Linear Methods for Nonlinear Inverse Problems

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INVERSE PROBLEM

$y_n = u_f + \frac{1}{\sqrt{n}} \dot{W}$, $\dot{W}$ white noise (or regression)

u_f solution to PDE that depends on parameter f

"inverse" $u_f \mapsto f$ not differentiable

"nonlinear" $f \mapsto u_f$ nonlinear EXAMPLE $\frac{1}{2}\Delta u_f = f u_f$

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"nonlinear" $f \mapsto u_f$ nonlinear EXAMPLE $\frac{1}{2}\Delta u_f = f u_f$

Regularisation $\hat{f} = \underset{f}{\operatorname{argmin}} \left[\| y - u_f \|^2 + \sigma^2 \,\mathrm{pen}(f) \right]$

Bayesian Regularisation $f \sim$ prior
(Stuart, 2010)

$\longrightarrow$ posterior $f \mid y_n$

$\longrightarrow$ posterior mean $E(f \mid y_n)$

LEM If $f \sim$ Gaussian with RKHS $\mathbb{H}$, then

$$E(f \mid y_n) = \underset{f}{\operatorname{argmin}} \left[\| y_n - u_f \|^2 + \sigma^2 \| f \|_{\mathbb{H}}^2 \right]$$

(Wahba 1978, Dashti et al 2013)

INVERSE PROBLEM - EXAMPLES

Schrödinger $\frac{1}{2}\Delta u = fu$

(Nickl 2020, & Wang 2022)

Heat with absorption $\partial_t u - \frac{1}{2}\Delta u = fu$

(Kekkonen 2022)

Non-abelian X ray transform $\log u(x,v) = \int_0^{\tau_v} f(x+tv)\,dt$

(Monard, Nickl, Paternain 2019, 21)

Divergence / Darcy $\nabla\cdot(f\nabla u) = g$

(Abraham, Nickl 2019, Bohr 2022)

Navier-Stokes

(Nickl, Titi 2023)

$$\begin{cases} \partial_t u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 \\ \nabla\cdot u = 0 \\ u(0,\cdot) = f \\ u = 0 \end{cases}$$

A.M. Stuart, 2010. Inverse problems: a Bayesian perspective.

R. Nickl, 2023, Bayesian Non-linear Statistical Inverse Problems

NONPARAMETRIC BAYESIAN INFERENCE

Computation

Frequentist Bayesian theory

- Contraction rate $E_{f_0} \Pi_n(f : \|f - f_0\| \lesssim \varepsilon_n \mid y_n) \to 1$
- Uncertainty quantification

  If $\Pi_n(C_n(y_n) \mid y_n) \geq 0.95$, for central set $C_n(y_n)$,

  then $P_{f_0}(f_0 \in C_n(y_n)) \gg 0$?

OUR CONTRIBUTION

Assume

$$\begin{cases} \mathcal{L} u_f = c(f, u_f) & \text{on } \mathcal{O} \subset \mathbb{R}^d \\ u_f = g & \text{on } \Gamma \subset \mathcal{O} \end{cases}$$

some operator

known

$$f = e(\mathcal{L} u_f)$$

some operator

- Recover $\mathcal{L} u_f$ from $Y_n = u_f + \frac{1}{\sqrt{n}} \dot{W} = K(\mathcal{L} u_f) + \frac{1}{\sqrt{n}} \dot{W}$, $\quad K = \mathcal{L}^{-1}$
- Recover f from $\mathcal{L} u_f$

EXAMPLE $\quad \frac{1}{2} \Delta u_f = f u_f \quad \longrightarrow \quad c(f, u_f) = 2 f u_f$

$\mathcal{L} = \Delta$

$$f = \frac{\Delta u_f}{2 u_f} = e(\Delta u_f)$$

LINEAR PROBLEM

Assume $\begin{cases} \mathcal{L} u_f = c(f, u_f) & \text{on } \mathcal{O} \subset \mathbb{R}^d \\ u_f = g & \text{on } \Gamma \subset \mathcal{O} \end{cases}$

c: some operator; g: known

$f = e(\mathcal{L} u_f)$

e: some operator

$\tilde{g}$ solves $\begin{cases} \mathcal{L} \tilde{g} = 0 & \text{on } \mathcal{O} \\ \tilde{g} = g & \text{on } \Gamma \subset \partial \mathcal{O} \end{cases}$

K solves $\begin{cases} \mathcal{L} K u = u & \text{on } \mathcal{O} \\ K u = 0 & \text{on } \Gamma \subset \partial \mathcal{O} \end{cases}$

$K = \mathcal{L}^{-1}$

$\rightarrow$ • $u_f = K \mathcal{L} u_f + \tilde{g}$

$\rightarrow$ • $\tilde{Y}_n := Y_n - \tilde{g} = K \mathcal{L} u_f + \frac{1}{\sqrt{n}} \dot{W}$

$v = \mathcal{L} u_f \qquad f = e(\mathcal{L} u_f) = e(v)$

L $\quad \tilde{\Pi}_n (v \in \cdot \mid \tilde{Y}_n)$ based on $\tilde{Y}_n = K v + n^{-1/2} \dot{W}$

N $\quad \Pi_n (f \in \cdot \mid Y_n)$ based on $Y_n = u_f + n^{-1/2} \dot{W}$

CONNECTING LINEAR AND NONLINEAR PROBLEMS

$$v = \mathcal{L} u_f \qquad f = e(\mathcal{L} u_f) = e(v)$$

L $\quad \tilde{\Pi}_n(v \in \cdot \mid \hat{Y}_n)$ based on $\hat{Y}_n = Kv + n^{-1/2}\dot{\mathbb{W}}$

N $\quad \Pi_n(f \in \cdot \mid Y_n)$ based on $Y_n = u_f + n^{-1/2}\dot{\mathbb{W}}$

PROP

If $\tilde{\Pi}_n(v \in V_n \mid \tilde{Y}_n) \to 1$ and $e : (V_n, \|\cdot\|) \to L_2$ is Lipschitz at v_0, then

$$E_{v_0} \tilde{\Pi}_n(\|v - v_0\| \lesssim \varepsilon_n \mid \hat{Y}_n) \to 1 \quad \text{implies} \quad E_{f_0} \Pi_n(\|f - f_0\|_{L_2} \lesssim \varepsilon_n \mid Y_n) \to 1$$

CONNECTING LINEAR AND NONLINEAR PROBLEMS

$$v = \mathcal{L}u_f \qquad f = e(\mathcal{L}u_f) = e(v)$$

$$\begin{array}{llll} \text{L} & \tilde{\Pi}_n(v \in \cdot \mid \hat{Y}_n) & \text{based on} & \hat{Y}_n = Kv + n^{-1/2}\dot{W} \\ \text{N} & \Pi_n(f \in \cdot \mid Y_n) & \text{based on} & Y_n = u_f + n^{-1/2}\dot{W} \end{array}$$

PROP

If $\tilde{\Pi}_n(v \in V_n \mid \tilde{Y}_n) \to 1$ and $e : (V_n, \|\cdot\|) \to L_2$ is Lipschitz at v_0, then

$E_{v_0}\tilde{\Pi}_n(\|v - v_0\| \lesssim \varepsilon_n \mid \hat{Y}_n) \to 1$ implies $E_{f_0}\Pi_n(\|f - f_0\|_{L_2} \lesssim \varepsilon_n \mid Y_n) \to 1$

$$\boxed{C_n(Y_n) := \{ f : \mathcal{L}u_f \in \hat{C}_n(Y_n - g) \} = \{ e(v) : v \in \tilde{C}_n(Y_n - g) \}}$$

PROP

- Credible and confidence level of $C_n(Y_n)$ are those of $\hat{C}_n(\hat{Y}_n)$
- If $\overline{V}_n \in \hat{C}_n(Y_n) \subset V_n$ and $e : V_n \to L_2$ uniformly Lipschitz at $\overline{V}_n$, then

$$O_{P_{f_0}}(\operatorname{diam}(C_n(Y_n), L_2)) \simeq O_{P_{v_0}}(\operatorname{diam}(\hat{C}_n(Y_n), \|\cdot\|)).$$

CONNECTING LINEAR AND NONLINEAR PROBLEMS

$v = \mathcal{L} u_f \qquad f = e(\mathcal{L} u_f) = e(v)$

L $\quad \tilde{\Pi}_n(v \in \cdot \mid \hat{Y}_n)$ based on $\hat{Y}_n = Kv + n^{-1/2}\mathring{W}$

N $\quad \Pi_n(f \in \cdot \mid Y_n)$ based on $Y_n = u_f + n^{-1/2}\dot{W}$

Connection works for any prior on f.

To make effective use of linearity,
we use "standard" priors on u_f or $\mathcal{L} u_f$.

CONNECTING LINEAR AND NONLINEAR PROBLEMS — EXAMPLE

$v = \mathcal{L} u_f \qquad f = e(\mathcal{L} u_f) = e(v)$

L $\quad \tilde{\Pi}_n(v \in \cdot \mid \hat{Y}_n)$ based on $\hat{Y}_n = Kv + n^{-1/2}\dot{W}$

N $\quad \Pi_n(f \in \cdot \mid Y_n)$ based on $Y_n = u_f + n^{-1/2}\dot{W}$

$$\begin{cases} \frac{1}{2}\Delta u_f = f\, u_f & \text{on } \mathcal{O} \\ u_f = g & \text{on } \partial\mathcal{O} \end{cases} \quad \longrightarrow \quad \mathcal{L} = \Delta, \quad K = \Delta^{-1} \qquad e(v) = \frac{v/2}{Kv + \tilde{g}}$$

PROP Assume $u_{f_0} = Kv_0 + \tilde{g} \geq c_0 > 0$ and $\|v_0\|_\infty < \infty$

- If posterior of Kv in (L) is $\|\cdot\|_\infty$-consistent, then contraction rate in (L) under v_0 is contraction rate in (N) under f_0.
- If $\hat{C}_n(Y_n) \subset \{v : \|Kv - Kv_0\|_\infty < c_0/2\}$ and $\exists\, v_n \in \hat{C}_n(Y_n)$ with $\|v_n\|_\infty = O(1)$, then $P_{f_0}\text{-diam}(C_n(Y_n)) \lesssim P_{v_0}\text{-diam}(\hat{C}_n(Y_n))$

LINEAR INVERSE PROBLEM

$\tilde{y}_n = Kv + \frac{1}{\sqrt{n}} \dot{\mathbb{W}}$, $\dot{\mathbb{W}}$ white noise $K : L_2(\mathcal{O}) \to L_2(\mathcal{O})$

$(h_i)_{i\in\mathbb{N}}$ orthonormal basis of $L_2(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^d$

$$\|v\|^2_{G^s} = \sum_{i=1}^{\infty} v_i^2 \, i^{2s/d}$$

$$\boxed{\text{Assume} \quad \|Kv\|_{L_2} \asymp \|v\|_{G^{-p}} \quad , v \in G^0}$$

THM (Yan et al, 2020)

If $\exists\, d_n \lesssim n\varepsilon_n^2$ and $\eta_n \gtrsim \varepsilon_n d_n^p \vee d_n^{-\beta}$ with

$$\Pi(v : \|Kv - Kv_0\|_{L_2} < \varepsilon_n) \gtrsim e^{-n\varepsilon_n^2}$$

$$\Pi(v : \|K^{-1} Q_{d_n} Kv - v\|_{L_2} < \eta_n) \lesssim e^{-4n\varepsilon_n^2}$$

Q_j projection onto $\mathrm{lin}(Kh_1, \dots, Kh_j)$

then $E_{v_0} \Pi_n(\|v - v_0\|_{L_2} \lesssim \eta_n \mid \hat{y}_n) \to 1$.

LINEAR INVERSE PROBLEM

$\tilde{y}_n = Kv + \frac{1}{\sqrt{n}} \dot{\mathbb{W}}$, $\dot{\mathbb{W}}$ white noise $\quad K: L_2(\mathcal{O}) \to L_2(\mathcal{O})$

$(h_i)_{i\in\mathbb{N}}$ orthonormal basis of $L_2(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^d$

$$\|v\|_{G^s}^2 = \sum_{i=1}^{\infty} v_i^2\, i^{2s/d}$$

$$\boxed{\begin{array}{ll} \text{Assume} & \|Kv\|_{L_2} \asymp \|v\|_{G^{-p}} \quad , v \in G^0 \\ & v = \sum_{i=1}^{\infty} v_i h_i \quad , v_i \overset{ind}{\sim} N(0, i^{-1-2\alpha/d}) \end{array}}$$

THM

- If $v_0 \in G^\beta$, then $E_{v_0} \Pi_n\left(\|v - v_0\|_{G^\delta} \lesssim n^{-\frac{(\alpha\wedge\beta - \delta)}{2\alpha+2p+\delta}} \mid \hat{y}_n\right) \to 1$, $\forall\ -p \le \delta < \alpha\wedge\beta$.

- If $v_0 \in G^\beta$ and $\sup_i i^{u/d} \|Kh_i\|_\infty < \infty$, some $u > \frac{d}{2} - \alpha\wedge\beta$, then $\Pi_n\left(\|Kv - Kv_0\|_\infty < \varepsilon \mid \hat{y}_n\right) \to 1 \quad \forall\ \varepsilon > 0$.

or $\begin{cases} \|\cdot\|_{H^\gamma} \lesssim \|\cdot\|_{G^\delta} , \text{ some } \gamma \in \left(\frac{d}{2}, \alpha\wedge\beta + p\right) \\ K: G^{\gamma p} \to G^\gamma \text{ continuous} \end{cases}$

(Yan et al, 2020, Koers et al 2023)

HB/FB if $\delta = 0$

LINEAR INVERSE PROBLEM — EIGEN PRIOR

$\hat{y}_n = Kv + \frac{1}{\sqrt{n}} \dot{\mathbb{W}}$, $\dot{\mathbb{W}}$ white noise $K: L_2(\theta) \to L_2(\theta)$

$(h_i)_{i\in\mathbb{N}}$ eigenbasis of K^TK, eigenvalues $\kappa_i^2 \asymp i^{-2p/d}$

$$\boxed{\text{Prior} \quad v = \sum_{i=1}^{\infty} v_i h_i \quad \begin{cases} v_i \mid \alpha \overset{ind}{\sim} N(0, i^{-1-2\alpha/d}) \\ \alpha \sim \lambda \quad \text{or} \quad \alpha = \text{EB } \hat{\alpha}_n \end{cases}}$$

$$\boxed{\text{Credible set} \quad \hat{C}_n(\hat{y}_n) = \{ v : \|v - \hat{v}_n\|_{L_2} \leq r\, \xi_{n,\gamma} \}}$$

THM

- If $v_0 \in G^\beta$, then $E_{v_0} \Pi_n \left(\|v - v_0\|_{G^\partial} \lesssim n^{-\frac{\beta-\partial}{2\beta+2p+d}} \ell_n^2 \mid \hat{y}_n \right) \to 1$, $\forall\, -p \leq \partial < \beta$
- If $v_0 \in G^\beta$ and $\sup_i i^{u/d} \|Kh_i\|_\infty < \infty$, $\frac{d}{2} < \beta + u$, or $\|\cdot\|_{H^\gamma} \lesssim \|\cdot\|_{G^\gamma}$, $\frac{d}{2} < \gamma < \beta + p$, then $E_{v_0} \Pi_n (\|Kv - Kv_0\|_\infty < \varepsilon \mid \hat{y}_n) \to 1 \;\; \forall\, \varepsilon > 0$.
- If v_0 polished tail, then $P_{v_0}(v_0 \in \hat{C}_n(\hat{y}_n)) \to 1$ and $\operatorname{diam}(\hat{C}_n) = O_{P_{v_0}}\left(n^{-\frac{(\beta-\partial)}{2\beta+2p+d}} \ell_n^2 \right)$

(Knapik et al., Szabo et al, Koers et al 2023)

LINEAR INVERSE PROBLEM – DISCRETE DESIGN

$\tilde{y}_n = (\tilde{y}_n(1), \dots, \tilde{y}_n(n))$ $\quad \tilde{y}_n(i) = Kv(X_{in}) + W_i$ $\quad W_i \overset{iid}{\sim} N(0,1)$

$K : L_2(\mathcal{O}) \to C(\mathcal{O})$

interpolation

> Assume $\forall n \ \exists \mathbb{L}_n$, $\dim(\mathbb{L}_n) = n$,
>
> $\|v\|_{L_2} \asymp \|v\|_n$, $\quad v \in \mathbb{L}_n$ $\qquad \|v\|_n^2 = \frac{1}{n} \sum_{i=1}^{n} v(X_{in})^2$
>
> $\|I_n v - v\|_{L_2} \lesssim n^{-\frac{\beta}{2\beta+2p+d}} \|v\|_{G^\beta}$ $\qquad I_n v \in \mathbb{L}_n$, $I_n v(X_{in}) = v(X_{in})$
>
> $i = 1, \dots, n$

THM

- Under interpolation THM 1 remains valid
- For random design and eigen prior THM 2 remains valid.

(Yan et al 2020, Koers et al 2023)

LINEAR INVERSE PROBLEM – DISTRIBUTED POSTERIOR

$\tilde{y}_n = (\tilde{y}_n(1), \dots, \tilde{y}_n(n))$ $\qquad \tilde{y}_n(i) = K\nu(x_{i,n}) + W_i$ $\qquad W_i \overset{iid}{\sim} N(0,1)$

$K: L_2(\mathcal{O}) \to C(\mathcal{O})$

- Divide data in m batches
- On each batch compute posterior with prior variance $\times m$.
- Compute law of $\nu = \frac{1}{m}\sum_{j=1}^{m} \nu_j$ for $\nu_j \sim \Pi_{2m}(\cdot \mid j\text{th batch})$

THM

- Retains contraction rates (Szabó, van Zanten 2017, Koers et al 2023)
- Spatial distribution makes FB / HB work (Hadji et al. 2023)
  (at least for estimating $K\nu$)

EXAMPLE – SCHRÖDINGER

$$\begin{cases} \frac{1}{2}\Delta u_f = f\, u_f & \text{on } \mathcal{O} = (0,1)^d \\ u_f = g & \text{on } \partial\mathcal{O} \end{cases}$$

Eigenbasis $h_{i_1 \cdots i_d}(x_1 \cdots x_d) = 2^{d/2} \prod_{j=1}^{d} \sin(i_j \pi x_j)$, $\quad \kappa_{i_1 \cdots i_d} = \frac{1}{\left(\sum_{j=1}^{d} i_j^2\right)\pi^2} \quad (i_1 \cdots i_d) \in \mathbb{N}^d$

Reordered $\kappa_\ell^2 \asymp \ell^{-\frac{2}{d}}$ $\quad \|v\|_{G^s}^2 = \sum_{\ell=1}^{\infty} v_\ell^2 \ell^{\frac{2s}{d}} \asymp \sum_{i_1} \cdots \sum_{i_d} v_{i_1 \cdots i_d}^2 \left(\sum_{j=1}^{d} i_j^2\right)^s$

THM Assume $\Delta u_{f_0} \in G^\beta$, $\beta + 2 > \frac{d}{2}$, $\inf u_{f_0} > 2c_0 > 0$

- L_2-contraction rate $n^{-\frac{\alpha \wedge \beta}{d+2\alpha+4}}$ or $n^{-\frac{\beta}{d+2\beta+4}} l_n^2$ for α-regular or HB/EB-prior
- Coverage and optimal diameter for α-regular prior with $\alpha = \beta - \frac{c}{\log n}$.

EXAMPLE – SCHRÖDINGER – DETAILS

$$\begin{cases} \frac{1}{2}\Delta u_f = f\, u_f & \text{on } \mathcal{O} = (0,1)^d \\ u_f = g & \text{on } \partial\mathcal{O} \end{cases}$$

Eigenbasis $h_{i_1 \cdots i_d}(x_1 \cdots x_d) = 2^{d/2} \prod_{j=1}^{d} \sin(i_j \pi x_j)$, $\quad \kappa_{i_1 \cdots i_d} = \frac{1}{\left(\sum_{j=1}^{d} i_j^2\right)\pi^2}$ $\quad (i_1 \cdots i_d) \in \mathbb{N}^d$

Reordered $\kappa_\ell^2 \asymp \ell^{-\frac{2}{d}}$ $\quad \|v\|_{G^s}^2 = \sum_{\ell=1}^{\infty} v_\ell^2 \ell^{\frac{2s}{d}} \simeq \sum_{i_1} \cdots \sum_{i_d} v_{i_1 \cdots i_d}^2 \left(\sum_{j=1}^{d} i_j^2\right)^s$

THM Assume $\Delta u_{f_0} \in G^\beta$, $\beta > \frac{d}{2}$, $\inf u_{f_0} > 2c_0 > 0$

- L_2-contraction rate for f_0 is $n^{-(\alpha \wedge \beta)/(d+2\alpha+4)}$ for α-prior on $v = \Delta u_f$, $\alpha \wedge \beta + 2 > \frac{d}{2}$
- " " is $n^{-\beta/(d+2\beta+4)}(\log n)^2$ for HB or EB, $\beta+2 > d/2$.
- $P_{f_0}\left(f_0 \in C_n(Y_n)\right) \to 1$ and $\operatorname{diam}_{f_0}\left(C_n(Y_n)\right) = O_{P_f}\left(n^{-\frac{\alpha}{d+2\alpha+4}}\right)$ if $\alpha = \beta - \frac{c}{\log n}$.

$$C_n(Y_n) = \left\{ e(v) : \|v - \hat{v}_n\|_{L_2} \le r\, \xi_{nj},\ \|Kv - K\hat{v}_n\|_\infty \le c_0 \right\}$$

eigen prior

$n=10^4$

$n=10^6$

$n=10^8$

$n=10$

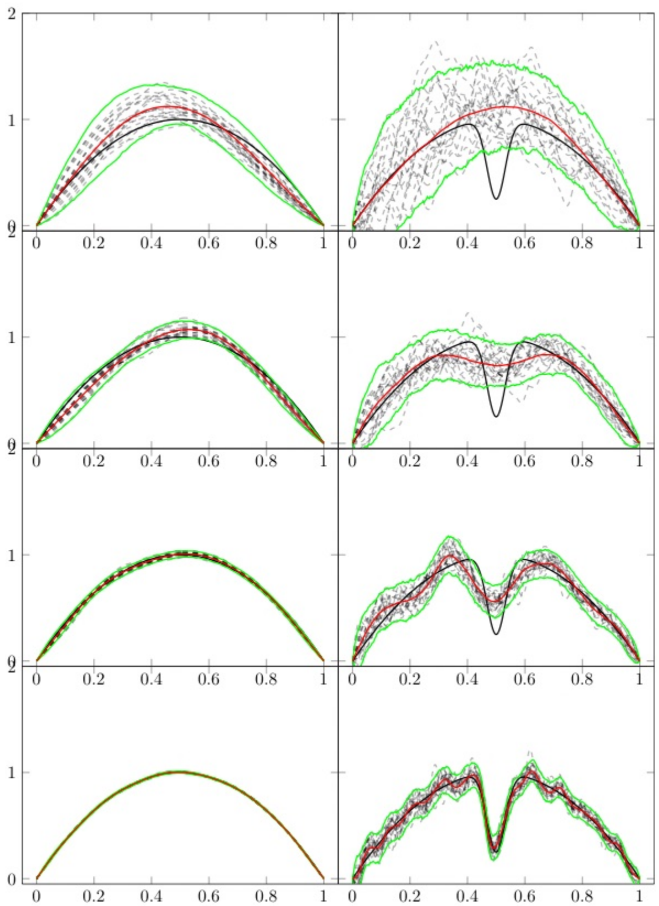

FIG 1. *Posterior distributions of f based on data $Y_n = u_f + n^{-1/2}\dot{\mathbb{W}}$ for $n = 10^4, 10^6, 10^8, 10^{10}$ (top to bottom) and prior based on eigenbasis. Black curve: true function f_0. Red curve: posterior mean. Green curves: pointwise 95% credible band. Dashed black curves: 20 draws from the posterior distribution. Left panels: $f_0(x) = 1 + 4(x-1/2)^2$, right panels: $f_0(x) = 1 - 4(x-1/2)^2 - \frac{3}{4}\exp(-500(x-1/2)^2)$.*

black f_0
red posterior mean
green 95% credible band
dashed posterior draws

eigen prior spline prior

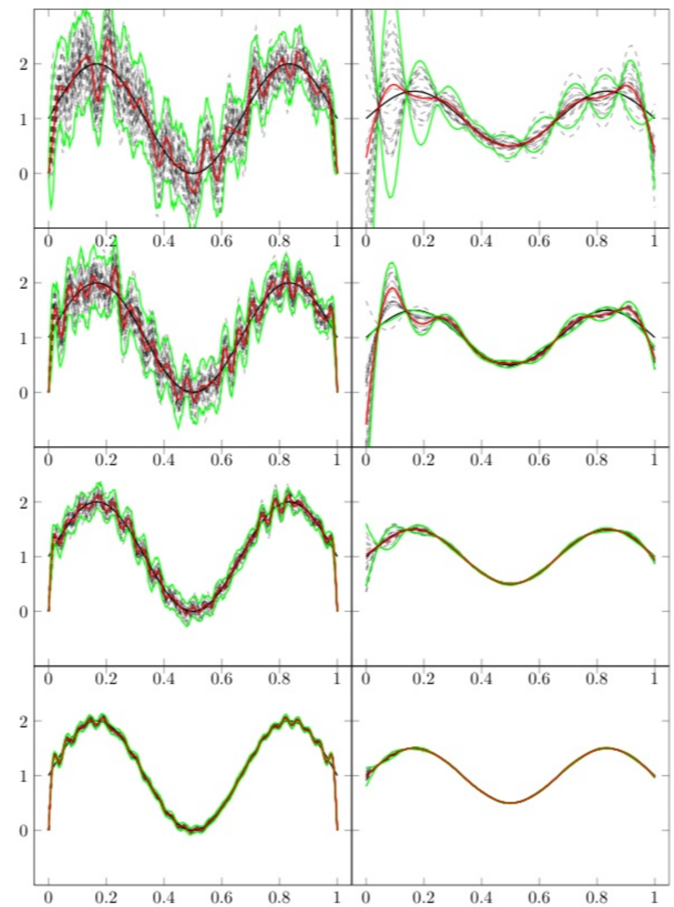

FIG 2. *Posterior distributions of f based on data $Y_n = u_f + n^{-1/2}\dot{\mathbb{W}}$ for $n = 10^4, 10^6, 10^8, 10^{10}$ (top to bottom). Black curve: true function $f_0(x) = \sum_{i=1}^{\infty} i^{-3/2}\sin(i)h_i$. Red curve: posterior mean. Green curves: pointwise 95% credible band. Dashed black curves: 20 draws from the posterior distribution. Left panels: prior based on eigenbasis. Right panels: prior based on spline basis.*

wavelet prior

eigen prior

eigen, distributed

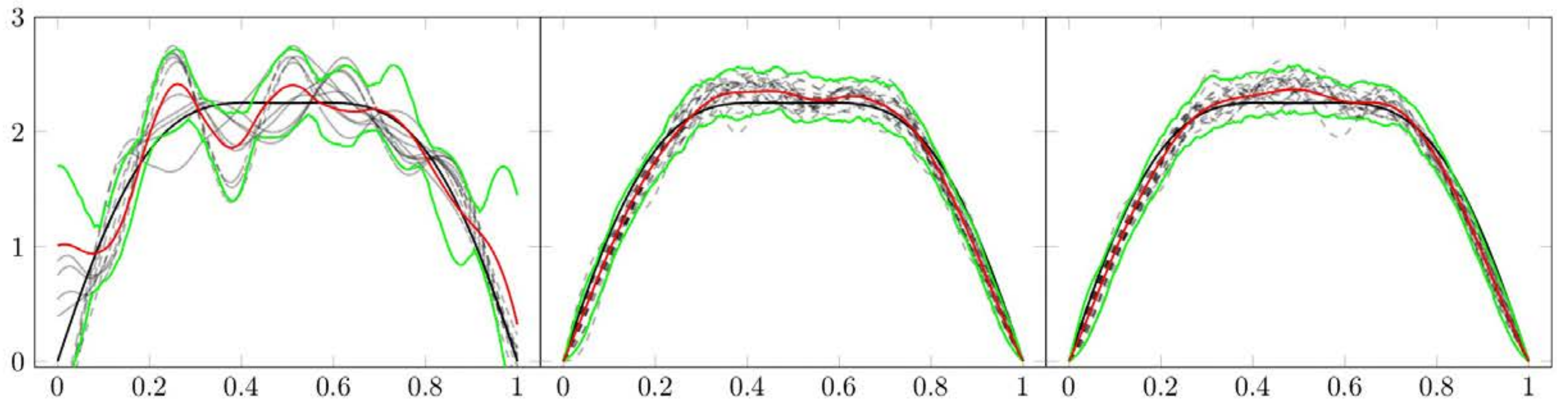

Figure 3.6.1: The true f_0 (black), the posterior mean (red), 95% credible bands (green) and 20 draws from the posterior (dashed grey) for $n = 2500$. Left: wavelet prior. Middle: SVD prior with $m = 1$. Right: SVD prior with $m = 10$.

≈ 2.5 days

≈ 3 minutes

≈ 17 s

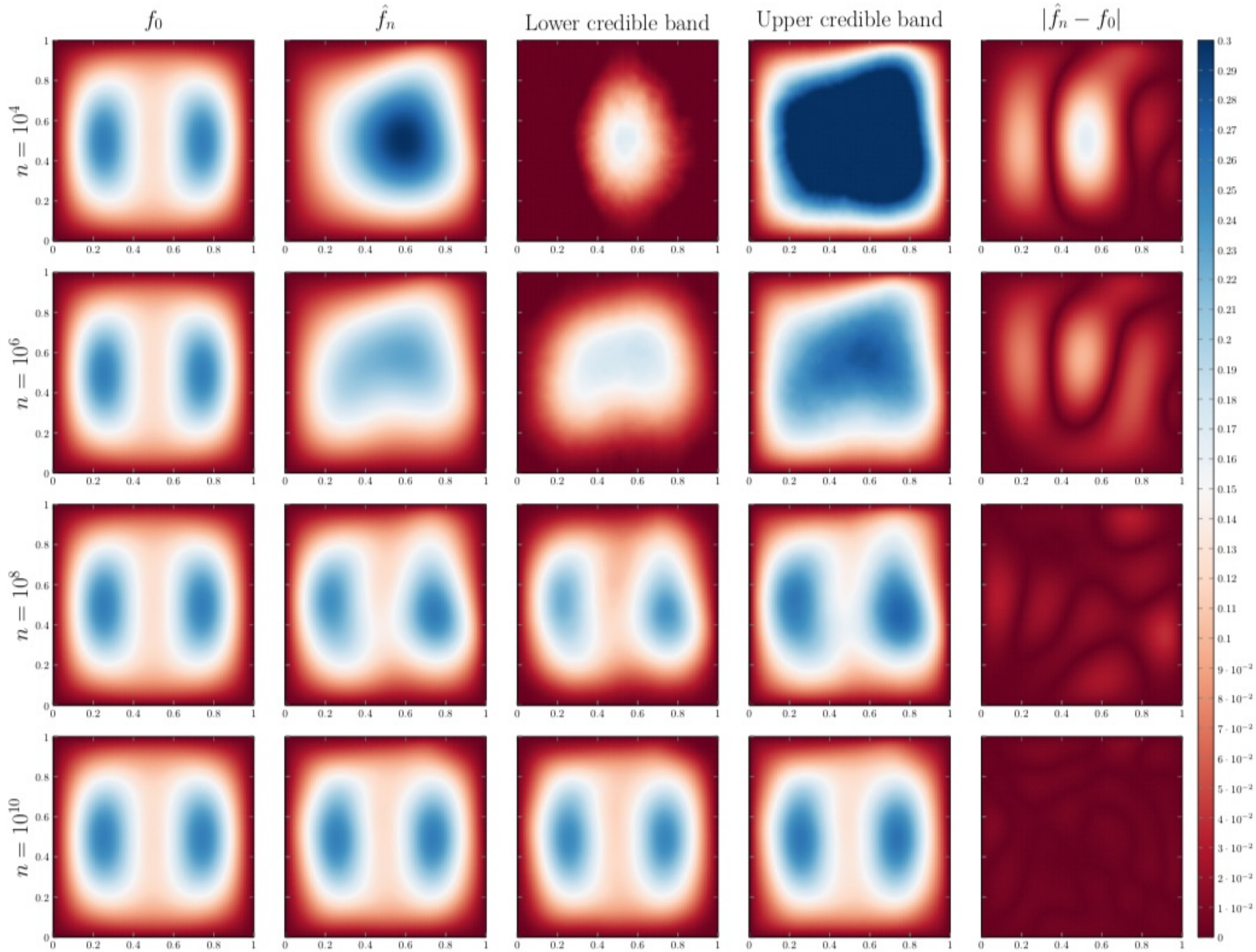

FIG 3. *The true function f_0 given in (9.1), the posterior mean, lower and upper 2.5% pointwise posterior quantiles, and absolute approximation error of posterior mean, resulting from the proposed Bayesian approach with signal-to-noise ratios $n = 10^4, 10^6, 10^8, 10^{10}$ in the PDE constrained Gaussian white noise model (1.1)-(1.3).*

$$
\begin{aligned}
f_0(x,y) &= 2x(x-1)y(y-1)(2+\sin(3+\pi x)\sin(\pi y)), \\
g(x,y) &= 3 + xy^2 + 2y\sin(2\pi x) + x\cos(3\pi y).
\end{aligned}
$$

EXAMPLE HEAT WITH ABSORPTION

$$\begin{cases} \frac{\partial}{\partial t} u - \frac{1}{2}\Delta u = f u & \text{on } [0,1)^d \times [0,1] \\ u = g & \text{on } \partial[0,1)^d \times [0,1] \\ u = u_0 & \text{on } [0,1)^d \times \{0\} \end{cases}$$

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta \qquad e(v) = \frac{v}{Kv + \tilde{g}}$$

$$K = \mathcal{L}^{-1}$$

Eigenbasis $(h_{i_1 \cdots i_d, k})$ eigenvalues $\lambda_{i_1 \cdots i_d, k} \simeq \frac{1}{\frac{1}{4}\sum_{j=1}^{d} i_j^2 + k^2\pi^2}$

THM Assume $\mathcal{L}u_{f_0} \in G^\beta$, $\beta > d+1$, $u_{f_0} \geq c_0 > 0$

- L_2-contraction rate $n^{-\frac{\alpha \wedge \beta}{d+1+2\alpha+4}}$ or $n^{-\frac{\beta}{d+1+2\beta+4}}$ for α-regular or HB/EB prior
- Coverage and optimal diameter for α-regular prior with $\alpha = \beta - \frac{c}{\log n}$.

EXAMPLE HEAT WITH ABSORPTION – DETAILS

$$\begin{cases} \frac{\partial}{\partial t}u - \frac{1}{2}\Delta u = fu & \text{on } [0,1)^d \times [0,1] \\ u = g & \text{on } \partial[0,1)^d \times [0,1] \\ u = u_0 & \text{on } [0,1)^d \times \{0\} \end{cases}$$

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta \qquad e(v) = \frac{v}{Kv + \tilde{g}}$$

$$K = \mathcal{L}^{-1}$$

Eigenbasis $(h_{i_1 \dots i_d, k})$ eigenvalues $\lambda_{i_1 \dots i_d, k} \simeq \frac{1}{\frac{1}{4}\sum_{j=1}^{d} i_j^2 + k^2\pi^2}$

THM Assume $\mathcal{L}u_{f_0} \in G^\beta$, $\beta > d+1$, $u_{f_0} \geq c_0 > 0$

- L_2-contraction rate for f_0 is $n^{-(\alpha \wedge \beta)/(d+1+2\alpha+4)}$ for α-regular prior, $\alpha > d+1$
- " " $n^{-\beta/(d+1+2\beta+4)}\ell_n^2$ for HB or EB
- $P_{f_0}(f_0 \in C_n(Y_n)) \to 1$ and $\mathrm{diam}_{f_0}(C_n(Y_n)) = O_{P_f}\left(n^{-\frac{\alpha}{d+2\alpha+4}}\right)$ if $\alpha = \beta - \frac{c}{\log n}$

EXAMPLE EXPONENTIATED VOLTERRA

$$\begin{cases} u_f' = f\,u_f & \text{on } (0,1) \\ u_f = g & \text{at } 0 \end{cases}$$

$$\mathcal{L}u = u'$$

$$e(v) = \frac{v}{Kv + g}$$

$$Kv = \int_0^{\cdot} v(s)\,ds$$

THM

- L_2-contraction at v_0 implies L_2-contraction at f_0.
- L_2-diameter of credible set retained

EXAMPLE DARCY

$$\begin{cases} \nabla \cdot (f \nabla u_f) = h & \text{on } \mathcal{O} \\ u_f = g & \text{on } \partial \mathcal{O} \end{cases}$$

$d=1$: $\begin{cases} f' u_f' + f u_f'' = h & \text{on } (0,1) \\ u_f = g & \text{on } \{0,1\} \end{cases} \longrightarrow f = \dfrac{\int_0^{\cdot} h(s)ds + f_0 u_f'(0)}{u_f'(x)}$

$d>1$: no explicit inverse

Nickl 2023 : inverse exists and is Lipschitz (on range) if $\Delta u_f + \|\nabla u_f\|^2 \gg 0$

Our approach should work if this is true for f_0

DISCUSSION

Idea is to isolate a linear operator $\mathcal{L}$ so that

$$f = e(\mathcal{L} u_f)$$

for nice operator e.

$$\boxed{\begin{array}{l}\text{Statistical problem becomes linear} \\ \text{Nonlinear inversion becomes deterministic}\end{array}}$$

- Not clear when this is possible.
- May need (expensive) numerical methods to compute e.
- Works best if (standard) prior is put on u_f.
- Boundary conditions not well understood.