

# Harmonic measure with lower dimensional boundaries

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# 1. Harmonic measure in the classical case

Before we talk about our degenerate case, we describe very rapidly the classical case with boundaries of co-dimension 1.

We start with (the Brownian path) definition for the harmonic measure.

Let  $\Omega$  be a nice domain in  $\mathbb{R}^n$ .

Just for the discussion below, say  $\Omega$  is bounded and smooth.

We define the **harmonic measure**  $\omega^X$  (centered at  $X \in \Omega$ ) by this:

For  $A \subset E$ ,  $\omega^X(A)$  is the probability for a Brownian trajectory starting from  $X$  to lie in  $A$  the first time it hits  $E$ .

This looks intuitive; this would require some work to write down, and appropriate assumptions too. And in practice we'll use another definition.

# Classical Harmonic measure ( $\omega^X(A)$ is harmonic)

Observe that as a function of  $X$ ,  $\omega^X(A)$  is harmonic in  $\Omega$ .

Proof by conditional expectation:

We will use the fact that for continuous functions  $f$  on  $\Omega$ ,  $f$  is harmonic on  $\Omega$  iff it satisfies the mean value property: for  $0 < r < \text{dist}(X, \partial\Omega)$ ,

$$f(X) = \frac{1}{\sigma(\partial B(X, r))} \int_{Y \in \partial B(X, r)} f(Y) d\sigma(Y).$$

To see this, pick a small circle  $\mathcal{C} = \partial B(X, r)$  centered at  $X$ . Before it leaves  $\Omega$ , each Brownian path must hit  $\mathcal{C}$  a first time.

This happens with equal probability to hit any  $Y \in \mathcal{C}$ .

And the Brownian path after  $Y$  is independent of the past.

So  $\omega^X(A)$  is the average on  $\mathcal{C}$  of  $\omega^Y(A)$ , as needed.

$\omega^X(A)$  is harmonic, so by [Harnack](#): for  $X, Y \in \Omega$  (connected),

$$C(X, Y)^{-1} \omega^Y(A) \leq \omega^X(A) \leq C(X, Y) \omega^Y(A).$$

# Classical Harmonic measure; other properties

We also expect that  $\omega^X(A) \rightarrow 1$  when  $X \in \Omega \rightarrow \xi \in A$   
and

$$\omega^X(A) \rightarrow 0 \text{ when } X \in \Omega \rightarrow \xi \in \partial B \setminus A.$$

And, roughly speaking, these properties (harmonicity and limits) should characterize  $\omega^X$ .

So  $\omega^X$  is a probability measure, and we want to relate its properties to the geometry of  $\partial\Omega$ .

Harnack above says that this depends only mildly on  $X$  (at least in the middle of a connected component of  $\Omega$ ).

# Classical Harmonic measure; simple examples

Especially with the definition above, it is not easy to define or compute  $\omega^X$ . Except in some simple cases below.

Example 1:  $\Omega = B(0, 1) \subset \mathbb{R}^n$ ,  $X = 0$ ; then  $\omega^X$  is the uniform probability measure on  $\partial B(0, 1)$ .

Example 2: when  $n = 2$ , it is not so hard to see that  $\omega$  is invariant under conformal mappings. Indeed, if  $\psi$  is an analytic mapping,  $f \circ \psi$  is harmonic when  $f$  is harmonic.

So if  $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$  is simply connected and we know a conformal mapping  $\psi : \Omega \rightarrow D(0, 1)$  such that  $\psi(X) = 0$ , we can compute  $\omega^X(A) = \text{length}(\psi(A))$ .

# Classical Harmonic measure; a more pleasant definition

Recall: for  $A \subset \partial\Omega$ ,  $\omega^X(A)$  is harmonic and “tends to”  $\mathbb{1}_A$  on  $\partial\Omega$ .

And  $\omega^X(A) = \int_{\partial\Omega} \mathbb{1}_A(\xi) d\omega^X(\xi)$ .

Next definition (as guessed): when  $\Omega$  is “regular,” for each  $g \in \mathcal{C}(\partial\Omega)$  (that is, continuous on  $\partial\Omega$ ), there is a unique harmonic extension  $f$  of  $g$  to  $\Omega$  (continuous on  $\overline{\Omega}$ ). (Perron method, say).

Then for each  $X \in \Omega$ ,  $g \rightarrow f(X)$  is a continuous linear form on  $\mathcal{C}(\partial\Omega)$  and by a theorem of Riesz there is a finite measure  $\omega^X$  on  $\partial\Omega$  such that

$$f(X) = \int_{\partial\Omega} g(\xi) d\omega^X(\xi) \quad \text{for } g \in \mathcal{C}(\partial\Omega).$$

This definition coincides with the previous one, with  $g = \mathbb{1}_A$ , in the good cases.

Easier to manipulate in general.

# Elliptic operators; elliptic measure

This was for harmonic functions and the Laplacian  $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ .  
But the definitions work also for some other “elliptic operators”.  
Let us consider only the operators in divergence form:

$$(1) \quad L = \operatorname{div} A \nabla$$

where  $A = A(X)$  is an  $n \times n$  real matrix (measurable in  $X$ ), and we require the usual boundedness property

$$(2) \quad |A(X)| \leq C \quad \text{for } X \in \Omega$$

and ellipticity condition

$$(3) \quad \langle A(x)\xi, \xi \rangle \geq C^{-1}|\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

Then it is possible to define elliptic measure  $\omega_L^X$  as above, but with solutions of  $Lf = 0$ . We rather call  $\omega_L^X$  **elliptic measure**. See later.

## 2. Some classical regularity results for Harmonic measure

How is the regularity of  $\omega^X$  connected to the geometry of  $E = \partial\Omega$ ?

For instance, when is  $\omega^X$  absolutely continuous with respect to surface measure (we'll also write  $\mathcal{H}^{n-1}$ , the Hausdorff measure)?

Very rich subject. Only a few examples here:

- F. and M. Riesz in 1916 (conformal mappings): If  $\Omega \subset \mathbb{R}^2$  is simply connected and  $\mathcal{H}^1(\partial\Omega) < +\infty$  (here  $\mathcal{H}^1$  is the length),  $\Phi : \mathbb{D} \rightarrow \Omega$  is conformal, then  $\int_{\partial\mathbb{D}} |\Phi'| < +\infty$  and  $\omega_\Omega^X \ll \mathcal{H}^1|_{\partial\Omega}$ .

- Lavrentiev in 1936: If  $E \subset \mathbb{R}^2$  is a chord-arc curve, then
$$\omega^X \in A_\infty(\mathcal{H}^1|_{\partial\Omega}).$$

Here  $A_\infty$  is the Muckenhoupt class, given by a nicely invariant quantitative condition of mutual absolute continuity, defined below.

- Dahlberg in 1977:  $\omega^X \in A_\infty(\mathcal{H}^{n-1}|_{\partial\Omega})$  (the surface measure) when  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$ .



# Ahlfors regular sets and measures

Before we continue, define **Ahlfors regular** sets and measures.

Let  $E \subset \mathbb{R}^n$  be closed. An Ahlfors regular measure of dimension  $d$  supported on  $E$  is a measure  $\mu$  such that

$$(4) \quad C^{-1}r^d \leq \mu(E \cap B(x, r)) \leq Cr^d \quad \text{for } x \in E \text{ and } r > 0.$$

Notation:  $\mu \in AR(d)$ . We say that the set  $E$  is Ahlfors regular when it supports some  $\mu \in AR(d)$ .

Then we can take  $\mu = \mathcal{H}^d|_E$ . That is,  $\mathcal{H}^d|_E \in AR(d)$  too.

We chose unbounded sets here (closer to what we will do in these lectures).

We'll restrict to AR sets below; this is not needed, but this will simplify our life.

# A need for geometric conditions

First, connectedness. If  $\Omega$  is not connected,  $\omega^X$  only charges a piece of  $\partial\Omega$  (the boundary of the component of  $X$  in  $\Omega$ ). But we'll assume  $\Omega$  is connected.

If  $\Omega$  is badly connected,  $\omega^X$  will be very small on some parts of  $\partial\Omega$  (and, iterating, this could prevent  $\mathcal{H}^{n-1} \ll \omega^X$ ).

If  $\Omega$  has a simple **cusp** at  $0 \in \partial\Omega$ , harmonic functions (such as  $\omega^X(A)$ ) may have singularities at 0.

If  $\Omega \subset \mathbb{C}$  and  $\Omega$  is a sector of angle  $\alpha$ , a typical analytic function near 0 is  $z^{\pi/\alpha}$ , and harmonic measure looks like  $|z|^{-1+\frac{\pi}{\alpha}} d\ell$  near 0.

Same sort of behavior when  $\Omega \subset \mathbb{R}^n$  is (looks like) a cone near 0.

Spherical harmonics: homogeneous harmonic functions come from eigenfunctions of the Laplacian on  $\Omega \cap \partial B(0, 1) \subset \partial B(0, 1)$ , and their homogeneity depends on eigenvalues of that Laplacian.

# Uniform connectedness and NTA

Harmonic (or elliptic) measure is not nice unless  $\Omega$  is sufficiently connected. Thus one often asks for :

- The existence of “**corkscrew balls**” in  $\Omega$ : there is  $C \geq 1$  such that for  $x \in E = \partial\Omega$  and  $r > 0$ , we can find

$$X_{x,r} \in \Omega \text{ such that } B(X_{x,r}, C^{-1}r) \subset \Omega.$$

- The existence of “**Harnack chains**” in  $\Omega$ : for  $X, Y \in \Omega$ , there is a thick path, not too long, that connects  $X$  to  $Y$  in  $\Omega$ . See the next page.
- Not always but often, we demand corkscrew balls in  $\mathbb{R}^n \setminus \overline{\Omega}$ .

**NTA** (non tangential access) is the conjunction of the three.  
One-sided NTA would be the first two only.  
Terminology from Jerison-Kenig.

# Definition of the Harnack condition in $\Omega$

A notation first (we always assume  $\emptyset \neq \Omega \neq \mathbb{R}^n$ ):

$$\delta(X) = \text{dist}(X, \mathbb{R}^n \setminus \Omega) = \text{dist}(X; \partial\Omega) \quad \text{for } X \in \Omega.$$

The standard definition with balls:  $\Omega$  has Harnack chains if for each  $A > 0$  there is  $N = N(A, \Omega)$  such that:

if  $X, Y \in \Omega$ , with  $|X - Y| \leq A \min(\delta(X), \delta(Y))$ , there is a chain of balls  $B_1, B_2, \dots, B_N \subset \Omega$ ,  $X \in B_1$ ,  $Y \in B_N$ , each  $2B_j \subset \Omega$ , and  $B_j$  meets  $B_{j+1}$  for  $1 \leq i < N$ .

Hence by Harnack we control  $u(X)/u(Y)$  when  $u$  is positive harmonic on  $\Omega$ .

Equivalent variant with thick paths: there exist constants  $c > 0$  and  $C \geq 1$  such that, for  $X, Y \in \Omega$  and setting  $R = |X - Y|$ , we can find a path  $\gamma = \gamma_{X,Y} : [0, R] \rightarrow \Omega$ , with  $\gamma(0) = X$ ,  $\gamma(R) = Y$ ,  $|\gamma'(t)| \leq C$  for  $t \in [0, R]$ ,  $\delta(\gamma(t)) \geq c(t + \delta(X))$  for  $0 \leq t \leq R/2$ , and  $\delta(\gamma(t)) \geq c((R - t) + \delta(Y))$  for  $R/2 \leq t \leq R$ .

# What does NTA yield?

Many domains satisfy the NTA condition without having an Ahlfors regular boundary of co-dimension 1. Think about snowflakes.

Yet some nice things can be said about  $\omega^X$  for those  $\Omega$ . For instance,  $\omega^X$  is doubling:

$$\omega^X(\partial\Omega \cap B(x, 2r)) \leq C\omega^X(\partial\Omega \cap B(x, r)) \text{ for } x \in \partial\Omega \text{ and } r > 0.$$

Also, positive harmonic functions on  $\Omega$  have extra nice properties: they are locally Hölder continuous up to  $\partial\Omega$ , and there is a comparison principle: if  $x \in \partial\Omega$  and  $r > 0$ , and  $u, v$  are harmonic and positive on  $\Omega \cap B(x, 2r)$ , and vanish on  $\partial\Omega \cap B(x, 2r)$ , then

$$(5) \quad C^{-1} \leq \frac{u(X) v(Y)}{u(Y) v(X)} \leq C$$

for  $X, Y \in \Omega \cap B(x, r)$ .

See Jerison, Kenig, and precursors (Ancona, etc.).

Techniques of potential theory and PDE's. Including Caccioppoli inequalities, Moser iteration, etc.

## Regularity results for Harmonic measure (2)

Now assume  $E = \partial\Omega$  is Ahlfors regular of codimension 1. We can try to compare  $\omega^X$  with the “surface measure”  $\mu = \mathcal{H}_{|E}^{n-1}$ .

The main breakthrough in dimensions  $n > 2$  is Dahlberg 1977:  $\omega^X \in A_\infty(\mathcal{H}_{|E}^{n-1})$  when  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$ .

The proof is more complicated; if I recall, Carleson measures, the Rellich identity that connects normal and tangential derivatives, and estimates.

## Regularity results for Harmonic measure (3)

Under our  $AR + NTA$  assumptions, things are all right:

D.-Jerison or Semmes 1990:  $\omega^X \in A_\infty(\mathcal{H}^{n-1})$  when  $E \in AR(n-1)$  and (Condition B) there exists  $C \geq 1$  such that for  $x \in E$  and  $r > 0$  we can find  $z_1, z_2 \in B(x, r)$ , with  $\text{dist}(z_i, E) \geq C^{-1}r$ ,  $z_1 \in \Omega$ , and  $z_2 \notin \overline{\Omega}$ .

Indeed, the NTA condition implies Condition B (corkscrew points on both sides).

Condition B (and  $E \in AR(n-1)$ ) implies the **uniform rectifiability** of  $E$ , and more:  $E$  contains big pieces of Lipschitz graphs, and even with the corresponding Lipschitz domains contained in  $\Omega$ .

Then  $\omega^X \in A_\infty(\mathcal{H}^{n-1})$  comes from Dahlberg's result and (essentially) the maximum principle.

## Regularity results for Harmonic measure (4)

Then there are weaker conditions that still imply  $\omega^X \in A_\infty(\mathcal{H}_{|E}^{n-1})$ , or a slightly weaker quantitative mutual absolute continuity result. For instance, one-sided NTA and uniform rectifiability. Optimal results by S. Hofmann and X. Tolsa, I think.

Also, very nice non uniform results by Azzam, Hofmann, Martell, Mayboroda, Mourougolou, Nyström, Uriarte-Tuero, Tolsa, Toro, Volberg, and others, involving rectifiability (not uniform) and density (instead of AR)!

And, with even more authors (Garnett, Pipher, etc.): a good part is still true for the nice enough elliptic operators.

Comment. For the converse part, important use of the result of Nazarov-Tolsa-Volberg on the boundedness of the Riesz transform (iff  $E \in UR$ ).



### 3. Elliptic measure in co-dimensions $> 1$

Why do we try to take  $E = \partial\Omega$  of dimension  $d < n - 1$ ?

- Curiosity: do some things go through? What is so special about codimension 1? What is the role of the Riesz transform?
- The urge to do the forbidden thing.
- Interesting things showed up eventually.

Why forbidden?

Harmonic measure (for  $\Delta$ ) is not always defined on  $\Omega$ : if  $d \leq n - 2$ , almost every Brownian trajectory avoids  $E$  altogether.

Same problem with the Dirichlet problem: harmonic functions extend harmonically through  $E$ , so we would be looking at restriction to  $E$  of harmonic functions near  $E$ .

What do we do?

Replace  $\Delta$  with well chosen degenerate elliptic operators  $L$  more suited to the geometry, but that depend on  $E$ .

Instead J. Lewis & K. Nyström replaced  $\Delta$  with a  $p$ -Laplacian (a non linear operator).

# Degenerate Elliptic Operators

From now on,  $E = \partial\Omega \in AR(d)$  for some  $d < n$ .

So we'll study elliptic measure, associated to degenerate elliptic operators of the form

$$(6) \quad L = -\operatorname{div}[A(X)\nabla],$$

where  $A$  is a  $n \times n$  matrix-valued function  $A$  with real coefficients, satisfying some degenerate elliptic condition. From homogeneity considerations, we chose the class  $\mathcal{M}_{dell}$  of matrices  $A$  of the form

$$(7) \quad A(X) = \operatorname{dist}(X, E)^{d+1-n} \mathcal{A}(X) =: w(X) \mathcal{A}(X)$$

where  $\mathcal{A}$  satisfies the usual boundedness and ellipticity conditions (2) and (3) above.

# Usual conditions/Comments

Recall the usual conditions:

$$(2) \quad |\mathcal{A}(X)\xi \cdot \zeta| \leq C_e |\xi| |\zeta| \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n,$$

$$(3) \quad \mathcal{A}(X)\xi \cdot \xi \geq C_e^{-1} |\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

## Comments.

The weight  $w(X)$  and the coefficients tend to infinity near  $E$  (in an ordered way).

This way, when  $E = \mathbb{R}^d \subset \mathbb{R}^n$ , and  $\mathcal{A}(X) \equiv Id$ , “radial” solutions  $u = u(x, |y|)$ , with  $x \in \mathbb{R}^d$  and  $y \in \mathbb{R}^{n-d}$ , come from harmonic functions on  $\mathbb{R}_+^{d+1}$ .

We’ll see later that things do not need to be so rigid.

When  $\mathcal{A} = Id$ , the associated process is a Brownian Motion with a drift in the direction of  $\nabla w$  (thus towards  $E$ ).

# Elliptic measure in this framework

With only the assumptions above, we can will reproduce the usual theory of elliptic measure, but not more precise absolute continuity results.

Notice that when  $d < n - 1$ , we don't need to assume the existence of Corkscrew Balls and Harnack chains in  $\Omega$  (the one sided NTA property), because this follows from  $AR(d)$ .

And the basic statement in this context is: for  $E \in AR(d)$ ,  $d < n - 1$ , and  $A \in \mathcal{M}_{dell}$ , we can define elliptic measures  $\omega_L^X$ ,  $X \in \Omega$ , with the usual properties.

That is, assuming  $E \in AR(d)$  and  $L = -\operatorname{div}[A(X)\nabla]$  for some  $A \in \mathcal{M}_{dell}$ , ...

# Properties of elliptic measure

- First we study weak solutions, essentially obtained by Lax-Milgram (see below).
- We prove Moser estimates, interior Hölder estimates, Harnack inequalities for solutions, Caccioppoli, Moser, Harnack, Hölder estimates on the boundary.
- We use this to prove a maximum principle, and solve the Dirichlet problem: for  $f \in C_0(E)$  we can find a solution of  $Lu = 0$  on  $\Omega$ , with  $Trace(u) = f$  on  $E$  (more about the trace soon).
- Hence by Riesz, we define, for  $X \in \Omega$ , a probability measure  $\omega^X = \omega_L^X$  such that  $u(X) = \int_E f d\omega^X$  for  $f$  and  $u$  as above.
- This measure does not depend wildly on  $X$  (by Harnack).
- We can define Green functions and estimate them.
- We prove comparison principles (at the boundary) and that  $\omega_L$  is doubling.

# Why does it still work? Elementary geometry

The main point (to me) is that, associated to the weight

$$(8) \quad w(X) = \text{dist}(X, E)^{d+1-n}$$

in  $\Omega$ , there is a weighted Sobolev space

$$(9) \quad W = \{f \in L^2_{loc}(\Omega); \nabla f \in L^2(\Omega, w(X)dX)\},$$

a good **trace operator**  $Tr : W \rightarrow H = H^{1/2}(E, d\mu)$ , and a good **extension operator**:  $Ext : H \rightarrow W$ , such that  $Tr \circ Ext = I$  on  $H$ .

Here  $\mu \in AR(E)$  and  $H$  is the analogue of the Sobolev space (half a derivative in  $L^2(E)$ ), defined by the condition

$$(10) \quad \|f\|_H^2 := \int_E \int_E \frac{|f(x) - f(y)|^2 d\mu(x) d\mu(y)}{|x - y|^{d+1}} < +\infty.$$

[For  $E = \mathbb{R}^d$ , this is the same space; proof by a Fourier transform computation.]

# Sobolev space and Poincaré estimates

The boundedness of  $Tr$  comes from integrating derivatives of  $u$  on line segments. Since  $d < n - 1$ , almost no line meets  $E$ .

The power in the weight  $w$  is chosen so that homogeneity is right.

The extension operator is the usual Whitney extension operator.

We use the same sort of estimates to evaluate  $f(x) - \text{Ext}(f)(X)$  when  $x \in E$  and  $X \in \Omega$ .

We replace barrier functions by a **weighted Poincaré estimate**

$$(11) \quad \int_{B(x,r)} |u(X)|^2 w(X) dX \leq Cr^2 \int_{B(x,r)} |\nabla u(X)|^2 w(X) dX$$

when  $x \in E$ ,  $r > 0$  and  $u \in W$  (or smoother) is **such that**  
 **$Tr(u) = 0$  on  $E \cap B(x, 2r)$ .**

Which is proved essentially as the boundedness of  $Tr$  (we chose  $w$  well, and we can find lots of paths from  $X \in \Omega \cap B$  to  $y \in E \cap B$ ).

# Weak solutions (1)

Return to a first definition of solutions for  $L = -\operatorname{div}[A(X)\nabla]$ , where  $A(X) = w(X)\mathcal{A}(X)$  satisfies the usual estimates (2) and (3) (as in (6) and (7)).

We'll write this  $A \in \mathcal{M}_{dell}(\Omega)$ , or even  $L \in \mathcal{M}_{dell}(\Omega)$ .

Define a form  $F$  on  $W \times W$ , by

$$\begin{aligned} (12) \quad F(u, v) &= \int_{\Omega} \langle A(X)\nabla u(X), \nabla v(X) \rangle w(X) dX \\ &= \int_{\Omega} \langle \mathcal{A}(X)\nabla u(X), \nabla v(X) \rangle dX. \end{aligned}$$

Notice that  $C^{-1}||u||_W^2 \leq F(u, u) \leq C||u||_W^2$ , by (6) and (7).



## Weak solutions (2)

We say that  $u$  is a **weak solution of  $Lu = 0$**  when

$$(13) \quad F(u, v) = 0 \quad \text{for } v \in W \text{ such that } \text{Tr}(v) = 0.$$

We could take  $v \in C_c^1(\Omega)$ ; it would be the same by density.  
Notice that this fits when  $u$  is smooth; integrate by parts.

Now we solve a Dirichlet-like problem with the extension operator.  
Consider  $f \in H$  and look for  $u \in W$  such that  $\text{Tr}(u) = f$  and that minimizes  $F(u, u)$ .

The trace condition defines a closed affine subspace.

Unique minimizer: by convexity and since  $F(u, u) \sim \|u\|_W^2$ .

This minimizer is a weak solution because

$$F(u + \lambda v, u + \lambda v) \geq F(u, u) \quad \text{for } v \in W; \text{Tr}(v) = 0 \text{ and } \lambda \in \mathbb{R}.$$

By the same computation, every weak solution  $u$  is a minimizer (for  $f = \text{Tr}(u)$ ).

Hence  $u$  is the only weak solution with  $\text{Tr}(u) = f$ .

Often one just says “Lax-Milgram.”

# Harmonic measure

Let us admit that the restrictions to  $E$  of  $C^1$  functions with compact support lie in  $C_0(E) \cap H$  and are dense in  $C_0(E)$  with the uniform norm. Not really hard, but let us not bother.

For such  $f$ , the solution  $u$  of  $Lu = 0$  with  $Tr(u) = f$  is continuous on  $\Omega$  (standard elliptic theory).

It is also bounded by  $\|f\|_\infty$ , because otherwise we may truncate, get a smaller energy  $F(u, u)$ , and contradict the uniqueness ( $\Omega$  is connected). This is a form of maximum principle.

Then for any given  $X \in \Omega$  the linear functional  $f \rightarrow u(X)$  is a bounded linear form on the dense class, By Riesz it extends to  $f \in C_0(E)$  and the extension is given by a finite measure  $\omega^X$ . That is,  $u(X) = \int_{y \in E} f(y) d\omega^X(y)$  as above.

This is also a way to solve the Dirichlet problem (but in fact we proceed slightly differently, and first prove good estimates for the weak solutions).

## What we'll skip here

Once we have weak solutions, we follow a standard route in elliptic theory.

We study the weak solutions, their regularity in  $\Omega$ , their behavior near  $E$ , prove the maximum principle, solve the Dirichlet problem for functions of  $C_0(E)$  (i.e., continuous that tend to 0), define  $\omega^X$  by Riesz as above, and prove additional estimates on  $\omega^X$  (such as the doubling property and change of poles properties), harmonic functions (like the comparison principle (5)), or the Green function. Barrier functions are not needed, because of the boundary Poincaré estimate (11).

## PART 2 : Absolute continuity and geometric properties

Now we worry about more specific operators, and the relations between the geometry of  $E$  and, for instance, the mutual absolute continuity of  $\omega^X$  and  $\mu$  (defined soon).

We wish to imitate Dahlberg's result, and a long collection of further ones, linking (under the NTA assumption, say) the uniform rectifiability of  $E$  and the  $A_\infty$  property of  $\omega^X$  (defined soon).

In the classical case, one either keeps  $L = \Delta$ , or one restricts to nice operators that can often be taken to be perturbations of  $\Delta$  in some way. Typically, one asks for Carleson conditions on the derivatives of  $A$ , so that  $L$  looks like a constant coefficient operator.

In particular there are counterexamples to absolute continuity when  $A$  is really matrix-valued. For instance obtained by applying quasiconformal but not bilipschitz change of variables (we'll see change of variables formulas below).

## 4. Our good operators $L$

So we take specific operators, to replace  $\Delta$  in our context.

We prefer to take scalar matrices  $A$ , i.e.,

$$(13) \quad L = -\operatorname{div}[A(X)\nabla] \quad \text{with} \quad A(X) = D(X)^{-n+d+1} \mathbf{I}_n$$

where  $D$  is a function, which we choose such that

$$(14) \quad C^{-1}\delta(X) \leq D(X) \leq C\delta(X), \quad \text{where } \delta(X) = \operatorname{dist}(X, E).$$

This way we avoid the examples with bad isotropy, and by (14)  $A$  lies in our good class  $A \in \mathcal{M}_{\text{dell}}(\Omega)$ .

The simplest would be to take  $D(X) = \delta(X)$ , but we shall prefer the smoother variants below (and in dimensions  $d > 1$  we don't have a proof of absolute continuity as below).

# The good distance $D_\alpha$

We pick  $\alpha > 0$ , choose  $\mu \in AR(d)$  on  $E$ , and set

$$(15) \quad D_\alpha(X) = R_\alpha(X)^{-1/\alpha} \quad \text{with} \quad R_\alpha(X) = \int_E |X - y|^{-d-\alpha} d\mu(y).$$

Think that  $D_\alpha(X)$  is a smoother substitute for  $\text{dist}(X, E)$ , defined as intrinsically as we could.

Homogeneity is correct :  $R_\alpha(X) \sim \delta(X)^{-\alpha}$ , so  $D_\alpha(X) \sim \delta(X)$  and  $A \in \mathcal{M}_{dell}(\Omega)$ .

With  $E = \mathbb{R}^d \subset \mathbb{R}^{d+1}$  and  $\mu = dx$ , we would get  $L = c\Delta$ .

So we take  $A$  is as nice as we can, and we expect  $L = L_\alpha$  to have good properties when the geometry of  $E$  is nice.

## 5. Classical sufficient conditions for $A_\infty$

We start again with the classical case and **uniform rectifiability**.

Recall: in the classical case, B. Dahlberg said that if  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain and  $L = \Delta$ , then  $\omega_\Delta \in A_\infty(\sigma)$ , where  $\sigma = \mathcal{H}_{|E}^{n-1}$  is the surface measure.

And D.-Jerison (and soon later S. Semmes) extended this to  $\Omega$  such that  $E \in AR(n-1)$  and the NTA condition holds.

In this case  $E$  is automatically (more than) **uniformly rectifiable**, just because of  $AR$  and the corkscrew in  $\Omega$  and the complement.

UR sets are a little more general than Lipschitz graphs or bilipschitz images of  $\mathbb{R}^d$  inside  $\mathbb{R}^n$ . One of the many definitions is by “Big Pieces of Lipschitz Images (BPLI)”: ...

# Uniform rectifiability

Let  $E \subset \mathbb{R}^d$  be Ahlfors regular of dimension  $d < n$ .

We say that  $E \in UR(d)$  when there exists  $\theta > 0$  and  $M \geq 1$  such that for  $x \in E$  and  $r > 0$ , there is a  $M$ -Lipschitz function  $f : B(0, r) \cap \mathbb{R}^d \rightarrow \mathbb{R}^n$  such that  $\mathcal{H}^d(E \cap f(B(0, r))) \geq \theta r^d$ .

- Notice: only integer dimensions  $d$  now.
- For sets  $E$  with finite diameters, just consider  $0 < \text{diam}(E)$ .
- Many other definitions exist. Characterizations by P. Jones  $\beta$ -numbers and by X. Tolsa  $\alpha$ -numbers; see below.

In general uniform rectifiability is a regularity condition on  $E$  that we like because of its invariance properties and different characterizations (geometric and analytic).



# The $A_\infty$ mutual absolute continuity condition

We give an ad hoc definition of  $A_\infty$  that corresponds to our geometric NTA case, with a basic measure  $\sigma = \mathcal{H}^d|_E$  and a collection of measures (the harmonic measures  $\omega^X$ ,  $X \in \Omega$ ), all doubling and with some change of poles properties.

The standard (true) definition of  $A_\infty$  would be slightly different but the basic properties would be the same. See the books of Journé or Garcia-Cuerva.

So here is our definition of  $\omega \in A_\infty(\sigma)$  (given the context):

$\forall \varepsilon > 0, \exists \delta > 0$  such that for  $B = B(x, r)$  centered on  $E$ ,  $X \in \Omega \cap B$  a corkscrew point for  $B$ , and  $A \subset E \cap B$  (measurable),

$$(16) \quad \mathcal{H}^d(A) \leq \delta \mathcal{H}^d(E \cap B) \implies \omega^X(A) \leq \varepsilon \omega^X(E \cap B).$$

Symmetric relation: for doubling measures as here,  $\omega \in A_\infty(\sigma)$  if and only if  $\sigma \in A_\infty(\omega)$ .

And the quantifiers  $\exists \varepsilon \in (0, 1), \exists \delta \in (0, 1)$  would work as well.

Or more precise decay estimates of  $\omega^X(A)/\omega^X(E \cap B)$  in terms of  $\mathcal{H}^d(A)/\mathcal{H}^d(E \cap B)$ .

# Sufficient conditions for $A_\infty$ when $d < n - 1$

Take  $E \in AR(d)$ ,  $\mu \in AR(d)$  on  $E$ ,  $\alpha > 0$ ,  $D_\alpha$  as in (9), and  $L = -\operatorname{div}[A(X)\nabla]$  our good operator as in (15). Consider  $\omega = \omega_L$ .

## THEOREM (DFM)

If  $E \subset \mathbb{R}^n$  is the graph of the Lipschitz function  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$  and  $\|\nabla F\|_\infty$  is small enough, then  $\omega \in A_\infty(\mu)$ .

[An extension of Dahlberg's result.]

## THEOREM (D+M, not yet written down, “beautiful” proof)

If  $E \subset \mathbb{R}^n$  is uniformly rectifiable, then  $\omega \in A_\infty(\mu)$ .

## THEOREM (J. Feneuil, not yet written either, simpler proof!)

If  $d < n - 2$  and  $E \subset \mathbb{R}^n$  is uniformly rectifiable, then  $\omega \in A_\infty(\mu)$ .

# Comments about the statements

Green = hope.

The three statements come from a statement for  $\mathbb{R}^d \subset \mathbb{R}^n$ , with more general (slightly non isotropic) coefficients, and then a change of variables. A standard idea, but we'll see that we have to respect some symmetries.

For the first theorem, the change of variables is easier to find because the geometry is simple.

For the second one, the method is a natural extension of the previous one (do suitable parameterizations of UR sets), but technically very complicated because we have to cut. Some comments later.

For the third one, a miracle occurs which simplifies the algebra, but so far does not work in co-dimension 2.

Anyway, we will first study operators  $L$  in the simple situation of  $E = \mathbb{R}^d \subset \mathbb{R}^n$ .

## 6. A sufficient condition on $A$ when $E = \mathbb{R}^d \subset \mathbb{R}^n$

So we take  $E = E_0 = \mathbb{R}^d \subset \mathbb{R}^n$ ,  $\Omega = \Omega_0 = \mathbb{R}^n \setminus \mathbb{R}^d$  and try to prove that  $\omega \in A_\infty(\sigma)$  for a large class of coefficients  $A \in \mathcal{M}_{dell}(\Omega_0)$ . Here  $\sigma$  is the Lebesgue measure on  $\mathbb{R}^d$ .

Write the variable  $(x, t) \in \mathbb{R}^n \times \mathbb{R}^{n-d}$ , and take  $L = -\operatorname{div}[A(X)\nabla]$  and  $A(x, t) = |t|^{-n+d+1}M(x, t)$ , with

$$(17) \quad M(x, t) = \begin{pmatrix} M^1(x, t) & C^2(x, t) \\ C^3(x, t) & b(x, t)I_{n-d} + C^4(x, t) \end{pmatrix},$$

where  $M^1(X)$  is an elliptic  $d \times d$  matrix, the  $|C^j|^2$  satisfy Carleson measure estimates,  $I_{n-d}$  is the identity matrix, and  $b$  is a function on  $\Omega_0$  such that  $C^{-1} \leq b \leq C$  and  $|t|^2|\nabla b|^2$  satisfies a Carleson measure estimate. [More general than usual even when  $d = n - 1$ .]

### THEOREM (DFM)

Under these assumptions,  $\omega_L$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}^d$ , with an  $A_\infty$  density.

# Definitions and comments

We say that  $f$  satisfies a **Carleson estimate** when

$dm = f(x, t) \frac{dxdt}{|t|^{n-d}}$  is a **Carleson measure on  $\Omega$** . That is,

$$(18) \quad m(\Omega \cap B(x, r)) \leq Cr^d \text{ for } x \in E \text{ and } r > 0.$$

The proof uses

- A (localized) estimate like  $\|Su\|_2^2 \leq C\|Nu\|_2^2$  for solutions: integrations by parts and Carleson measure estimates;
- As a consequence, Carleson measure estimates on the square function of solutions of the Dirichlet problem with bounded boundary values on  $\mathbb{R}^d$ ;
- A general argument, roughly as in Kenig-Kirchheim-Pipher-Toro or Dindos-Petermichl-Pipher, to go from there to  $A_\infty$ .

Carleson measures are usual to measure “small most of the time”.

It is important in the proof that  $b(x, t)/|t|^{n-d}$  acts the same way in all directions orthogonal to  $\mathbb{R}^d$ . We'll see some part.

# Proof (1) : Maximal functions and Carleson's theorem

Here easier geometry:  $E = \mathbb{R}^d \subset \mathbb{R}^n$ ; but this part goes through.  
For  $F : \Omega \rightarrow \mathbb{R}$ , define a nontangential maximal function  $\mathcal{M}F$  by

$$(19) \quad \mathcal{M}F(x) = \sup \{ |F(y, t)| ; |t| > |x - y| \} \quad \text{for } x \in E.$$

Carleson's theorem says that if  $\mu$  is a Carleson measure on  $\Omega$ ,

$$(20) \quad \int_{\Omega} |F(X)|^p d\mu(X) \leq C_{\mu} \int_E \mathcal{M}F(x)^p dx$$

This is true for any  $p > 0$ ; in fact

$$\mu(\{X ; F(X) > \lambda\}) \leq C_{\mu} |\{x ; \mathcal{M}F(x) > \lambda\}| \quad \text{for } \lambda > 0.$$

Easy proof with a covering of  $\{X ; F(X) > \lambda\}$  and the geometry of tents.

Often  $\mathcal{M}F(x)$  is itself bounded by the Hardy-Littlewood maximal function of some  $g$ . For instance if  $F$  is a Poisson extension of  $g$ .

## Proof (2) : Square function and Maximal function

Two important quantities for solutions  $u$  of  $L$ , that one likes to estimate in term of each other. Assume here that the functions are smooth and all the integrals converge.

Call  $\gamma(x) = \{(y, t) \in \Omega; |t| > |x - y|\}$  the access cone above  $x$ .

The square function (of  $u$ ) is given by

$$(21) \quad Su(x)^2 = \int_{(y,t) \in \gamma(x)} |\nabla u(y, t)|^2 \frac{dydt}{|t|^{n-2}}$$

And the maximal function is

$$(22) \quad Nu(x) = \mathcal{M}u(x) = \sup_{(y,t) \in \gamma(x)} u(y, t)$$

Normally we should play with apertures and truncate, but let us just cheat.

## Proof (3) : The “usual” estimates

The  $A_\infty$  estimate follows from a “well-known” general plan, where we first prove that

$$(23) \quad \|Su\|_2 \leq C \|Nu\|_2 \quad \text{or equivalently} \quad \|Su\|_2^2 \leq C \|Nu\|_2 \|Su\|_2$$

and in fact a local Carleson measure version of that.

This is our main estimate and we'll try to give hints.

Then one proves a weaker version of a Carleson measure estimate for solutions of a Dirichlet problem (with bounded boundary value  $f$ ), except that we concentrate on characteristic functions  $f = \mathbb{1}_H$ ,  $H \subset E$ , and require that the extension  $u_H(X) = \omega^X(H)$  satisfies the Carleson condition

$$(24) \quad |t|^{-2} |\nabla u_H(y, t)|^2 \frac{dy dt}{|t|^{n-d}} \text{ is a Carleson measure.}$$

....



## The “usual” estimates (2)

... Finally one proves that  $\omega \in A_\infty(\sigma)$  as soon as we have the estimate (24).

The second and third part are less surprising to the experts. Owing to Kenig, Kirchheim, Koch, Pipher, Toro, and then Dindoš, Petermichl, and Pipher. I'll pretend to understand, skip, and concentrate on the part (23) that contains the main assumptions and Carleson estimates!

Except for the fact that I don't believe this leads to a proof, you can see (23), or rather its version (25) below (localized on a cube  $Q$ ) as connected to the desired harmonic measure estimate. We are given  $E \subset Q$  and we consider the solution  $u = \omega^X(E)$ . We know that if  $\mu(E)/\mu(Q)$  is small the integral of its boundary values is small, and we can hope the same for  $\|N^Q u\|_2$ . If we have nice small bounds for  $S^Q$  as in (25), we can expect to integrate  $\nabla u$  on lots of paths between  $Q \setminus E$  and a small ball  $B$  centered on a corkscrew point  $X_0$  for  $Q$ , and get that  $u(X_0)$  is small, as needed.

# The square function estimate (1)

Now watch me integrate by parts!

Recall we want a local version of (23), localized by a cube  $Q \subset \mathbb{R}^d$ , of the form

$$(25) \quad \|S^Q u\|_2 \leq C \|N^Q u\|_2$$

where  $S^Q u$  is defined by

$$(26) \quad S^Q u(x)^2 = \int_{(y,t) \in \gamma(x) \text{ and } |t| \leq \text{diam}(Q)} |\nabla u(y, t)|^2 \frac{dy dt}{|t|^{n-2}}$$

for  $x \in Q$  and  $S^Q u(x) = 0$  otherwise (say).

We integrate on  $x \in Q$  to get  $\|S^Q u\|_2^2$ , and use Fubini.

Notice that  $y \in CQ$  because  $|y - x| \leq |t| \leq \text{diam}(Q)$ .

Integrating in  $x$  adds an extra  $\text{diam}(Q)^d$ ; thus...

## The square function estimate (2)

$$\begin{aligned} \|Su\|_2^2 &= \int_{x \in Q \subset E} \int_{(y,t) \in \gamma(x); |t| \leq \text{diam}(Q)} |\nabla u(y, t)|^2 \frac{dy dt dx}{|t|^{n-2}} \\ &\leq C \int_{y \in CQ, |t| \leq \text{diam}(Q)} |\nabla u(y, t)|^2 \frac{dy dt}{|t|^{n-d-2}} \\ &\leq C \int |\nabla u(y, t)|^2 \psi(y, t)^2 \frac{dy dt}{|t|^{n-d-2}} \end{aligned}$$

where  $\psi = \psi_Q$  is an appropriate cut-off function that forces  $y \in CQ$  and  $|t| \leq \text{diam}(Q)$ .

The square in  $\psi(X)^2$  is more convenient in computations later.

Also, we can make  $\psi$  compactly supported in  $\Omega$ , at almost no cost (but  $\psi$  will have a large derivative in a thin region near  $E$ ).

Finally, we will cheat, give the same name to different cut-off functions  $\psi$ , and not worry about the difference.

## The square function estimate (3)

Recall that  $\mathcal{A}(X) = \delta(X)^{n-d-1} A(X)$  is elliptic. So

$$\begin{aligned} |\nabla u(y, t)|^2 &\leq C \langle \mathcal{A}(y, t) \nabla u(y, t), \nabla u(y, t) \rangle \\ &= C |t|^{n-d-1} \langle A(y, t) \nabla u(y, t), \nabla u(y, t) \rangle. \end{aligned}$$

Also recall that  $b(y, t) \sim 1$  by assumption. Hence...

$$\|Su\|_2^2 \leq C \int |\nabla u(y, t)|^2 \psi(y, t)^2 \frac{dy dt}{|t|^{n-d-2}} \leq CJ,$$

with

$$(27) \quad J = \int \langle A(y, t) \nabla u(y, t), \nabla u(y, t) \rangle \psi(y, t)^2 |t| \frac{dy dt}{b(y, t)}.$$

In due time, in some estimate  $\frac{1}{b(y, t)}$  will cancel with a  $b(y, t)$  coming from  $A$ .

## The square function estimate (4)

Now a soft integration by parts.

Set  $\varphi = u\psi^2|t|\frac{1}{b(y,t)}$ . This is a test function, so

$$(28) \quad \int \langle A(y, t) \nabla u(y, t), \nabla \varphi(y, t) \rangle dy dt = 0$$

We write  $\nabla \varphi$  as a sum of terms; (28) shows that a sum of integrals is 0, and the first one will be  $J$ .

$$\nabla \varphi = \nabla u \psi^2 |t| \frac{1}{b} + 2u\psi \nabla \psi \frac{1}{b} + u\psi^2 \nabla |t| \frac{1}{b} + u\psi^2 \nabla b \frac{|t|}{b(y, t)^2}$$

This yields

$$J \leq J_1 + J_2 + J_3,$$

and we have three integrals to estimate (and we used one assumption).

## The square function estimate (5)

We start with  $J_3$ . We will use the Carleson condition on  $\nabla b$ . We have to estimate

$$\begin{aligned} J_3 &= \int \left| \langle A \nabla u, \nabla b \rangle u \psi^2 \frac{|t|}{b^2} \right| dy dt \\ &\leq C \int |A| |\nabla u| u \psi(y, t)^2 |\nabla b| |t| dy dt \\ &\leq C \int |\nabla u| u \psi^2 |\nabla b| |t|^{2+d-n} dy dt \end{aligned}$$

We apply Cauchy Schwarz (we always do):

$$J_3 \leq C \left\{ \int |\nabla u|^2 \psi^2 |t|^{2+d-n} dy dt \right\}^{1/2} \left\{ \int u^2 \psi^2 |\nabla b|^2 |t|^{2+d-n} dy dt \right\}^{1/2}.$$

The first part is almost the same as  $\|S^Q u\|$  (slightly larger set of integration, but we can manage). To be eaten by left-hand side.

## The square function estimate (6)

Recall that  $|t|^2|\nabla b|^2|t|^{d-n}dydt$  is a Carleson measure. So by the Carleson theorem (20) the second part is at most

$$C\left\{\int_E |\mathcal{M}(u\psi)|^2\right\}^{1/2} \leq C\|N^Q u\|.$$

This yields the desired estimate for  $J_3$  (compare with (23)).

The term  $J_1$  is almost the same. This time we use the fact that  $|\nabla\psi|^2|t|^{d-n}dydt$  is a Carleson measure, which one can prove by hand with the definition of  $\psi$ .

# The square function estimate (7)

We are left with

$$J_2 = \left| \int \langle A \nabla u, \nabla |t| \rangle \frac{u \psi^2}{b} dy dt \right|$$

and here we use the decomposition of  $A$ .

First there are the three Carleson pieces  $A^i$  coming from the  $\mathcal{C}^i$ . That is,  $|A^i| = |t|^{d+1-n} |\mathcal{C}^i|$  and  $t^{d-n} |\mathcal{C}^i|^2$  is a Carleson measure. Then

$$J_2^i \leq C \int |A^i| |\nabla u| u \psi^2 dy dt = C \int |\mathcal{C}^i| |\nabla u| u \psi^2 |t|^{1+d-n} dy dt$$

because  $|\nabla |t|| = 1$ . We apply Cauchy-Schwarz as usual:

$$J_2^i \leq C \left\{ \int |\nabla u|^2 \psi^2 |t|^{2+d-n} dy dt \right\}^{1/2} \left\{ \int u^2 \psi^2 |\mathcal{C}^i|^2 |t|^{d-n} dy dt \right\}^{1/2}.$$

The first part is almost like  $\|S^Q u\|$ , and for the second we apply Carleson's theorem as above!



## The square function estimate (8)

We are left with  $J^0$  coming from the roughly diagonal terms in  $A$ . The top  $d \times d$  piece  $M^1$  does not contribute because  $\nabla|t| = t/|t|$  lives in  $\mathbb{R}^{n-d}$ . We are left with the piece  $|t|^{d+1-n} b l_{n-d}$  on the bottom. This part of  $\langle A \nabla u, \nabla|t| \rangle$  is equal to  $b|t|^{d+1-n} \sum_j \frac{\partial u}{\partial t_j} \frac{t_j}{|t|}$ , the two  $b$  cancel as promised, and

$$J_2 = \left| \int u \psi^2 \sum_j \frac{\partial u}{\partial t_j} \frac{t_j}{|t|} |t|^{d+1-n} dy dt \right|.$$

Time to use the (important) radial invariance!

Notice that the radial derivative of  $u^2$  is  $\partial_\rho(u^2) = 2u \sum_j \frac{\partial u}{\partial t_j} t_j / |t|$ , so the integrand looks like  $|t|^{d+1-n}$  times a radial derivative.

We claim that

$$J_2^0 := \int \partial_\rho(u^2 \psi^2) |t|^{d+1-n} dy dt = 0.$$

## The square function estimate (9)

Recall the claim :  $J_2^0 := \int \partial_\rho(u^2\psi^2)|t|^{d+1-n}dydt = 0$ .

Fix  $y$  and integrate only on  $t \in \mathbb{R}^{n-d}$ . Then integrate in “spherical” coordinates; the weight disappears and we get

$$J_2^0(y) = c_{n-d} \int_{\xi \in \partial B(0,1)} \int_{s>0} \partial_\rho(u^2\psi^2)(t\xi) ds d\xi.$$

Now the interior integral is the integral of  $\partial_\rho(u^2\psi^2)$  on a ray. Vanishes because  $\psi$  is compactly supported away from  $\mathbb{R}^d$ . Whence the claim.

Compare  $J_2$  with  $J_2^0$ ; we get

$$J_2 = \left| \int u^2\psi \sum_j \frac{\partial\psi}{\partial t_j} \frac{t_j}{|t|} |t|^{d+1-n} dydt. \right|$$

Controlled as usual because  $\nabla\psi$  gives a Carleson measure.

End of proof (almost). All assumptions were used.

Some approximate rotation invariance in  $\mathbb{R}^{n-d}$  seems needed.

## 7. The general case; first a generic change of variable

We want to see how the equation  $Lu = 0$  on  $\Omega$  transforms.

Write  $u = v \circ g$ , where  $g : \Omega \rightarrow \Omega_0$  is a (local) diffeomorphism.

For the derivative,  $Du(Y) = Dv(g(Y)) \circ Dg(Y)$ ; we identify the derivatives with matrices, so  $Du$  and  $Dv$  are line matrices, and  $\nabla u$  and  $\nabla v$  are their transposed matrices. Say,  $\nabla u = Du^*$ .

Also let  $\varphi$  be a test function on  $\Omega$  and  $\psi = \varphi \circ g$  the corresponding test function on  $\Omega_0$ . Set  $X = g(Y)$ ; then

$$\begin{aligned}\langle A(Y)\nabla u(Y), \nabla \varphi(Y) \rangle &= \nabla \varphi(Y)^* A(Y) \nabla u(Y) \\ &= D\varphi(Y) A(Y) Du(Y)^* \\ &= D\psi(X) Dg(Y) A(Y) Dg(Y)^* Dv(X) \\ &= \langle \tilde{A}(X) \nabla v(X), \nabla \psi(X) \rangle\end{aligned}$$

with  $\boxed{\tilde{A}(X) = Dg(Y) A(Y) Dg(Y)^*}$ .

## Formula for the changes of variable (2)

Let  $f : \Omega_0 \rightarrow \Omega$  be the inverse of  $g$ . So  $Y = f(X)$ .

Recall that  $u$  is a weak solution when for every  $\varphi$ ,

$$\begin{aligned} 0 &= \int_{\Omega} \langle A(Y) \nabla u(Y), \nabla \varphi(Y) \rangle dY = \int_{\Omega} \langle \tilde{A}(X) \nabla v(X), \nabla \psi(X) \rangle dY \\ &= \int_{\Omega_0} \langle \tilde{A}(X) \nabla v(X), \nabla \psi(X) \rangle Jf(X) dX \end{aligned}$$

where  $Jf$  is the Jacobian determinant of  $f : \Omega_0 \rightarrow \Omega$ . So  $u$  is a weak solution for  $L$  when  $v$  is a weak solution for  $L_0$ , and the relation between  $A$  and the matrix  $A_0$  for  $L_0$  is

$$\begin{aligned} A_0(X) &= Jf(X) Dg(Y) A(Y) Dg(Y)^* \\ &= Jf(X) Df(X)^{-1} A(f(X)) (Df(X)^{-1})^*. \end{aligned}$$

# Comments on the changes of variable formula

- If we start from a scalar matrix  $D(Y)^{d+1-n}I_n$  as we like, we still get a complicated matrix

$$(29) \quad A_0 = (D \circ f)^{d+1-n} Jf \left[ Df^{-1}(Df^{-1})^* \right]$$

which is symmetric but not scalar!

Logical: quasiconformal mappings do not hurt the ellipticity of  $A$ , but mess up its symmetry and can be bad for the absolute continuity of  $\omega_L$ .

- We start from the nice domain  $\Omega$ , and need to find a good mapping  $f = \Omega_0 \rightarrow \Omega$ , so that when we compute  $A_0$  in (29), we find one of the good matrices of our previous theorem!

So we'll be need to be careful, in particular so that  $f$  almost acts in an isotropic way (i.e., almost like a dilation) in the orthogonal variables  $t \in \mathbb{R}^{n-d}$ . Trivial when  $d = n - 1$ , but not here!

## 8. The proof of the Dahlberg theorem

Recall we claim that for  $L = \operatorname{div} D_\alpha^{d+1-n} \nabla$ ,  $\omega_L \in A_\infty(\mu)$  when  $D_\alpha$  is one of our nice distance functions in (15),  $E$  is the graph of a small Lipschitz function  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ , and  $\mu = \mathcal{H}_E^d$ .

You guessed the idea: find a nice bilipschitz change of variable  $f : \Omega_0 \rightarrow \Omega$  (where  $\Omega_0 = \mathbb{R}^n \setminus \mathbb{R}^d$  as before), and prove that  $A_0$  in (29) satisfies the assumptions of the previous theorem.

And it fits just right.

Bilipschitz ensures that  $A_0$  has the right size, and also that  $A_\infty$  transfers through  $f$ , because  $f_*\mu$  is AR on  $E$ .

The main point is that, with errors whose squares satisfy a Carleson condition, the mapping  $f$  respects the directions  $t$  perpendicular to  $E$  and acts as a dilation there.

Very amusing that in the geometry below, Square Carleson conditions are the right thing too.

# How do we parameterize $E$ and $\Omega$ ? (1)

Still think about the graph of  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ . In the past, changes of variables with simple parameterizations of  $\Omega$  by  $\Omega_0$  were used to solve elliptic problems.

The simplest is  $f(x, t) = (x, t + F(x))$ . Works some times, but not so smooth, which creates problems.

More careful :  $f(x, t) = (x, t + \varphi_{|t|} * F(x))$ , where the convolution with  $\varphi_{|t|}$  is a smooth approximation of identity (less smooth near the boundary, as it should be).

But this is not good enough yet. Even when  $F$  is a linear mapping (important case), these two mappings are the same and move points sideways. They do not try to preserve the tangent and normal directions, or act as a dilatation in the  $t$ -variables.

## How do we parameterize $E$ and $\Omega$ ? (2)

Fortunately, there is another standard way to parameterize sets  $E$  (and their complement  $\Omega$ ), that comes from the [Reifenberg topological disk theorem](#).

Usually the set  $E$  is just Reifenberg flat ( $\varepsilon r$ -good approximation of  $E$  by a  $d$ -plane in each ball  $B(x, r)$  centered on  $E$ , with a small fixed  $\varepsilon$ ), and the parameterization is not bilipschitz. But when  $E$  is nice the constructed mapping is bilipschitz too.

The parameterization is constructed as a composition of finer and finer approximations (at scales  $10^{-k}$ ), using slowly moving approximations by planes, the associated projections, and partitions of unity.

Here  $E$  is sufficiently nice for this to work, and we can even find a formula (using rotations  $R_{|t|}(x)$ ) without iterations.

Anyway, thinking about the approximate tangent direction and forgetting about horizontal and vertical helps.



# Where is the geometry of $E$ ?

Formulas are simpler when  $E$  is a Lipschitz graph, but most of the Carleson estimates on  $Df$  come from estimates on the [P. Jones  \$\beta\$ -numbers](#), that actually work for general uniformly rectifiable sets. Here is a definition.

Let  $E \in AR(d)$  and  $\mu \in AR(d)$  be supported on  $E$ . Take  $E$  unbounded for convenience. Let  $1 \leq q < +\infty$  be given (unfortunately,  $p = +\infty$  does not work well here when  $d \geq 2$ ).

For  $x \in E$  and  $r > 0$  set

$$(30) \quad \beta_q(x, r) = \inf_P \left\{ r^{-d} \int_{E \cap B(x, r)} [r^{-1} \operatorname{dist}(y, P)]^q d\mu(y) \right\}^{1/q}$$

where the infimum is taken over all affine  $d$ -planes  $P$  (through  $x$  if you really insist).

Dimensionless numbers. You may also account for holes with a bilateral variant of  $\beta_q$ .

## Beta numbers (2)

It turns out [D.-Semmes] that for  $q > \frac{2d}{d-2}$ ,  $E$  is uniformly rectifiable if and only if the  $\beta_q(x, r)^2$  satisfy a Carleson condition. That is, if

$$(31) \quad \int_{x \in B(x_0, r_0)} \int_{0 < r < r_0} \beta_q(x, r)^2 \frac{d\mu(x) dr}{r} \leq Cr_0^d$$

for  $x_0 \in E$  and  $r_0 > 0$ . So  $E$  is often close to a  $d$ -plane.

For Lipschitz graphs and  $q = 2$  (hence  $1 \leq p \leq 2$  by Hölder), this is easy and comes from a Fourier computation [J. Verdera, I think]. And here  $q = 1$  is enough I think.

## X. Tolsa's Alpha numbers

But many of our computations also concern the variations and approximation of the smooth distance  $D_\alpha$  by distances to  $d$ -planes, or it turns out, approximation in balls  $B(x, r)$  of our  $AR$  measure  $\mu$  by flat measures  $\lambda_P$ .

A flat measure is a positive multiple of  $\mathcal{H}^d|_P$  (or the Lebesgue measure on  $P$ ), for an affine  $d$ -plane  $P$ .

This comes in computations where, in the formula (15) for  $R_\alpha$  and  $D_\alpha$ , one replaces  $\mu$  with a  $\lambda_P$  to get an approximation of  $D_\alpha$ .

The distance is measured by a variant of the Wasserstein 1-distance of optimal transportation: Given measures  $\mu, \lambda$  and a ball  $B(x, r)$ , we measure the distance

$$(32) \quad d_{x,r}(\mu, \sigma) = r^{-d-1} \sup_f \left| \int f d\mu - \int f d\sigma \right|,$$

where the sup is taken over all 1-Lipschitz functions  $f$  that are supported on  $B(x, r)$ , and  $r^{-d-1}$  is a normalization to make  $d_{x,r}(\mu, \sigma)$  and  $\alpha(x, r)$  below dimensionless.

## Alpha numbers (2)

Xavier Tolsa sets

$$(33) \quad \alpha(x, r) = \alpha_\mu(x, r) = \inf_{\lambda flat} d_{x,r}(\mu, \sigma).$$

And again proves (among other things) that if  $E \in AR(d)$  and  $\mu \in AR$  is supported by  $E$ ,  $E$  is uniformly rectifiable if and only if the  $\alpha_\mu(x, r)^2$  satisfy a Carleson condition (same definition as above).

It looks like this could have depended on  $\mu$ , but not really. But the case of a density measure on the plane is already interesting!

Again, easier for small Lipschitz graphs. And notice the square!

Here we only use one direction, but it is good to know that we have the right condition.

In summary: estimates on the  $\beta$  and  $\alpha$  numbers are used a lot in the proof and they hold for general uniformly rectifiable sets. The small Lipschitz graph property is used to find a formula that gives a bilipschitz mapping.

## $\omega \in A_\infty$ for the uniformly rectifiable sets

The case of a general uniformly rectifiable set looks much harder. At least when  $d = n - 2$  when J. Feneuil does not have an incredible trick.

The plan is to follow the same general route, but it becomes technical: we only have good parameterizations in “stopping time regions”, so we need arguments to glue results from different regions. Some geometry (we need some good control on the algorithm of [D.-Toro]), plus a (complicated to me) trick called “extrapolation,” plus a way to compare domains because sawtooth domains do not have a large co-dimensional boundary.

So, not for here [and also, to be written!]

This feels good because the UR condition seems to be optimal. It was (modulo finesses with the NTA condition) in codimension 1.

## PART 3. Converse, magic, other quantities

Taking the  $A_\infty$  result for UR for granted, [what about the converse?](#)

That is, take  $D_\alpha$  and assume that  $\omega_L \in A_\infty(\mu)$ ; can we say that  $E$  is uniformly rectifiable?

Can  $\omega_L \in A_\infty(\mu)$  ever happen when  $d \notin \mathbb{N}$ ?

Recall: this is OK but quite hard in co-dimension 1, with the help of NTV on the boundedness of the Riesz transform.

Even assume that  $\mu = \sigma := \mathcal{H}^d|_E$  if you want.

A statement of order 0, corresponding to a blow-up limit at a point of density of  $\frac{d\omega}{d\sigma}$  would be welcome: suppose  $\omega^\infty = \sigma$ ; is  $E$  flat?

Later in the lectures: other relevant properties, either geometric or connected to solutions  $L = L_\alpha$ .

## 9. Magic ALPHA (D.-Engelstein-Mayboroda)

We (with Max Engelstein) found a special case where things are simpler, and we expected that the converse would be easier to prove in this case, i.e., when

$$(34) \quad \alpha = n - d - 2 \quad \text{hence } n > d + 2, \text{ but } d \notin \mathbb{N} \text{ is allowed}.$$

Then  $R_\alpha(X) = \int_E \frac{d\mu(y)}{|X - y|^{n-2}}$  is harmonic, and by a computation

$$(35) \quad D_\alpha(x) = R_\alpha(x)^{-1/\alpha} \text{ satisfies } Lu = 0.$$

Assume to simplify that  $E$  is rectifiable and  $\mu = \sigma$ .

By another computation,  $\frac{\partial D}{\partial n}$  (the normal derivative) is constant on  $E$ . Hence  $D$  is the Green function for  $L$ , with pole at  $\infty$ .

That is,  $\omega^\infty = \sigma$ .

Very strange! (Think about paint and the Brownian paths.)

Even when  $E$  is merely Ahlfors regular and  $\mu \in AR(d)$ , we get that  $\omega \ll \sigma$ , with a density  $h$  such that  $C^{-1} \leq h \leq C$ .

## Ask other questions then?

Not at all what we expected!

Does what happens when  $\alpha = n - d - 2$  mean something special?

Now we (maybe) conjecture that  $\omega_L \in A_\infty(\sigma)$  implies  $E \in UR$  as soon as  $\alpha \neq n - d - 2$ . But we have a hard time doing this.

In the mean time, try to characterize  $UR$  in terms of other properties connected to  $D_\alpha$ , or  $L$ .

Examples below, but probably we'll concentrate on the Green function because it is connected to  $L_\alpha$ , even though we don't have full answers.

But all that follows about the Green function is subject to at least more checking!

Feneuil above ( $E \in UR$  implies  $\omega_L \in A_\infty(\sigma)$  when  $n > d + 2$ ) uses information from the magic exponents!



## 10. The Green function

Let  $E \in AR(d)$ ,  $\mu \in AR(d)$  supported on  $E$ , and  $L = \operatorname{div} D_\alpha^{d+1-n} \nabla$  be given. For  $X \in \Omega = \mathbb{R}^n \setminus E$ , there is a Green function  $G^X$ , associated to  $\omega^X$ .

Let us only say that  $G^X$  is  $L_\alpha$ -harmonic in  $\Omega \setminus \{X\}$ , positive, has a vanishing trace on  $E$ , and has a reasonable singularity at  $X$ , i.e., looks like  $-\log(|Y - X|)$  when  $n = 2$  and  $|Y - X|^{2-n}$  when  $n > 2$ .

This makes  $G^X$  unique modulo a multiplicative constant. And it has the same sort of estimates as in the classical case, in particular relative to changes of pole  $X$ .

And by a limiting process, we can define a Green function with pole at  $\infty$ ,  $G = G^\infty$ , which is harmonic on  $\Omega$ , positive, with a reasonable size at  $\infty$ , and vanishes on  $E$ .

Essentially unique too (see below for more about this and  $G^\infty$ ).

Classically, in the smooth case, the density of  $\omega^X$  against  $d\sigma$  is a constant times the normal derivative  $\frac{\partial G^X}{\partial n}$ .

# The Green function and $D_\alpha$

There are very few cases where we can compute  $G$ . Here are the two main ones, where in addition  $G = D_\alpha$ .

Case 1. When  $E = \mathbb{R}^d$ ,  $\mu = \mathcal{H}_{|E}^d$ , and for any  $\alpha > 0$ . Then  $G(Y) = cD_\alpha(Y) = c' \text{dist}(Y, E)$ .

Just by invariance and essential uniqueness.

Case 2. When  $d < n - 2$  and  $\alpha = n - d - 2$  is magic, because then  $D_\alpha$  is  $L_\alpha$ -harmonic and by essential uniqueness.

Natural Question 1: conversely, if  $D_\alpha$  is  $L_\alpha$ -harmonic, are we in Case 1 or Case 2?

Comment: if  $D_\alpha$  is  $L_\alpha$ -harmonic, it is the Green function.

So we really ask when the Green function  $G^\infty$  is a multiple of  $D_\alpha$ .

## 10. When is $D_\alpha$ a $L_\alpha$ -harmonic function?

We can at least start to compute.

Suppose  $L_\alpha D_\alpha = 0$ , i.e.,  $\operatorname{div}[D_\alpha^{d+1-n} \nabla D_\alpha] = 0$ .

Suppose  $d \neq n - 2$ , let  $\gamma = d + 2 - n$ , and consider  $D_\alpha^\gamma$ .

Then  $\nabla D_\alpha^\gamma = \gamma D_\alpha^{\gamma-1} \nabla D_\alpha = \gamma D_\alpha^{d+1-n} \nabla D_\alpha$  hence  $\Delta D_\alpha^\gamma = 0$ .

So we found a plain harmonic function,  $D_\alpha^\gamma$ , to work with!

Two main cases:  $d < n - 2$  and  $d > n - 2$ .

[We expect  $d = n - 2$  to be more complicated.]

## When $d < n - 2$ and $\gamma$ is negative

[The following needs to be checked!]

Set  $D = D_\alpha = D_{\alpha,\mu}$ . So  $\Delta D^\gamma = 0$ , with  $\gamma = d + 2 - n < 0$ .

Given the size of  $D^\gamma$  (controlled singularity near  $E$ ), we claim that there is a representation of  $D^\gamma$  as a potential from charges on  $E$ ,

$$(36) \quad D^\gamma(X) = \int_{x \in E} \frac{d\nu(x)}{|X - x|^{n-2}},$$

and even with a measure  $\nu \in AD(d)$  supported on  $E$ . [At least we can easily check that the left-hand side is harmonic and has about the right size; then (36) should be a matter of taking weak limits and integrating by parts.]

Recall that the magic number is  $\alpha_0 = n - d - 2 = -\gamma$  and call  $D_0 = D_{\alpha_0,\nu}$  the distance associated to  $\alpha_0$  and  $\nu$ ; (36) says that

$$D(X) = D_\alpha(X) = \left\{ \int_E \frac{d\nu(x)}{|X - x|^{n-2}} \right\}^{-1/\alpha_0} = D_0(X).$$

This naturally leads to our second question:

## Question 2

Natural Question 2: Suppose that  $D_{\alpha,\mu} = D_{\beta,\nu}$ , with  $\mu, \nu \in AD(d)$  and  $\alpha \neq \beta$ , can we say that  $d$  is an integer and  $E$  is a  $d$ -plane?

Here we are mostly interested in the case when  $d < n - 2$  and  $\beta = n - d - 2 = -\gamma$  is magic. Also, if this happens, we can continue the argument, compute the asymptotics of  $D_\alpha, D_\beta$  near  $E$ , and compute  $\nu$  in terms of  $\mu$ , so that for instance if  $\mu = \mathcal{H}_{|E}^d$  we get  $\nu = C\mu$ .

As for Question 1, this seems quite plausible but we have no clue! Both question are about different expressions involving convolutions of  $\mu, \nu$  with the usual Green kernel (and maybe its derivatives), and “miraculous algebraic relations” between those, to be satisfied on  $\mathbb{R}^n \setminus E$ .

## When $d > n - 2$ and $\gamma$ is positive

Here  $D^\gamma$ , with a positive exponent, is Hölder near  $E$  and is (a multiple of) the Green function for  $\Delta$ . This uniqueness is, as for  $G^\infty$  above, a refinement of the comparison principle that says that when two positive harmonic functions  $u, v$  are defined on a much larger ball than  $B$ , then  $u/v$  varies very little on  $B$ .

We claim that, thanks to a relation between the Green function and the harmonic measure, we have the estimate

$$C^{-1}r^d \leq \omega^X(B(y, r)) \leq Cr^d \text{ when } X \notin B(y, 10r),$$

with  $C$  independent of  $X, y, r$ . That is, the (standard!) harmonic measure is locally AR, and is absolutely continuous with respect to  $\mu$ , with a weight (locally) bounded and bounded from below.

# Summary of claims

At this point:

When  $n - 2 < d < n - 1$ , I thought I had an argument, but I get confused about dimensions so let's not claim anything, except that having an  $A^\infty$  harmonic measure like that for a low dimensional boundary is strange.

When  $d = n - 1$ , the classical case, recent results show that since  $\omega \in A^\infty(\mu)$ , so  $E$  is uniformly rectifiable. This will be enough for the converse later. Probably, having  $G^\infty = CD_\alpha$  implies that  $E$  is flat, but no proof that I recall.

When  $n - 1 < d < n$ , we assume one-sided NTA, and then our initial problem also makes sense. And we claim a contradiction, because the Green function  $G^\infty$  is not allowed to be that small at a point of contact of a ball  $B$  with  $E$ . See below.

# Some Green function estimates (1)

We start with the following estimate, for  $B = B(x, r)$  centered on  $E$ ,  $X_0$  a corkscrew point for that ball, and  $X \in \Omega \setminus B(x, 10r)$ :

$$\omega^X(B \cap E) \simeq r^{n-2} G^X(X_0)$$

with an equivalence constant that depends only on the geometric constants. In the case of our  $L = L_\alpha$ , it would be

$$\omega^X(B \cap E) \simeq r^{d-1} G^X(X_0).$$

Fix  $X$  for simplicity, and use the comparison principle on the functions  $G^X$  and  $G^\infty = CD^\gamma$ ; this gives

$$\omega^X(B \cap E) \simeq r^{n-2} G^\infty(X_0) \simeq r^{n-2+\gamma} = r^d$$

with constants that depend on  $X$  and the choice of  $G^\infty$ , but not on  $x, r$  as long as  $B(x, r)$  stays in a fixed ball depending on  $X$ .

As we claimed. And with  $L_\alpha$ , a similar computation would have given  $\omega_L^X(B \cap E) \simeq r^d$  too.



## Green function estimates (2)

Now pick a touching ball  $B = B(X_0, \delta(X_0))$ .

Consider the Green function  $G_B^{X_0}$  (for the operator  $\Delta$ ) and  $G = D^\gamma$  (we assume it is harmonic).

Apply the maximum principle to these functions on the domain  $A = B \setminus B(X_0, \delta(X_0)/2)$ . On the interior boundary,  $G_B^{X_0} \leq CG$ , just by Harnack. We don't care if  $C$  is huge.

On  $\partial B$ ,  $G_B^{X_0} \leq CG$  too because  $G_B^{X_0} = 0$  and  $G \geq 0$ .

So  $G_B^{X_0} \leq CG$ .

Now assume that  $G = D^\gamma$  with  $\gamma > 1$ . Then near the point  $\xi$  of  $E \cap \partial B$ ,  $G(X) \leq C|X - \xi|^\gamma$ , while  $G_B^{X_0} \geq C^{-1} \text{dist}(X, \partial B)$ .

The needed contradiction.

## 11. Positive results for the Green function

Again, everything here should be checked.

Take  $d < n$ , but if  $d \geq n - 1$  add the assumption that  $\Omega$  is one sided NTA (then our definitions makes sense).

Or just assume  $d < n - 1$  as before.

We claim that if  $d$  is an integer and  $E$  is uniformly rectifiable, the Green function  $G^\infty$  is very close to a multiple of  $D_\alpha$  on most balls!

Let us first state closeness in weak terms. Let  $\tau, \varepsilon$  be small parameters. We say that the ball  $B(x, r)$  (centered on  $E$ ) is good, and we write  $(x, r) \in \mathcal{G}(\tau, \varepsilon)$  when there is a positive constant  $\lambda$  such that

$$(37) \quad (1 - \varepsilon)\lambda D_\alpha(X) \leq G^\infty(X) \leq (1 + \varepsilon)\lambda D_\alpha(X)$$

for every  $X \in B(x, \tau^{-1}r)$  such that  $\text{dist}(X, E) \geq \tau r$ .

## Positive results for the Green function (2): weakly nice

And we say that  $G^\infty$  is weakly nice when for each choice of  $\varepsilon, \tau$ , the complement  $\mathcal{B}(\tau, \varepsilon)$  of  $\mathcal{G}(\tau, \varepsilon)$  satisfies a Carleson Packing condition. That is, there exists  $C = C(\tau, \varepsilon)$  such that

$$(38) \quad \int_{(x,r) \in B(x_0, r_0) \times (0, r_0]} \mathbb{1}_{\mathcal{B}(\tau, \varepsilon)}(x, r) \frac{d\mu(x)dr}{r} \leq C(\tau, \varepsilon) r_0^d$$

Fans of UR should love this!

This fits the notion of being close to a  $\lambda D_\alpha$  most of the time.

And the first positive result is that

### THEOREM (DM)

If  $E$  is uniformly rectifiable and  $L_\alpha, D_\alpha$  are as usual, then  $G^\infty$  is weakly nice.

## Positive results for the Green function (3): comments

- Since  $\mu$  is close to a flat measure Carleson-most of the time, this means that  $G^\infty$  is close to an affine function most of the time!

But for the converse it seems more natural to compare  $G^\infty$  with  $D_\alpha$ . because both vanish on  $E$ .

- We also claim a stronger result, which says that for each  $\tau > 0$ , the quantity

$$(39) \quad b(x, r) = \inf_{\lambda > 0} \int_{X \in \Omega \cap B(x, \tau^{-1}r; \text{dist}(X, E) \geq \tau r)} |\lambda \nabla G^\infty - D_\alpha|^2 dX$$

satisfies the quadratic Carleson condition

$$(40) \quad \int_{(x, r) \in B(x_0, r_0) \times (0, r_0]} b(x, r)^2 \frac{d\mu(x) dr}{r} \leq C(\tau) r_0^d.$$

More complicated proof (and to be checked) though!

# Idea of proof for the weak result

By limits and compactness. There are a few believable ingredients.

- Compactness 1: given a sequence  $\{E_k\}$  of sets of  $AR(d)$  and for each  $E_k$  a measure  $\mu_k$  supported on  $E_k$  and uniformly AR, assuming  $0 \in E_k$ , there is a subsequence so that  $E_k$  tends to some  $E_\infty$ ,  $\mu_k$  tends weakly to  $\mu_\infty$ , and  $\mu_\infty$  is AR and supported on  $E_\infty$ .
- Compactness 2: In the situation above, for a subsequence we also have that after renormalization,  $G_k^\infty$  converges uniformly on compact sets to a positive harmonic function  $G_\infty^\infty$  and...
- Uniqueness of the Green function: ... and  $G_\infty^\infty$  is the Green function at infinity for the operator  $L_\alpha$  associated to the limit of the  $D_\alpha = D_{\alpha,k}$ .

Then the proof. Let  $\tau$  and  $\varepsilon$  be given. Let  $\kappa > 0$  be very small, to be chosen depending on  $\tau$  and  $\varepsilon$ .

## Idea of proof for the weak result (2)

Consider the new good set  $\mathcal{G}'(\kappa)$  of pairs  $(x, r)$  such that  $x \in E$  and  $r > 0$  are such that on  $B(x, \kappa^{-1}r)$ ,  $E$  is  $\kappa r$  close to a  $d$ -plane  $P$  and  $\mu$  is  $\kappa^2$  close (in Wasserstein distance as above) to a flat measure on  $P$ .

And the bad set  $\mathcal{B}'(\kappa) = E \times R_+^* \setminus \mathcal{G}'(\kappa)$ . If we prove that  $\mathcal{G}'(\kappa) \subset \mathcal{G}(\tau, \varepsilon)$  and hence  $\mathcal{B}(\tau, \varepsilon) \subset \mathcal{B}'(\kappa)$ , then (38) follows because UR and Tolsa say that  $\mathcal{B}'(\kappa)$  satisfies a Carleson condition.

And if this fails, start from a counterexample  $(E_k, \mu_k)$  with  $\kappa = 2^{-k}$ , with  $x = 0$  and  $r = 1$  (allowed by invariance) and extract converging subsequences as above.

By definition of  $\mathcal{G}'(\kappa)$ ,  $E_\infty$  is a  $d$ -plane and  $\mu_\infty$  is flat.

Then we compute that  $G_\infty^\infty(X) = C \operatorname{dist}(X, P) = CD_\infty(X)$ .

But  $G_k^\infty$  converges uniformly on compact sets to  $G_\infty^\infty$ .

And  $(x, r) = (0, 1) \in \mathcal{B}(\tau, \varepsilon)$  is impossible for  $k$  large!



# What about the converse?

We only have a partial converse, and this is related to Question 1 above, about when can we have  $L_\alpha D_\alpha = 0$ ? Let us explain.

For  $\kappa > 0$  small, define the good set  $\mathcal{G}''(\kappa)$  of the pairs  $(x, r)$ , with  $x \in E$  and  $r > 0$ , such that  $E$  is  $\kappa r$ -close to a  $d$ -plane  $P$  in  $B(x, r)$  (in Hausdorff distance).

We know that if for each  $\kappa > 0$ , the complement of  $\mathcal{G}''(\kappa)$  satisfies a Carleson measure condition as in (38), then  $E$  is uniformly rectifiable.

Hence to prove that if  $G^\infty$  is weakly nice, then  $d$  is an integer and  $E$  is uniformly rectifiable, it is enough to prove that for each  $\kappa > 0$ , we can find  $\tau > 0$  and  $\varepsilon > 0$  such that  $\mathcal{G}(\tau, \varepsilon) \subset \mathcal{G}''(\kappa)$ .

We start the proof by compactness. Suppose not. Let  $\kappa > 0$  be given. Let  $E_k$  and  $\mu_k$  provide a counterexample with  $\tau = \varepsilon = 2^{-k}$ . By invariance, we may assume that this happens for  $B(0, 1)$ . Take a convergent subsequence as above.

## Converse (2)

... Since  $(0, 1) \in \mathcal{G}(\tau, \varepsilon)$ ,  $G_k^\infty$  is closer and closer to a multiple of  $D_{\alpha, \mu_k}$  in any compact set of  $\Omega$ .

Then  $G_\infty^\infty = D_{\alpha, \mu_\infty}$  and so  $G_\infty^\infty$  is  $L_\infty$ -harmonic, where  $L_\infty$  is associated with  $E_\infty$  and  $\mu_\infty$ .

If the answer to Question 1 is positive, and if  $\alpha$  is not magic (otherwise, we actually expect that  $G^\infty = D_\alpha$  for all  $E$ , and we know it for sure for rectifiable sets), then  $E_\infty$  is a plane, this contradicts our assumption that  $(0, 1) \notin \mathcal{G}''(\kappa)$ , and we get the desired UR result.

When  $d \leq n - 2$  we don't know this but would bet on it.

When  $d = n - 2$ , a little less clear, agreed.

When  $n - 2 < d < n - 1$ , this is the case that I left out. Good hopes too.

When  $d > n - 1$ , we got a contradiction.

When  $d = n - 1$ , we do not exactly get that  $E_\infty$  is a plane, but it is uniformly rectifiable....



## Converse (3)

In this case we modify the argument, and instead of asking  $E$  to be close to a plane we ask for a corkscrew ball in the complement (Condition B).

The fact that  $E_\infty$  is uniformly rectifiable and one-sided NTA (by another limiting argument) implies that we can find a corkscrew ball in the complement for  $E_\infty$ , and then we prove that this ball also works for  $E_k$ ,  $k$  large.

We skip the details.

Provisional conclusion: using the Green function  $G^\infty$  rather than the harmonic measure  $\omega$  seems to give a better chance for a converse.

But we don't really know how to go from  $\omega$  to  $G^\infty$ , so this may not help for  $\omega$ .

# The square functions (DEM)

Can we characterize something else than  $\omega_L \in A_\infty$ ?

For instance, the USFE? The relevant **square function** here is

$$F(X) = \delta(X) |(\nabla |\nabla D_\alpha|^2(X))|,$$

where  $\delta(X) = \text{dist}(X, E)$ , and the USFE means that

$$F(X)^2 \delta(X)^{-n+d} dX \text{ is a Carleson measure on } \Omega.$$

That is, the measure of a ball of radius  $r$  centered on  $E$  is  $\leq Cr^d$ .

And we (DEM) show that for  $E \in AR(d)$ ,  $d < n$ , the USFE holds if and only if  $d$  is an integer and  $E$  is uniformly rectifiable.

Here the value of  $\alpha > 0$  does not matter.

But the connection with  $\omega$  or a Riesz transform is not obvious.

Idea of proof here?

# Nontangential limits of $|\nabla D_\alpha|$

Similarly, for  $E$  Ahlfors regular of dimension  $d < n$ , the function  $|\nabla D_\alpha|(X)$  has a nontangential limit  $\mu$ -almost everywhere on  $E$  if and only if  $d$  is an integer and  $E$  is uniformly rectifiable.

Nice, explains why we like  $D_\alpha$ , but again only geometry here, no real operator.

Proofs by  $\alpha$ -numbers in one direction, by compactness and the case when  $|\nabla D_\alpha|$  is constant in the other direction.

Hope for more geometric characterizations, valid only when  $\alpha$  is not magic.

A good example being with the Green function above.

# Doubling measures $\mu$ and sets $E$ of varying dimensions

Just a short description. In higher co-dimensions described above, it seems harder to copy the sawtooth domains arguments. How do we approximate a domain  $\Omega$  with a larger, possibly more regular one?

Let us just take an example. Let  $E$  be a Lipschitz graph of dimension  $d < n - 1$  in  $\mathbb{R}^n$ . Then for some reason build a sawtooth region around some part of  $E$ . That is, a domain  $\tilde{\Omega} \subset \Omega$ , bounded in part by a subset  $Z \subset E$ , and in part by conical surfaces  $S$  of co-dimension 1 around  $E \setminus Z$ .

Keep the same operator  $L$  as for  $E$ . Thus, less singular near  $S$ . Can we discuss the associated elliptic measure?

There is a natural measure on  $\tilde{E} = \tilde{\Omega} = Z \cup S$ : keep  $\mu$  on  $Z$ , and take something like  $\delta(X)^{n-d-1} d\sigma$  on  $S$ .

This measure  $\tilde{\mu}$  is doubling, and behaves in intermediate dimensional ways on intermediate balls.

Can we study the elliptic measures  $\omega_L$  relative to the size of that measure, and prove results as above?

## Doubling measures and varying dimensions (2)

Turns out, yes. With more work but ideas similar to the above. Need to be more careful at the beginning, with Sobolev spaces  $W = \{u \in L^2_{loc}(\Omega); \nabla u \in L^2(wdX)\}$  and  $H = H^{1/2}(E, d\mu)$ . And we use arguments from Heinonen, Kilpeläinen, Martio, Hajlasz, Koskela, and others. The theory needs:

- A doubling measure on  $E$ , such as  $\mu$  before or  $\tilde{\mu}$  just above.
- A doubling measure  $m = w(X)dX$  on  $\Omega$  (a component of  $\mathbb{R}^n \setminus E$ ), such as  $\delta(X)^{d+1-n}dX$  as above, but for instance slightly different exponents would be allowed, and  $m$  could have mild singularities inside  $\Omega$ .
- A relation between the two measures, where one does not grow much faster than the other. See below.
- When  $n - 1 \leq d < n$  (this is allowed now), require as usual the existence of Corkscrew balls and Harnack chains (one-sided NTA).
- A Weak Poincaré estimate away from  $E$ , see below.

## Doubling measures and varying dimensions (3)

Let me not explain much about the weak Poincaré estimate. It consists mostly in an estimate like

$$\frac{1}{m(B)} \int_B |u - m_B(u)| \leq Cr \left( \frac{1}{m(B)} \int_B |\nabla u|^2 dm \right)^{1/2}$$

when  $B$  is a ball of radius  $r$  such that  $2B \subset \Omega$ .

For the relation between  $m$  and  $\mu$ , set, for  $x \in E$  and  $r > 0$ ,  
 $\rho(x, r) = r^{-1} \mu(E \cap B(x, r))^{-1} m(\Omega \cap B(x, r))$ .

Notice that this would be about constant in the basic example above. One requires that for  $x \in E$  and  $0 < s < r$ ,

$$\frac{\rho(x, r)}{\rho(x, s)} \leq C(r/s)^{1-\varepsilon}.$$

A way to say that their “local” dimensionalities are not so different.

Comments: surprised it works so well, in a context where I would not have dared to study  $L$  (even in the classical case).

Even some  $A_\infty$  results seem to hold here too.