mardi 15h30

Woib

- ~~Test 1~~ .

  L on va commencer la séance par ce test .

- TDs : aujourd'hui = à partir de 16h + r

  dispos semaine prochaine ?

Mon groupe

MSV Elo

CORRIGE TEST 1

1. (a) $X \sim Bin(n,p)$ $\qquad X \in [0,\dots,n]$ ps.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \forall k \in [0,\dots,n].$$

(b) Quelle est la loi de $n-X$?

si $0 \leq n-k \leq n \Leftrightarrow 0 \leq k \leq n$.

$$P(n-X=k) = P(X=n-k) = \binom{n}{n-k} p^{n-k} (1-p)^k$$

$$= \binom{n}{k} (1-p)^k p^{n-k}$$

$$\binom{n}{n-k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

on reconnait $Bin(n, \underline{1-p})$

2. $F(t) = (1 - e^{-t})\mathbb{1}_{\{t \geq 0\}} = \int_{-\infty}^{t} e^{-x}\mathbb{1}_{\{x \geq 0\}}\,dx$

densité de la loi Exp(1)

(a) $\underset{\lambda > 0}{\mathbb{P}(\lambda X \leq t)} = \mathbb{P}\left(X \leq \frac{t}{\lambda}\right) = F\left(\frac{t}{\lambda}\right) = \left(1 - e^{-t/\lambda}\right)\mathbb{1}_{\{t \geq 0\}}$

$= \int_{-\infty}^{t} \frac{1}{\lambda} e^{-x/\lambda}\mathbb{1}_{\{x \geq 0\}}\,dx$

on reconnaît la FdR de Exp(1/λ).

pour $t \geq 0$:

(b) $\mathbb{P}(\sqrt{X} \leq t) = \mathbb{P}(X \leq t^2) = \left(1 - e^{-t^2}\right)$

bien définie

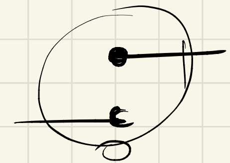
pour $t < 0$:

$\mathbb{P}(\sqrt{X} \leq t) = 0$

$\forall t \in \mathbb{R}$, $\quad \mathbb{P}(\sqrt{X} \leq t) = (1 - e^{-t^2}) \mathbb{1}_{\{t \geq 0\}} = \int_{-\infty}^{t} 2x\, e^{-x^2} \mathbb{1}_{\{x \geq 0\}} \, dx$

Loi de Rayleigh.

3. (a) $\quad t \to F(1-t)$



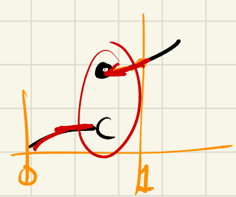
NON : × $\lim_{-\infty} F(1-t) = \lim_{+\infty} F(t) = 1$

× continuité à droite non garantie.

× décroissance $\Rightarrow$ si fonction de répartition, elle serait aussi croissante, donc constante, absurde!

F càd

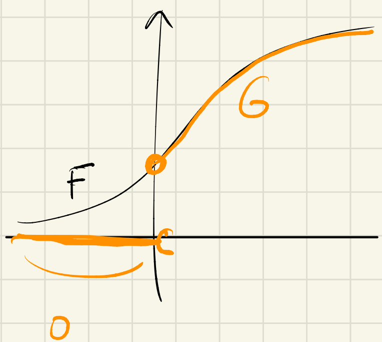
$\rightarrow$ $(t \mapsto F(1-t))$ est càg



(b) $t \mapsto F(t)\,\mathbb{1}_{\{t \geq 0\}} =: G(t)$

C'est bien une FdR car :

- $\lim_{+\infty} G = 1$ $\qquad \lim_{-\infty} G = 0$
- G reste croissant
- G reste continue à droite.



redonne les 3 propriétés

De quelle VAR est-elle la FdR ?

$$\mathbb{P}(\max\{X, 0\} \leq t) = \begin{cases} 0 & \text{si } t < 0 \\ \mathbb{P}(X \leq t) & \text{si } t \geq 0 \end{cases}$$

$\mathbb{P}(X \leq t) = F(t)$

$\{\max\{X, 0\} \leq t\} = \{X \leq t\}$ $t \geq 0$

$$= F(t)\, \mathbb{1}_{\{t \geq 0\}}$$

(c) $\quad t \longmapsto \left(F(t) - F((-t)-) \right) \mathbb{1}_{\{t \geq 0\}}$

est bien une FdR :

$$\underbrace{\mathbb{P}(|X| \leq t)}_{} = \begin{cases} 0 & \text{si } t < 0 \\ \mathbb{P}(-t \leq X \leq t) & \text{si } t \geq 0 \end{cases}$$

$$\mathbb{P}(-t \leq X \leq t) = \mathbb{P}(X \leq t) - \mathbb{P}(X < -t) = F(t) - F((-t)-).$$

$$= \left(F(t) - F((-t)-) \right) \times \mathbb{1}_{\{t \geq 0\}}.$$

$$\mathbb{P}(X_{(n-1)} \le t) = t^n + n(1-t)\,t^{n-1} = F(t)$$

$$\Rightarrow \quad \underline{X_{(n-1)} \sim \text{Beta}} \quad ?$$

$$\frac{1}{\text{Beta}(a,b)} \times x^{a-1}(1-x)^{b-1}\,\mathbb{1}_{\{0<x<1\}}$$

$$F'(t) = n\,t^{n-1} - n\,t^{n-1} + n(1-t)(n-1)\,t^{n-2}$$

$$= n(n-1)\,t^{n-2}(1-t), \quad \forall\, t \in [0,1)$$

$$\mathbb{P}(X_{(n-1)} \le t) = \int_{-\infty}^{t} \underbrace{n(n-1)\,x^{n-2}(1-x)^{1}\,\mathbb{1}_{\{0<x<1\}}}_{\text{Beta}(n-1,\,2)}\,dx$$

$X_{(n)} \sim \text{Beta}(n, 1)$

$X_{(n-1)} \sim \text{Beta}(n-1, 2)$

3. $\left[\begin{array}{l} X_{(1)} \sim \text{Beta}(1, n) \\ X_{(2)} \sim \text{Beta}(2, n-1). \end{array}\right.$?

$X_{(1)} = \min\{X_1, \dots, X_n\} \overset{\text{loi}}{=} 1 - X_{(n)}$

$\overset{\text{loi}}{=} \min(1-X_1, \dots, 1-X_n)$

$= 1 - \max\{X_1, \dots, X_n\}$

$X_1 \overset{(\text{loi})}{=} 1 - X_1$

si $X_1 \sim \text{Unif}(0, 1)$

$X_{(2)} \overset{(\text{loi})}{=} 1 - X_{(n-1)}.$

si $X \sim \mathrm{Beta}(a,b)$, $1-X$?

$$\mathbb{E}[\varphi(1-X)] = \int_0^1 \varphi(1-x) \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} \mathbb{1}_{]0<x<1[} \, dx.$$

$$= \int_0^1 \varphi(y) \frac{1}{B(a,b)} (1-y)^{a-1} y^{b-1} \mathbb{1}_{]0<y<1[} \, dy.$$

loi $\mathrm{Beta}(b,a)$

$$\mathbb{P}(X_{(1)} > t) = \mathbb{P}(\min\{X_1, \dots, X_n\} > t)$$

$$= \mathbb{P}\left(\bigcap_{i=1}^n \{X_i > t\}\right)$$

ind.

$$= \prod_{i=1}^n \mathbb{P}(X_i > t) = (1-t)^n.$$

$t \in [0,1]$

$$\underbrace{P(X_{(1)} \leq t)}_{t \in (0,1)} = 1 - (1-t)^n = \int_{-\infty}^{t} n(1-x)^{n-1} \mathbb{1}_{\{0<x<1\}} \, dx$$

$$\downarrow$$

$X_{(2)} \ldots$

Beta $(1, n)$

$t > 0$

Exercice 4.

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} \, dx$$

1. $\Gamma(1) = \int_0^{+\infty} e^{-x} \, dx = 1.$

$$\Gamma(n+1) = \int_0^{+\infty} x^n e^{-x} \, dx$$

$$= \underbrace{\left[-x^n e^{-x}\right]_0^{+\infty}}_{\parallel \atop 0} + n \underbrace{\int_0^{+\infty} x^{n-1} e^{-x} \, dx}_{\Gamma(n)}$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

$\Gamma(2) = 1 \cdot \Gamma(1) = 1$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2$

$\Gamma(4) = 3 \cdot 2 \cdot$

par récurrence $\Gamma(n) = (n-1) \times (n-2) \times \ldots \times 2 = (n-1)!$

2. $\mathbb{E}[X^n] = \int_0^{+\infty} x^n \beta e^{-\beta x} dx$

$y = \beta x$

$\beta > 0$

$= \int_0^{+\infty} \left(\frac{y}{\beta}\right)^n e^{-y} dy.$

$= \frac{1}{\beta^n} \underbrace{\int_0^{+\infty} y^n e^{-y} dy}_{\Gamma(n+1)} = \frac{n!}{\beta^n}.$

3. $Y \sim \Gamma(\alpha, \beta)$: densité $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{\{x > 0\}}$

$$\mathbb{E}[Y^n] = \int_0^{+\infty} x^n \boxed{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{n+\alpha-1} e^{-\beta x} dx$$

$y = \beta x$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{y}{\beta}\right)^{n+\alpha-1} e^{-y} \frac{dy}{\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^n} \underbrace{\int_0^{+\infty} y^{n+\alpha-1} e^{-y} dy}_{\Gamma(n+\alpha)} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \times \frac{1}{\beta^n}.$$

$/ \alpha = 1$ (4)

$$\frac{\Gamma(n+1)}{\Gamma(1)} = n!$$

(5)

1. $$\mathbb{E}[X^{2n+1}] = \int_{-\infty}^{+\infty} x^{2n+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$X \sim N(0,1)$

intégrable

densité $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$x^{2n+1} e^{-x^2/2}$

$= 0$ car intégrale d'une fct impaire intégrable

$f(-x) = -f(x)$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} f(-x)\,dx$$

$$= \int_{-\infty}^{+\infty} -f(x)\,dx$$

$$= -\int_{-\infty}^{+\infty} f(x)\,dx$$

$$\int_{\mathbb{R}} f(x)dx = 0$$

2. a) $\displaystyle\int_{-\infty}^{+\infty} x^{2n} e^{-x^2/2}\,dx$

x^{2n-1} $\quad x e^{-x^2/2}$

$$= \underbrace{\left[-x^{2n-1} e^{-x^2/2}\right]_{-\infty}^{+\infty}}_{0} + (2n-1)\int_{-\infty}^{+\infty} x^{2n-2} e^{-x^2/2}\,dx$$

$$= (2n-1)\int_{0}^{+\infty} x^{(2n-2)} e^{-x^2/2}\,dx$$

$$E[X^{2n}] = (2n-1)\,E[X^{2n-2}] \quad , \quad X \sim N(0,1)$$

$E[X]$

(b) $E[X^0] = E[1] = 1$.

$E[X^2] = 1 \cdot E[X^0] = 1$

$\text{Var}(X) = 1 - 0^2 = 1$.

$$\mathbb{E}[X^4] = 3 \quad \mathbb{E}[X^2] = \underline{\underline{3}}.$$

$$\mathbb{E}[X^6] = 5 \times 3.$$

$$\mathbb{E}[X^{2n}] = (2n-1)\, \mathbb{E}[X^{2n-2}]$$

$$= (2n-1) \times (2n-3) \times \dots \times 1$$

$$= \prod_{k=1}^{n} (2k-1)$$

$$= \frac{2n \times (2n-1) \times (2n-2) \times (2n-3) \times (2n-4) \quad 2\ 1}{2n\ \underbrace{(2n-2)}_{2(n-1)}\ \underbrace{(2n-4)}_{2(n-2)} \dots 2}$$

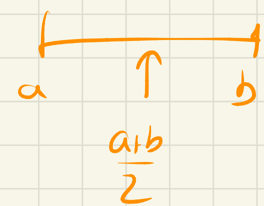
$$= \frac{(2n)!}{2^n\ n!}$$

② Feuille 5.

1. $\mathrm{Unif}(a,b)$: $\dfrac{\mathbb{1}_{\{a<x<b\}}}{b-a}$ densité.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \,\frac{\mathbb{1}_{\{a<x<b\}}}{b-a}\, dx = \int_a^b \frac{x\, dx}{b-a} = \frac{\left[\frac{x^2}{2}\right]_a^b}{b-a}$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$



$$\mathbb{E}[X^2] = \int_a^b \frac{x^2\, dx}{b-a} = \frac{\left[\frac{x^3}{3}\right]_a^b}{b-a} = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

car $b^3-a^3 = (b-a)(b^2+ab+a^2)$.

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(a^2+ab+b^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \boxed{\frac{(b-a)^2}{12}}.$$

2. $\mathrm{Unif}(0,1) : \mathbb{1}_{\{0<x<1\}}$

$$\mathbb{E}[X] = \int_0^1 x\,dx = \frac{1}{2}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

si $X \sim Unif(0,1)$, $\underbrace{a+(b-a)X}_{=Y} \sim Unif(a,b)$.

$$\underline{\mathbb{E}[Y]} = \mathbb{E}[a+(b-a)X] = a+(b-a)\underbrace{\mathbb{E}[X]}_{1/2} = \frac{a+b}{2}$$

$$Var(Y) = Var(a+(b-a)X) = (b-a)^2\, Var(X) = \frac{(b-a)^2}{12}.$$

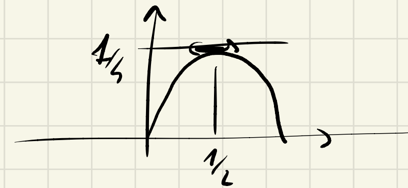
1 1. $X \in \{0,1\}$ $\begin{cases} \mathbb{P}(X=1) = p \\ \mathbb{P}(X=0) = 1-p \end{cases}$

$X = X^2$ ici

$$\begin{cases} \mathbb{E}[X] = p\times 1 + (1-p)\times 0 = p \\ \mathbb{E}[X^2] = p\times 1^2 + (1-p)\times 0^2 = p \end{cases}$$

$$Var(X) = p - p^2 = p(1-p)$$

est maximale pour $p = 1/2$.



2. $$\boxed{k\binom{n}{k} \overset{?}{=} n\binom{n-1}{k-1}}$$

$$k\,\frac{n!}{k!(n-k)!} \qquad n\,\frac{(n-1)!}{(k-1)!\,(n-(k-1))!} = \frac{n!}{(k-1)!\,(n-k)!}$$

OK

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

$$k(k-1)\,\frac{n!}{k!\,(n-k)!} \qquad n(n-1)\,\frac{(n-2)!}{(k-2)!\,(n-k)!}$$

OK

$X \sim Bin(n,p)$: $\quad \mathbb{E}[X] = \sum_{k=\cancel{0}\,1}^{n} \boxed{k \binom{n}{k}} p^k (1-p)^{n-k}$ $\quad n\binom{n-1}{k-1}$

$\mathbb{E}[X(X-1)] = \sum_{k=\cancel{0}\,2}^{n} \boxed{k(k-1) \binom{n}{k}} p^k (1-p)^{n-k}$

$n(n-1)\binom{n-2}{k-2}$

SUITE DES CALCULS (APRÈS TD)

(1) . 3. $\mathbb{E}[X] = \sum_{k=\cancel{0}\,1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k (1-p)^{n-k}$

$= np \sum_{j=0}^{m} \binom{n-1}{j} p^j (1-p)^{n-1-j}$

$j = k-1$

$$= np \left(p + (1-p) \right)^{n-1}$$

$$= \boxed{np} \qquad \left(n \times \mathbb{E}[Ber(p)] \right)$$

$$\mathbb{E}[X(X-1)] = \sum_{k=\cancel{0}\,2}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=2}^{n} n(n-1) \binom{n-2}{k-2} p^k (1-p)^{n-k}$$

$$= n(n-1) p^2 \sum_{j=0}^{n-2} \binom{n-2}{j} p^j (1-p)^{n-2-j}$$

$j = k-2$

$$= n(n-1) p^2 \left(p + 1 - p \right)^{n-2} = n(n-1) p^2$$

d'où $\quad Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \boxed{np(1-p).} \qquad \left(= n \, Var(Ber(p)). \right)$$

4. On veut maintenant calculer $\mathbb{E}$ et Var de $Y \sim Poi(\lambda)$.

$$\mathbb{E}[Y] = \sum_{k=\cancel{0}\,1}^{+\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \lambda \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}}_{e^{\lambda}} = \boxed{\lambda}$$

$$\mathbb{E}[Y(Y-1)] = \sum_{k=\cancel{0}\,2}^{+\infty} k(k-1)\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=2}^{+\infty} \frac{\lambda^k}{(k-2)!} = e^{-\lambda} \lambda^2 \underbrace{\sum_{j=0}^{+\infty} \frac{\lambda^j}{j!}}_{e^{\lambda}} = \lambda^2.$$

donc $\mathrm{Var}(Y) = \mathbb{E}[Y(Y-1)] + \mathbb{E}[Y] - \mathbb{E}[Y]^2$

$$= \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

(logique si on pense à $Poi(\lambda)$ comme cas limite de $Bin(n, \frac{\lambda}{n})$).

$\downarrow$

CAR $\begin{cases} \mathbb{E} = np_n = \lambda \\ Var = np_n(1-p_n) \xrightarrow[n\to\infty]{} \lambda \end{cases}$

③ 1. $X \sim Ber(p)$

$$\varphi_X(t) = \mathbb{E}[e^{tX}] = e^{t0}(1-p) + e^{t1}(p) = (pe^t + 1-p)$$

$Y \sim Bin(n,p)$

puissance n

$$\varphi_Y(t) = \mathbb{E}[e^{tY}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = (pe^t + 1-p)^n$$

sûrement cela a quelque chose à voir avec $Y = \sum_{i=1}^{n} X_i$

si $X_i \sim Ber(p)$
indépendantes

2. On retrouve les valeurs de $\mathbb{E}[X]$, $Var(X)$, $\mathbb{E}[Y]$, $Var(Y)$ en calculant les $DL_2(0)$ de $\varphi_X(t)$ et $\varphi_Y(t)$:

$$\varphi_X(t) = pe^t + 1-p = p\left(1+t+\frac{t^2}{2}+o(t^2)\right)+1-p$$

$$= 1 + \underbrace{p}_{=\mathbb{E}[X]}\, t + \underbrace{p}_{=\mathbb{E}[X^2]}\, \frac{t^2}{2} + o(t^2).$$

ce qui était évident !

donc $Var(X) = p - p^2 = p(1-p)$.

$$\varphi_Y(t) = (pe^t + 1-p)^n = \left(p\left(1+t+\frac{t^2}{2}+o(t^2)\right)+1-p\right)^n$$

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2}u^2 + o(u^2)$$

$$= \left(1 + pt + \frac{p}{2}t^2 + o(t^2)\right)^n$$

$$= 1 + n\left(pt + \frac{p}{2}t^2\right) + \frac{n(n-1)}{2}\left(pt + p\frac{t^2}{2}\right)^2 + o(t^2)$$

$$= 1 + \underbrace{(np)}_{E[Y]}t + \underbrace{\left(np + n(n-1)p^2\right)}_{E[Y^2]}\frac{t^2}{2} + o(t^2)$$

donc $\mathrm{Var}(Y) = np + n(n-1)p^2 - (np)^2 = np(1-p)$

comme attendu.