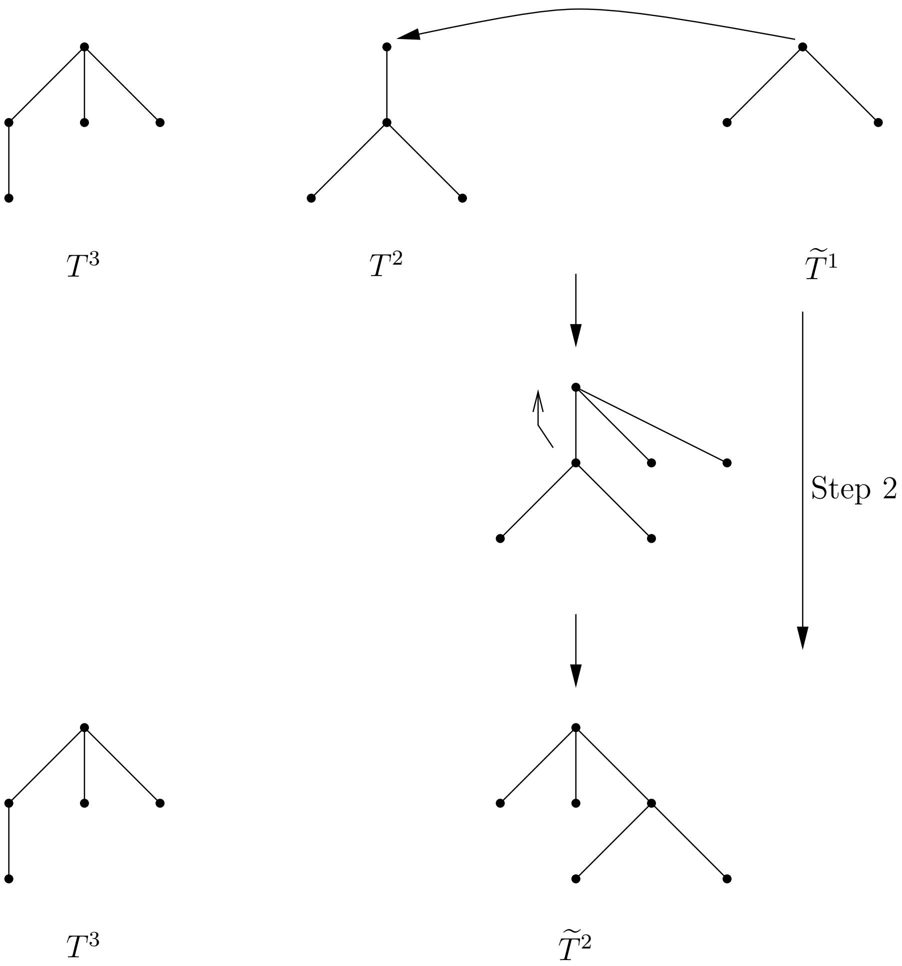
Note that we wrote in the theorem that $Ca_{n_1,\ldots,n_t}$ is a *subset* of $Ca_{n_1+\cdots+n_t}$ containing as much elements as $Ca_{n_1} \times \cdots \times Ca_{n_t}$. Thus, we will have to prove that the transformation H is injective.

The proof of this theorem shall be illustrated later, but as in the previous section, let us give a simple graphical interpretation of the above construction, For 3 trees as below, let us give the three steps leading to a tree $\widetilde{T}^3$.

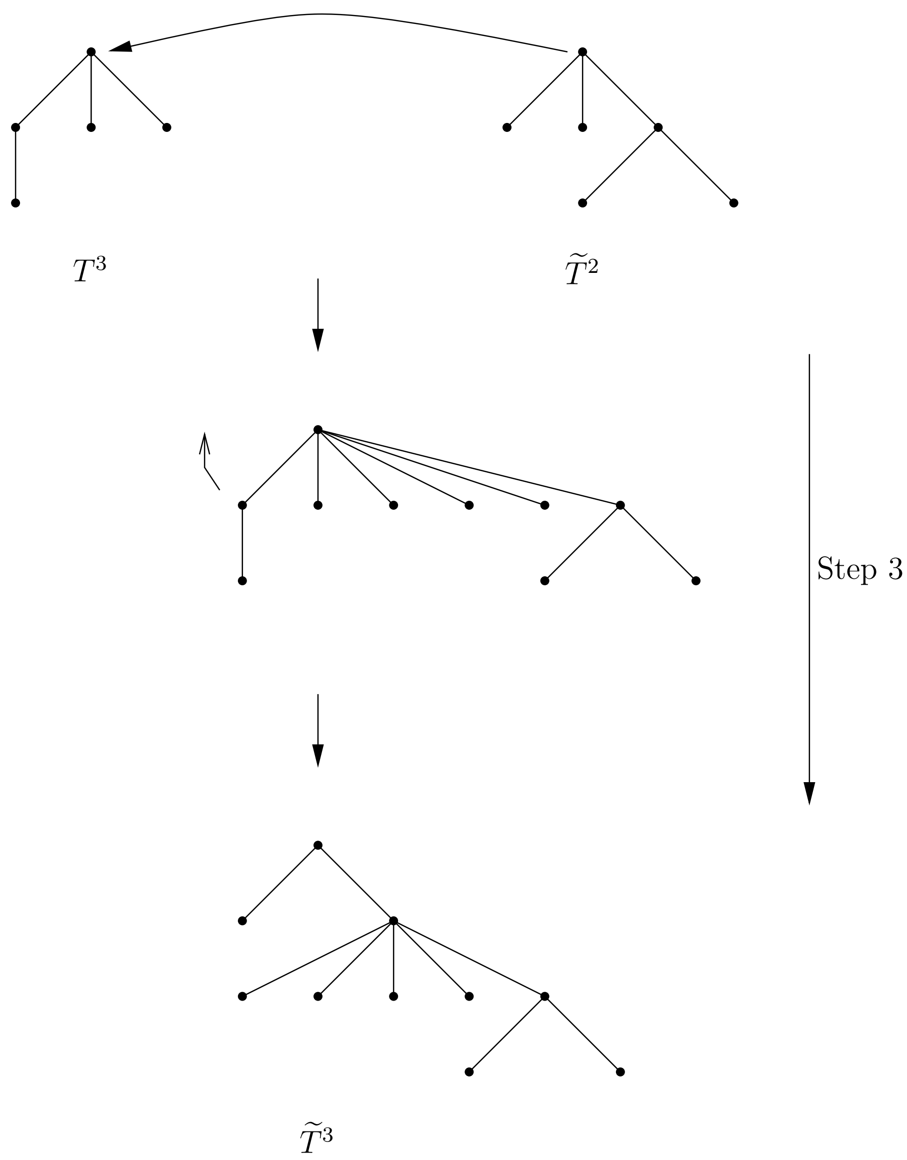
Step 1 :



Step 2 :



Step 3 :



We give now the same kind of result for the weights of an average induced by exponential diffusion.

5.5 Tree-decomposition for the weights of an average.

Once again, let us define the weights (see section 4) :

$$\mathbf{m}^{n_1,\ldots,n_t} = \mathbf{m}^{(\pm)^{n_1}(\mp)^{n_2}\ldots(\varepsilon_t)^{n_t}\bar{\varepsilon}_t} \tag{5.26}$$

for $t \geq 1$ and $n_i \geq 1$ $(1 \leq i \leq t)$.

We remind that it is sufficient to know these weights to determine all the weights of the average **m**. Using the previous theorem and the proposition 5 :

Theorem 4 *For* $(n_1, \ldots, n_t) \in (\boldsymbol{N})^t$ $(t \geq 1)$ *:*

$$\boldsymbol{m}^{n_1,\ldots,n_t} = \sum_{T \in Ca_{n_1,\ldots,n_t}} \boldsymbol{m}^T \tag{5.27}$$

with

$$\boldsymbol{m}^T = \sum_{i=1}^{d} \frac{a_i}{\lambda_i} R_i^T \tag{5.28}$$

In this case, the theorem is a very simple consequence of the previous one.

We can now give a partial answer to the following question : Which are the exponential diffusion inducing the Catalan average ?

6 Exponential diffusions inducing the Catalan average.

Let us first give some precise statements for the Catalan diffusion.

6.1 The Catalan diffusion.

The Catalan diffusion is :

$$fa(x) = \frac{1}{2} e^{-|x|} \tag{6.1}$$

and, because of the homogeneity of the construction, we can generate the Catalan average by any of the following diffusions :

$$f(x) = a e^{-\lambda |x|} \quad \frac{a}{\lambda} = \frac{1}{2} \ , \ \lambda > 0 \tag{6.2}$$

These diffusions define the same average **man** with weights :

$$\mathbf{man}^{\varepsilon_1,\ldots,\varepsilon_n} \equiv 4^{-n} ca_{n_1} ca_{n_2} \ldots ca_{n_s} (1 + n_s) \tag{6.3}$$

with the classical Catalan numbers :

$$ca_n \stackrel{def}{=} \frac{(2n)!}{n! \, (n+1)!} \quad (ca_n \in \mathbf{N}) \tag{6.4}$$

which in this case are indexed by the integers $n_1, n_2, \ldots, n_s$ which denote the numbers of identical consecutive signs within the address $(\varepsilon_1, \ldots, \varepsilon_n)$:

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\pm)^{n_1} (\mp)^{n_2} \ldots (\varepsilon_n)^{n_s} \quad (\textit{of course } n_1 + \cdots + n_s = n) \tag{6.5}$$

Note that (see the previous section) for a sequence of integers $(n_1, \dots, n_t)$:

$$\mathbf{man}^{n_1,\dots,n_t} = \mathbf{man}^{(\pm)^{n_1}(\mp)^{n_2}\dots(\varepsilon_t)^{n_t}\bar{\varepsilon}_t} = \frac{1}{2}4^{-(n_1+\dots+n_t)}ca_{n_1}ca_{n_2}\dots ca_{n_t} \tag{6.6}$$

This average depends only on the stacks of identical signs and this property already exists for the weighted functions associated to the Catalan diffusion :

$$Fa^{(\pm)^{n_1}(\mp)^{n_2}\dots(\varepsilon_n)^{n_s}} = 4^{-n+n_s}ca_{n_1}ca_{n_2}\dots ca_{n_{s-1}}Fa^{(\varepsilon_n)^{n_s}}(x) \tag{6.7}$$

Once again note that :

$$\begin{aligned} Fa^{n_1,\dots,n_t} &= T^f_{n_t}\dots T^f_{n_1}Fa \\ &= Fa^{(\pm)^{n_1}(\mp)^{n_2}\dots(\varepsilon_t)^{n_t}\bar{\varepsilon}_t} \\ &= 4^{-(n_1+\dots+n_t)}ca_{n_1}ca_{n_2}\dots ca_{n_t}Fa^{\bar{\varepsilon}_t} \end{aligned} \tag{6.8}$$

for $t \geq 1$, $n_i \geq 1$ $(1 \leq i \leq t)$ and, of course, $Fa^{\varepsilon} = fa$.

We don't go further in the description of the Catalan average and the reader can refer to [6] for details.

6.2 The strong and the weak Catalan property.

We shortly described the Catalan diffusion and the Catalan average. If we are looking for diffusions f inducing the Catalan average, the most natural way to do it is to compare the weights of the average $\mathbf{m}$ induced by f to the weights of $\mathbf{man}$. If $\mathbf{m} = \mathbf{man}$ then the diffusion f has the "weak Catalan property".

But these equations are rather difficult to solve. Thus, we will restrict ourselves to a more simple case. Instead of working on the weights induced by a given diffusion f, we can rather consider the weighted functions. Using the equation (6.7), we get :

$$\forall n \geq 1, \quad T^{fa}_n fa(x) = \frac{ca_n}{4^n} fa(x) \tag{6.9}$$

This property is noticeable because it's a sufficient property for fa to generate the average $\mathbf{man}$, as a consequence of the proposition 1. Thus, if a diffusion f verifies the equations (6.9), it automatically induces the Catalan average. We call this property the "strong Catalan property".

6.3 Exponential diffusions having the strong Catalan property.

Let us consider an exponential diffusion :

$$f(x) = \sum_{i=1}^{d} a_i e^{-\lambda_i |x|}$$

Using the proposition 5, this diffusion can be written $\vec{a} = (a_1, \ldots, a_d)$ in the basis $e_{1,0}, \ldots, e_{d,0}$. If f has the strong Catalan property, then we must have at least :

$$\forall n \geq 1, \quad \widetilde{P}_n^f \vec{a} = \vec{a^n} = \gamma_n \vec{a} \tag{6.10}$$

For $n = 1$, $1 \leq i \leq d$,

$$a_i^1 = \sum_{j=1}^{d} \beta_{i,j}(0) a_j = \sum_{j=1}^{d} \frac{a_i a_j}{\lambda_i + \lambda_j} = a_i \sum_{j=1}^{d} \frac{a_j}{\lambda_i + \lambda_j} = a_i S(\lambda_i) \tag{6.11}$$

with

$$S(X) = \sum_{j=1}^{d} \frac{a_j}{X + \lambda_j} \tag{6.12}$$

Thus, to obtain the equation (6.10) for $n = 1$, we must have :

$$\forall i, j \ ; \quad S(\lambda_i) = S(\lambda_j) \tag{6.13}$$

and then,

$$\gamma_1 = S(\lambda_i)$$

We can note that

$$S(0) = \sum_{j=1}^{d} \frac{a_j}{\lambda_j} = \frac{1}{2}$$

If we write

$$S(X) = \frac{P(X)}{Q(X)}$$

then

$$S(X) - S(\lambda_1) = \frac{P(X)Q(\lambda_1) - P(\lambda_1)Q(X)}{Q(X)Q(\lambda_1)} = \frac{P_1(X)}{Q(X)Q(\lambda_1)}$$

and, because of equation (6.13), we must have

$$P_1(\lambda_1) = \cdots = P_1(\lambda_d) = 0$$

But this means that

$$P_1(X) = -P(\lambda_1) \prod_{i=1}^{d} (X - \lambda_i)$$

and then

$$S(X) = -S(\lambda_1) \frac{\prod_{i=1}^{d}(X - \lambda_i) - \prod_{i=1}^{d}(X + \lambda_i)}{\prod_{i=1}^{d}(X + \lambda_i)}$$

The equation :

$$S(0) = \frac{1}{2} = -S(\lambda_1)((-1)^d - 1)$$

proves that d must be odd and that $S(\lambda_1) = 1/4$.

But if we decompose the fraction $S(X)$, we get :

$$\begin{aligned} S(X) &= \sum_{j=1}^{d} \frac{a_j}{X+\lambda_j} \\ &= -\frac{1}{4}\frac{\prod_{i=1}^{d}(X-\lambda_i) - \prod_{i=1}^{d}(X+\lambda_i)}{\prod_{i=1}^{d}(X+\lambda_i)} \\ &= \frac{1}{4}\sum_{j=1}^{d} \frac{\prod_{i=1}^{d}(\lambda_j+\lambda_i)}{\prod_{\substack{i=1\\ i\neq j}}^{d}(\lambda_j-\lambda_i)} \frac{1}{X+\lambda_j} \end{aligned}$$

This means that if f has the strong Catalan property, it must at least have the following properties :

$$\begin{cases} d \text{ is odd} \\ \\ \forall 1 \leq j \leq d, \quad a_j = \dfrac{1}{4}\dfrac{\prod_{i=1}^{d}(\lambda_j+\lambda_i)}{\prod_{\substack{i=1\\ i\neq j}}^{d}(\lambda_j-\lambda_i)} \end{cases} \tag{6.14}$$

and this means that $\gamma_1 = 1/4 = ca_1/4$.

We can now prove that this is also a sufficient condition. Using theorem 2 and proposition 5, we have :

$$\gamma_1 = \sum_{j=1}^{d} \beta_{j,i}^{1}(0) = \sum_{j=1}^{d} R_i^j(0) = \frac{1}{4} \tag{6.15}$$

but for $n > 1$ the function $T_n^f f$ is represented by the vector $\vec{b^n} = (b_1^n, \ldots, b_d^n)$ and, see theorem 3 :

$$b_i^n = a_i \sum_{T\in Ca_n} R_i^T$$

and,

$$\begin{aligned} R_i^T &= \sum_{(l_0,l_1,\ldots,l_{n-1})\in\{1,\ldots,d\}^n} R^{T_{i:l_0,\ldots,l_{n-1}}}(0) \\ &= \sum_{(l_0,l_1,\ldots,l_{n-1})\in\{1,\ldots,d\}^n} \prod_{l_k\in fc(T)} R_i^{l_k} \prod_{l_k\in c(T)/fc(T)} R_{l_k^-}^{l_k} \\ &= \sum_{(l_0,l_1,\ldots,l_{n-1})\in\{1,\ldots,d\}^n} \prod_{l_k\in c(T)} R_{l_k^-}^{l_k} \end{aligned}$$

But if we sum on the indices $l_{n-1}, l_{n-2}, \ldots, l_1, l_0$ in this specific order, then, using equation (6.15), we get :

$$b_i^n = a_i \sum_{T\in Ca_n} \frac{1}{4^n} = a_i \frac{Card(Ca_n)}{4^n} = a_i \frac{ca_n}{4^n}$$

and that proves that f has the strong Catalan property.

We can give the :

Theorem 5 *A diffusion f :*

$$f(x) = \sum_{i=1}^{d} a_i e^{-\lambda_i |x|}$$

*has the strong Catalan property (and thus induces **man**) if and only if :*

$$\begin{cases} d \text{ is odd} \\ \\ \forall 1 \leq i \leq d, \quad a_i = \dfrac{1}{4} \dfrac{\prod_{j=1}^{d} (\lambda_i + \lambda_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{d} (\lambda_i - \lambda_j)} \end{cases} \tag{6.16}$$

6.4 What about the weak property ?

We won't go further here but it seems that :

Conjecture : *The only diffusions inducing **man** are those with the strong Catalan property.*

We shall give some ideas to prove this in the conclusion.

7 Key examples to prove theorems 2 and 3.

We won't give completely exhaustive proofs for the above theorems. For each one, we will give key examples. Nonetheless, the reader should easily understand that these examples somehow *contains* the proof, as all the necessary arguments are exposed. Rigorous proofs are left to the reader willing to deal with great amounts of heavy notations.

7.1 Lemma 1

Let us first give in sections 7.1.1 and 7.1.2 some remarks about the transformation $\mathbf{U}$ induced by equation (5.12). Once this work is done, we shall justify the combinatorial part of Lemma 1 on a very illustrative example.

7.1.1 Remarks about U.

First of all, the transformation $\mathbf{U}$ can be trivially extended to a linear operator on matrices $d \times d$ with coefficients in $\mathbf{C}[[x]]$ (see eq. (5.12)).

Due to this, we can look at its action on a matrix $\left(\left(R^T_{l_0,j}(x)\right)\right)_{1 \leq l_0, j \leq d}$ for any tree $T \in Ca_n$ $(n \geq 1)$.

If we remind the following notations :

$$R^T_{l_0,j}(x) = \sum_{(l_1,\ldots,l_{n-1})\in\{1,\ldots,d\}^{n-1}} R^{T_{j:l_0,l_1,\ldots,l_{n-1}}}(x)$$

and, if T has k first children,

$$T_{j:l_0,l_1,\ldots,l_{n-1}} = \mathcal{T}^k_{j:l_0,\ldots,l_{k-1}} \ \cup_{p=0}^{k-1} T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}}$$

then we can write the following equations :

$$\begin{aligned}
\left(\mathbf{U}R^T\right)_{l_0,j} &= \sum_{\substack{m=1\\m\neq j}}^{d} R^T_{l_0,m}(0)\alpha_{m,j}(x) \ + \ \Delta(R^T_{l_0,j}(x)\alpha_{j,j}(x)) \\
= \sum_{l_1,\ldots,l_{n-1}} & \left(\left(\sum_{\substack{m=1\\m\neq j}}^{d} R^{\mathcal{T}^k_{m:l_0,\ldots,l_{k-1}}}(0)\alpha_{m,j}(x) \ + \ \Delta(R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\alpha_{j,j}(x))\right)\prod_{p=0}^{k-1} R^{T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}}}\right)
\end{aligned}$$

And, thanks to the decomposition of trees, we can focus on the following coefficients :

$$\mathcal{R}^{j:l_0,l_1,\ldots,l_{k-1}} \overset{def}{=} \sum_{\substack{m=1\\m\neq j}}^{d} R^{\mathcal{T}^k_{m:l_0,\ldots,l_{k-1}}}(0)\alpha_{m,j}(x) \ + \ \Delta(R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\alpha_{j,j}(x)) \quad (k\geq 1) \tag{7.1}$$

7.1.2 Decomposition of U.

The action of the operator **U** is resumed in the above coefficient. This one can be decomposed in a very interesting way. Using the definitions of Δ and $\alpha_{i,j}(x)$, we get :

$$\Delta\left(R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\alpha_{j,j}(x)\right) = \Delta\left(R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\right)\alpha_{j,j}(0) + R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\Delta(\alpha_{j,j}(x))$$

and

$$\begin{aligned}
\alpha_{j,j}(0) &= a_j \\
\Delta(\alpha_{j,j}(x)) &= \sum_{m=1}^{d}\beta_{m,j}(x) - \sum_{\substack{m=1\\m\neq j}}^{d}\alpha_{m,j}(x)
\end{aligned}$$

thus we can write $\mathcal{R}^{j:l_0,l_1,\ldots,l_{k-1}} = \mathcal{R}_k^{j:l_0,l_1,\ldots,l_{k-1}} + \widetilde{\mathcal{R}}^{j:l_0,l_1,\ldots,l_{k-1}}$:

$$\begin{aligned}
\mathcal{R}_k^{j:l_0,l_1,\ldots,l_{k-1}} &= \sum_{m=1}^{d} R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)R^{\mathcal{T}^1_{j:m}}(x) \qquad (7.2)\\
\widetilde{\mathcal{R}}^{j:l_0,l_1,\ldots,l_{k-1}} &= \sum_{\substack{m=1\\m\neq j}}^{d}\left(R^{\mathcal{T}^k_{m:l_0,\ldots,l_{k-1}}}(0) - R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x)\right)\alpha_{m,j}(x)
\end{aligned}$$

$$+\ a_j \Delta \left(R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x) \right) \tag{7.3}$$

and, after some easy computations, we get :

$$\widetilde{\mathcal{R}}^{j:l_0,l_1,\ldots,l_{k-1}} = \sum_{t=0}^{k-1} \sum_{m=1}^{d} R^{\mathcal{T}^{t+1}_{j:l_0,\ldots,l_t}}(x) R^{\mathcal{T}^1_{l_t:m}}(0) R^{\mathcal{T}^{k-1-t}_{m:l_{t+1},\ldots,l_{k-1}}}(0) \tag{7.4}$$

We have decomposed $\mathcal{R}^{j:l_0,l_1,\ldots,l_{k-1}}$ in $k+1$ terms :

$$\mathcal{R}_t^{j:l_0,l_1,\ldots,l_{k-1}} = \sum_{m=1}^{d} R^{\mathcal{T}^{t+1}_{j:l_0,\ldots,l_t}}(x) R^{\mathcal{T}^1_{l_t:m}}(0) R^{\mathcal{T}^{k-1-t}_{m:l_{t+1},\ldots,l_{k-1}}}(0) \quad (0 \le t \le k-1) \tag{7.5}$$

$$\mathcal{R}_k^{j:l_0,l_1,\ldots,l_{k-1}} = \sum_{m=1}^{d} R^{\mathcal{T}^k_{j:l_0,\ldots,l_{k-1}}}(x) R^{\mathcal{T}^1_{j:m}}(x) \tag{7.6}$$

We gave in the two above sections the computations that are necessary to prove Lemma 1. To finish the proof we will first give some graphical and combinatorial interpretations of the above results. This step will really simplify the induction of the proof.

7.1.3 Combinatorial interpretation.

Let us graphically resume the two previous sections.

First we considered in section 7.1.1 an elementary coefficient $R^T_{l_0,j}(x)$ for a tree T. Let us give the example illustrated by figure 6.

T

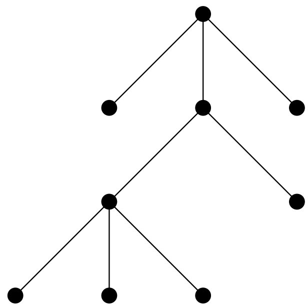


Figure 6: Example of tree.

We reminded then that this coefficient was a sum over indices $l_1, \ldots, l_{n-1}$ of some elementary coefficients indexed by a labeled tree $T_{j:l_0,l_1,\ldots,l_{n-1}}$:

$T_{j:l_0,l_1,\ldots,l_{n-1}}$

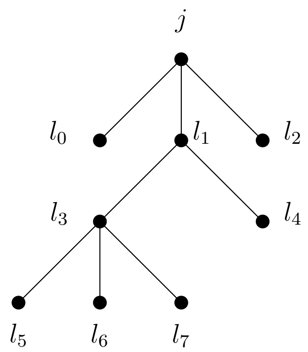


Figure 7: Associated labeled tree $T_{j:l_0,l_1,\ldots,l_{n-1}}$.

Using the decomposition of a labeled tree, see figure 8 below,we were then able to isolate the most important part in the action of $\mathbf{U}$, that is the expression $\mathcal{R}^{j:l_0,l_1,\ldots,l_{k-1}}$. See equation (7.1).

In section 7.1.2 we decomposed this expression in $k+1$ terms, each of them having a new summation index m. Putting this together with :

- the decomposition of our tree,
- our previous summations on $l_1, \ldots, l_{n-1}$,
- and the convention that the branch $\mathcal{T}^1_{l_t:m}$ (resp. $\mathcal{T}^1_{j:m}$) will stay on the left of the other branches attached to the summit l_t (resp. j),

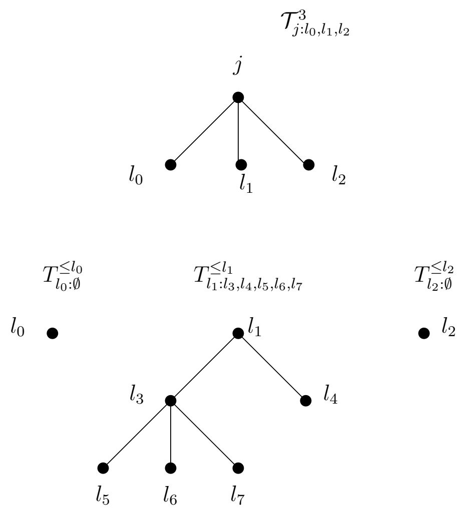


Figure 8: Decomposition of the labeled tree $T_{j:l_0,l_1,\dots,l_{n-1}} = \mathcal{T}^k_{j:l_0,\dots,l_{k-1}} \cup_{p=0}^{k-1} T^{\leq l_p}_{l_p:\boldsymbol{l}^{<l_p}}$.

we can graphically understand that, using the transformation $\mathbf{U}$ each coefficient $R^T_{l_0,j}$ will yield, for a tree of Ca_n with k first children, $k+1$ similar coefficients indexed by trees of Ca_{n+1}. Let us now illustrate these transformations for our tree. Let us first remind that, in our example :

- We are dealing at the beginning with the coefficient (see figure 7 and eq. (5.4)) :

$$R^T_{l_0,j} = \sum_{(l_1,\dots,l_7)\in\{1,\dots,d\}^7} R^{l_0}_j(x) R^{l_1}_j(x) R^{l_2}_j(x) R^{l_3}_{l_1} R^{l_4}_{l_1} R^{l_5}_{l_3} R^{l_6}_{l_3} R^{l_7}_{l_3}$$

- Our tree $T_{j:l_0,\dots,l_7}$ can be decomposed :

$$T_{j:l_0,l_1,\dots,l_7} = \mathcal{T}^3_{j:l_0,\dots,l_2} \cup T^{\leq l_0}_{l_0:\emptyset} \cup T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6} \cup T^{\leq l_2}_{l_2:\emptyset}$$

  and

$$R^{T_{j:l_0,l_1,\dots,l_7}} = R^{\mathcal{T}^3_{j:l_0,\dots,l_2}}(x) R^{T^{\leq l_0}_{l_0:\emptyset}}(0) R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6}}(0) R^{T^{\leq l_2}_{l_2:\emptyset}}(0)$$

- Finally, see eq. (7.1), (7.5) and (7.6), we first just deal with the coefficient :

$$\mathcal{R}^{j:l_0,l_1,\dots,l_2} = \sum_{t=0}^{3} \mathcal{R}^{j:l_0,l_1,\dots,l_2}_t$$

- We can thus write :

$$(\mathbf{U}R^T)_{l_0,j} = \sum_{t=0}^{3} (\mathbf{U}R^T)^t_{l_0,j} \sum_{t=0}^{3} \sum_{(l_1,\dots,l_7)\in\{1,\dots,d\}^7} \mathcal{R}^{j:l_0,l_1,\dots,l_2}_t R^{l_3}_{l_1} R^{l_4}_{l_1} R^{l_5}_{l_3} R^{l_6}_{l_3} R^{l_7}_{l_3}$$

Let us now use eq. (7.5) and (7.6) for a fixed t ($0 \leq t \leq 3$).
For $t = 0$, see eq. (7.5) :

$$(\mathbf{U}R^T)^0_{l_0,j} = \sum_{\substack{(l_1,\dots,l_7)\in\{1,\dots,d\}^7 \\ m\in\{1,\dots,d\}}} R^{\mathcal{T}^1_{j:l_0}}(x) R^{\mathcal{T}^1_{l_0:m}}(0) R^{\mathcal{T}^2_{m:l_1,l_2}}(0) R^{T^{\leq l_0}_{l_0:\emptyset}}(0) R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6}}(0) R^{T^{\leq l_2}_{l_2:\emptyset}}(0)$$

and, see figure below, this corresponds to a new coefficient $R^{0T}_{l_0,j}$:

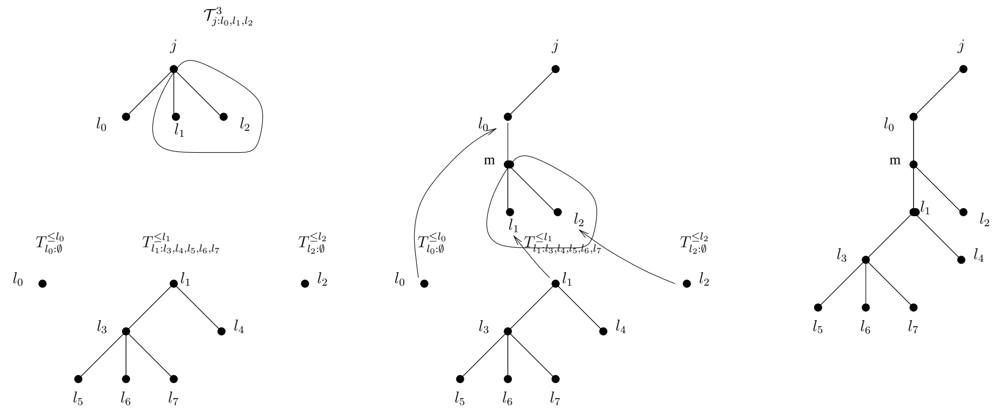


Figure 9: Interpretation of the equation (7.5) for $t = 0$: $T_{j:l_0,l_1,\dots,l_7} \to {}^0T_{j:l_0,m,l_1,\dots,l_7}$.

For $t = 1$, see eq. (7.5) :

$$(\mathbf{U}R^T)^1_{l_0,j} = \sum_{\substack{(l_1,\dots,l_7)\in\{1,\dots,d\}^7 \\ m\in\{1,\dots,d\}}} R^{\mathcal{T}^2_{j:l_0,l_1}}(x) R^{\mathcal{T}^1_{l_1:m}}(0) R^{\mathcal{T}^1_{m:l_2}}(0) R^{T^{\leq l_0}_{l_0:\emptyset}}(0) R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6}}(0) R^{T^{\leq l_2}_{l_2:\emptyset}}(0)$$

and, see figure below, this corresponds to a new coefficient $R^{1T}_{l_0,j}$:

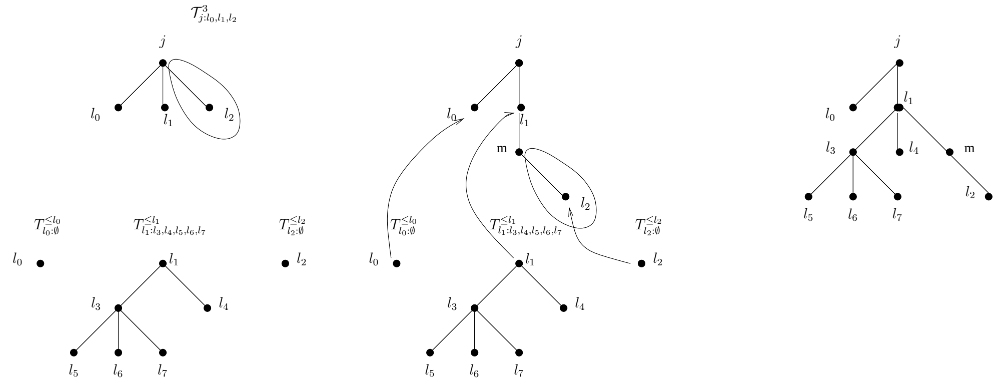


Figure 10: Interpretation of the equation (7.5) for $t = 1$: $T_{j:l_0,l_1,\dots,l_7} \to {}^1T_{j:l_0,l_1,l_3,l_4,m,l_5,l_6,l_7,l_2}$
.

For $t = 2$, see eq. (7.5) :

$$(\mathbf{U}R^T)^2_{l_0,j} = \sum_{\substack{(l_1,\ldots,l_7)\in\{1,\ldots,d\}^7 \\ m\in\{1,\ldots,d\}}} R^{\mathcal{T}^3_{j:l_0,l_1,l_2}}(x) R^{\mathcal{T}^1_{l_2:m}}(0) R^{\mathcal{T}^0_{m:\emptyset}}(0) R^{T^{\leq l_0}_{l_0:\emptyset}}(0) R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6}}(0) R^{T^{\leq l_2}_{l_2:\emptyset}}(0)$$

and, see figure below, this corresponds to a new coefficient $R^{2T}_{l_0,j}$

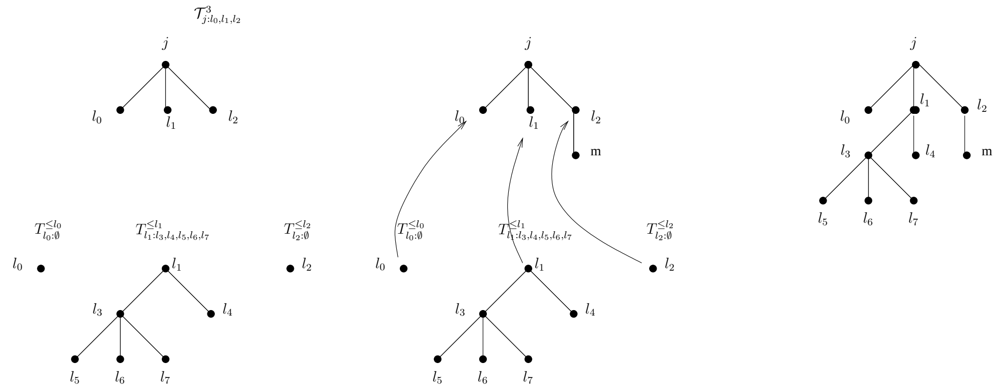


Figure 11: Interpretation of the equation (7.5) for $t = 2$: $T_{j:l_0,l_1,\ldots,l_7} \to {}^2T_{j:l_0,l_1,l_2,l_3,l_4,m,l_5,l_6,l_7}$.

For $t = 3$, see eq. (7.6) :

$$(\mathbf{U}R^T)^3_{l_0,j} = \sum_{\substack{(l_1,\ldots,l_7)\in\{1,\ldots,d\}^7 \\ m\in\{1,\ldots,d\}}} R^{\mathcal{T}^3_{j:l_0,l_1,l_2}}(x) R^{\mathcal{T}^1_{j::m}}(x) R^{T^{\leq l_0}_{l_0:\emptyset}}(0) R^{T^{\leq l_1}_{l_1:l_3,l_4,l_5,l_6}}(0) R^{T^{\leq l_2}_{l_2:\emptyset}}(0)$$

and, see figure below, this corresponds to a new coefficient $R^{3T}_{l_0,j}$:

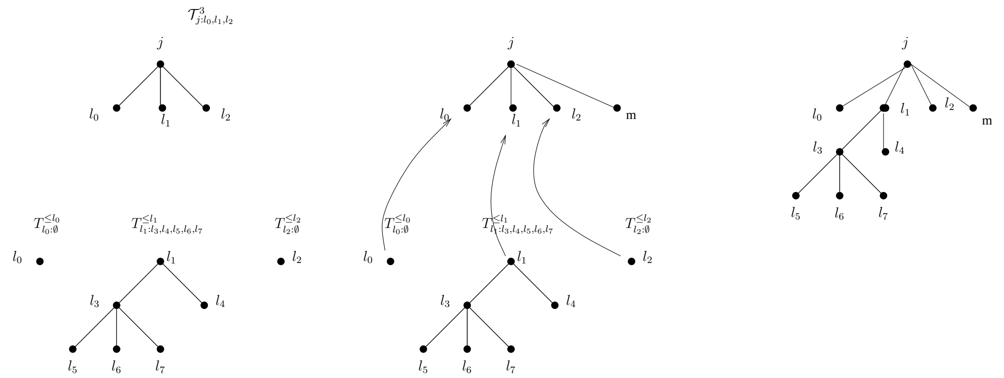


Figure 12: Interpretation of the equation (7.6) : $T_{j:l_0,l_1,\ldots,l_7} \to {}^3T_{j:l_0,l_1,l_2,m,l_3,l_4,l_5,l_6,l_7}$.

This example clearly illustrate the operations described in Lemma 1.

To end with theorem 2, it remains to prove Lemma 2. This will be done in the following section.

7.2 Lemma 2.

7.2.1 The transformation $\mathbf{U}^*$.

Let us first remind some definitions. For a given non-negative integer n we defined in section 5.3 the set :

$$\widetilde{Ca}_n \stackrel{def}{=} \{(k,T)\ ;\ T \in Ca_n\ ;\ 0 \le k \le \mathrm{Card}(fc(T))\} \tag{7.7}$$

And using lemma 1 the transformation $\mathbf{U}$ induces an application $\mathbf{U}^*$ (eq (5.20)) :

$$\begin{array}{rccl} \mathbf{U}^*\ : & \widetilde{Ca}_n & \to & Ca_{n+1} \\ & (k,T) & \mapsto & {}^kT \quad \text{(see Lemma 1)} \end{array}$$

Let us remind its definition. For an element (k,T) of $\widetilde{Ca}_n$, define $\mathbf{U}^*(k,T)$ as follows (see lemma 1) : Let $0 \le n_0 \le n$ the cardinal of first children of the root in T (note that $0 \le k \le n_0$).

$\mathbf{U}^*(k,T)$ stem from well-defined transformation of the tree T :

- If $0 \le k \le n_0 - 1$, then "cut" the tree T at the root, just after the edge of the $(k+1)^{th}$ first child (first children counted from left to right). We have now a "left" tree and a "right" one. On the left tree (with $k+1$ first children), add an edge to the $(k+1)^{th}$ first child, to the right of its other edges, and then identify ("glue") the lower extremity of this edge to the root of the right tree. This new tree is ${}^kT$.
- If $k = n_0$ then simply add an edge to the root, to the right of its other edges

We shall now define an inverse to $\mathbf{U}^*$.

7.2.2 The transformation $\mathbf{V}^*$.

Let us define the application :

$$\begin{array}{rccl} \mathbf{V}^*\ : & Ca_{n+1} & \to & \widetilde{Ca}_n \\ & T & \mapsto & (k,\widetilde{T}) \end{array} \tag{7.8}$$

Let be given a tree $T \in Ca_{n+1}$ with n_1 first children.

- If the right first child of the root has no children, then consider the tree $\widetilde{T} \in Ca_n$ obtained by cutting the edge bearing this first child. In this case, $\mathbf{V}^*(T) = (n_1 - 1, \widetilde{T})$.
- Otherwise, omit the edge linking the right first child of the root to its own right first child. Two trees are obtained. Let $\tilde{n}$ be the cardinal of first children of the tree containing the original root of T and $\widetilde{T} \in Ca_n$ be the tree obtained by identifying the roots of both obtained trees, tree containing the original root of T being located on the left. In this case, $\mathbf{V}^*(T) = (\tilde{n} - 1, \widetilde{T})$.

Then it is easy to prove that :

$$\mathbf{U}^* \circ \mathbf{V}^* = Id_{Ca_{n+1}} \tag{7.9}$$
$$\mathbf{V}^* \circ \mathbf{U}^* = Id_{\widetilde{Ca_n}} \tag{7.10}$$

This clearly implies that $\mathbf{U}^*$ is a bijection and it ends the proof of Lemma 2. The reader should convince himself of this proof by looking at the following figure for $n = 2$.

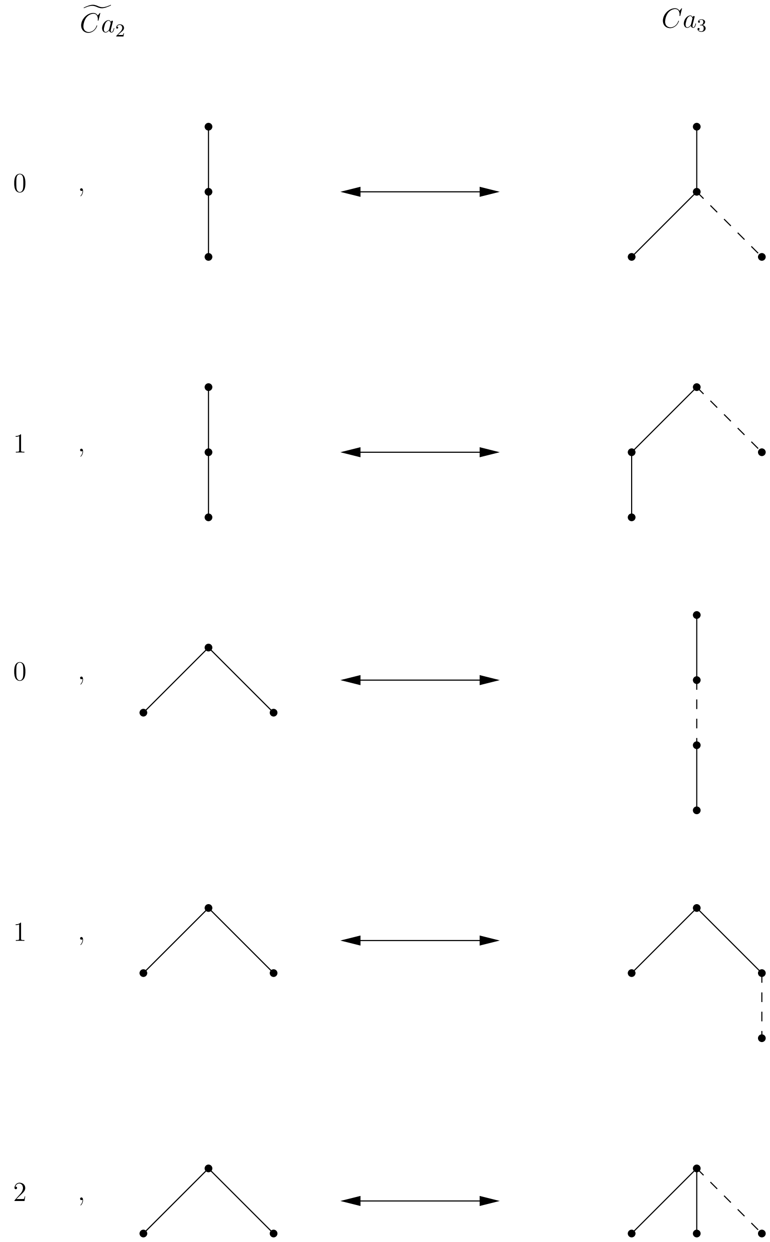


Figure 13: Bijection between $\widetilde{Ca}_2$ and Ca_3.

As theorem 4 is a consequence of theorem 3, it remains to give a proof to this one. Once again we should restrict ourself to key example.

7.3 Proof of theorem 3.

7.3.1 Theorem 2 and its consequences.

Because of theorem 2, we remind that, for a given integer n, the $d \times d$ matrix $\beta^n(0)$ is :

$$\beta^n_{i,j}(0) = \sum_{T \in Ca_n} R^T_{i,j}$$

with coefficients :

$$R^T_{i,j} = \sum_{(l_1,\ldots,l_{n-1}) \in \{1,\ldots,d\}} R^{T_{j:i,l_1,\ldots,l_{n-1}}} = \sum_{(l_1,\ldots,l_{n-1}) \in \{1,\ldots,d\}^{n-1}} \prod_{u \in \{i,l_1,\ldots,l_{n-1}\}} R^u_{u^-}$$

and u^- is the father of u in T. For example :

If we consider the tree T pictured in figure 14, then, for $j : i, l_1, \ldots, l_7$:

$$R^{T_{j:i,l_1,\ldots,l_7}} = R^i_j R^{l_1}_j R^{l_2}_j R^{l_3}_{l_1} R^{l_4}_{l_1} R^{l_5}_{l_3} R^{l_6}_{l_3} R^{l_7}_{l_3} \tag{7.11}$$

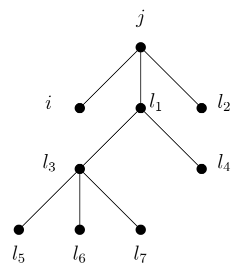


Figure 14: Example of tree.

Let us just remind finally that :

$$R^i_j = \frac{a_i}{\lambda_i + \lambda_j}$$

To prove theorem 3, we have to compute, for $t \geq 1$, for a sequence of positive integers $(n_1, \ldots, n_t)$, the vector :

$$\vec{b}^{n_1,\ldots,n_t} = \beta^{n_t}(0) \ldots \beta^{n_1}(0) \, . \, \vec{a} \quad (\vec{a} = (a_1, \ldots, a_d))$$

Thus, using theorem 2, the i^{th} coordinate of $\vec{b}^{n_1,\ldots,n_t}$ is :

$$\vec{b}^{n_1,\ldots,n_t}_i = \sum_{(k_1,\ldots,k_t) \in \{1,\ldots,d\}^t} \beta^{n_t}_{i,k_t}(0) \beta^{n_{t-1}}_{k_t,k_{t-1}}(0) \ldots \beta^{n_1}_{k_2,k_1}(0) a_{k_1}$$

$$= \sum_{\substack{(T^1,\dots,T^t)\in Ca_{n_1}\times\dots\times Ca_{n_t}\\(k_1,\dots,k_t)\in\{1,\dots,d\}^t}} R^{T^t}_{i,k_t} R^{T^{t-1}}_{k_t,k_{t-1}} \dots R^{T^1}_{k_2,k_1} a_{k_1} \tag{7.12}$$

In the two following sections, a careful examination of the coefficient :

$$R^{T^t}_{i,k_t} R^{T^{t-1}}_{k_t,k_{t-1}} \dots R^{T^1}_{k_2,k_1} a_{k_1}$$

for the cases $t = 1$ and $t = 2$ will allow us to define some operations on trees. These ones are the transformations that are involved in the definition of the sets $Ca_{n_1,\dots,n_t}$ and the end of the proof will be straightforward.

7.3.2 Case $t = 1$: Left-pulling and right-pulling.

Note that in this case, for a given positive integer n_1, the set "Ca_{n_1}" in theorem 3 should coincide with the previously defined set Ca_{n_1}.

Because of eq. (7.12), for $1 \le i \le d$:

$$b_i^{n_1} = \sum_{T^1\in Ca_{n_1}} \sum_{k_1\in\{1,\dots,d\}} R^{T^1}_{i,k_1} a_{k_1}$$

and, for a given tree T^1 and a given index k_1 :

$$R^{T^1}_{i,k_1} a_{k_1} = \sum_{(l_1,\dots,l_{n_1-1})\in\{1,\dots,d\}^{n_1-1}} R^i_{k_1} \prod_{u\in\{l_1,\dots,l_{n_1-1}\}} R^u_{u^-}\ a_{k_1}$$

but

$$R^i_{k_1} a_{k_1} = a_i R^{k_1}_i$$

thus

$$R^{T^1}_{i,k_1} a_{k_1} = a_i \sum_{(l_1,\dots,l_{n_1-1})\in\{1,\dots,d\}^{n_1-1}} R^{k_1}_i \prod_{u\in\{l_1,\dots,l_{n_1-1}\}} R^u_{u^-}$$

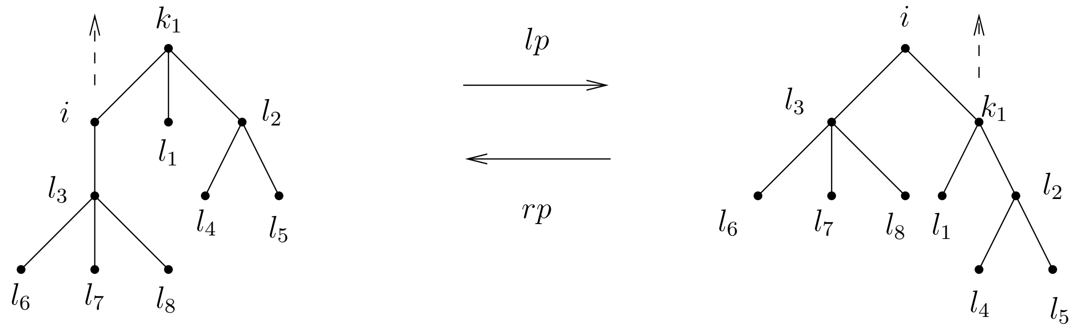
Let us have a "tree" interpretation of this identity. We did not change the fatherhood of any of the vertices labelled by $l_1, \dots, l_{n_1-1}$. We just invert the the fatherhood between i and k_1. Assuming, by convention, that the root k_1 became the *right* first child of i, then, see figure 15,

$$R^{T^1}_{i,k_1} a_{k_1} = a_i R^{\widetilde{T}^1}_{k_1,i}$$

and $\widetilde{T}^1$ is obtained by pulling up the left first child of the root of the tree T^1 the original root remaining on the right :

We call this transformation lp for left pulling :

$$\begin{array}{rccc} lp : & Ca_{n_1} & \to & Ca_{n_1} \\ & T^1 & \mapsto & \widetilde{T}^1 \end{array} \tag{7.13}$$



$a_{k_1}\ R^{i}_{k_1} R^{l_1}_{k_1} R^{l_2}_{k_1} R^{l_3}_{i} R^{l_4}_{l_2} R^{l_5}_{l_2} R^{l_6}_{l_3} R^{l_7}_{l_3} R^{l_8}_{l_3}$ $\qquad$ $a_i\ R^{k_1}_{i} R^{l_1}_{k_1} R^{l_2}_{k_1} R^{l_3}_{i} R^{l_4}_{l_2} R^{l_5}_{l_2} R^{l_6}_{l_3} R^{l_7}_{l_3} R^{l_8}_{l_3}$

Figure 15: Transformation $T^1 \leftrightarrow \widetilde{T}^1$.

This is a bijection because, if we define, similarly to lp, a *right* pulling rp, then, obviously :

$$rp \circ lp = lp \circ rp = Id_{Ca_{n_1}}$$

and this ends the proof of theorem 3 for $t = 1$:

$$\begin{aligned}
b_i^{n_1} &\underset{\text{def}}{=} \sum_{k_1 \in \{1,\ldots,d\}} \beta^{n_1}_{i,k_1}(0) a_{k_1} \\
&\underset{\text{theorem 2}}{=} \sum_{T^1 \in Ca_{n_1}} \sum_{k_1 \in \{1,\ldots,d\}} R^{T^1}_{i,k_1} a_{k_1} \\
&= \sum_{T^1 \in Ca_{n_1}} \sum_{k_1 \in \{1,\ldots,d\}} a_i\, R^{lp(T^1)}_{k_1,i} \\
&\underset{lp \text{ bijective}}{=} a_i \sum_{T^1 \in Ca_{n_1}} \sum_{k_1 \in \{1,\ldots,d\}} R^{T^1}_{k_1,i} \\
&\underset{\text{def}}{=} a_i \sum_{T^1 \in Ca_{n_1}} R^{T^1}_{i}
\end{aligned}$$

Let us have a look to the case $t = 2$

7.3.3 Case $t = 2$: Product of two trees.

Let us consider two positive integers (n_1, n_2). Because of eq. (7.12), for $1 \leq i \leq d$:

$$b_i^{n_1,n_2} = \sum_{(T^1,T^2) \in Ca_{n_1} \times Ca_{n_2}} \sum_{(k_1,k_2) \in \{1,\ldots,d\}^2} R^{T^2}_{i,k_2} R^{T^1}_{k_2,k_1} a_{k_1}$$

and, because of the previous section :

$$b_i^{n_1,n_2} = \sum_{\substack{(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}\\ (k_1,k_2)\in\{1,\dots,d\}^2}} R_{i,k_2}^{T^2} R_{k_1,k_2}^{lp(T^1)} a_{k_2}$$

Thus, we have to deal with coefficients :

$$\sum_{k_1\in\{1,\dots,d\}} R_{i,k_2}^{T^2} R_{k_1,k_2}^{lp(T^1)}$$

Once again, going back to the definition of the coefficients $R_{i,j}^T$, it is not difficult to give a tree interpretation of this operation : Consider two trees $(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}$, one can easily define a non commutative product $\pi(T^2,T^1)\in Ca_{n_1+n_2}$ where $\pi(T^2,T^1)$ is the tree obtained by identifying the roots of T^2 and T^1, T^1 being on the right of T^2. Then :

$$\sum_{k_1\in\{1,\dots,d\}} R_{i,k_2}^{T^2} R_{k_1,k_2}^{T^1} = R_{i,k_2}^{\pi(T^2,T^1)} \tag{7.14}$$

and

$$\begin{aligned} b_i^{n_1,n_2} &= \sum_{\substack{(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}\\ k_2\in\{1,\dots,d\}}} R_{i,k_2}^{\pi(T^2,lp(T^1))} a_{k_2} \\ &= a_i \sum_{\substack{(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}\\ k_2\in\{1,\dots,d\}}} R_{k_2,i}^{lp(\pi(T^2,lp(T^1)))} \\ &= a_i \sum_{(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}} R_i^{lp(\pi(T^2,lp(T^1)))} \end{aligned}$$

The last equation comes from the results for $t=1$ and it remains to prove the assumption on the product π in equation (7.14), but this is clarified by the figure 16.

To end the proof for $t=2$, we can note that, for two given positive integers n_1,n_2, the product π, as an application from $Ca_{n_1}\times Ca_{n_2}$ into $Ca_{n_1+n_2}$ is *injective* : Consider two trees $(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}$ and then their product $\pi(T^2,T^1)$. Their is clearly only *one* way to cut the tree at the root, between two edges, such that the left tree is in Ca_{n_2}, and once we cut the tree, we obviously recovered T^2 on the left and T^1 on the right.

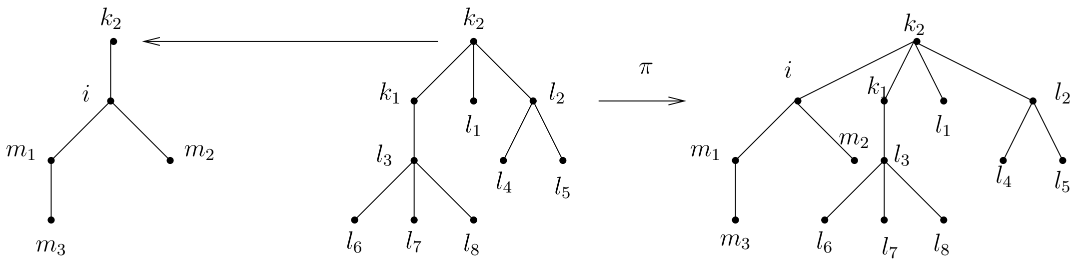
As the application lp is bijective, the application :

$$\begin{aligned} H_2 \ : \quad Ca_{n_1}\times Ca_{n_2} &\to Ca_{n_1+n_2} \\ (T^1,T^2) &\mapsto lp(\pi(T^2,lp(T^1))) \end{aligned}$$

is injective : The set $Ca_{n_1,n_2}=H_2(Ca_{n_1}\times Ca_{n_2})$ is a subset of $Ca_{n_1+n_2}$ having the same cardinal as $Ca_{n_1}\times Ca_{n_2}$, that is $ca_{n_1}ca_{n_2}$. Moreover the application H_2 is exactly the one described in theorem 3. Finally :

$$b_i^{n_1,n_2} = a_i \sum_{(T^1,T^2)\in Ca_{n_1}\times Ca_{n_2}} R_i^{lp(\pi(T^2,lp(T^1)))} = a_i \sum_{T\in Ca_{n_1,n_2}} R_i^T \tag{7.15}$$

Let us have a look to the general case.



$$R^{T^2}_{i,k_2} = \sum_{(m_1,m_2,m_3)\in\{1,\ldots,d\}^3} R^i_{k_2} R^{m_1}_i R^{m_2}_i R^{m_3}_{m_1}$$

$$R^{T^1}_{k_1,k_2} = \sum_{(l_1,\ldots,l_8)\in\{1,\ldots,d\}^8} R^{k_1}_{k_2} R^{l_1}_{k_2} R^{l_2}_{k_2} R^{l_3}_{k_1} R^{l_4}_{l_2} R^{l_5}_{l_2} R^{l_6}_{l_3} R^{l_7}_{l_3} R^{l_8}_{l_3}$$

$$R^{\pi(T^2,T^1)}_{i,k_2} = \sum_{\substack{(l_1,\ldots,l_8)\in\{1,\ldots,d\}^8\\(m_1,m_2,m_3)\in\{1,\ldots,d\}^3\\k_1\in\{1,\ldots,d\}}} R^i_{k_2} R^{k_1}_{k_2} R^{l_1}_{k_2} R^{l_2}_{k_2} R^{m_1}_i R^{m_2}_i R^{l_3}_{k_1} R^{l_4}_{l_2} R^{l_5}_{l_2} R^{m_3}_{m_1} R^{l_6}_{l_3} R^{l_7}_{l_3} R^{l_8}_{l_3}$$

Figure 16: Product of T^2 and T^1.

7.3.4 Conclusion

For a given $t \geq 3$ it is easy to end the proof by induction, using the transformations introduced for $t = 1$ and $t = 2$. Let $(n_1, \ldots, n_t)$ be t positive integers. For $u \leq t$ we can build the following transformations by induction :

For $u = 1$:

$$\begin{array}{cccc} H_1 : & Ca_{n_1} & \rightarrow & Ca_{n_1} \\ & T^1 & \mapsto & lp(T^1) \end{array}$$

For $u = 2$:

$$\begin{array}{cccc} H_2 : & Ca_{n_1} \times Ca_{n_2} & \rightarrow & Ca_{n_1+n_2} \\ & (T^1, T^2) & \mapsto & lp(\pi(T^2, H_1(T^1))) \end{array}$$

For $u > 2$:

$$\begin{array}{cccc} H_u : & Ca_{n_1} \times Ca_{n_2} \cdots \times Ca_{n_u} & \rightarrow & Ca_{n_1+n_2+\ldots n_u} \\ & (T^1, T^2, \ldots, T^u) & \mapsto & lp(\pi(T^u, H_{u-1}(T^1, T^2, \ldots, T^{u-1}))) \end{array}$$

But then, using the same method as in the case $t = 2$, it becomes obvious that :

- H_t is injective, thus $Ca_{n_1,\ldots,n_t} = H_t(Ca_{n_1} \times \cdots \times Ca_{n_t})$ is a subset of $Ca_{n_1+\cdots+n_t}$ of cardinal $ca_{n_1} \ldots ca_{n_t}$.
- The transformation H_u are identical to the ones described in theorem 3 to build the set $Ca_{n_1,\ldots,n_t}$.

- By induction, we finally obtain, as in equation (7.15), that, for $1 \leq i \leq d$

$$b_i^{n_1,\ldots,n_t} = a_i \sum_{T \in Ca_{n_1,\ldots,n_t}} R_i^T \tag{7.16}$$

This ends the proof of theorem 3 and thus of theorem 4.

8 Conclusion.

One of the main results of this paper is the tree-decomposition for an average induced by exponential diffusion. This results can be easily extended to other diffusions, using some density arguments. This can also be done explicitly, by a combinatorial procedure, independent of the diffusion. We will explain this in a forthcoming paper. Moreover, this will lead us to an inverse formula for tree-decomposition.

These coming results shall yield a complete answer to the question : When does an exponential diffusion induce the Catalan average (Weak Catalan Property). In fact, we should get an answer for a more general problem. Let be given two “exponential diffusion” f and g and their induced averages $\mathbf{m}_f$ and $\mathbf{m}_g$, is there a simple necessary and sufficient condition on f and g for inducing the same average, i.e. $\mathbf{m}_f = \mathbf{m}_g$?

References

[1] Louis Comtet. *Analyse Combinatoire*, volume 1. P.U.F., 1970.

[2] J. Ecalle. Well-behaved convolution averages and their applications to real resummation. To appear.

[3] J. Ecalle et F. Menous. Well-behaved convolution averages and the non-accumulation theorem for limit-cycles. *Prépublications d'Orsay.*, 1995.

[4] C. Even. Etude d'une fonction remarquable associée aux moyennes de convolution. *Annales de l'Institut Fourier*, 49(2):687–705, 1999.

[5] H.M. Finucan. Some elementary aspects of the Catalan numbers. In *Combinat. Math. IV*, volume 560 of *LNM*, pages 41–45. Springer, 1975.

[6] F. Menous. *Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle.* Thèse de doctorat, 1996.

[7] F. Menous. Les bonnes moyennes uniformisantes et une application à la resommation réelle. *Annales de la Faculté des Sciences de Toulouse*, 2000. To appear.