An introduction to Uniform Rectifiability

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Abstract

This text is a set of notes on a series of three lectures that were given at a Westlake University winter school in February 2026. The plan is to give general background on the notion of uniform rectifiability, and insist on one important technique, the corona construction.

Contents

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1 Introduction and first definitions

The notion of uniform rectifiability roughly dates from the early 1990's, and a good part of the material in these notes will roughly date from this period, although some recent applications and connections will be mentioned here and there.

The author (also denoted by I in the sequel) will try to explain the basic notions and the simplest or most useful results, often with just a hint concerning the method of proof, and then will concentrate on advertising for a method of proof, called the corona construction, which has been very successfull in the subject. I will try to be reasonably clear, but the reader should know that this text is full of simplifications, possibly lies, mostly designed not to overwhelm them with notations.

Some motivations will come along the way (subsection 1.5 and later), but we start with the main definitions.

1.1 Rectifiable sets

For us a measurable set E (and in fact all our sets will be Borel subsets of $\mathbb{R}^n$) with σ-finite Hausdorff measure $\mathcal{H}^d$, will be called d-rectifiable (or just rectifiable) when we can write

$$E \subset Z \cup \big(\cup_{j \in J} A_j \big), \tag{1.1}$$

where Z is $\mathcal{H}^d$-negligible, i.e., $\mathcal{H}^d(Z) = 0$, J is (at most) countable, and each A_j, $j \in J$, is a C^1 surface of dimension d.

Here $d \in [0, n]$ is an integer (we don't define d-rectifiable otherwise). Of course "countable" is important here (as E is always the union of its elements). There are slight variants of this definition, but this will not matter here. Also, sets of σ-finite Hausdorff measure (i.e., countable unions of sets of finite measure) will be enough for us. Recall that we would have obtained an equivalent definition by requiring only the A_j to be (images under rotations of) Lipschitz graphs, or even Lipschitz images of subsets of $\mathbb{R}^d$, because one can almost-cover C^1 surfaces by such sets.

There is an opposite notion of "total unrectifiability", which characterizes sets E such that $\mathcal{H}^d(E \cap A) = 0$ for all C^1 surfaces (or equivalently all d-rectifiable sets) A.

And there are beautiful characterizations of the rectifiability of E by the existence of an **approximate tangent d-plane** at $\mathcal{H}^d$-almost every point of E, or by the positivity of $\mathcal{H}^d(\pi(E))$, where π is the **orthogonal projection** onto a plane P, for a set positive measure of d-planes P (and in fact, for all directions but at most one), and also by the fact that at $\mathcal{H}^d$-almost every point of E, E has **the same density as a d-plane**. Some of our characterizations of uniform rectifiability will of course correspond to these results. But we will first give definitions that make (1.1) more quantitative.

1.2 Ahlfors regular sets and measures

We will make our life simpler by assuming that all our sets E are Ahlfors regular of some dimension d. Here d does not need to be an integer (and it is not hard to see that $d \leq n$

because $E \subset \mathbb{R}^n$). We say that $E \subset \mathbb{R}^n$ is an **Ahlfors regular set of dimension** d (in short, $E \in AR(d)$) when E **is closed** and

$$C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C r^d \quad \text{for } x \in E \text{ and } 0 < r < \operatorname{diam}(E). \tag{1.2}$$

We phrased the condition so that E is allowed to be bounded or unbounded. And of course it is important that the Ahlfors constant C may depend on E, but not on x or r.

An **Ahlfors regular measure** of dimension d is a positive measure μ such that

$$C^{-1} r^d \leq \mu(E \cap B(x,r)) \leq C r^d \quad \text{for } x \in E \text{ and } 0 < r < \operatorname{diam}(E), \tag{1.3}$$

where E is the closed support of μ. In short, we write $\mu \in AR(d)$.

It is easy to check that then the support E of μ is Ahlfors regular, and that $C^{-1}\mathcal{H}^d_{|E} \leq \mu \leq C\mathcal{H}^d_{|E}$ (cover $E \cap B(x,r)$ by balls of radius ηr, η small, and use both sides of (1.3) to count the balls). Thus AR sets are just the supports of AR measures, but it may be convenient to use AR measures on E which are not exactly Hausdorff measures.

We decided to restrict to AR sets in these notes, but of course it happens that one has to deal with non AR, and not even doubling, measures. See [DM, NTrV, Hy, To5, To6] and others. There are ways to do so, but often at the price of technical complications that we shall avoid here.

In spite of its name (coming from a paper of Ahlfors [Ah] on covering maps), AR is about size, not regularity.

A small advantage of AR sets is that the notion of approximate tangent simplifies for them: is is easy to check that if $E \in AR(d)$ has an approximate tangent d-plane P at $x \in E$ (we refer to [Ma], for instance, for the definition), then P is a tangent plane to E at x in the usual acception (no density needed). See Figure 1.

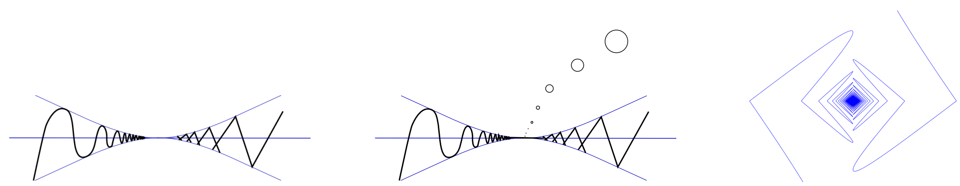

Figure 1: Left: A true tangent plane. Middle: An approximate tangent line (but $E \notin AR(1)$). Right: This set (drawn by M. Badger) is not $AR(1)$: too much length in the middle.

1.3 Uniformly rectifiable sets (UR) of dimension 1

We start with one-dimensional uniformly rectifiable sets because their definition is simpler.

Definition 1.1. *We call* **regular parameterization** *a function* $z : I \to \mathbb{R}^n$ *which is* 1-*Lipschitz, i.e., such that* $|z(x) - z(y)| \leq |x - y|$ *for* x, y *in the (nontrivial) interval* $I \subset \mathbb{R}$, *and*

$$\left|z^{-1}(B(x,r))\right| \leq Cr \quad \textit{for } x \in \mathbb{R}^n \textit{ and } r > 0. \tag{1.4}$$

Then $E = z(I)$ *is called an* **Ahlfors regular curve**.

We could also have required z to be a parameterization of the curve $\Gamma = z(I)$ by arclength, i.e., that $|z'(t)| = 1$ for almost every $t \in I$. This is because, if we make z run at maximal speed, then the sets $z^{-1}(B(x,r))$ is smaller (and (1.4) holds even more). And if z is a parameterization by arclength, then (1.4) measures the total length of the pieces of curve inside of $B(x,r)$. Note that we do not require z to be injective; it may even run through the same piece of curve twice (and then this piece is counted twice in (1.4)).

Also observe that the set $\Gamma = z(I)$ is Ahlfors regular: the upper bound comes from (1.4) (and generalities on lengths of rectifiable curves, see for instance [Fa]), and the lower bound comes from the fact that Γ is connected. Indeed, for $x \in \Gamma$ and $0 < r < \mathrm{diam}(\Gamma)$, we can find $y \in \Gamma$ such that $|y - x| \geq r/2$, and then a connected piece of $\Gamma \cap \overline{B}(x, r/2)$ between x and a point of $z \in \partial B(x, r/2)$, and the projection of $\Gamma \cap \overline{B}(x, r/2)$ on a line with direction $z - x$ is such that $\mathcal{H}^1(\pi(\Gamma \cap \overline{B}(x, r/2))) \geq \mathcal{H}^1(\pi([x,z])) = r/2$, so $\mathcal{H}^1(\Gamma \cap \overline{B}(x, r/2)) \geq \mathcal{H}^1(\pi(\Gamma \cap \overline{B}(x, r/2)) = r/2$ because π is 1-Lipschitz. So the name of Ahlfors regular curve is not shocking.

Lines, Lipschitz curves, and chord-arc curves if you like them, are obviously Ahlfors regular. But the union X of two orthogonal lines through 0 is not, because a parameterization of X by $\mathbb{R}$ would have to go through 0 infinitely many times, which would violate (1.4). Yet this is only a minor difficulty, if we allow $z(I)$ to get out of X from time to time, and it is easy to prove that every **connected Ahlfors regular set** (of dimension 1) is contained in a regular curve. This relies on a simple lemma that says that any connected finite graph can be parameterized, with a parameterization that runs exactly twice along each edge; see for instance Lemma 30.3 in [Da6].

Definition 1.2. *The set* $E \in AR(1)$ *is called* **uniformly rectifiable** *(in short,* $E \in UR(1)$*) when there is an Ahlfors regular curve* Γ *(or, equivalently, by the discussion above, a connected set* $\Gamma \in AR(1)$*) such that* $E \subset \Gamma$.

We like this because parameterizations always seem simpler (and in this case, they are). But I would argue that the definition by AR connected sets is better. And we should observe that for sets of $AR(1)$, connectedness is a strong condition; this is no longer true for sets of larger dimensions.

The Garnett-Ivanov Cantor set K depicted in Figure 2 (left) is probably the most emblematic totally unrectifiable Ahlfors regular set of dimension 1 (due to the length ratio of 1/4 between consecutive generations). The reader will very easily find all the relevant information concerning this; let me just mention that the AR property is easy because there is a natural probability measure on K, which is easily seen to be AR; we leave it as an exercise to check that K is unrectifiable using the different characterizations by (approximate) tangent planes, projections, and density. The name with Garnett and Ivanov comes from the fact that they showed that this set has (positive $\mathcal{H}^1$-measure but) vanishing analytic capacity; see [Ga]. Also, and this is related, the Cauchy integral does not define a bounded operator on $L^2(d\mu)$, where μ is the natural probability measure on K. I added a triangular version of K on the right of K, just to insist that K is not so special: it is just simpler.

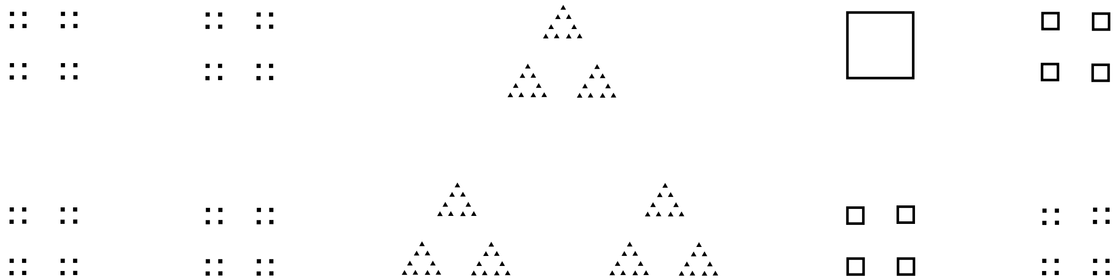

Figure 2: Left: Four generations of the construction of the 1/4 Cantor set K (one should continue and take the intersection). Middle: same thing for a triangular shape. Right: a partially stopped 1/4 Cantor set which is rectifiable, but not UR.

The picture on the right depicts (modulo the right bottom square where more detail needs to be added) a set which is rectifiable (simply because it is a countable union of squares) but is not uniformly rectifiable, at least if when we continue the picture longer than shown here, we make sure that for all $k \geq 1$, there is at least one square of the picture (somewhere very near the bottom right) where we can see k full generations (with 4^k squares) of the construction of K.

1.4 UR sets, definition by Big Pieces of Lipschitz Images

We now give a definition of uniform rectifiability (for any integer d) which is as simple as possible. Of course there will be other (equivalent) ones, often easier to use.

Definition 1.3. *Let $d \geq 1$ be an integer. We say that the set $E \in AR(d)$* **contains big pieces of Lipschitz images of $\mathbb{R}^d$,** *(in short $E \in BPLI(d)$) when there exist constants $\theta > 0$ and $M \geq 1$ such that, for $x \in E$ and $0 < r < \mathrm{diam}(E)$, we can find an M-Lipschitz mapping $\Phi : rQ_0^d \to \mathbb{R}^n$ such that*

$$\mathcal{H}^d(E \cap B(x,r) \cap \Phi(rQ_0^d)) \geq \theta r^d. \tag{1.5}$$

Here Q_0^d denotes the unit cube in $\mathbb{R}^d$.

And we say that that E is **uniformly rectifiable** *(in short $E \in UR(d)$) when ($E \in AR(d)$ and) $E \in BPLI(d)$.*

Obviously this is a uniform version of the fact that E is rectifiable when, up to a negligible set, E is covered by a countable union of images of $\mathbb{R}^d$ (or cubes of $\mathbb{R}^d$) by Lipschitz maps. Notice also that our sets are closed, which makes sense because we no longer want (nor can afford) our notion to be stable under countable unions. Similarly, we don't add negligible sets; this would not fit with AR.

Let us mention here an apparently stronger notion that uses parameterizations. We do this because the corresponding definition showed up earlier (in [Da2]) and looks natural, but this will also be an opportunity to criticize definitions by parameterizations. We start with a generalization of the regular parameterizations of Definition 1.1.

Definition 1.4. *Let ω be an A_1 weight in $\mathbb{R}^d$. This means that ω is a positive locally integrable function on $\mathbb{R}^d$ such that, for some constant $C \geq 1$,*

$$\frac{1}{B}\int_B \omega(x)dx \leq C \inf_{x\in B} \omega(x) \tag{1.6}$$

for every ball $B \subset \mathbb{R}^d$. [Here the infimum should probably be an essential infimum]. We say that $z : \mathbb{R}^d \to \mathbb{R}^n$ **is an ω-regular mapping** *when there is $C \geq 1$ such that*

$$|z(x) - z(y)| \leq C \int_{B((x+y)/2,|x-y|)} \omega(u)^{1/d}\, du \quad \text{for } x, y \in \mathbb{R}^d \tag{1.7}$$

and, for $x \in \mathbb{R}^n$ and $r > 0$,

$$\left|z^{-1}(B(x,r))\right|_{\omega du} := \int_{u\in\mathbb{R}^d; z(u)\in B(x,r)} \omega(u)du \leq Cr^d. \tag{1.8}$$

We say that $E \in AR(d)$ **is an ω-regular set** *when there is a weight $\omega \in A_1(\mathbb{R}^d)$ and an ω-regular mapping $z : \mathbb{R}^d \to \mathbb{R}^n$ such that $E = z(\mathbb{R}^d)$.*

The simplest generalization of Definition 1.1 would have been to take $\omega = 1$, but this will not be general enough for us (and in general we won't be able to re-parameterize). In this respect, allowing $\omega \in A_1$ is the next best thing. One can still check easily that $E = z(\mathbb{R}^d)$ is Ahlfors regular. And also (a little harder because a maximal function argument is needed) that $E \in UR$, because it contains big pieces of Lipschitz images of $\mathbb{R}^d$. It turns out that the converse is true, modulo a minor twist.

Theorem 1.5. *[DS1] Let $E \subset \mathbb{R}^n$ be a uniformly rectifiable set of dimension d. Set $n^* = \min(n+1, 2d)$. Then there is a weight $\omega \in A_1(\mathbb{R}^d)$ and an ω-regular mapping $z : \mathbb{R}^d \to \mathbb{R}^{n^*}$ such that $E \subset z(\mathbb{R}^d)$.*

The reason for n^* is that during the proof, we need to make sure that the pieces of surfaces that we add (so that we have a parameterization by the whole $\mathbb{R}^d$) do not cross each other too much. This looks like a very nice result, because parameterizations are often hard to construct, but this does not make it so easy to apply. The notion of ω-regular mapping and surfaces was introduced in [Da2], in order to prove the L^2-boundedness of singular integral operators on E through the existence of big pieces of Lipschitz graphs (BPLG) in E (see Definition 2.9) but even establishing BPLG in this case was not as easy as it should have been.

We shall discuss the proof of Theorem 1.5 later in Subsection 3.7, to show the strenght of the corona construction.

1.5 Why should we care?

We end this introduction with a rapid discussion of motivations for uniform rectifiability. Here and below, I try to give some references, because it is better than nothing, but no to be exhaustive: I will certainly forget many relevant results.

• **1**. The main one, at the beginning of the story, was to decide for which $E \subset \mathbb{R}^n$ it is true that the usual **singular integrals** define bounded operators on $L^2(E, d\mu)$, where μ is the relevant surface measure on E.

The emblematic example is when $n = 2$, $d = 1$, and the singular integral is given by the Cauchy kernel. In this case the L^2 boundedness on Lipschitz graphs was proved by Coifman, Mc Intosh, and Meyer [CMM] after numerous efforts, including the proof by A. Calderón in the special case of Lipschitz graphs with small constants [Cal]. A simple averaging trick (known as the rotation method) allows one to deduce from [CMM] the case of d-dimensional Lipschitz graphs in R^n, when (for instance) the singular integral is given by any of the Riesz kernel coordinates

$$K_{i,d}(u) = u_i |u|^{-d-1} \quad \text{for } u \in \mathbb{R}^n \setminus \{0\}, \tag{1.9}$$

$1 \leq i \leq n$. Let us be a little more specific. Given the integers $1 \leq d < n$ and any odd kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that

$$|K(u)| \leq C|u|^{-d} \text{ and } |\nabla K(u)| \leq C|u|^{-d-1}, \quad \text{for } u \in \mathbb{R}^n \setminus \{0\}, \tag{1.10}$$

we consider d-dimensional sets E equipped with a measure μ, and the simplest is to take them Ahlfors regular and equipped with an Ahlfors regular measure μ, and we ask whether a formula like

$$Tf(x) = \int_E K(x-y) f(y) d\mu(y) \tag{1.11}$$

allows one to define a bounded linear operator on $L^2(E, d\mu)$. In fact the integral in (1.11) does not converge in general, so we require instead that the truncated operators T_ε defined by

$$T_\varepsilon f(x) = \int_{E \setminus B(x,\varepsilon)} K(x-y) f(y) d\mu(y) \quad \text{for } x \in E \tag{1.12}$$

be uniformly bounded in L^2, and if this is the case, we can replace (1.11) with a principal value integral or some other limiting process. We said earlier that by [CMM], the answer is yes when $n = 2$ and $d = 1$, $K(x, y) = (x + iy)^{-1}$ is the Cauchy kernel, and E is a Lipschitz graph. And in fact, the answer is also yes for Lipschitz graphs of dimension d in $\mathbb{R}^n$ when the kernel is the (vector-valued) Riesz kernel above. Essentially by the averaging trick mentioned above, this extends to a much larger (and fairly natural) class $\mathcal{K}$ of kernels that satisfy (1.10). And for this class it was observed in [Da2] that the result stays true when E is an ω-regular surface.

The question was then to find all the Ahlfors-rectifiable sets E for which all the K in the class $\mathcal{K}$ above define bounded operators. And it turns out that the answer is given by the class of subsets of ω-regular surfaces (i.e., UR sets) [DS1].

But this was only partially satisfactory: a more interesting question is whether the L^2 boundedness of the Cauchy integral operator alone (when $d = 1$ and $n = 2$), or of the Riesz transforms alone (in the general case), is enough to imply that E (assumed to be Ahlfors regular) is uniformly rectifiable. And a positive answer was given by two spectacular results. The first one is [MMV], for 1-dimensional sets in the plane and the Cauchy kernel. This relies on the discovery of a relation between the Menger curvature of three points and the Cauchy kernel, and it had many applications to bounded harmonic functions and analytic capacity. The second one, in [NToV], solves the problem when $d = n-1$, and had important applications to harmonic measures on rough domains. The case of intermediate dimensions is still undecided. We refer to (other articles in this volume and) the book [To6] for details and complements.

• **2**. **Geometry and analysis** directly on $E \in UR$. A good motivation for studying uniform rectifiablilty as such was to convince ourselves that we had the right notion. So we decided to prove that as many natural definitions as possible give the same class of uniformly rectifiable sets. In this respect we did quite well: see the long list of acronyms in [DS3], for instance. We shall insist here on conditions based on flatness, but we also tried (with a little less success) conditions related to density, or projections (rapidly discussed below too, because of recent progress), or even properties of Lipschitz functions on E or layer potentials defined from E.

• **3**. There are instances where some **sets of interest** (such as the singular sets of minimizers of the Mumford-Shah functional, or some free boundaries, or sets that almost minimize $\mathcal{H}^d(E)$ under given boundary constraints) turned out to be uniformly rectifiable. In this case, the tools developed for UR were useful, but these may be less striking examples, because UR is just a first step, and some additional regularity is often expected from those sets.

• **4**. It turns out that (uniform) rectifiability is a key condition when we study the regularity of the **harmonic measure** ω on domains Ω such that $\partial\Omega$ is of co-dimension 1, and more precisely the (mutual) absolute continuity of ω with respect to the surface measure $\sigma = \mathcal{H}^{-1}_{|\partial\Omega}$. In the direct direction, the A_∞ mutual absolute continuity of ω and σ was proved in [Dah] for Lipschitz domains, in [DJ] and [Se2] in the case of one-sided NTA domains with large balls in the complement (and incidentally the proof in [Se2] use a corona decomposition, as far as I know for the first time in the context of harmonic measure), and finally in [AHMNT, HM, HMU], for general one-sided NTA domains with UR boundaries. It turned out that, thanks to the Nazarov-Tolsa-Volberg result on the Riesz transforms [NToV], the converse is also true; see [AHM3TV].

Thanks are due to the organizers of the winter school and Westlake university for the friendly welcome, and Inkscape, which I used for the pictures.

2 Carleson measures and Flatness conditions

We turn to (geometric) necessary and sufficient conditions for UR. Most of these will be written in terms of Carleson measures, so we first discuss Carleson measures on the set of balls.

2.1 Carleson measures on $\mathcal{E} = E \times (0, \mathrm{diam}(E))$

Let $E \in AR(d)$ be given. We will use the set

$$\mathcal{E} = E \times (0, \mathrm{diam}(E)), \tag{2.1}$$

which we think of as the set of balls centered on E. We will measure flatness, for instance by the size of quantities like $\beta(x,r), (x,r) \in \mathcal{E}$, measured as follows.

Definition 2.1. *Let $E \in AR(d)$ and $\mathcal{E}$ be as above. A Carleson measure on $\mathcal{E}$ is a measure ν on $\mathcal{E}$ such that*

$$\nu(B(X,R) \times (0,R) \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \mathrm{diam}(E). \tag{2.2}$$

The smallest constant C that works for the above will be denoted by $||\nu||_{CM}$.

Carleson measures are not to be feared: the condition above is the logical condition that is invariant under translations and dilations (with the right homogeneity), and where the "same" condition is required from all balls of $\mathcal{E}$. It is also known that Carleson measure speak well with non-tangential maximal functions (with balls centered on E), but we shall not directly use this fact here.

Observe that the "invariant" measure $d\nu = \frac{d\mu dr}{r}$ is **not** a Carleson measure; in fact $\nu(B(X,R)\times(0,R) = +\infty$ because $\int \frac{dr}{r}$ diverges at 0. However, for instance, $d\nu = \mathbb{1}_{r_0 \leq r \leq Mr_0} \frac{d\mu dr}{r}$ is a Carleson measure for every $M \geq 1$, with norm $||\nu||_{CM} \leq C\log(M)$. Often our measures will be of the form $f(x,r)^2 \dfrac{d\mu dr}{r}$, whence the following definition.

Definition 2.2. *Let $E \in AR(d)$ and $\mathcal{E}$ be as above. We say that the function $f : \mathcal{E} \to \mathbb{R}$* **satisfies a square Carleson condition** *when $f(x,r)^2 \dfrac{d\mu(x)dr}{r}$ is a Carleson measure on $\mathcal{E}$. That is, when*

$$\int_{B(X,R)\times(0,R)} f(x,r)^2 \frac{d\mu(x)dr}{r} \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \mathrm{diam}(E). \tag{2.3}$$

This is a way to say that f is small most of the time, since $\frac{d\mu(x)dr}{r}$ is locally infinite. Often the square comes from a form of orthogonality somewhere.

2.2 Flatness, the P. Jones β-numbers, and the geometric lemma

Recall that rectifiable d-sets E have approximate tangent d-planes almost everywhere, and that for $E \in AR(d)$, approximate tangent planes are automatically tangent planes (an exercice on densities); see Figure 1 above. We want to measure this, i.e., how close E is to d-planes.

Let $E \in AR(d)$ be given. For $x \in E$ and $0 < r < \mathrm{diam}(E)$, define the Peter Jones number

$$\beta(x,r) = \beta_{E,\infty}(x,r) = \inf_P \Big\{\frac{1}{r} \sup_{y \in E \cap B(x,r)} \mathrm{dist}(y,r)\Big\}, \tag{2.4}$$

where the infimum is taken over all the affine d-planes (and it is enough to consider those that meet $B(x,r)$). Thus, by normalization, $0 \leq \beta(x,r) \leq 1$. Also define the L^q version, $1 \leq q < +\infty$, by

$$\beta_q(x,r) = \beta_{E,q}(x,r) = \inf_P \Big\{ r^{-d} \int_{y \in E \cap B(x,r)} (r^{-1}\,\mathrm{dist}(y,r))^q d\mu(y) \Big\}^{1/q}, \tag{2.5}$$

where μ is a (given) AR measure on E, so that $r^{-d} \sim \mu(B(x,r)^{-1}$, thus the normalisation is still such that $0 \leq \beta_q(x,r) \leq C$. We do not require P to go through x, but if I recall correctly, but if we did not, we would have an equivalent definition, in the sense that $\beta_q(x,r)$ could be much larger at some points, but (as it turns out) essentially the same on average. It is easy to deduce from Hölder's inequality that

$$\beta_p(x,r) \leq C\beta_k(x,r) \ \text{ for } 1 \leq p < q \leq +\infty \tag{2.6}$$

(where C depends on the AR constant). These numbers are no so different when $E \in AR(d)$, because it is not hard to check that

$$\beta_\infty(x,2r) \leq C(q)\, \beta_q(x,r)^{\frac{q}{d+q}} \ \text{ for } 1 \leq q < +\infty \tag{2.7}$$

(with a constant $C(q)$ that depends also on the AR constant for E; see (1.73) in [DS3].

We need the L^q versions because (for $d \geq 2$), the L^∞-norm is a little too precise to be controlled nicely by the uniform rectifiability, already in the case of Lipschitz graphs. Let us give the logical quantitative definition that goes with these numbers.

Definition 2.3. *We say that $E \in AR(d)$* **satisfies the geometric lemma** *with exponent $q \in [1,+\infty]$ (in short, $E \in LG(q)$) when the coefficients $\beta(x,r)$ satisfy a square Carleson condition. That is, when*

$$\int_{B(X,R)\times(0,R)} \beta_q(x,r)^2 \frac{d\mu(x)dr}{r} \leq CR^d \ \text{ for } X \in E \text{ and } 0 < R < \mathrm{diam}(E). \tag{2.8}$$

The name comes from a paper of P. Jones [Jo1] where he introduced the $\beta(x,r)$ to control the Cauchy operator in terms of the geometry, and where a "geometric lemma" was used to estimate them when E is a Lipschitz curve. Jones [Jo2] and [Ok] then characterized sets in $\mathbb{R}^n$ that are contained in a curve with finite length, in terms of square summability of (the discretized variant of) the β numbers. We will discuss the strength of this condition, compared to other ones to come, in the upcoming subsection on snow flakes. The relation with UR is as follows.

Theorem 2.4. *Let $E \in AR(d)$. If $E \in GL(q)$ for any $q \geq 1$, then $E \in UR$. Conversely if $E \in UR$ and $d = 1$, then $E \in GL(q)$ for every $q \in [1, +\infty]$, while, when $d \geq 2$, if $E \in UR$, then $E \in GL(q)$ for every $q \in [1, \frac{2d}{d-2})$.*

When $d \geq 2$ and $E \in UR$, the $GL(q)$ usually fails for $q \geq \frac{d}{d-2}$; this is related to Sobolev inclusions, there is an example of Fang [Fa] to this effect, with lots of little vertical tents.

For the fact that $E \in GL(q)$ when E is a Lipschitz graph, the result is about approximation of Lipschitz functions by affine function, and the proof is by Littlwood-Paley theory: write down a reasonable affine approximation $a_{x,r}$ on each $B(x, r)$ (choose the average and slopes by convolution with bump functions), and then use the Fourier transform to estimate a multiple integral. This is a very special case of a theorem of Dorronsoro [Do]. For the general case, the proof is a beautiful example of the corona construction; we will discuss this later.

Since then, many interesting results generalized aspects of Theorem 2.4 (and the initial results of Jones and Okikiolu) in various metric spaces, and in particular Carnot groups where even getting the right definition is a challlenge.

2.3 Joint flatness and density (Tolsa's α-numbers)

Let μ be an AR measure on $E \in AR(d)$, d integer. We want to say that when $E \in UR$, not only E is often close to a d-plane P in $B(x, r)$, but also μ looks like a multiple of the Lebesgue measure on P. For this we will use a Wasserstein distance (and the duality with the Lipschitz functions) to measure the distance. Denote by

$$\mathcal{F} = \left\{\lambda \mathcal{H}^d_{|P}\,;\, \lambda > 0 \text{ and } P \text{ is an affine } d\text{-plane}\right\} \tag{2.9}$$

the set of (nontrivial) flat measures (i.e., positive multiples of Lebesgue measures on d-plane). Then set

$$\alpha(x, r) = \inf_{\nu \in \mathcal{F}} d^W_{x,r}(\mu, \nu), \tag{2.10}$$

where we use the local Wasserstein distance $d^W_{x,r}$ defined by

$$d^W_{x,r}(\mu, \nu) = r^{-d-1} \sup_{\varphi} \Big| \int_{B(x,r)} \varphi d\mu - \int_{B(x,r)} \varphi d\nu \Big|, \tag{2.11}$$

where the supremum is over 1-Lipschitz functions φ supported in $B(x, r)$. Thus $\alpha(x, r) \leq C$ and α has the usual scale invariance. It is not hard to see that $\beta_1(x, r) \leq C\alpha(x, 2r)$ (exercise: prove this; try $\varphi(y) = \theta(y) \operatorname{dist}(y, P)$, with a bump function θ). Then $\alpha(x, 2r)$ also controls a power of $\beta_q(x, r)$ for each $q \leq +\infty$, either directly or by (2.7), so altogether the α tends to be more precise than the β.

Theorem 2.5. *[Thm 1.2 in [To2]] Let $E \in AR(d)$, d integer. Then $E \in UR(d)$ if and only if $(x, r) \mapsto \alpha(x, r)$ satisfies a square Carleson condition. That is, when*

$$\int_{B(X,R)\times(0,R)} \alpha(x, r)^2 \, \frac{d\mu(x)dr}{r} \leq CR^d \quad \textit{for } X \in E \textit{ and } 0 < R < \operatorname{diam}(E). \tag{2.12}$$

A special case of this is when $E = \mathbb{R}^d$ and $\mu(x) = f(x)dx$, for a density f such that $C^{-1} \leq f(x) \leq C$ on $\mathbb{R}^d$. In this case the theorem gives some control on the numbers $\fint_{B(x,r)} |f(y) - m_{B(x,r)} f|^2 dy$. Recall also that for rectifiable sets, $\mathcal{H}^d_{|E}$ has a density at almost all points $x \in E$; in a sense Theorem 2.5 controls the variations of this density.

Theorem 2.5 is very useful when we do constructions based on integrating measures on E, and we want to get some control on what we get (typically, do cancellations that occur for flat measures also occur for our AR measure μ?).

The proof of Theorem 2.5 is, if I recall correctly, by Littlewood-Paley estimates and stopping times, or a variant of the corona construction.

2.4 Big projections

We rapidly mention here results concerning the size of projections of E on d-planes. Recall that when E is totally unrectifiable, $\mathcal{H}^d(\pi_\theta(E)) = 0$ for almost every direction θ of a d-plane, while rectifiable sets have the opposite behavior. Again we want to localize and quantify some of this. We start with an attempt with only one big projection (for each ball).

Definition 2.6. *Let $E \in AR(d)$, d integer. We say that E has* **big projections** *when there is a constant $\tau > 0$ such that for $x \in E$ and $0 < r < \operatorname{diam}(E)$, there is a (vector) d-plane P such that*

$$\mathcal{H}^d(\pi_P(E \cap B(x,r))) \geq \tau r^d. \tag{2.13}$$

Here π_P denotes the orthogonal projection on P.

This is not enough to imply even simple rectifiability, because there are four obvious lines P for which the projection $\pi_P(K)$ of the Garnett-Ivanov Cantor set K of Figure 2 above contains an nontrivial line segment; yet this set is AR of dimension 1, but totally unrectifiable. However, together with a weaker version of the geometric lemmas, we reach the uniform rectifiability, and even the slightly stronger $BPLG$. We need definitions.

Definition 2.7. *Let $E \in AR(d)$ be given, set $\mathcal{E} = E \times (0, \operatorname{diam}(E))$. We say that the set $\mathcal{B} \subset \mathcal{E}$* **is a Carleson set** *when $\mathbb{1}_{\mathcal{B}} \frac{d\mu(x)dr}{r}$ is a Carleson measure on $\mathcal{E}$. That is, when there is a constant $C \geq 0$ such that*

$$\int_{(x,r) \in \mathcal{B} \cap B(X,R) \times (0,R)} \frac{d\mu(x)dr}{r} \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \operatorname{diam}(E). \tag{2.14}$$

Definition 2.8. *Let $E \in AR(d)$, d integer. We say that E* **satisfies the weak geometric lemma** *when for each (small) $\varepsilon > 0$, the set*

$$\mathcal{B}(\varepsilon) = \big\{(x,r) \in \mathcal{E}\,;\, \beta(x,r) > \varepsilon\big\} \tag{2.15}$$

is a Carleson set.

This condition is weaker than $GL(q)$. Indeed, if $(x,r) \in \mathcal{B}(\varepsilon)$, then by (2.7) $\beta_q(x,r/2)^{\frac{q}{d+q}} \geq C(q)^{-1}\beta(x,r) \geq C(q)^{-1}\varepsilon$, so $\beta_q(x,r/2)^2 \geq C^{-1}\varepsilon^{\frac{2(d+q)}{q}}$ and

$$\int_{B(X,R)\times(0,R)} \mathbb{1}_{\mathcal{B}} \frac{d\mu(x)dr}{r} \leq C\varepsilon^{-\frac{2(d+q)}{q}} \int_{B(X,R)\times(0,R)} \mathbb{1}_{\mathcal{B}} \beta_q(x,r/2)^2 \frac{d\mu(x)dr}{r} \leq C\varepsilon^{-\frac{2(d+q)}{q}} R^d, \tag{2.16}$$

by Chebyshev and as needed for the WGL. But the WGL is in principle much weaker, since we do not require any control on the way the Carleson constant for $\mathcal{B}(\varepsilon)$ depends on ε. And in practice, when we use the WGL, only one (very small) value of ε is used.

Definition 2.9. *Let $E \in AR(d)$, d integer. We say that E* **contains big pieces of Lipschitz graphs** *(in short, $E \in BPLG$) when there exists $\tau > 0$ and $M \geq 0$ such that for $x \in E$ and $0 < R < +\infty$, there is (an image by a rotation of) a Lipschitz graph Γ with constant $\leq M$ such that*

$$\mathcal{H}^d(E \cap B(x,r)) \geq \tau r^d. \tag{2.17}$$

Of course $BPLG$ is stronger than $BPLI$, which is our definition of uniform rectifiability. It is even strictly stronger, because one can find examples of E, based on Venitian blinds, that are uniformly rectifiable, but do not have big projections (certainly needed for $BPLG$).

Theorem 2.10. *[DS2] Let $E \in AR(d)$, d integer. If E has big projections and $E \in WGL$, then $E \in BPLG$.*

These conditions will be slightly illustrated in the next subsection (on snowflakes).

One of the reasons to mention projections is that there was very interesting new progress in this domain. See [CT, DV, Or] for just a few; we refer to D. Dąbrowski's lectures for more details.

2.5 Examples of snowflakes with holes

We want to describe a small class of examples in the plane, to illustrate the definitions above. We consider the following variant of the Van Koch snowflake construction where we add holes in the middle to preserve the length of K_n at each stage of the construction. See Figure 3.

We start with the unit interval $K_0 = I_0$ in the plane, and then each subsequent set K_{k+1}, $k \geq 0$ will be composed of 4^{k+1} intervals of equal length 4^{-k-1}, obtained by replacing each of the 4^k intervals I that compose K_k by 4 intervals $I_1, \ldots, I_4$ of length 4^{-k-1}. For this we choose a small angle $\alpha_k < 10^{-1}$ (to be safe) and choose I_1, I_2, I_3, and I_4 making respective angles with I of 0, α_k, $-\alpha_k$, and 0. We do this so that, for instance, I_1 starts at the same point as I, I_4 ends at the same point of I, and then between I_j and I_j+1, $1 \leq j \leq 3$, there are two hole parallel to I, of equal size (about $\frac{1}{2}(1-\cos(\alpha_k)4^{-k-1} \sim 4^{-k-2}\alpha_k^2$, if I compute right). We want to study the limiting set K. The properties listed below should all be easy to check, and we skip the details.

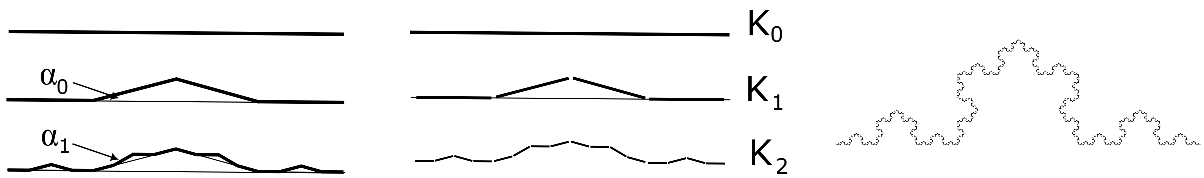

Figure 3: Figure 2.5. Left: The first two steps in the standard Koch construction, with $\alpha_0 = \alpha_1 = 15°$. Middle: The first two steps in our construction with holes (same angles). Right: A few steps in the Koch construction with $60°$ angles.

First, K was constructed to be Ahlfors regular of dimension 1, and there is an obvious AR probability measure μ on K, such that $\mu(I) = 4^{-k}$ for each of our intervals I of generation k. (As usual, μ can be used to prove the first part).

It is easy to see that if $\sum_k \alpha_k^2 < +\infty$, then the sum of the lengths of the holes is finite, so (by filling the holes), K is contained in regular curve. Then $K \in UR(1)$.

Conversely, if $\sum_k \alpha_k^2 = +\infty$, it is easy to imagine that no curve of finite length contains K, but the direct verification may be painful. But we can also trust P. Jones and compute the β-numbers for K. Observe first that

(2.18) each point of K_{k+1} is $4^{-k}\alpha_k$-close to (the corresponding interval of) K_k,

but we will also use the fact that, if we denote by x_I the central point of an interval I of generation k, then

$$\mathrm{dist}(x_I, K_{k+1}) \geq 4^{k-1}\alpha_k^2 \text{ and } \mathrm{dist}(x_I, K) \geq 4^{k-1}\alpha_k^2 \tag{2.19}$$

because of the central hole.

We estimate the $\beta(x,r) = \beta_\infty(x,r)$ because it is easier (and we know that it is enough when $d = 1$), but we would have analogous results for the $\beta_q(x,r)$. With the help of (2.18) to take care of the future generations, it is easy to see that for $x \in K$ and r such that $4^{-k-3} \leq r \leq 4^{-k-1}$, say,

$$100^{-1} \sum_{l \geq k+4} 4^{l-k}\alpha_k \leq \beta(x,r) \leq 100^{-1} \sum_{l \geq k-4} 4^{l-k}\alpha_k. \tag{2.20}$$

Thus we have roughly the same estimate for every point $x \in K$, namely that

$$\int_0^1 \beta(x,r)^2 \frac{dr}{r} \sim \sum_k \alpha_k^2 \tag{2.21}$$

(square the inequality above and sum over k and $l > k$). So, if we trust Theorem 2.4, $K \in UR$ if and only if $\sum_k \alpha_k^2 < +\infty$.

The situation for the Weak geometric lemma is even easier: we claim that

$$K \in WGL \text{ if and only if } \lim_{k \to +\infty} \alpha_k = 0. \tag{2.22}$$

If $\lim_{k\to+\infty} \alpha_k = 0$, then for each $\varepsilon > 0$, (2.20) says that $\beta(x,r) \leq \varepsilon$ for r small enough, and the the WGL follows easily. The converse is essentially as easy: if there are infinitely many k such that $\alpha_k \geq 100\varepsilon$, say, we can use (2.20) to find infinitely many integers j such that $\beta(x,r) > \varepsilon$ for $4^{-j-1} \leq r \leq 4^{-j}$, and violate the definition of the WGL.

Let us also discuss tangent lines. If $\sum_k \alpha^2 < +\infty$, we know that K is rectifiable and Ahlfors regular, so it has tangent planes almost everywhere. Yet it is not completely obvious to see precisely at which points. We would need to write every point $x \in K$ as an element of $4^{\mathbb{N}}$, i.e., give it coordinates in $\{1,2,3,4\}$, define a current angle in terms of these coordinates x_k (add α_k when $x_k = 2$, subtract α_k when $x_k = 3$, and leave it alone otherwise), and check when the current angle has a limit. And, for instance, if we want a tangent line at every $x \in K$, we need the stronger condition $\sum \alpha_k < +\infty$. That is, at most points the size of the $\beta(x,r)$ is not enough to say that K has a tangent point at x, but yet this happens almost always, by a combination of orthogonality and cancellations.

We can also check Theorem 2.5 with the α-numbers; since $\alpha(x,2r) \geq C^{-1}\beta_1(x,r)$, and modulo checking the computation above also for β_1, we just need to check the square Carleson condition for $\alpha(x,r)$ when $\sum_k \alpha_k^2 < +\infty$. And in addition to the (already taken care of) flatness of K, $\alpha(x,r)$ is also large when K has a hole (so that μ is not very uniformly distributed). Let us just evaluate the effect, but without precise computations. We consider $(x,r) \in K \times (0,1)$, and if $K \cap B(x,r)$ has a hole of size roughly τr, $\tau < 1$, but is otherwise flatter than this (small β), we expect $\alpha(x,r) \sim \tau$. The case where $\tau \sim 1$ occurs, when x lies close to a hole, but for a given r of generation k, it only happens with probability α_k^2 (the relative size of holes) in x. The total contribution of these events is thus roughly $\sum \alpha_k^2$.

But we also need to check the contribution of the pairs (x,r) such that $\alpha(x,r) \sim \tau \ll 1$ because of holes; this event typically happens τ^{-1} times more often, but then $\alpha(x,r)^2$ is multiplied by τ^2, so we expect a geometric series in j. The computation is complicated a bit by the fact that the holes in this case are of size τr, so they belong to a slightly earlier generation than $\log_4(1/\tau)$, and so we leave the details to the reader.

2.6 Apparently weak conditions

We already saw a weak condition, the WGL of Definition 2.8, and in this case we know that this condition is strictly weaker than UR. But to our surprise at the time, other weak conditions that don't look so different are actually equivalent to UR. Let us give our best example. We start with the definition of the bilateral β, where we care for the full local Hausdorff distance between E and a plane. Set, for $x \in E$ and $0 < r < \mathrm{diam}(E)$,

$$b\beta(x,r) = \inf_P \mathrm{dist}_{x,r}(E,P), \tag{2.23}$$

where the infimum is taken over d-planes (through $B(x,r)$) and, for two sets E, F that meet $B(x,r)$, their local normalized Hausdorff distance is

$$\mathrm{dist}_{x,r}(E,F) = r^{-1} \sup_{y\in F\cap B(x,r)} \mathrm{dist}(y,E) + r^{-1} \sup_{y\in E\cap B(x,r)} \mathrm{dist}(y,F). \tag{2.24}$$

Thus $b\beta(x,r)$ is large also when E is flat but has large holes in $B(x,r)$.

Definition 2.11. *Let $E \in AR(d)$, d integer. We say that E* **satisfies the bilateral weak geometric lemma** *(in short, $E \in BWGL$) when for every $\varepsilon > 0$, the set*

$$b\mathcal{B}(\varepsilon) = \big\{(x,r) \in E \times (0, \mathrm{diam}(E))\,;\, b\beta(x,r) > \varepsilon\big\} \tag{2.25}$$

is a Carleson set (as in Definition 2.7).

As for the WGL, we do not keep track of the Carleson constant for $b\mathcal{B}(\varepsilon)$, and in practice, only one (very small) ε is used. The reader may check that in our Koch flakes with holes, $E \in BWGL$ if and only if $\sum \alpha_k^2 < +\infty$. The proof is the same as for the α-numbers.

Here is the surprise.

Theorem 2.12. *[DS3] Let $E \in AR(d)$, d integer. Then $E \in UR$ if and only if it satisfies the bilateral weak geometric lemma.*

The proof is again by corona construction (and Reifenberg's theorem). We will say just a few words about it later.

There are other similar weak conditions that are equivalent to UR (see [DS3]), but the most used is really the $BWGL$, so we skip.

3 The Corona Construction

Since many proofs concerning UR (and even further) use a corona construction it sounds reasonable to try to describe how this works, mostly on one example (how do we compare $BPLI$ and $GL(q)$), and comment on other options. We start with a tool that is useful when we want to implement stopping time arguments.

3.1 Dyadic pseudocubes

The following is a technical construction. Mostly, it is neither too surprising, just useful. Let $E \in AR(d)$ be given, with any $d > 0$. We can find, for $k \in \mathbb{Z}$ such that $2^{-k} < \mathrm{diam}(E)$, say, partitions of E into "pseudocubes" Q, $Q \in \Delta_k$, which are measurable subsets of E, with the following properties:

• **1.** For each $Q \in \Delta_k$, there is a center $x_Q \in E$ such that

$$E \cap B(x_Q, C^{-1}2^{-k}) \subset Q \subset E \cap B(x_Q, C2^{-k}). \tag{3.1}$$

• **2.** If $R \in \Delta_k$ and $Q \in \Delta_{k+1}$, then either $Q \subset R$ or $Q \cap R = \emptyset$; (this is the usual nesting properties of dyadic cubes in $\mathbb{R}^n$)

• **3** (the "small boundary" property). For $Q \in \Delta_k$ and $\tau < 1$,

$$\mu\big(\big\{x \in Q\,;\ \mathrm{dist}(x, E\backslash Q) \le \tau 2^{-k}\big\}\big) + \mu\big(\big\{x \in E\backslash Q\,;\ \mathrm{dist}(x, Q) \le \tau 2^{-k}\big\}\big) \le C\tau^a 2^{-kd}. \tag{3.2}$$

Here the constants $a > 0$ depends on the AR constants (and the dimension). The last condition is often useful to control boundary effects (typically between stopping time regions), we will not see this here because we will hide all the technicalities.

There are a few constructions of pseudocubes, with I think similar ideas of proof, but some of them better organized; see for instance [Da3], Section 3 (the initial construction), [Ch2], which works for spaces of homogeneous type [Da4] which is a simplified (more dyadic) version of the one in [Da3], [Hy, HK], more flexible in some cases, and even [DM] for non doubling measures (this was needed in relation to to Lipschitz harmonic capacity).

We'll use those sets like one uses dyadic cubes, and even call them cubes even though their precise shape may be quite complicated. A difference with the usual dyadic cubes is that with this construction, a given cube $Q \in \Delta_k$ may have only one child (i.e., even though we know that Q is the disjoint union of all cubes of Δ_{k+1} that are contained in Q, there may be only one cube in this collection, namely, Q seen as a cube of Δ_{k+1}.

We will denote by $\Delta = \cup_k \Delta_k$ the collection of all dyadic cubes, and for each $Q \in \Delta$, by $\Delta(Q)$ the set of cubes $Q \in \cup_{j \geq k} \Delta_j$ that are contained in Q.

The general idea of the proof is not so complicated. Things are easier when E is bounded, of diameter comparable to 1 and then we decide that there is a top cube Q_0 of generation 0 equal to E. Then we try cutting Q_0 into subcubes of generation 1 (cover E by balls of radius $1/2$, make the covering disjoint by removing the previous balls from the current one, and also choose the balls carefully so that they still have a center and do not have too much mass of E near their boundary. Then cut again, and so on.

The technical difficulty is we proceed exactly like this, by successively cutting the same balls, it will be very hard to choose the initial cutting balls so well that no serious problem ever happens at the smallest scales, so we don't do this. Instead, for each scale 2^-k, $k \geq 0$, we can choose a covering of Q_0 by difference of balls, that we choose carefully by a Chebyshev argument, so that they satisfy (3.2) for $C^{-1} \leq \tau \leq 1$. But then our cubes don't satisfy the nesting property, so we need to revise the first generations, depending on the later ones. The idea is to modify slightly what we did at earlier generations so that the large cubes are now unions of cubes of generation k. And we do this at all scales, either by going back and forth and taking a limit, or by a combinatoric formula. Once we do this and get (3.1) and (3.2), we get the extra advantage that the small property (3.2) for every small τ can be deduced from its version with $C^{-1} \leq \tau \leq 1$ and the nesting property.

3.2 Small sets of cubes

For the moment we tried to write the geometric properties of E in terms of our space of balls $\mathcal{E} = E \times (0, \mathrm{diam}(E))$, but for stopping time arguments it is more convenient to use the dyadic cubes above. Thus, for instance, instead of $\beta(x,t)$, we could use the numbers

$$\beta_Q = \inf_P \frac{1}{\mathrm{diam}(Q)} \sup\big\{ \mathrm{dist}(y, P)\,;\, y \in E \cap B(x_Q, 2\mathrm{diam}(Q))\big\},\ Q \in \Delta, \tag{3.3}$$

where the infimum is taken over the d-planes. It is easy to see that the $\beta(x,r)$ and the β_Q control each other, so that properties like the GL, WGL, $BWGL$ can be written in terms of cubes (and packing conditions like below). Thus, for the proofs below, we will use the analogue for cubes of the Carleson condition for sets $\mathcal{B} \subset \mathcal{E}$, which is the following.

Definition 3.1. *We say that the set $\mathcal{B} \subset \Delta$ of cubes satisfies a* **(Carleson) packing condition** *(in short, $\mathcal{B} \in Capac$) when there is a constant $C \geq 0$ such that, for each cube $Q_0 \in \Delta$,*

$$\sum_{Q \in \Delta; Q \subset Q_0} \mu(Q) \leq C\mu(Q_0). \tag{3.4}$$

Or equivalently, since $E \in AR(d)$, we demand that for $X \in E$ and $0 < R < \mathrm{diam}(E)$,

$$\sum_{Q \in \Delta; Q \subset B(X,R)} \mu(Q) \leq CR^d. \tag{3.5}$$

Thus for instance the weak geometric lemma is equivalent to saying that for every $\varepsilon > 0$, the set $\mathcal{B}(\varepsilon) = \big\{Q \in \Delta\,;\, \beta_Q > \varepsilon\big\}$ satisfies a Carleson packing condition. Of course, when $\mathcal{B} \subset \Delta$ satisfies a Capac, we get that most cubes of Δ are not in $\mathcal{B}$ (and certainly Δ does not satisfy a Capac).

As before, the notion is unavoidable because it has the right scale invariance and we like conditions that also contain information on $E \cap Q$ for every cube Q.

We will often use the following simple consequence of the definition: if $\mathcal{B} \in Capac$ and $Q_0 \in \Delta$, and if for $x \in Q_0$ we denote by $N(x) = N_{Q_0}(x)$ the number of cubes $Q \in \mathcal{B}$ such that $x \in Q \subset Q_0$, then

$$\int_{x \in Q_0} N(x) d\mu(x) \leq C\mu(Q_0). \tag{3.6}$$

In particular, μ-almost every $x \in Q_0$ lies in only finitely many $Q \in \mathcal{B} \cap \Delta(Q_0)$. This is easy, because by Fubini,

$$\int_{x \in Q_0} N(x) d\mu(x) = \int_{x \in Q_0} \sum_{Q \in \mathcal{B} \cap \Delta(Q_0)} \mathbb{1}_Q(x) d\mu(x) = \sum_{Q \in \mathcal{B} \cap \Delta(Q_0)} \mu(Q) \leq C\mu(Q_0). \tag{3.7}$$

3.3 The stopping time region $\mathcal{S}(Q)$ below a cube Q

There are many different ways to start a corona construction, and we'll concentrate on a specific one (from $GL(2)$, say, to "the existence of a corona decomposition", a condition that we'll explain later).

For the moment the game consists in constructing a "stopping time region" $\mathcal{S}(Q) \subset \Delta$ below Q, where $Q \in \Delta$ is any given cube, by a simple procedure; we will see later why this can be useful. Also, the reader should be warned that everyone has their vocabulary and notation, so we may not use exactly the same names as other people.

Even before we start, we give ourselves a collection $\mathcal{B}$ of "bad cubes" (we'll call them red cubes, and typically we assume that $\mathcal{B} \in Capac$). If $Q \in \mathcal{B}$, then we decide to stop at once and take $\mathcal{S}(Q) = \{Q\}$.

Otherwise, we consider the children of Q, i.e. the cubes $R \in \Delta_{k(Q)+1}$ that are contained in Q, where $k(Q)$ denotes the generation of Q. If one of them is red, we decide to stop for this one, and also for the other children of Q (this does not cost us much, and this simplifies some estimates in the accounting). In this case $\mathcal{S}(Q)$ is composed of Q, plus its children.

We continue like this. For each generation $k \geq k(Q)$, we have a list of "not yet stopped cubes" R, $R \in \mathcal{S}_k(Q)$, and for each such R we consider the children of R, and stop them all if one of the children of R is bad. We still include these children in $\mathcal{S}(Q)$, but we will never consider their children or descendants. The other children are kept as not yet stopped (at least, for this moment in the discussion).

We iterate this construction and get the desired region $\mathcal{S}(Q)$. See Figure 4 (left) for a symbolic picture of $\mathcal{S}(Q)$. Note that $\mathcal{S}(Q)$ has a useful heredity property, namely the fact that when $R \in \mathcal{S}(Q)$, then all the intermediate cubes $S \in \Delta$ such that $R \subset S \subset Q$ lie in $\mathcal{S}(Q)$ as well. [Because of the fact that cubes may only have one children, it would be more precise to say "the cubes of intermediate generations". But we will not bother.]

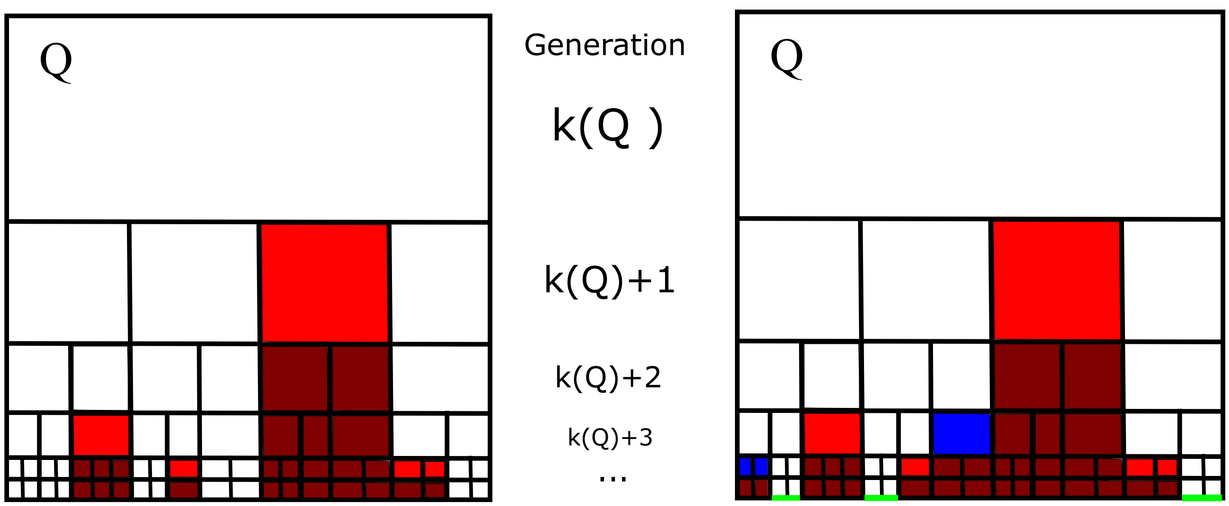

Figure 4: Symbolic pictures of $\mathcal{S}(Q) \subset \Delta(Q)$. Left: With only red stopped cubes (we also remove the subcubes, in brown). Right: With additional blue stopped cubes and, in green, the access region $A(Q)$ of (3.17).

In our main example, $\mathcal{B} = \mathcal{B}(\varepsilon) = \{Q \in \Delta\,;\, \beta_Q > \varepsilon\}$ for some very small $\varepsilon > 0$, to be chosen at the end of the proof, but of course you can choose other classes, depending on your assumptions. Yet we only do this for $\mathcal{B}$ that satisfy a Capac. Also, in our main example, or if we choose $\mathcal{B} \supset \mathcal{B}(\varepsilon)$, then for every cube Q in the good set $\mathcal{G} = \Delta \setminus \mathcal{B}$, we can choose a d-plane $P(Q)$ such that

$$\operatorname{dist}(y, P(Q)) \leq \varepsilon \operatorname{diam}(Q) \quad \text{for every } y \in E \cap B(x_Q, M\operatorname{diam}(Q)). \tag{3.8}$$

The initial definition of β_Q has a 2 instead of M, but using a large M gives an equivalent Capac condition, and is much easier to use in practice. There will be lots of other small lies like this below.

Now in our main example, we introduce another reason to stop at a cube, which will marked with the blue color. Again we may invent many reasons to stop; typically we stop when we think that we are beginning to lose control, and want to avoid this at all costs. In the case of the GL, we decide to color R in blue and stop when (R is not red and)

$$Angle(P(Q), P(R)) \geq \delta, \tag{3.9}$$

where δ is some other small constant that we choose at the end of the argument. We just need to choose $\delta \geq C\varepsilon$, where C depends on the AR constant for E; it is not clear that there is a point for choosing δ even larger, but we could. And choosing $\delta < \varepsilon$ would not make sense, because the directions of $P(Q)$ and $P(R)$ of these cubes are only determined with an error of roughly ε, by the fact that none of these cubes is red.

So we color all these new cubes in blue, decide to stop (as before) for al these cubes, as well as their siblings (the children of their parent), and to remove all the subcubes of these cubes from $\mathcal{S}$. So this is as with the red cubes, but now the blue cubes can decided some time along the construction (here by a very simple rule). See Figure 4 (right).

This completes the definition of $\mathcal{S}(Q)$; it is still hereditary, and of course it is smaller than without the blue condition, but we hope (and will need to prove) that it will not be so much smaller.

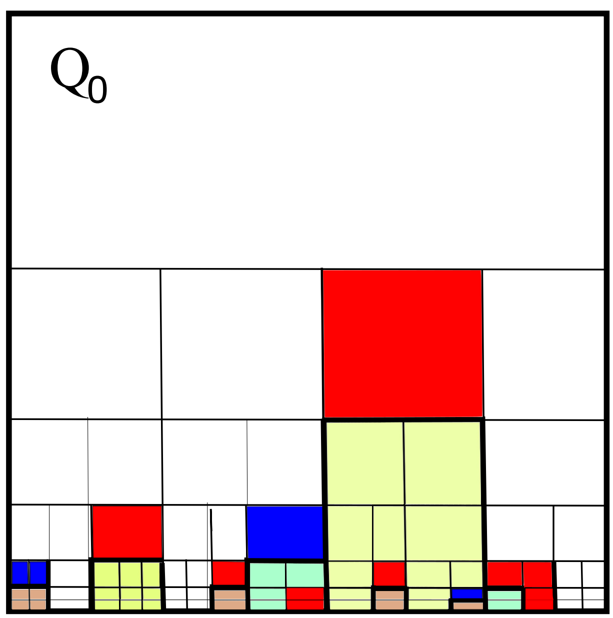

Figure 5: Symbolic picture of the covering (3.10) of $\Delta(Q_0)$ by regions $\mathcal{S}(Q)$. The main region is mostly in white, the next ones in pale colors. We kept the colors of the red and blue cubes, which lie at the bottom of their region.

Notice that once we have a systematic way to construct $\mathcal{S}(Q)$, as here, we automatically have a covering of $\Delta(Q_0)$ (the subcubes of a given cube Q_0) by disjoint regions, i.e.,

$$\Delta(Q_0) = \bigcup_{Q \in W} \mathcal{S}(Q) \tag{3.10}$$

(see Figure 5). Indeed, we start from $\mathcal{S}(Q_0)$, which at least contains Q_0, then for all the children Q of any stopped cube of $\mathcal{S}(Q_0)$, we add the region $\mathcal{S}(Q)$, then add all the regions $\mathcal{S}(Q)$ associated to a child Q of one of the previous regions, and so on. With this notation, W is a set of cubes composed of Q_0, and all the children of stopped cubes of the (previous) regions.

Some times it will be useful to denote by $Stop(\mathcal{S}(Q))$ the set of stopped cubes (for either the red or blue reason) of a region $\mathcal{S}(Q)$. Here we just said that every cube $Q \in W$, except Q_0, is a child of some $Stop(\mathcal{S}(R))$ for an $R \in W$ of a previous generation.

3.4 Corona decomposition

We give a name to the sets E for which, when we do the construction above, there are not too many cubes in W. Then we will see how to get such a condition, and how to use it.

Definition 3.2. *Let $E \in AR(d)$, with d integer. We say that E has* **corona decompositions** *when for all choice of $0 < \varepsilon < \delta < 10^{-2}$, we can find $C(\varepsilon, \delta) \geq 0$ so that for every choice of $Q_0 \in \Delta$, we can find a decomposition (3.10) of $\Delta(Q_0)$ as in Subsection 3.3, such that*

$$\sum_{Q \in W} \mu(Q) \leq C(\varepsilon, \delta)\mu(Q_0). \tag{3.11}$$

We do not need to be so precise about "a decomposition as in Subsection 3.3", in the sense that even if we used a slightly different condition to define the stopping time, we would be able to use the corona decomposition in the same way. And indeed the original definition in [DS1] is a little bit more flexible.

Notice that since the $Q \in W$ satisfy a Capac, (3.6) says that for almost every $x \in Q_0$, the number of cubes $Q \in W$ such that $x \in Q$ is finite. Then, if Q is the smallest such cube, x does not lie in any of the stopped cubes of $\mathcal{S}(Q)$. That is, we found a cube $Q \in W$ such that

$$R \in \mathcal{S}(Q) \text{ for all the cubes } R \text{ such that } x \in R \subset Q. \tag{3.12}$$

This will be useful later. We will discuss part of the proof of the following.

Theorem 3.3. *[DS1] Let $E \in AR(d)$, d integer. Then $E \in UR$ if and only if it has corona decompositions.*

For the direct part, the most convenient is to use the fact E satisfies a geometric lemma (and $GL(1)$ is enough). For the converse, we will explain how we can use corona decompositions to parameterize a superset of E. The construction will be surprisingly easy, and we'll take this as a hint that corona decompositions are easy to use.

3.5 The Lipschitz graph associated to a region $\mathcal{S}(Q)$

A good thing is that whenever $\mathcal{S}(Q)$ is any stopping time region, as in Subsection 3.3, we can construct a Lipschitz graph $\Gamma = \Gamma(\mathcal{S}(Q))$, with the following properties:

$$\Gamma \text{ is the graph of a } C\delta\text{-Lipschitz function } A : P(Q) \to P(Q)^{\perp}, \tag{3.13}$$

and E is close to Γ in the sense that for $x \in Q$ (and even near Q),

$$\mathrm{dist}(x, \Gamma) \leq C\varepsilon d(x), \tag{3.14}$$

where

$$d(x) = \inf_{R \in \mathcal{S}(Q)} (\mathrm{diam}(R) + \mathrm{dist}(x, R)) \tag{3.15}$$

estimates the scale of the smallest cubes of $\mathcal{S}(Q)$ near x.

Here we are lying a bit: it is better to require that $\beta(x_R, M\mathrm{diam}(R)) \leq \varepsilon$ for the good cubes R (and say that R is a red cube when this fails), with a very large M that allows to go somewhat beyond R and control how $P(R)$ varies. The existence of Γ is not much more than an exercise on partitions of unity (at a scale $d(x)$ that varies slowly), and (3.14) uses the fact that the stopping time condition (3.9) did not occur yet to control the slope of Γ.

It is very convenient that we can find Γ, and work on it, as soon as we constructed $\mathcal{S}(Q)$, and that the precision is at the scale of the smallest cubes of $\mathcal{S}(Q)$. In particular, when x is such that all the cubes R such that $x \in R \subset Q$ lie in $\mathcal{S}(Q)$, then $d(x) = 0$ and so $x \in \Gamma$. Because of (3.6), we expect this to happen often.

One of the main point of the construction is that once we have $\mathcal{S}(Q)$ and Γ, we can use (3.14) to go back and forth between Γ and E, and estimate quantities relative to E using the corresponding quantities for Γ, and vice versa. For instance, we can prove results that look like the following: for q in the same range as for the geometric lemma,

$$\begin{aligned}\sum_{R \in \mathcal{S}(Q)} \int_{x \in B(c_R, \mathrm{diam}(R))} \int_{C^{-1}\mathrm{diam}(R) \leq t \leq C\mathrm{diam}(R)} & \beta_{E,q}^2(x,t) \frac{d\mu(x)dt}{t} \\ & \simeq \int_{x \in \Gamma} \int_{t>0} \beta_{\Gamma,q}^2(x,t) \frac{dxdt}{t}.\end{aligned} \tag{3.16}$$

Here we cheat a lot, in particular we because we should play with localization constants (and for instance use $x \in B(c_R, k\mathrm{diam}(R))$ with different values of k, depending on which inequality we are proving), but the fact that the last double integral extends also away from Q is not a mistake: our Γ can be taken to be very flat away from Q.

At some point of time, if we want to prove that E has corona decomposition, we need to control how many times we stop for the blue reason (the red reason is taken care of by our assumption that $\mathcal{B}(\varepsilon)$ satisfies a Capac). And similarly, if we want to get control on E, and for instance prove that $E \in GL(q)$, we need to use the fact that W satisfies a Capac. But in the mean time, we can still work and prove estimates like (3.16) that relate the two.

We claim that the corona construction works like a linearization (but with sets): we almost get to assume that E is a small Lipschitz graph like Γ, and then estimate the errors. We will try to explain how this works in specific instances.

3.6 The Geometric lemma yields corona decompositions

Let us consider the part of Theorem 3.3 where we go from $GL(q)$ and go to corona decompositions. We construct stopping time regions $\mathcal{S}(Q)$ as in Subsection 3.3, and use the decomposition (3.10), so we only need to show that the top cubes Q, $Q \in W$, satisfy the packing condition (3.11).

Of course we'll use something like (3.16) to control the number of regions of a certain type, but let us start with the easy estimates. For $Q \in W$, we define the access set as

$$A(Q) = \big\{x \in Q\,;\ \text{all the cubes } R \text{ such that } x \in R \subset Q \text{ lie in } \mathcal{S}(Q)\big\}. \tag{3.17}$$

We also denote by $Stop(Q)$ the collection of all the stopped cubes of $\mathcal{S}$; these cubes are disjoint by construction (we never continue with $\mathcal{S}(Q)$ under a stopped cube) and it is easy to see that

$$\bigcup_{R \in Stop(Q)} R = Q \setminus A(Q). \tag{3.18}$$

We first consider the set

$$W_1 = \big\{Q \in W\,;\, \sigma(A(Q) \geq 10^{-1}\sigma(Q)\big\}, \tag{3.19}$$

and prove a Capac for this set. Simply observe that the set $A(Q)$, $Q \in W$, are disjoint (because if $x \in A(Q)$, Q is the smallest cube that contains x). Hence

$$\sum_{Q \in W_1} \mu(Q) \leq 10 \sum_{Q \in W_1} \mu(A(Q)) \leq 10\mu(Q_0), \tag{3.20}$$

where Q_0 is the root cube in (3.10) and because all the $A(Q)$ are contained in Q_0. This is good enough for (3.11). The next set is

$$W_2 = \big\{Q \in W\,;\, \sum_{R \in RStop(Q)} \mu(R) \geq 10^{-1}\sigma(Q)\big\}, \tag{3.21}$$

where $RStop(Q)$ is the collection of stopped cubes $R \in Stop(Q)$ that were stopped for the red reason (i.e., R is either a red cube or a sibling of a red cube). This is also good, because $RStop$ is also composed of disjoint cubes, and we have a packing condition on the red cubes:

$$\begin{aligned} \sum_{Q \in W_1} \mu(Q) \leq 10 \sum_{Q \in W_1} \sum_{R \in RStop(Q)} \mu(R) &\leq C \sum_{Q \in W_1} \sum_{R \in RStop(Q) \cap \mathcal{B}(\varepsilon)} \mu(R) \\ &\leq C \sum_{R \in \mathcal{B}(\varepsilon); R \subset Q_0} \mu(R) \leq C\mu(Q_0), \end{aligned} \tag{3.22}$$

as needed. We are left with the most interesting collection

$$W_3 = W \setminus (W_1 \cup W_2) \subset \big\{Q \in W\,;\, \sum_{R \in BluStop(Q)} \mu(R) \geq 10^{-1}\sigma(Q)\big\}, \tag{3.23}$$

where $BluStop(Q)$ is the collection of cubes that were stopped for the blue reason (3.9), including the siblings of blue cubes, but then as before

$$\sum_{R \in Blue(Q)} \mu(R) \geq C^{-1} \sigma(Q), \tag{3.24}$$

where now we sum only over the blue cubes (not their siblings). Now we need a packing condition all these cubes R; we just give an idea.

For each $Q \in W_3$, we consider the region $\mathcal{S}(Q)$ and the graph $\Gamma = \Gamma(\mathcal{S}(Q)$. Say that Γ is the graph of the $C\delta$-Lipschitz function $G : P(Q) \to P(Q)^{\perp}$. We have an additional information on Γ, because we stopped on a good portion Q because $Angle(P(R), P(Q) \geq \gamma$, which, if ε is significantly smaller than δ, implies that Γ makes an angle at least $C^{-1}\delta$ with $P(Q)$. That is, $|\nabla G(x)| \geq C^{-1}\delta$ for x in a large subset of $P(Q) \cap B(x_Q, \mathrm{diam}(Q))$. This forces, by Littlewood-Paley theory, a lower bound like

$$\int_{x \in \Gamma \cap B(x_Q, C\mathrm{diam}(Q))} \int_{0<t<C\mathrm{diam}(Q)} \beta_{\Gamma,q}^2(x,t) \frac{dxdt}{t} \geq C(\delta)^{-1} \mu(Q), \tag{3.25}$$

which we can transfer to E itself, as hinted near (3.16), to get

$$\int_{x \in E \cap B(x_Q, C\mathrm{diam}(Q))} \int_{0<t<C\mathrm{diam}(Q)} \beta_{E,q}^2(x,t) \frac{dxdt}{t} \geq C^{-1} \mu(Q) \tag{3.26}$$

(for a larger C, but let us not record). Then we sum all this and get that

$$\sum_{Q \in W_3} \mu(Q) \leq C \sum_{Q \in W_3} \int_{x \in E \cap B(x_Q, C\mathrm{diam}(Q))} \int_{0<t<C\mathrm{diam}(Q)} \beta_{E,q}^2(x,t) \frac{dxdt}{t}, \tag{3.27}$$

prove that the domains of integration above have a bounded covering property, and get that

$$\sum_{Q \in W_3} \mu(Q) \leq C \int_{x \in E \cap B(x_Q, C\mathrm{diam}(Q_0))} \int_{0<t<C\mathrm{diam}(Q_0)} \beta_{E,q}^2(x,t) \frac{dxdt}{t} \leq C\mu(Q_0); \tag{3.28}$$

this completes the proof that E has corona decompositions (modulo some details). □

3.7 Corona decompositions give ω-regular parameterizations

Let us now illustrate the fact that Corona decompositions are easy to use. We explain in this subsection how we can use them to find an ω-regular parameterization $z : \mathbb{R}^d \to \mathbb{R}^{n^*}$ such that $E \subset z(\mathbb{R}^d)$. This property that $E \subset z(\mathbb{R}^d)$ is a little stronger a priori than our initial definition of UR by Big pieces of Lipschitz Images, even though some argument is needed. Roughly, we need to know that when z is ω-regular and $B \subset \mathbb{R}^d$ is a ball, there is a very large subset of B where z is $C(\fint_B \omega)^{1/d}$-Lipschitz, and this is not too hard.

The existence of z is simpler when E is bounded, or else when we only want to show that $Q_0 \subset z(\mathbb{R}^d)$ for some cube $Q_0 \subset E$, so let us only discuss this.

We use the decomposition of Subsection 3.3, and assume the packing condition (3.4) on the top cubes $Q \in W$, as in Definition 3.1. For each $Q \in W$, we have a graph $\Gamma = \Gamma(\mathcal{S}(Q))$, and by (3.7), μ-almost every $x \in Q_0$ lies in finitely many top cubes $Q \in W$. Let $Q(x)$ be the smallest one; then x lies in the access region $A(Q)$ of (3.17), and in particular $d(x) = 0$ and (3.14) says that $x \in \Gamma$. Thus we just need to find an ω-regular mapping z such that all the $\Gamma(\mathcal{S}(Q))$, or even just the $A(Q)$, are contained in $F = z(\mathbb{R}^d)$; indeed we do not fear the remaining set of vanishing μ-measure, because F will be closed by definition, and then this set is contained in F too, since $E \in AR$. The details of the construction of z are not really complicated, but we only give some ideas.

For any cube $Q \in W$ which is not red, we associated a region $\mathcal{S}(Q)$ and a Lipschitz graph $\Gamma(Q)$, which is the graph of some function G_Q defined on $P(Q)$. We don't need to worry about the red cubes, because they have no $A(Q)$ that we would need to cover. Obviously we will only need to use the part of the graph of G_Q over $B_+(Q) = P(Q) \cap B(\pi(x_Q), C\text{diam}(Q))$, where π is the orthogonal projection onto $P(Q)$ and C is taken large enough. But also, we can dispense with the part which is above all the small balls $B_0(R) = P(Q) \cap B(\pi(x_R), 2c\text{diam}(R))$, where R runs over the class $Next(Q)$ of children of the stopped cubes of Q (and c is small enough). Indeed, with a little bit of geometry one proves that the part of $\Gamma(Q)$ above any of these balls $B_0(R)$ does not meet the access region $A(Q)$. So in fact it will be enough to cover all the $\Gamma'(Q)$, $Q \in W$ (and Q is not red), where $\Gamma'(Q)$ is the part of $\Gamma(Q)$ above

$$B'(Q) = B_+(Q) \setminus \cup_{R \in Next(Q)} B_0(R) \tag{3.29}$$

We call this class $Next(Q)$ because these are the top cubes of the stopping time regions just blow Q. Notice that $B'(Q)$ is a ball of radius $2\text{diam}(Q)$, from which we remove a collection of very small balls that are far from the boundary and each other, and $\Gamma'(Q)$ is a Lipschitz graph over $B'(Q)$. Let us call $z_Q : B'(Q) \to \Gamma'(Q)$ the obvious parameterization of $\Gamma'(Q)$. See Figure 6 for this and the following.

For the sake of definiteness, we also define $B_+(Q)$, $\Gamma'(Q)$ and z_Q when Q is red: we pick any plane $P(Q)$ through x_Q and proceed as above, with $\Gamma = P(Q)$ and the children R of Q. This time we just take small balls $B_0(R) \subset B(x_Q, \text{diam}(Q)$ of diameter $\text{diam}(Q)/C$, such that the $2B_0(R)$ are disjoint.

We still need to glue all these $\Gamma'(Q)$. We start with Q_0. We construct $\Gamma'(Q_0)$ and $B'(Q_0)$ as above, and now try to glue the mappings z_R, $R \in Next(Q_0)$; recall that the $R \in Next(R)$ are also the top cubes of the construction after Q_0, so the sets $A(R)$ are the next ones that we need to cover. We want a final parameterization defined on $B'(Q_0)$, so we re-parameterize each z_R to make it start from a small ball in $B'(Q_0)$. That is, we keep $z = z_{Q_0}$ on $B'(Q_0)$, but on the small balls $B_1(R) = \frac{1}{2}B_0(R) = P(Q) \cap B(\pi(x_R), c\text{diam}(R))$, we decide to take $z = z_R \circ \psi_R$, where ψ_R is a (fairly brutal) dilation that sends $B_1(R)$ to $B_+(R)$. This leaves open the definition of z in the intermediate annuli $B_0(R) \setminus B_1(R)$, $R \in Next(Q_0)$, but for the moment we don't care and continue our construction.

In fact we do not define z by $z = z_R \circ \psi_R$ on the whole $B_1(R)$ as promised, but only on $\psi_R^{-1}(B'(R))$, because in fact z_R was only defined on $B'(R)$). This leaves a lot of tiny holes, the $\psi_R^{-1}(B_0(S))$, where $S \in Next(R)$. The reader already guessed how we are going to define z on the $\psi_R^{-1}(B_1(S))$: we take $z = \psi_R \circ \psi_S$, except of course on tiny holes of the next generation where we proceed by induction. When we iterate this construction, we have a mapping z which is defined everywhere on $B_+(Q_0)$, except on a collection of smaller and smaller disjoint annuli coming from the annuli $B_0(S) \setminus B_1(S)$, $S \in W$, which are not needed to cover E but where we need to extend z, essentially by independent applications of Whitney's extension theorem, to complete the definition.

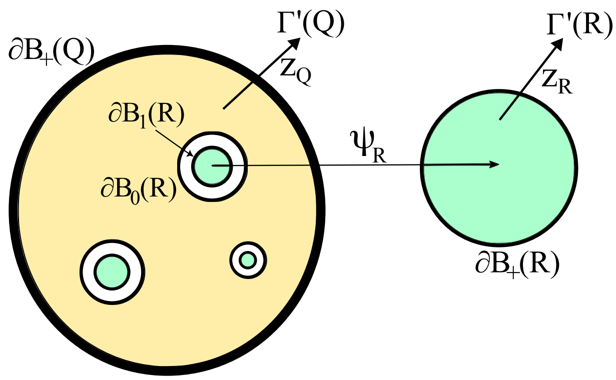

Figure 6: Definition of a part of z on a ball $B_+(Q)$ (minus some annuli, and, except when $Q = Q_0$, we will need to precompose this by ψ_Q and previous ones).

At this stage, we just need to check that we can extend z so that it is ω-regular. The fact that we need a weight is fairly clear now, even before we extend z, because each time we cross a generation (i.e., go from a top cube Q to a cube $R \in Next(Q)$), we compose with a mapping ψ_R that expands z by a fixed, but large factor. The packing condition on W says that this does not happen too often, and at the end we get an A_1 weight which is large at points x such that $N(x)$ in (3.6) is large. Also, for (1.8) we need to control the size of inverse images $z^{-1}(B(x,r))$ of balls; a good part of this comes from the Ahlfors regularity of E itself, but when we extend z to the annuli, we need to make sure that the connecting tubes do not cross too much, and this is why we work in the larger $\mathbb{R}^{n^*} = \mathbb{R}^{n+1}$ when $2d > n$, to allow general position arguments. □

3.8 Other proofs with corona decompositions

First we say a few words about how to prove that the bilateral weak geometric lemma (BWGL) implies that $E \in UR$. Here we work with the larger class $\mathcal{B}(\varepsilon)$ of red cubes for which the bilateral beta number is large, namely, $b\beta(x_Q, C\text{diam}(Q)) > \varepsilon$. We construct the regions $\mathcal{S}(Q)$ as before, except that we only stop for red cubes (i.e., we do not have blue cubes this time). Then we cannot expect to have a good approximation by Lipschitz graphs, as in (3.13)-(3.15), because the d-plane $P(Q)$ may turn a lot when Q goes down many generations. But instead we have a good approximation by a Reifenberg-flat set $\mathcal{G}$, with flatness controlled by ε, and this is almost as good as above, especially since, by pushing the estimates from E to $\mathcal{G}$, we can prove that $\mathcal{G}$ is also Ahlfors regular. Then there are different ways to show that $\mathcal{G}$ is uniformly rectifiable, and if I recall correctly we chose in [DS3] to use a variant of Semmes' Condition B that requires the existence of $(n-d)$-dimensional spheres in the complement that are linked to $\mathcal{G}$; initially, Condition B was defined in co-dimension 1, where it requires the local existence of large balls in different connected components of the complement.

Once we have this good set $\mathcal{G}$, it is not so hard to deduce the uniform rectifiability of E.

The fact that one can go from corona decompositions to properties like Geometric lemmas (GL(q)), should not shock the reader, because the Lipschitz graphs Γ in the constriction satisfies these properties (however, as in Subsection 3.7, covering and gluing estimates are needed).

With time, the corona decomposition has been a very convenient way to connect the geometry of E (typically, its uniform rectifiability) with various facets of the analysis on E. The most striking examples probably concern elliptic PDE's on domains bounded by E. I believe the first example is the proof by Semmes [Se2] of the fact that if $\Omega \subset \mathbb{R}^n$ is a domain with nontangential access (one also says "uniform"), its boundary is Ahlfors regular of dimension $n-1$, and its complement has corkscrew points (large balls), the harmonic measure on Ω is absolutely continuous with respect with the AR measure, with a density which is an A_∞ Muckenhoupt weight. An independent proof of this was given at the same time [DJ], using a slightly more direct approach by geometry and big pieces of Lipschitz graphs, but it turns out that corona decompositions, and more generally geometric stopping times, have been used many times with success in connection with PDE's (see many of the references below, by Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa in particular).

A similar remark applies to problems related to analytic capacity (see [To6] for a detailed account), projections (see for instance [CT, DV, Or], or better refer to other lecture notes for this winter school).

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