Cauchy-Kovalevska Theorem, Characteristics and Holmgren Theorem

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The proofs in these notes are not original and are in the ideas taken from the references. They are just written and organized in the way I understand them.

There may be errors in these notes, hence if you have any questions or remarks about these notes, I will be happy to discuss with you by mail or in person.

Good reading !

Contents

References

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1. Introduction

We consider Ω an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, with coordinates $(x_1, \dots, x_n) = (x', x_n)$. Let f be an analytic function on Ω, $u_0, \dots, u_{m-1}$ be analytic functions on $\Omega \cap \{x_n = 0\}$ and $P(x, D_x)$ be a diffenrential operator of degree m with analytic coefficients in Ω. First, we want to discuss about the existence and unicity of a solution u to the following Cauchy problem:

$$\begin{cases} P(x, D_x)u = f(x) \\ D_{x_n}^j u_{|x_n=0} = u_j(x') \end{cases}$$

In order to do that, we will start by the first order, and after we will show that we can reduce a problem of order m to a problem of order 1.

2. Cauchy-Kovalevska Theorem and order 1

In this section, we consider $\Omega \subset \mathbb{C}^n$ a bounded open subset, and we denote $(z_1, \dots, z_n)$ the coordinates in $\mathbb{C}^n$.

Definition 2.1. We denote by $H(\bar{\Omega}, \mathbb{C}^m)$ the space of continuous functions in $\bar{\Omega}$ which are holomorphic in the set Ω and valued in $\mathbb{C}^m$. We can equip this space with the maximum norm

$$|f|_\Omega = \sup_{z\in\Omega} \|f(z)\|$$

Proposition 2.2. *$H(\bar{\Omega}, \mathbb{C}^m)$ equipped with this norm is a Banach space.*

Proof. Let $(u_j)_j$ be a Cauchy sequence in $H(\bar{\Omega}, \mathbb{C}^m) \subset C^0(\bar{\Omega}, \mathbb{C}^m)$. Then it converges uniformly on $\bar{\Omega}$ to u a continuous function in $\bar{\Omega}$. For any function φ with compact support, $0 \le k \le n$, we have in Ω :

$$0 = \langle \frac{\partial u_j}{\partial \bar{z}_k}, \varphi \rangle = -\langle u_j, \frac{\partial \varphi}{\partial \bar{z}_k} \rangle \underset{j\to\infty}{\longrightarrow} -\langle u, \frac{\partial \varphi}{\partial \bar{z}_k} \rangle = \langle \frac{\partial u}{\partial \bar{z}_k}, \varphi \rangle$$

Hence $\frac{\partial u}{\partial \bar{z}_k} = 0$ in the distribution sense and u is holomorphic in Ω. □

Proposition 2.3. *Let Ω_1 be an bounded open subset such that $\bar{\Omega} \subset \Omega_1$ and let $d := d(\Omega, \partial\Omega_1)$ denote the distance from Ω to the boundary of Ω_1. The partial differentiation $\frac{\partial}{\partial z_j}$ for $1 \le j \le n$ defines a bounded linear operator $H(\bar{\Omega}_1, \mathbb{C}^m) \to H(\bar{\Omega}, \mathbb{C}^m)$ with norm*

$$\left\| \frac{\partial}{\partial z_j} \right\| \le d^{-1}$$

Proof. Let f be any element of $H(\bar{\Omega}_1, \mathbb{C}^m)$, $z \in \bar{\Omega}$. By Cauchy's inequalities, we have

$$\left| \frac{\partial f}{\partial z_j} \right| \le \frac{1}{d} \sup_{|z'_j - z_j| < d} |f(z_1, \dots, z_{j-1}, z'_j, z_{j+1}, \dots, z_n)| \le \frac{|f|_{\Omega_1}}{d}$$

Hence

$$\left| \frac{\partial f}{\partial z_j} \right|_\Omega \le \frac{|f|_{\Omega_1}}{d}$$

□

Notation 2.4. Let E and F be two Banach spaces. We denote $B(E, F)$ the space of bounded linear operators from E to F.

Proposition 2.5. *Let $M \in H(\bar{\Omega}, M_m(\mathbb{C}))$. Then $f \in H(\bar{\Omega}, \mathbb{C}^m) \mapsto Mf \in H(\bar{\Omega}, \mathbb{C}^m)$ is a bounded linear operator with norm*

$$\|M\|_\Omega = \sup_{z\in\Omega} \|M(z)\|$$

Proof. For any $f \in H(\bar{\Omega}, \mathbb{C}^m)$, we have

$$|Mf|_\Omega = \sup_{z\in\Omega} \|M(z)f(z)\| \le \sup_{z\in\Omega} \|M(z)\| |f|_\Omega$$

□

Let $T > 0$.

Proposition 2.6. *Let $t \in \,]-T, T[\, \mapsto M(\cdot, t) \in H(\bar{\Omega}, M_m(\mathbb{C}))$ be a continuous function. Then $t \mapsto (f \mapsto M(\cdot, t)f)$ is a continuous function in $]-T, T[$ valued in $B(H(\bar{\Omega}, \mathbb{C}^m), H(\bar{\Omega}, \mathbb{C}^m))$. For each t, the operator norm of $f \mapsto M(\cdot, t)f$ is not larger than $\|M(\cdot, t)\|_\Omega$.*

Proof. For $t \in]-T, T[$, it is clear that this function is valued in $B(H(\bar{\Omega}, \mathbb{C}^m), H(\bar{\Omega}, \mathbb{C}^m))$ and furthermore

$$|M(\cdot, t)f|_\Omega \leq \|M(\cdot, t)\|_\Omega |f|_\Omega$$

The continuity follows directly from the continuity of the function $t \mapsto M(\cdot, t)$. □

We look at the Cauchy problem given by the two equations

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j(z,t)\frac{\partial u}{\partial z_j} + A_0(z,t)u(z,t) + f(z,t) \tag{2.7}$$

where for any j, $t \mapsto A_j(\cdot, t)$ is a continous function for $|t| < T$ valued in $H(\bar{\Omega}, M_m(\mathbb{C}))$, $t \mapsto f(\cdot, t)$ a continuous function for $|t| < T$ valued in $H(\bar{\Omega}, \mathbb{C}^m)$, and

$$u(z, 0) = u_0(z) \tag{2.8}$$

where $u_0 \in H(\bar{\Omega}, \mathbb{C}^m)$.

Proposition 2.9. *For $n+1$ functions A_j as in the equation (2.7), set*

$$A(t) := \sum_{j=1}^{n} A_j(\cdot, t)\frac{\partial}{\partial z_j} + A_0(\cdot, t)$$

Then A is a continuous function for $|t| < T$ valued in $B(H(\bar{\Omega}_1, \mathbb{C}^m), H(\bar{\Omega}, \mathbb{C}^m))$. For each t, the norm of $A(t)$ is not larger than

$$\frac{1}{d}\sum_{j=1}^{n} \|A_j(\cdot, t)\|_\Omega + \|A_0(\cdot, t)\|_\Omega$$

Proof. For any function $f \in H(\bar{\Omega}_1, \mathbb{C}^m)$, we have

$$\begin{aligned} |A(t)f|_\Omega &= \left|\sum_{j=1}^{n} A_j(\cdot, t)\frac{\partial f}{\partial z_j} + A_0(\cdot, t)f\right|_\Omega \\ &\leq \sum_{j=1}^{n} \left|A_j(\cdot, t)\frac{\partial f}{\partial z_j}\right|_\Omega + |A_0(\cdot, t)f|_\Omega \\ &\leq \sum_{j=1}^{n} \|A_j(\cdot, t)\|_\Omega \left|\frac{\partial f}{\partial z_j}\right|_\Omega + \|A_0(\cdot, t)\|_\Omega |f|_\Omega \\ &\leq \frac{1}{d}\sum_{j=1}^{n} \|A_j(\cdot, t)\|_\Omega |f|_{\Omega_1} + \|A_0(\cdot, t)\|_\Omega |f|_{\Omega_1} \end{aligned}$$

Hence A is valued in $B(H(\bar{\Omega}_1, \mathbb{C}^m), H(\bar{\Omega}, \mathbb{C}^m))$ with the ineaquality of norms we want. The continuity follows easily from the continuity of the functions $t \mapsto A_j(\cdot, t)$. □

We will prove the following important statement :

Theorem 2.10 (Cauchy-Kovalevska)**.** *Let Ω_0 be a nonempty connected open subset of $\mathbb{C}^n$ such that $\bar{\Omega}_0 \subset \Omega_1$. Then there exists $0 < \delta_0 \leq T$ such that there exists a unique C^1 function $t \in]-\delta_0, \delta_0[\mapsto u(\cdot, t) \in H(\bar{\Omega}_0, \mathbb{C}^m)$ satisfying the equations (2.7) and (2.8) for any $z \in \Omega_0$ and $|t| < \delta_0$.*

Remark 2.11. Even if we are interesting in a 'real' version of the Cauchy-Kovalevska theorem, it has sense to consider holomorphic functions regarding the 'spatial' variables, because what we need in this theorem is analycity, which is equivalent to be holomorphic in the complex case. If we take f a real-analytic function in an interval $|t| < T$, it is easy to see f as a complex-analytic function in the disk $D(0, T)$ and so holomorphic in this disk.

For such an open subset Ω_0, let d_0 be the distance between Ω_0 and the boundary $\partial\Omega_1$. We can without loss of generality assume that

$$\Omega_1 = \{z \in \mathbb{C}^n | d(z, \Omega_0) < d_0\}$$

We introduce the one-parameter family of connected open sets

$$\Omega_s := \{z \in \mathbb{C}^n | d(z, \Omega_0) < sd_0\} \quad , \quad 0 \leq s \leq 1$$

Definition 2.12. Let $(E_s)_{0\leq s\leq 1}$ be a one-parameter family of Banach spaces. We say that it is a *scale of Banach spaces* if for all $0 \leq s' < s \leq 1$, $E_s \subset E_{s'}$ and the norm in E_s is not smaller than the norm on E_s induced by $E_{s'}$.

We set $E_s := H(\bar{\Omega}_s, \mathbb{C}^m)$ for $0 \leq s \leq 1$.

Proposition 2.13. *$(E_s)_{0\leq s\leq 1}$ is a scale of Banach spaces.*

Proof. If $s' < s$, we have a natural injection $f \in E_s \hookrightarrow f_{|\bar{\Omega}_{s'}} \in H(\bar{\Omega}_{s'}, \mathbb{C}^m)$ given by $\bar{\Omega}_{s'} \subset \Omega_s$. Furthermore :

$$|f|_{\Omega_s} = \sup_{z\in\Omega_s} |f(z)| \leq \sup_{z\in\Omega_{s'}} |f(z)| = |f|_{\Omega_{s'}}$$

□

By applying Proposition 2.9, replacing Ω_1 by Ω_s and Ω by $\Omega_{s'}$, we have that if $s' < s$, then A is a continuous function valued in $B(E_s, E_{s'})$. We can notice that $d(\Omega_{s'}, \mathbb{C}^n \backslash \Omega_s) = (s - s')d_0$. Hence, for t such that $|t| < T$, the norm of $A(t) \in B(E_s, E_{s'})$ as operator is not greater than $\frac{C(t)}{s-s'}$ where

$$C(t) := \frac{1}{d_0} \sum_{j=1}^{n} \|A_j(\cdot, t)\|_{\Omega_1} + \|A_0(\cdot, t)\|_{\Omega_1}$$

As C is bounded on any closed subinterval of $]-T, T[$, if we decrease T, we can assume C is bounded by a constant and therefore for each t, the operator norm of $A(t) : E_s \to E_{s'}$ is not greater than $\frac{C}{s-s'}$ where $C > 0$ a constant that doesn't depend on s, s' nor t. Moreover, we can assume that $(Ce)^{-1} \leq T$ and f is continous in $[-T, T]$ valued in E_1. We can now state the following result :

Theorem 2.14 (Abstract version of the Cauchy-Kovalevska theorem)**.** *Under the preceding hypotheses, given $u_0 \in E_1$ any continous function $f \in C^0([-T, T], E_1)$:*

1. *There is a function u in the interval $|t| < (Ce)^{-1}$, valued in E_0, which for any $0 \leq s \leq 1$ is a C^1 function in the interval $|t| < (Ce)^{-1}(1 - s)$ valued in E_s and which satisfies (2.7) and (2.8) when $|t| < (Ce)^{-1}$.*
2. *If, for some $0 < T' \leq T$ and $0 < s \leq 1$, there are two C^1 functions in the interval $|t| < T'$ valued in E_s and satisfying (2.7) and (2.8) in this interval, then they must be equal.*

Proof. (1) *Existence of the solution* :

Let $(v_k)_{k\in\mathbb{N}}$ be the sequence of continuous functions in $[-T,T]$ valued in E_s for any $0\leq s<1$ defined as follows:

$$v_0(t)=u_0+\int_0^t f(t')dt'$$

$$v_{k+1}(t)=v_0(t)+\int_0^t A(t')v_k(t')dt'$$

If v_k is a continuous function in the interval $|t|<T$ valued in E_s, then, for any $s'<s$, $t\mapsto A(t)v_k(t)$ is a continous function in the same interval valued in $E_{s'}$, and so it is also the case of v_{k+1}. For any $0\leq s<1$, we can take a sequence $s<s_{k-1}<\cdots<s_0<1$ such that for each $0\leq j\leq k-1$, v_j is valued in E_{s_j} and so v_k is valued in E_s. We set

$$w_0:=v_0 \quad , \quad w_k:=v_k-v_{k-1}=\int_0^t A(t')w_{k-1}(t')dt'$$

Each w_k is a continuous function in the interval $|t|<T$ valued in E_s. We denote N_s the norm in E_s. We want to show, by induction on k, that :

$$N_s(w_k(t))\leq M(t)\left(\frac{Ce|t|}{1-s}\right)^k \quad , \quad |t|<T \tag{2.15}$$

where

$$M(t)=N_1(u_0)+\left|\int_0^t N_1(f(t'))dt'\right|$$

M is a nondecreasing function of $|t|$ on $[0,T]$ and $[-T,0]$ by the positivity of the norm N_1.

When $k=0$, we have

$$N_s(w_0(t))=N_s(v_0(t))\leq N_s(u_0)+\left|\int_0^t N_s(f(t'))dt'\right|\leq N_1(u_0)+\left|\int_0^t N_1(f(t'))dt'\right|=M(t)$$

Suppose now that it holds up to k and let us prove it for $k+1$. If $0\leq s'<s<1$,

$$\begin{aligned} N_{s'}(w_{k+1}(t)) &\leq \left|\int_0^t \|A(t')\|N_s(w_k(t'))dt'\right| \\ &\leq \frac{C}{s-s'}\left|\int_0^t N_s(w_k(t'))dt'\right| \\ &\leq \frac{C}{s-s'}\left|\int_0^t M(t')\left(\frac{Ce|t'|}{1-s}\right)^k dt'\right| \\ &\leq \frac{C}{s-s'}M(t)\left(\frac{Ce}{1-s}\right)^k\frac{|t|^{k+1}}{k+1} \end{aligned}$$

If we take $s=s'+\frac{1-s'}{k+1} \iff 1-s=\frac{k}{k+1}(1-s')$, then

$$N_{s'}(w_{k+1}(t))\leq M(t)\left(\frac{Ce|t|}{1-s'}\right)^{k+1}$$

f is continuous in $[-T,T]$, hence there exists M a constant such that for any t, $M(t) \leq M$.

$$N_s(w_k(t)) \leq M \left(\frac{Ce|t|}{1-s}\right)^k$$

Hence $\sum_{k=0}^{+\infty} w_k$ converges uniformly in E_s in every compact of the interval $|t| < \frac{1-s}{Ce}$. We denote u its limit. By definition of $(w_k)_k$, the sequence $(v_k)_k$ converges uniformly in E_s in every compact in $|t| < \frac{1-s}{Ce}$ such that

$$u(t) = u_0 + \int_0^t f(t')dt' + \int_0^t A(t')u(t')dt' \tag{2.16}$$

For $0 < \epsilon < 1-s$, u is a continuous function in $|t| < \frac{1-s-\epsilon}{Ce}$ valued in $E_{s+\epsilon}$, and hence $t \mapsto A(t)u(t)$ is a continuous function in $|t| < \frac{1-s-\epsilon}{Ce}$ valued in E_s. The equality (2.16) makes u a C^1 function in $|t| < \frac{1-s-\epsilon}{Ce}$ valued in E_s. By taking $\epsilon \to 0^+$, u is a C^1 function in $|t| < \frac{1-s}{Ce}$ valued in E_s. Furthermore, we have $u_t = f(t) + A(t)u(t)$ and $u(0) = u_0$.

(2) *Uniqueness* :

If u_1 and u_2 are two functions satisfying (2.7) and (2.8), then $h := u_1 - u_2$ is such that $h_t = A(t)h(t)$ and $h(0) = 0$ and conversely. Hence it is sufficient to prove our point for a such function h. We have $h^{-1}(0)$ a nonempty closed subset of $]-T',T'[$. Let t_0 such that $h(t_0) = 0$. We have

$$h(t) = \int_{t_0}^t A(t')h(t')dt'$$

Let $0 \leq s' < s$. We want to prove that, for $M(t) := \sup\limits_{[t_0,t]} N_s(h(t'))$, we have

$$N_{s'}(h(t)) \leq M(t)\left(\frac{Ce|t-t_0|}{s-s'}\right)^k$$

It is trivial for $k=0$. If we assume that it is true for some k, then by taking $\epsilon = \frac{s-s'}{k+1}$ and seeing $A(t) : E_s \to E_{s'+\epsilon}$, we have

$$\begin{aligned}
N_{s'}(h(t)) &\leq \left|\int_{t_0}^t \frac{C}{s-s'-\epsilon} N_{s'+\epsilon}(h(t'))dt'\right| \\
&\leq \frac{C}{s-s'-\epsilon}\left|\int_{t_0}^t M(t')\left(\frac{Ce|t'-t_0|}{s-s'-\epsilon}\right)^k dt'\right| \\
&\leq M(t)\frac{C|t-t_0|^{k+1}}{(k+1)(s-s'-\epsilon)}\left(\frac{Ce}{s-s'-\epsilon}\right)^k
\end{aligned}$$

Hence our inequality is true by remarking $\left(1+\frac{1}{k}\right)^k \leq e$.

If $|t-t_0| < \frac{s-s'}{Ce}$, then $N_{s'}(h(t)) = 0$ by taking $k \to +\infty$ and so $N_s(h(t)) = 0$ ($s' < s$). We can conclude that $h^{-1}(0)$ is a nonempty closed open subset of a connected space, and hence $h(t) = 0$ for each $|t| < T'$.

□

Remark 2.17. We can formulate a holomorphic version of Theorem 2.14:

Theorem 2.18. *Under almost the same hypotheses, assuming this time that t is a complex variable varying in the disk $\bar{D}(0,T)$, that if $s' < s$, then A is a holomorphic function of t on the open disk valued in $B(E_s, E_{s'})$, and that f is an element of $H(\bar{D}(0,T), E_1)$, the following is true :*

(1) There is a function u in the disk $D(0,(Ce)^{-1})$, valued in E_0, which for any $0 \le s < 1$ is a holomorphic function in $D\left(0, \frac{1-s}{Ce}\right)$ valued in E_s, and which satisfies (2.7) and (2.8) in the disk $D(0,(Ce)^{-1})$.

(2) If, for some $0 < T' \le T$ and $0 < s \le 1$, there are two holomorphic functions in $D(0,T')$ valued in E_s and satisfying (2.7) and (2.8), they must be equal.

The proof is essentially the same as the real version.

Example 2.19 (Lewy's example)**.** Consider in $\mathbb{R} \times \mathbb{C}$ with variables (t, z) the operator

$$L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}$$

We can apply the Cauchy-Kovalevska theorem to this operator i.e. if we consider f a real-analytic function near 0, there exists a solution u that is real-analytic near 0 satisfying $L(u) = f$. But there exists f a smooth non-analytic function such that $L(u) = f$ has no solution near 0 [3], even if u is taken as a distribution.

The same fact occurs for the Mizohata operator :

$$M = \frac{\partial}{\partial x} + ix\frac{\partial}{\partial y}$$

near any point $(0, y)$, with 2 real variables x, y.

3. Reduction from Higher Orders Problem to First Order Problem

In the precedent section, we have proved the Cauchy-Kovalevska theorem for first-order linear PDEs or systems. We will prove in this section that extend the result to higher orders for some convenient systems. We deal here with a system of order $m > 1$ given by

$$D_t^m u = \sum_{\substack{\alpha_0 + |\alpha| \le m \\ \alpha_0 < m}} c_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} u + f(x,t) \tag{3.1}$$

with Cauchy conditions

$$D_t^k u_{|t=0} = v^k(x) \quad , \quad 0 \le k \le m-1 \tag{3.2}$$

where our variables (x, t) are taken in some open subset Ω of $\mathbb{R}^{n+1}$, or $\mathbb{C}^{n+1}$, or $\mathbb{C}^n \times \mathbb{R}$, and $f \in C^\infty(\Omega, \mathbb{C}^N)$, $v^k \in C^\infty(p_x(\Omega), \mathbb{C}^N)$ and $c_{\alpha_0,\alpha} \in M_N(C^\infty(\Omega))$ (we will discuss of analycity later).

Remark 3.3. We can see in this kind of systems that the variable t has a privileged role assigned by the restriction $\alpha_0 < m$ in the summation. In fact, there is some systems that can't be put into the form (3.1).

Set

$$u_0 := u \quad u_j := D_{x_j} u \quad u_{n+1} := D_t u \quad U = (u_0, \ldots, u_{n+1})$$

Observe that

$$D_t^{m-1} u_0 = D_t^{m-2} u_{n+1} \quad D_t^{m-1} u_j = D_t^{m-2} D_{x_j} u_{n+1}$$

and that we can rewrite (3.1) as

$$D_t^{m-1}u_{n+1} = \sum_{j=0}^{n+1} \sum_{\substack{\alpha_0+|\alpha|\leq m-1 \\ \alpha_0<m-1}} c_{\alpha_0,\alpha,j}(x,t) D_t^{\alpha_0} D_x^{\alpha} u_j + f(x,t)$$

Combining all of this, we can write

$$D_t^{m-1}U = \sum_{\substack{\alpha_0+|\alpha|\leq m-1 \\ \alpha_0\leq m-2}} C_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} U + F \tag{3.4}$$

with $F = (0, \dots, 0, f)$.

Now, we have to transform the Cauchy conditions (3.2) in a convenient way for our new system. It is easy to see that

$$D_t^k u_{0|t=0} = v^k \quad D_t^k u_{j|t=0} = D_{x_j} v^k \quad D_t^k u_{n+1|t=0} = v^{k+1}$$

for every $0 \leq k \leq m-2$, $1 \leq j \leq n$. Hence we can rewrite (3.2) as

$$D_t^k U_{|t=0} = V^k \quad , \quad 0 \leq k \leq m-2 \tag{3.5}$$

with $V^k := (v^k, D_{x_1}v^k, \dots, D_{x_n}v^k, v^{k+1})$.

Proposition 3.6. *The existence and unicity of a solution u to a system given by (3.1) and (3.2) is equivalent to the existence and unicity of a solution U to a system given by (3.4) and (3.5).*

Proof. (1) We take a system given by (3.1) and (3.2). If we assume that any system given by (3.4) and (3.5) admits a solution U, then it is the case for the system given by $F = (0, \dots, 0, f)$ and $V^k = (v^k, D_{x_1}v^k, \dots, D_{x_n}v^k, v^{k+1})$. We take $U = (u_0, \dots, u_{n+1})$ a solution to this system, hence $u = u_0$ is a solution to the system given by (3.1) and (3.2).

(2) Conversely, we take a system given by (3.4) and (3.5) (we don't assume that the first components of F are zero). We set

$$U' := U - \sum_{k=0}^{m-2} \frac{t^k}{k!} V^k$$

For any $0 \leq k \leq m-2$, we have

$$D_t^k U'_{\ |t=0} = 0$$

and

$$D_t^{\alpha_0} D_x^{\alpha} U' = D_t^{\alpha_0} D_x^{\alpha} U - \sum_{k=\alpha_0}^{m-2} \frac{t^{k-\alpha_0}}{(k-\alpha_0)!} D_x^{\alpha} V^k$$

Hence U is a solution for F and V^k if and only if U' is a solution for $F - \sum C_{\alpha_0,\alpha}(x,t) \sum_{k=\alpha_0}^{m-2} \frac{t^{k-\alpha_0}}{(k-\alpha_0)!} D_x^{\alpha} V^k$ and 0. Hence we can assume that for any k, $V^k = 0$. We write $F = (F_0, \dots, F_{n+1})$ and we take, for any $0 \leq j \leq n$, w_j be some functions we will precise after such that

$$D_t^k w_{j|t=0} = 0$$

and we set $\tilde{U} := U - (w_0, \ldots, w_n, 0)$. Hence $D_t^k \tilde{U}_{\,|t=0} = D_t^k U_{\,|t=0} = 0$ and

$$\begin{aligned}
D_t^{m-1}\tilde{U} &= D_t^{m-1}U - (D_t^{m-1}w_0, \ldots, D_t^{m-1}w_n, 0) \\
&= \sum C_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} U + F - (D_t^{m-1}w_0, \ldots, D_t^{m-1}w_n, 0) \\
&= \sum C_{\alpha_0,\alpha}(x,y) D_t^{\alpha_0} D_x^{\alpha} \tilde{U} + \sum C_{\alpha_0,\alpha}(x,t)(D_t^{\alpha_0} D_x^{\alpha} w_0, \ldots, D_t^{\alpha_0} D_x^{\alpha} w_n, 0) \\
& \quad -(D_t^{m-1}w_0, \ldots, D_t^{m-1}w_n, 0) + F \\
&= \sum C_{\alpha_0,\alpha}(x,y) D_t^{\alpha_0} D_x^{\alpha} \tilde{U} + (0, \ldots, 0, F_{n+1})
\end{aligned}$$

for a good choice of the w_k. We obtain the transform of the problem (3.1)-(3.2) with $f = F_{n+1}$ and $v^k = 0$ for all k. If we take u a solution to this problem, then the associated U is a solution for our basis problem.

(3) We assume that U_1 and U_2 are two solutions to (3.4)-(3.5) given F and V^k. Wet set $U := U_1 - U_2$. Hence

$$\begin{aligned}
D_t^{m-1}U &= D_t^{m-1}U_1 - D_t^{m-1}U_2 \\
&= \sum C_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} U_1 + F - \sum C_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} U_2 - F \\
&= \sum C_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} U
\end{aligned}$$

and $D_t^k U_{\,|t=0} = 0$ for all k. In this case, if we write $U = (u_0, \ldots, u_{n+1})$, then we have a solution u_0 to the associated problem (3.1)-(3.2) with $f = 0$ and $v^k = 0$. We suppose that we have the unicity to the problem (3.1)-(3.2). Necessarily, we have $u_0 = 0$. In this case :

$$D_t^{m-2} u_{n+1} = D_t^{m-1} u_0 = 0$$

and so u_{n+1} is polynomial in the variable t of order $\leq m-3$. Moreover

$$D_t^k u_{n+1\,|t=0} = D_t^{k+1} u_{0\,|t=0} = v^{k+1} = 0 \quad , \quad 1 \leq k \leq m-2$$

Thus $u_{n+1} = 0$ and in the same way, $D_t^{m-1} u_j = D_t^{m-2} D_{x_j} u_{n+1} = 0$ and for all $k \leq m-2$, we have

$$D_t^k u_{j\,|t=0} = D_{x_j} v^k = 0$$

that implies $u_j = 0$. Hence $U = (u_0, \ldots, u_{n+1}) = 0$.

(4) Conversely, we assume u a solution to (3.1)-(3.2) with $f = 0$ and $v^k = 0$. Then it is easy to verify that U given by $u = u_0$ is a solution to (3.4)-(3.5) with $F = 0$ and $V^k = 0$. By unicity, we have $U = 0$ and so $u = 0$.

□

Remark 3.7. We have now proved that existence and unicity of a solution for a system of order m is equivalent to the existence and unicity of a solution for a system of order $m-1$, and so for any system of order m with analytic coefficients (3.1)-(3.2) has a unique solution by the Cauchy-Kovalevska theorem.

4. Reminders on Characteristics and Link with Cauchy-Kovalevska Theorem

We consider a differential operator

$$P(x,t,D_x,D_t) = \sum_{\alpha_0 + |\alpha| \leq m} a_{\alpha_0,\alpha}(x,t) D_t^{\alpha_0} D_x^{\alpha} \in D_{\Omega}^{\leq m}$$

of order m with analytic coefficients in an open subset $\Omega \subset \mathbb{R}^{n+1}$ (or $\mathbb{C}^{n+1}$).

Definition 4.1. Let Ω be a open subset of $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We can associate to any differential operator $P(y, D_y) \in D_\Omega^{\leq m}$, with analytic coefficients, a polynomial

$$P(y,\eta) = \sum_{|\alpha|\leq m} a_\alpha(y)\eta^\alpha \in \mathscr{O}_\Omega\left[\eta_1,\dots,\eta_n\right]$$

called the *total symbol* of P (we denote it with the same letter as the differential operator). From the total symbol, we can define the *principal symbol*

$$P_m(y,\eta) := \sum_{|\alpha|=m} a_\alpha(y)\eta^\alpha$$

Definition 4.2. For $y \in \Omega$, we denote

$$C_P(y) := \{\eta \in \mathbb{K}^n | P_m(y,\eta) = 0\}$$

the *characteristic cone of* P *at the point* $y \in \Omega$. Every covector $\eta \in C_P(y)$ different from zero is said to be *characteristic with respect to* P *at the point* y. We denote

$$C_P := \sqcup_{y\in\Omega}\{y\} \times C_P(y) = \{(y,\eta) \in T^*\Omega | P_m(y,\eta) = 0\}$$

the *characteristic variety*.

Let $\Sigma \subset \Omega$ be a C^1 hypersurface. Σ is said to be *characteristic at the point* $y \in \Omega$ *with respect to* P if any normal covector η to Σ at y is characteristic with respect to P at y i.e. $N_y^*\Sigma \subset C_P(y)$. Σ is said to be *characteristic with respect to* P if it is at every point i.e. $N^*\Sigma \subset C_P$.

If we assume that for all $(x,t) \in \Omega$, $0 \neq P_m(x,t,\xi,\tau) = a_{m,0}(x,t)$, i.e. the covector $(0,1)$ is not characteristic with respect to P at any point of Ω, we can change any equation of the form $P(x,t,D_x,D_t)u = g$ in an equation of the form (3.1) with $c_{\alpha_0,\alpha} = -\frac{a_{\alpha_0,\alpha}}{a_{m,0}}$ and $f = \frac{g}{a_{m,0}}$.

Definition 4.3. A differential operator $P(y,D_y)$ in Ω is said to be *elliptic at* $y_o \in \Omega$ if $C_P(y_0) = \{0\}$.

Example 4.4. We consider the Laplace operator $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ on $\mathbb{R}^n$. Hence its principal symbol is $p(x,\xi) = |\xi|^2$. We can easily see that Δ is everywhere elliptic.

Example 4.5. We consider the operator $L = \partial_t - a(x,t)\partial_x$ in $\mathbb{R}^2$ where a is smooth with real values (our definitions hold for smooth functions). At $(x_0,t_0) \in \mathbb{R}^2$, we have

$$P(x_0,t_0,\xi,\tau) = \tau - a(x_0,t_0)\xi$$

and $C_L(x_0,t_0) = \{(\xi,\tau) \in \mathbb{R}^2 | \tau = a(x_0,t_0)\xi\}$. We consider x the curve of points $(x(t),t)$ defined by

$$\frac{dx}{dt} = -a(x(t),t) \quad , \quad x(t_0) = x_0 \tag{4.6}$$

We have $T_{(x_0,t_0)}x = \mathbb{R}(-a(x_0,t_0),1)$ and $N_{(x_0,t_0)}x = \mathbb{R}(1,a(x_0,t_0)) = C_L(x_0,t_0)$. Thus x is characteristic with respect to L.

Conversely, let γ be a C^1 curve in $\mathbb{R}^2$. We assume γ is characteristic with respect to L. Let (x_0,t_0) be a point of γ, U_0 be a neighborhood of (x_0,t_0) and φ a C^1 function such that $U_0 \cap \gamma = \varphi^{-1}(0)$. We can assume that $\nabla_\varphi = (\varphi_x,\varphi_t)$ does not vanish in U_0. The conormal to γ in any point of $U_0 \cap \gamma$ is then spanned by (φ_x,φ_t), and so $\varphi_t - a(x,t)\varphi_x \simeq 0$ in $U_0 \cap \gamma$.

∇_φ does not vanish in U_0, hence φ_x does not vanish in a neighborhood of $U_0 \cap \gamma$ which can be assumed to be U_0. The vector field

$$\left(-\frac{\varphi_t}{\varphi_x}, 1\right) = (-a(x,t), 1)$$

is tangent to $U_0 \cap \gamma$. Thus, the integral curve of this vector field, that is defined by (4.6), is exactly $U_0 \cap \gamma$, i.e. (4.6) defines locally every characteristic curve with respect to L. We can conclude in the following way: the characteristic curves of L are the integral curves of L.

Remark 4.7. There is a way to salvage the preceeding statement for characteristic hypersurfaces with a notion of bicharacteristic curve [5].

To conclude, we will state a 'coordinates independent' version of the Cauchy-Kovalevska theorem:

Theorem 4.8. *Let Ω be an open subset of $\mathbb{R}^{n+1}$, $P(y, D_y)$ be a linear differential operator of order $m > 0$ with analytic coefficients in Ω. Let $\Sigma \subset \Omega$ be an analytic hypersurface which is nowhere characteristic with respect to $P(y, D_y)$ and whose exterior normal at every point is well-defined. Let f be an analytic function in Ω, $(u_j)_{0 \le j \le m-1}$ be m analytic functions in Σ. There exists a neighborhood $\Sigma \subset \mathscr{V} \subset \Omega$ and a unique analytic function u in $\mathscr{V}$ such that*

$$P(y, D_y)u = f \text{ in } \mathscr{V} \tag{4.9}$$

$$\frac{\partial}{\partial \nu}^{j} u = u_j \text{ in } \Sigma \text{ , for every } 0 \le j \le m-1 \tag{4.10}$$

where we have denoted by $\frac{\partial}{\partial \nu}$ the partial differentiation in the exterior normal direction to Σ.

Proof. It suffices to prove this statement locally. Let $y_0 \in \Sigma$. Let $\mathscr{V}(y_0)$ be a neighborhood of y_0 sufficiently small such that we can perform a change of variables $x^1, \ldots, x^n, t$ in which $\Sigma \cap \mathscr{V}(y_0)$ corresponds to $t = 0$ and ∂_t is the partial differentiation in the exterior normal direction. In these coordinates, P has an expression of the form

$$P(x, t, D_x, D_t) = a(x,t) D_t^m + \sum_{j=1}^{m} Q_j(x, t, D_x) D_t^{m-j}$$

where a is an analytic function in $\mathscr{V}(y_0)$, and the Q_j are of ordrer at most j with analytic coefficients. As we showed earlier, the fact that Σ is assumed to be nowhere characteristic implies that a does not vanish in a neighborhood of Σ and our problem can be written :

$$D_t^m u + \sum_{j=1}^{m} \frac{1}{a} Q_j(x, t, D_x) D_t^{m-j} u = \frac{f}{a} \quad , \quad \frac{\partial}{\partial t}^{j} u = u_j$$

Applying the Cauchy-Kovalevska theorem, we have a unique analytic solution u to (4.9)-(4.10) in $V(y_0)$. □

Remark 4.11. We have stated what we could call a 'real-analytic' version of the Cauchy-Kovalevska theorem. We can also state a 'holomorphic' version in the following sense.

Theorem 4.12 (Holomorphic version of Cauchy-Kovalevska theorem)**.** *Let Ω be an open subset $\mathbb{C}^n$ and Σ be an analytic submanifold of codimension 1 of Ω. Let $P(z, D_z)$ be a differential operator of order m with holomorphic coefficients in Ω. We assume that Σ is nowhere characteristic with respect to $P(z, D_z)$ and that we have a holomorphic vector field $\frac{\partial}{\partial \nu}$ in Ω, normal to Σ at each one of its points. Then there is an open neighborhood $\Omega' \subset \Omega$ of Σ such that for any $f \in \mathscr{O}(\Omega)$, for any $u_1, \dots, u_{m-1} \in \mathscr{O}(\Sigma)$, there exists a unique $u \in \mathscr{O}(\Omega')$ such that*

$$P(z, D_z)u = f \text{ in } \Omega' \tag{4.13}$$

$$\frac{\partial}{\partial \nu}^{j} u = u_j \text{ in } \Sigma \tag{4.14}$$

Proof. Let z_0 be any point of Σ and $V(z_0)$ be a neighborhood of z_0 in Ω. If $V(z_0)$ is small enough, then there exists φ_0 a holomorphic function in $V(z_0)$ such that $V(z_0) \cap \Sigma = \varphi_0^{-1}(0)$ and ∇_{φ_0} does not vanish in $V(z_0)$. Furthermore, $V(z_0)$ can be assumed to be sufficiently small such that we have local coordinates $(w^1, \dots, w^n)$ with $w^n = \varphi_0(z)$. In $V(z_0) \cap \Sigma$, there exists λ a holomorphic function such that

$$\frac{\partial}{\partial \nu} = \lambda(z) \nabla_{\varphi_0}(z)$$

Hence there exists $\mu \in \mathscr{O}(V(z_0) \cap \Sigma)$ such that

$$\frac{\partial}{\partial \nu} = \mu(w) \partial_{w^n}$$

In this neighborhood $V(z_0)$, we can write

$$P(z, \partial_z) = a(w) D_{w^n}^m + \sum_{j=1}^{m} Q_j(w, D_{w^1}, \dots, D_{w^{n-1}}) D_{w^n}^{m-j}$$

where a is a holomorphic function and the Q_j are of order $\leq j$ with holomorphic coefficients. As Σ is assumed to be nowhere characteristic with respect to P, a can be assumed to be nowhere vanishing in $V(z_0)$. We have now to solve the problem

$$D_{w^n}^m u + \sum_{j=1}^{m} \frac{1}{a} Q_j(w, D_{w^1}, \dots, D_{w^{n-1}}) D_{w^n}^{m-j} u = \frac{f}{a} \text{ in } V(z_0) \tag{4.15}$$

$$\frac{\partial}{\partial w^n}^{j} u = \frac{1}{\mu} u_j \text{ in } V(z_0) \cap \Sigma \tag{4.16}$$

□

5. Holmgren Theorem

In this section, we will study a unicity result that we can call the dual form of the Cauchy-Kovalevska theorem. For instance, let $(E_s)_{0 \leq s \leq 1}$ be a scale of Banach spaces and A be a function in $|t| < T$ such that

$$\text{if } s' < s \text{, then } A \text{ is a continuous function valued in } B(E_s, E_{s'}) \tag{5.1}$$

$$\exists C > 0, \forall 0 \leq s' < s \leq 1, \forall |t| < T, \|A(t)\| \leq \frac{C}{s - s'} \tag{5.2}$$

Furthermore, we consider this additionnal hypothesis :

$$\text{If } s' < s \text{, then } E_s \text{ is dense in } E_{s'} \tag{5.3}$$

We set, for each $0 \leq s \leq 1$, $F_s := E'_{1-s}$ as dual of Banach space. First, with (5.3), if we denote $i_{s,s'} : E_s \hookrightarrow E_{s'}$ the natural injection associated to our two Banach spaces, then ${}^t i_{s,s'} : F_{1-s'} \to F_{1-s}$ is an injective continuous linear map of norm ≤ 1. Moreover

Lemma 5.4. *(F_s) is a scale of Banach spaces and (5.1)-(5.2) hold for the scale (F_s) and the continous map ${}^tA$.*

Proof. If $s' < s$, it is clear that $F_s = E'_{1-s} \subset E'_{1-s'} = F_{s'}$. Moreover, for any $f \in F_s$, we have

$$|f|_{F_{s'}} = \sup_{v \in E_{1-s'} \setminus \{0\}} \frac{|f(v)|}{|v|_{E_{1-s'}}} \leq \sup_{v \in E_{1-s} \setminus \{0\}} \frac{|f(v)|}{|v|_{E_{1-s}}} \leq |f|_{F_s}$$

If $s' < s$, then $A(t) \in B(E_{1-s'}, E_{1-s})$ and hence ${}^tA(t) \in B(F_s, F_{s'})$.

$$\begin{aligned}
\|{}^tA(t)\| &= \sup_{|f|_{F_s} \leq 1} |{}^tA(t)f|_{F_{s'}} \\
&= \sup_{|f|_{F_s} \leq 1} \sup_{|u|_{E_{1-s'}} \leq 1} |\langle {}^tA(t)f, u\rangle| \\
&\leq \sup_{|f|_{F_s} \leq 1} \sup_{|u|_{E_{1-s'}} \leq 1} |\langle f, A(t)u\rangle| \\
&\leq \sup_{|u|_{E_{1-s'}} \leq 1} |A(t)u|_{E_{1-s}} \\
&\leq \|A(t)\| \\
&\leq \frac{C}{s-s'}
\end{aligned}$$

□

The important consequence of this lemma is that we can apply the theorem 2.14 in this new context.

Lemma 5.5. *Under the preceding hypotheses, let v be a C^1 function in $|t| < T$, valued in F_s for some $s < 1$ and satisfying the equation*

$$\frac{dv}{dt} = {}^tA(t)v$$

If $v(0) = 0$, then $v \equiv 0$.

Proof. Applying the uniqueness part of the theorem (2.14) concludes. □

Lemma 5.6. *We assume the following fact:*

$$0 \leq s' < s \leq 1 \implies A \in C^\infty(]-T, T[, B(E_s, E_{s'})) \tag{5.7}$$

Then ${}^tA$ satisfies the analogous property :

$$0 \leq s' < s \leq 1 \implies {}^tA \in C^\infty(]-T, T[, B(F_{1-s'}, F_{1-s})) \tag{5.8}$$

Proof. For any $f \in F_{1-s'}$ and $u \in E_s$, we have

$$\begin{aligned}
\left\langle \lim_{t \to t_0} \frac{{}^tA(t) - {}^tA(t_0)}{t - t_0} f, u \right\rangle &= \lim_{t \to t_0} \frac{1}{t-t_0} \langle f, (A(t) - A(t_0))u\rangle \\
&= \langle f, A'(t_0)u\rangle
\end{aligned}$$

We can deduce of this equality that ${}^tA$ is C^∞ as A. □

Remark 5.9. The lemma is in fact the same replacing C^∞ by C^m or analytic.

We suppose that (5.8) holds. If v is a distribution in $|t| < T$, valued in F_1, then ${}^tAv$ is a distribution in the same interval valued in F_s for any $s < 1$. We interest in distributions which can be written in the form:

$$v = \partial_t^k g$$

where g is a continuous function valued in F_1 and vanishing for $t < 0$. We have for any smooth function φ with compact support:

$$\begin{aligned}
\langle {}^tA(t)\partial_t^k g, \varphi\rangle &= \langle \partial_t^k g, A(t)\varphi\rangle \\
&= (-1)^k\langle g, \partial_t^k(A\varphi)\rangle \\
&= (-1)^k\left\langle g, \sum_{j=0}^{k}\binom{k}{j}\partial_t^j A\partial_t^{k-j}\varphi\right\rangle \\
&= (-1)^k\sum_{j=0}^{k}\binom{k}{j}\left\langle {}^tA^{(j)}g, \partial_t^{k-j}\varphi\right\rangle \\
&= (-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left\langle \partial_t^{k-j}({}^tA^{(j)}g), \varphi\right\rangle
\end{aligned}$$

i.e.

$${}^tA(t)\partial_t^k g = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\partial_t^{k-j}({}^tA^{(j)}g)$$

Lemma 5.10. *Under the preceding hypotheses and given v a distribution as preceeds, if $\partial_t v - {}^tAv$ is a C^∞ function in $|t| < T$, valued in F_1, then v is a C^∞ function in the same interval valued in F_s for any $s < 1$.*

Proof. We set, for $0 \le j \le k$, $g_j := (-1)^j\binom{k}{j}({}^tA^{(k-j)}g)$ that are continuous function valued in F_s for any $s < 1$ by (5.8).

$$f := \partial_t v - {}^tA(t)v = \partial_t^{k+1}v - \sum_{j=0}^{k}\partial_t^j g_j$$

We can write

$$g_j = \partial_t^{k-j+1}h_j \quad , \quad f = \partial_t^{k+1}f_1$$

with h_j and f_1 vanishing for $t < 0$. Furthermore, f_1 is a C^∞ function valued in F_1.

If g is a C^μ function valued in every F_S ($s < 1$), then $h := \sum_{j=0}^k h_j$ is a $C^{\mu+1}$ such function. We have

$$\partial_t^{k+1}(g - h - f_1) = \partial_t^{k+1}g - \sum_{j=0}^{k}\partial_t^{k+1}h_j - \partial_t^{k+1}f_1 = \partial_t^{k+1}g - \sum_{j=0}^{k}\partial_t^j g_j - f = 0$$

Hence $g - h - f_1$ is a polynomial of t of degree at most k. But for $t < 0$, we have $(g - h - f_1)(0) = 0$. It follows that $g = h + f_1$ is a $C^{\mu+1}$ function valued in every F_s ($s < 1$). □

Corollary 5.11. *Under the same hypotheses as in the Lemma 5.10, if $\partial_t v = {}^tAv$ in $]-T, T[$, then $v \equiv 0$ in the same interval.*

Proof. $\partial_t v - {}^t A v = 0$ is a C^∞ function. By Lemma 5.10, v is a C^∞ function valued in F_s for any $s < 1$. Moreover, v vanishes when $t < 0$, then $v(0) = 0$. By Lemma 5.5, we have $v \equiv 0$. □

We consider now $r_0, d_0 > 0$, and we set for every $0 \leq s \leq 1$:

$$\Omega_s := \left\{ z \in \mathbb{C}^n \middle| |z^j| < r_0 + s d_0 \right\}$$

and

$$E_s := H(\bar{\Omega}_s, \mathbb{C}^m)$$

Lemma 5.12. $(E_s)_{0 \leq s \leq 1}$ *is a scale of Banach spaces satisfying (5.3).*

Proof. We already know that this kind of construction makes (E_s) a scale of Banach spaces. We have to show that if $s' < s$, then $H(\bar{\Omega}_s)$ is dense in $H(\bar{\Omega}_{s'})$. Let h be an element of $H(\bar{\Omega}_{s'})$ and, for each $0 < \delta < 1$, set

$$h_\delta : z \mapsto h((1-\delta)z)$$

h is uniformly continuous in $\bar{\Omega}_{s'}$, hence h_δ converges to h uniformly on $\bar{\Omega}_{s'}$, when δ converges to 0^+. h_δ can be extended to $(1+\delta)\bar{\Omega}_{s'}$ as a continuous function that is holomorphic in the interior of the polydisk and

$$\forall z \in \bar{\Omega}_{s'}, h_\delta((1+\delta)z) = h((1-\delta^2)z)$$

Let $P_{\delta,n}$ ($n \in \mathbb{N}$) be the polynomial Taylor expansion of degree n of h_δ at the point 0. $P_{\delta,n}$ converges uniformly to h_δ on every compact of $(1+\delta)\Omega_{s'}$. Hence it is the case on $\bar{\Omega}_{s'}$. By a diagonal argument, we obtain a sequence of polynomials (and so of elements of $H(\bar{\Omega})$) converging to $h \in H(\bar{\Omega}_{s'})$. □

We interest to distributions of the form

$$u = \sum_{\alpha_0 + |\alpha| \leq k} \partial_t^{\alpha_0} D_x^\alpha f_{\alpha_0,\alpha}(x,t)$$

where the $f_{\alpha_0,\alpha}$ are continuous functions, valued in $\mathbb{C}^n$, and whose support has a x-projection contained in a compact K of $\Omega_0 \cap \mathbb{R}^n$. We may choose the $f_{\alpha_0,\alpha}$ vanishing when $x \notin K'$ with K' a compact neighborhood of K in $\Omega_0 \cap \mathbb{R}^n$. We set

$$g_{\alpha_0}(x,t) := \sum_{|\alpha| \leq k - \alpha_0} D_x^\alpha f_{\alpha_0,\alpha}(x,t)$$

Lemma 5.13. *The distribution* g_{α_0} *can be seen as a continuous function of* t *valued in* $H(\bar{\Omega}_0, \mathbb{C}^m)'$ *by*

$$G_{\alpha_0}(t) : h \mapsto \sum_{|\alpha| \leq k - \alpha_0} (-1)^{|\alpha|} \int_{K'} f_{\alpha_0,\alpha}(x,t) D_x^\alpha h(x) dx \tag{5.14}$$

Proof. We denote $d := d(K', \partial\Omega_0)$. For any $h \in H(\bar{\Omega}_0, \mathbb{C}^m)$, we have

$$\begin{aligned}
|\langle G_{\alpha_0}(t) - G_{\alpha_0}(t'), h\rangle| &\leq \sum_{|\alpha|\leq k-\alpha_0} \int_{K'} |f_{\alpha_0,\alpha}(x,t) - f_{\alpha_0,\alpha}(x,t')||D_x^\alpha h(x)|dx \\
&\leq \sum_{|\alpha|\leq k-\alpha_0} \int_{K'} |f_{\alpha_0,\alpha}(x,t) - f_{\alpha_0,\alpha}(x,t')|dx \sup_{x\in K'} |D_x^\alpha h(x)| \\
&\leq \sum_{|\alpha|\leq k-\alpha_0} \alpha! d^{-|\alpha|} \int_{K'} |f_{\alpha_0,\alpha}(x,t) - f_{\alpha_0,\alpha}(x,t')||h|_{\Omega_0} dx
\end{aligned}$$

Moreover

$$\begin{aligned}
|\langle G_{\alpha_0}(t), h_1 - h_2\rangle| &\leq \sum_{|\alpha|\leq k-\alpha_0} \int_{K'} |f_{\alpha_0,\alpha}(x,t)||D_x^\alpha h_1(x) - D_x^\alpha h_2(x)|dx \\
&\leq \sum_{|\alpha|\leq k-\alpha_0} \int_{K'} |f_{\alpha_0,\alpha}(x,t)|dx \sup_{x\in K'} |D_x^\alpha (h_1 - h_2)(x)| \\
&\leq \sum_{|\alpha|\leq k-\alpha_0} \alpha! d^{-|\alpha|} \int_{K'} |f_{\alpha_0,\alpha}(x,t)||h_1 - h_2|_{\Omega_0} dx
\end{aligned}$$

which implies that $G_{\alpha_0}(t)$ is continuous on $H(\bar{\Omega}_0, \mathbb{C}^m)$. □

We write

$$v = \sum_{\alpha_0 \leq k} \partial_t^{\alpha_0} G_{\alpha_0}$$

If the G_{α_0} vanish for $t \leq 0$, we may write $G_{\alpha_0} = \partial_t^{k-\alpha_0} H_{\alpha_0}$ where the H_{α_0} are continuous functions valued in $H(\bar{\Omega}_0, \mathbb{C}^m)'$ and vanish for $t < 0$. If we set

$$g := \sum_{\alpha_0=0}^{k} H_{\alpha_0}$$

then we obtain the same type of distribution than before. We can evaluate v on test functions of the form $\varphi(t)h$ where $h \in H(\bar{\Omega}_0, \mathbb{C}^m)$ and $\varphi \in C_c^\infty(]-T, T[\,, \mathbb{C})$.

$$\begin{aligned}
\langle v, \varphi h\rangle &= \sum_{\alpha_0 \leq k} \langle \partial_t^{\alpha_0} G_{\alpha_0}, \varphi h\rangle \\
&= \sum_{\alpha_0 \leq k} (-1)^{\alpha_0} \langle G_{\alpha_0}, h\partial_t^{\alpha_0}\varphi\rangle \\
&= \sum_{\alpha_0 \leq k} (-1)^{\alpha_0} \langle g_{\alpha_0}, h\partial_t^{\alpha_0}\varphi\rangle \\
&= \langle u, h\varphi\rangle
\end{aligned}$$

Hence, if $v = 0$, then u vanishes on products of the form $h\varphi$ where φ is a smooth function with compact support and h is a restriction to $\Omega_0 \cap \mathbb{R}^n$ of an element of $H(\bar{\Omega}_0, \mathbb{C}^m)$. In particular, it is the case when h is a polynomial. The linear combinations of such products are denses in $C^\infty(\mathbb{R}^n \times]-T, T[)$. Hence $u = 0$ and the correspondence $u \mapsto v$ is injective.

We consider a differential operator

$$A(t) = \sum_{j=1}^{n} A_j(z,t)\partial_{z^j} + A_0(z,t)$$

where $A_j \in M_m(\mathscr{O}(\Omega))$ with Ω an open neighborhood $\Omega_0 \times]-T,T[\subset \Omega \subset \mathbb{C}^{n+1}$. The distribution $A(t)v$ is associated to a distribution $\mathfrak{A}(t)u$ where

$$\mathfrak{A}(t) = \sum_{j=1}^{n} A_j(x,t)\partial_{x^j} + A_0(x,t)$$

If $\varphi \in C_c^\infty(]-T,T[)$ and $h \in E_0$, then

$$\begin{aligned} \langle A(t)v, h\varphi(t)\rangle &= \langle A(t)u, h\varphi(t)\rangle \\ &= \langle \sum_{j=1}^{n} A_j \partial_{x^j} u + A_0 u, h\varphi(t)\rangle \\ &= \sum_{j=1}^{n} \langle \partial_{x^j} u, \, {}^t A_j (h\varphi)\rangle + \langle u, \, {}^t A_0(h\varphi)\rangle \\ &= \sum_{j=1}^{n} -\langle u, \varphi \partial_{x^j}({}^t A_j h)\rangle + \langle u, \varphi \, {}^t A_0 h\rangle \\ &= \langle v, \varphi B(t) h\rangle \end{aligned}$$

with

$$B(t) := -\sum {}^t A_j \partial_{z^j} + {}^t A_0 - \sum \frac{\partial \, {}^t A_j}{\partial z^j}$$

Hence $B(t) \in B(E_s, E_{s'})$ is the transpose of $A(t)$.

First, we see that B satisfies (5.7) for our scale of Banach spaces E_s. By Lemma 5.6, A satisfies (5.8). With our definition of the Ω_s, we can apply 2.9 to B and so B satisfies (5.2). Hence the hypotheses needed to apply our preceeding results are fulfilled and we are in conidtion to state our first version of Holmgren's theorem :

Theorem 5.15. *Let u be a distribution in $(\Omega_0 \cap \mathbb{R}^n) \times]-T,T[$ vanishing when $x \notin K$ for a certain compact subset $K \subset \Omega_0 \cap \mathbb{R}^n$. Suppose that*

$$\frac{du}{dt} = \mathfrak{A}(t)u$$

and that u vanishes for $t < 0$. Then $u = 0$ in $(\Omega_0 \cap \mathbb{R}^n) \times]-T,T[$.

Proof. We may decrease T if needed. Then we can write

$$u = \sum_{\alpha_0 + |\alpha| \le k} \partial_t^{\alpha_0} D_x^\alpha f_{\alpha_0,\alpha}(x,t)$$

and associate to u a distribution v of t valued in E_0' such that $v_t = A(t)v$ with A associated to $\mathfrak{A}$ and vanishing for $t < 0$. Hence with the same notations, B satifies (5.7) and (5.2), A satisfies (5.8). The hypotheses of Lemma 5.10 are fulfilled, then we can apply its corollary to v. Hence $v = 0$ and it follows that $u = 0$ by injectivity. $\square$

Remark 5.16. This version of Holmgren's theorem can be named the abstract version of this theorem in the same way that the Cauchy-Kovalevska theorem. In the following, we will show the classical version of Holmgren's theorem, and use it and the notion of characteristics to state a geometric version of it.

Let $\Omega \subset \mathbb{R}^n$ be an open subset. We consider the following partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j(x,t)\frac{\partial u}{\partial x^j} + A_0(x,t)u \tag{5.17}$$

where $A_j \in M_m(\mathscr{O}(\Omega \times]-T,T[)$.

Theorem 5.18. *There is an open neighborhood $\Omega \times \{0\} \subset U \subset \Omega \times]-T,T[$ such that every distribution u in $\Omega \times]-T,T[$, satisfying (5.17) and vanishing for $t < 0$, must also vanish in U.*

Proof. As usual in these notes, it is sufficient to prove that u is vanishing in a neighborhood of every point $(x_0, 0) \in \Omega \times \{0\}$ independent of u. Without loss of generality, we can suppose that $x_0 = 0$ and Ω is a neighborhood of the origin. We set the following changement of variables :

$$y := x \quad , \quad s := t + |x|^2$$

We have then

$$\partial_{x^j} = \partial_{y^j} + 2y^j\partial_s \quad , \quad \partial_t = \partial_s$$

which implies

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial t} \\ &= \sum_{j=1}^{n} A_j(x,t)\frac{\partial u}{\partial x^j} + A_0(x,t)u \\ &= \sum_{j=1}^{n} A_j(x,t)\frac{\partial u}{\partial y^j} + \sum_{j=1}^{n} 2y^j A_j(x,t)\frac{\partial u}{\partial s} + A_0(x,t)u \end{aligned}$$

Then we have an equation

$$M(y,s)\frac{\partial u}{\partial s} = \sum_{j=1}^{n} B_j(y,s)\frac{\partial u}{\partial y^j} + B_0(y,s)$$

where $M(y,s) = 1 - 2\sum_j y^j B_j(y,s)$ and $B_j(y,s) = A_j(y, s-|y|^2)$. As $M(0,s) = 1$, we can suppose that if Ω is small enough and $y \in \Omega$, M is invertible with an analytic inverse. Then (5.17) is equivalent to the equation

$$\frac{\partial u}{\partial s} = \sum_{j=1}^{n} C_j(y,s)\frac{\partial u}{\partial y^j} + C_0(y,s)$$

where $C_j = M^{-1}B_j$, and u vanishing for $t < 0$ is equivalent to u vanishing for $s < |y|^2$. Hence our problem is the same if we suppose $u = 0$ for $t < |x|^2$ which implies

$$\operatorname{supp} u \subset \{(x,t) \in \Omega \times]-T,T[| t \geq |x|^2\}$$

If T is small enough, then the x-projection of $\operatorname{supp} u$ can be assumed to be contained in a compact subset $K \subset \Omega$. Contracting Ω, it can be assumed to be contained in a polydisk Ω_0

as in Theorem 5.15, such that the A_j can be extended as holomorphic functions to an open neighborhood of

$$\{(z,t) \in \mathbb{C}^{n+1} | z \in \bar{\Omega}_0, |t| \leq T\}$$

Now, applying Theorem 5.3, we conclude that $u = 0$ in $\Omega\times] - T, T[$ (our restrictions on this set during the proof were independent of u as wanted). □

Corollary 5.19. *Let u be a C^1 function satisfying (5.17) in $\Omega\times] - T, T[$. If for all $x \in \Omega$, $u(x, 0) = 0$, then u vanishes in a neighborhood of $\Omega \times \{0\}$.*

Proof. We denote H the Heaviside's function and consider $\tilde{u} : (x, t) \mapsto H(t)u(x, t)$. For any test function φ, we have

$$\begin{aligned}
\langle \partial_t \tilde{u}, \varphi \rangle &= -\langle Hu, \partial_t \varphi \rangle \\
&= -\int_\Omega \int_0^T u(x,t) \partial_t \varphi dt dx \\
&= \int_\Omega \int_0^T \partial_t u \varphi dt + [u\varphi]_0^T dx \\
&= \langle H \partial_t u, \varphi \rangle
\end{aligned}$$

Hence $\tilde{u}$ satisfies (5.17) and vanishes for $t < 0$. By applying Theorem 5.18, $\tilde{u} = 0$ in a neighborhood $U \subset \Omega \times \{0\}$ i.e. $u = 0$ in $U \cap \Omega \times \mathbb{R}^+$. We obtain the part in $\Omega \times \mathbb{R}^-$ with the same argument using $1 - H$. □

Remark 5.20. The last statement is what it is called the 'classical' version of Holmgren's theorem. To conclude those notes, we will use the notion of characteristics to state an 'invariant' version of Holmgren's theorem, in the same way of Cauchy-Kovalevska theorem.

Theorem 5.21. *Let $P(y, D_y)$ be a differential operator of order m in an open subset $\mathscr{U} \subset \mathbb{R}^N$. Suppose that the coefficients of $P(y, D_y)$ are analytic in $\mathscr{U}$. Let Σ be a C^1 hypersurface in $\mathscr{U}$ such that*

$$\mathscr{U} = \mathscr{U}^+ \sqcup \Sigma \sqcup \mathscr{U}^-$$

and Σ is nowhere characteristic with respect to $P(y, D_y)$.

Then there exists a neighborhood $\Sigma \subset \mathscr{N} \subset \mathscr{U}$ such that every distribution u in $\mathscr{U}$, satisfying $P(y, D_y)u = 0$ and vanishing in $\mathscr{U}^-$, vanishes in $\mathscr{N}$.

Proof. It is sufficient to prove it in a neighborhood of every point of Σ which can be assumed to be the origin in $\mathbb{R}^N$. Let φ be a C^1 function in a neighborhood $0 \in \mathscr{V} \subset \mathscr{U}$ such that $\Sigma \cap \mathscr{V} = \varphi^{-1}(0)$ and ∇_φ does not vanish in $\mathscr{V}$. We can suppose chosen coordinates $y = (y^1, \ldots, y^N)$ such that

$$\varphi(y) = y^N + \circ(|y|)$$

in $\mathscr{V}$. We set $y' := (y^1, \ldots, y^{N-1})$ and we have $\delta > 0$ such that

$$\mathscr{V}^+ := \mathscr{V} \cap \mathscr{U}^+ = \{y \in \mathscr{V} | \varphi(y) > 0\} \subset \{y \in \mathscr{V} | y^N > -\delta |y'|\}$$

Let $\epsilon > 0$ and

$$\mathscr{W}_\epsilon := \{y \in \mathbb{R}^N | |y'| < 2\epsilon\delta \ , \ |y^N| < 4\epsilon^2\delta^2\}$$

If ϵ is small enough, then $\mathscr{W}_\epsilon \subset \mathscr{V}$ and we set

$$\psi(y) := y^N + \frac{1}{\epsilon}|y'|^2$$

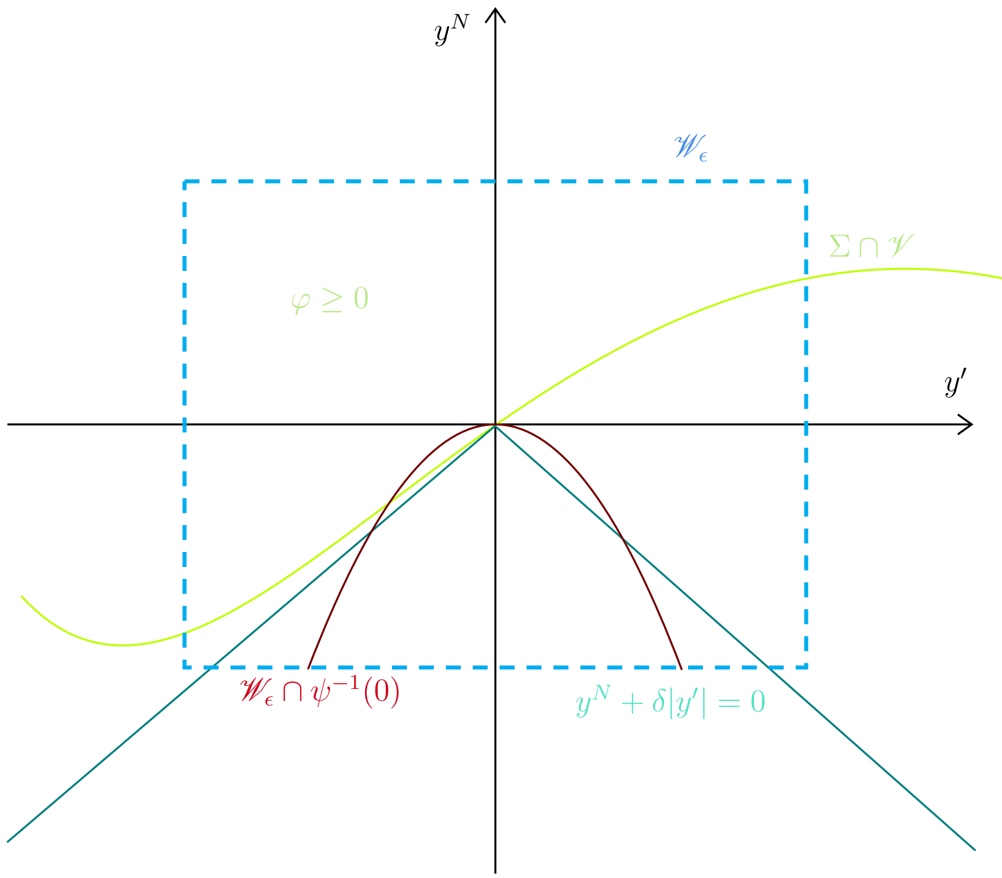
and

$$T_\epsilon := \psi^{-1}(0) \cap \mathscr{W}_\epsilon$$

(1)
$$|\nabla_\psi - (0, \dots, 0, 1)| = |\frac{2}{\epsilon} y'| < 4\delta$$
in $\mathscr{W}_\epsilon$. Moreover $P_m(0, (0, \dots, 0, 1)) \neq 0$, hence, assuming that δ is small enough, $P_m(y, \nabla_\psi)$ does not vanish in $\mathscr{W}_\epsilon$, and in particular T_ϵ is nowhere characteristic.

(2) If $y \in \partial T_\epsilon$, then $|y'| = 2\epsilon\delta$ and $|y^N| = 4\delta^2\epsilon^2$, which implies
$$y^N = -\frac{1}{\epsilon}|y'|^2 = -4\epsilon\delta^2 = -2\delta|y'| < -\delta|y'|$$
Hence if $y \in \partial T_\epsilon$, then $y^N < -\delta|y'|$.

(3) $y \mapsto (x, t) = (y', \psi(y))$ defines a diffeomorphism of $\mathscr{W}_\epsilon$ onto $\mathscr{W}$ a neighborhood of the origin in $\mathbb{R}^N$.



Trough the last change of variables, the operator becomes

$$P(x, t, D_x, D_t) = a(x, t)D_t^m + \sum_{j=1}^{m} Q_j(x, t, D_x)D_t^{m-j}$$

where a and the coefficients of the Q_j are analytic functions in a neighborhood of the origin, and Q_j is of order $\leq j$. Moreover, a does not vanish in a neighborhood of the origin.

Let u be a distribution in $\mathscr{W}_\epsilon$ satisfying $P(y, D_y)u = 0$ and vanishing in $\mathscr{W}_\epsilon \cap \mathscr{V}^- = \{y \in \mathscr{W}_\epsilon | \varphi(y) < 0\}$. After the change of variables, we have $P(x, t, D_x, D_t)u = 0$ in $\mathscr{W}$ of the form

$$\mathscr{W} = \{(x,t) \in \mathbb{R}^N | |x| < r\ ,\ -t_0 < t < t_1\}$$

and there is $0 < r' < r$ such that $u(x,t) = 0$ when $|x| > r'$ and there is $0 < t_2 < t_0$ such that $u(x,t) = 0$ when $-t_0 < t < -t_2$. We know that we can transform the equation $P(x,t,D_x,D_t)u = 0$ into a first order-system. We can now apply the Holmgren theorem to conclude that $u = 0$ in the neighborhood $\mathscr{W}$.

□