The contour for the irrationality of $L(2,\chi_{-3})$

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The goal of the talk is to present the final choice of contour $\varphi : (\overline{\mathbb{D}}, 0) \to (\mathbb{C}, 0)$ that appears in the holonomy bounds in [2] to prove the irrationality of the special L-function value $L(2, \chi_{-3})$.

For a well-chosen contour φ, we have a 14-dimensional $\mathbb{Q}(X)$ vector space

$$H(\varphi)$$

that is the span of 14 functions $(G_i)_{1\leq i\leq 14}$ (assuming the linear dependence of 1, $\zeta(2)$, and $L(2, \chi_{-3})$ over $\mathbb{Q}$) on the modular curve $Y_0(2)$ that are linearly independent over $\mathbb{Q}(X)$ and that satisfy the following conditions:

(1) $\varphi^* G_i \in \mathcal{O}(\overline{\mathbb{D}})$
(2) G_i has fixed denominator type.

Under some conditions on φ, we have the holonomy bound

$$\dim_{\mathbb{Q}(X)} H(\varphi) \leq \frac{\iint_{T^2} \log |\varphi(z) - \varphi(w)| \; \mu(z)\mu(w)}{\log|\varphi'(0)| - \tau}$$

where τ is a fixed real number that corresponds to the condition (2), T is the unit circle, and μ is the Haar measure on T.

We want to optimize the right hand size so that it is smaller than 14 to get a contradiction.

1. Some context

1.1. **Potential theory and Bost-Charles double integral.** Let (V,P) be a Riemann surface with boundary pointed at P as defined by Bost and Charles in [1].

Definition 1. *A classical Green function associated to (V, P) is a function $g_{V,P}$: $V \setminus P \to \mathbb{R}^+$ that satisfies the following conditions:*

(1) *$g_{V,P}$ is continuous on $V \setminus P$, equal to zero on ∂V*
(2) *$g_{V,P}$ is harmonic on $\overset{o}{V} \setminus P$*
(3) *near P, $g_{V,P} - \log|z - P|^{-1}$ is harmonic*

Proposition 1. *Let (V,P) be a compact, connected, Riemann surface with non-empty boundary pointed at P. Then there exists a unique classical Green function on (V, P).*

Remark 1. *Conditions 2 and 3 imply that the classical Green function is locally L^2 on V and we define the harmonic measure associated to (V, P) as*

$$\mu_{V,P} := \frac{i}{\pi}\partial\overline{\partial} g_{V,P} + \delta_P \; .$$

It is a probability measure supported on the smooth curve ∂V.

Example 1. *In the case of interest of* [2]*, we have* $(V, P) = (\overline{\mathbb{D}}, 0)$*, the classical Green function is* $g_{V,P} = \log^+ |z|^{-1}$ *and the harmonic measure* $\mu_{V,P}$ *is the Haar measure* μ *on the circle* T*.*

We can now define the *Overflow* as introduced by Bost and Charles in [1].

Definition 2. *Let* N *be a Riemann surface, and let* $\alpha : (V, P) \to N$ *be a non constant analytic morphism, étale at* 0*. We define the Overflow of* α *as*

$$\mathrm{Ex}(\alpha : (V, P) \to N) := \int_V g_{V,P}(\alpha^* \alpha_* \mu_{V,P} + \delta_{\alpha^{-1}(\alpha(P))-P}) \in \mathbb{R}^+ \ .$$

In the case of interest in [2], the following theorem proven in a more general framework (theorem 5.1, [1]) explains the link between Bost-Charles double integral and the overflow.

Theorem 1. *Let* $\varphi : (\overline{\mathbb{D}}, 0) \to (\mathbb{C}, 0)$ *be an analytic morphism, étale at* 0*. Then we have the formula*

$$\mathrm{Ex}(\varphi : (\overline{\mathbb{D}}, 0) \to (\mathbb{C}, 0)) = \iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z)\mu(w) - \log |\varphi'(0)| \ .$$

Remark 2. *In the different holonomy bounds, the other expression with the rearrangement integral*

$$J := \int_0^1 2t \cdot (\log |\varphi(e^{2i\pi t})|)^* dt$$

appears and has been compared with Bost-Charles double integral

$$I := \iint_{T^2} \log |\varphi(z) - \varphi(w)| \mu(z)\mu(w)$$

by Nazarov. For any continuous map $\varphi : T \to \mathbb{C}$*, we have*

$$I \leq J \ .$$

For example, in the proof of the irrationality of products of logarithms, the final estimate gives $I \simeq 9,963$ *and* $J \simeq 9,972$*.*

1.2. **Explicit conditions on the optimization problem.** The 14 functions $(G_i)_{1 \leq i \leq 14}$ are defined on the modular curve $Y_0(2)$ and have a singularity at $-1/72$. To apply the holonomy bound, we need that

$$\varphi^* G_i \in \mathcal{O}(\overline{\mathbb{D}}) \ .$$

The modular function $h := \lambda + \frac{\lambda}{\lambda - 1}$ is the function that ensures that we have this extension property. We will define $\varphi := h \circ \psi$ as the composition of a conformal isomorphism $\psi : \mathbb{D} \to \Omega$ such that $\overline{\Omega} \subset \mathbb{D}$ with the modular function h.

Let h be the Hauptmodul for $Y_0(2)$ defined as

$$h : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}, \ q \mapsto -256 \prod_{n=1}^{+\infty} (1 + q^n)^{24} \ .$$

If

$$\psi : (\overline{\mathbb{D}}, 0) \to (\overline{\Omega}, 0)$$

is a conformal isomorphism that avoids all the preimages of $-1/72$ of h (which is the only singularity of the G_i's on $\mathbb{D}$) except for the one real preimage

$$\{0.000054... =: x_0\}\ ,$$

then the point x_0 is an "overconvergent" point for the $(G_i)_{1\leq i\leq 14}$ in the sense that

$$(h\circ\psi)^*G_i \in \mathcal{O}(\overline{\mathbb{D}})\ .$$

Therefore, it suffices to choose a contour $\partial\psi : \partial\Omega \to \mathbb{D}$ (the conformal isomorphism $\psi : \mathbb{D} \to \Omega$ will exist by the Riemann mapping theorem) that satisfies:

(1) $h^{-1}(-1/72)\ \cap \psi(\overline{\mathbb{D}}) \subset \{x_0\}$ (to get convergence on $\overline{\mathbb{D}}$)
(2) $\log|256\psi'(0)| - \tau > 0$ (to apply the holonomy bound).

2. The final choice of contour

(1) To optimize the bound, we want to have a large conformal radius $|\psi'(0)|$. But the problem is that near any cusp $\alpha = e^{2i\pi\frac{a}{b}}$ with b odd of $Y_0(2)$, the function h grows as

$$\log|h(z)| \sim \frac{1}{2\pi^2 b^2(1-|z|)}$$

and therefore, the numerator

$$\iint_{T^2} \log|h(\psi(z)) - h(\psi(w))|\ \mu(z)\mu(w)$$

is large near the cusps. The contour needs to be a large circle inside $\mathbb{D}$ but not too large to avoid the horoballs near the cusps.
(2) The preimages of $-1/72$ under h different from $\{x_0\}$ are distributed on horoballs near the boundary T. If we exclude the horoball at -1, these horoballs are contained in the ring $\{z \in \mathbb{D},\ 77/100 < |z| < 1\}$.

If we choose the contour as the circle of radius $R := 77/100$, the only preimages of $-1/72$ that are in the disk of radius R are the ones in the horoball near -1, and we have only four of them. For the problem of cusps, the only cusp α for which the "large h values set" horoball intersects the disk of radius R is the cusp 1.

Therefore, we will define the domain Ω as the disk of radius R inside $\mathbb{D}$ and remove the following parts of the disk that won't change the conformal radius so much:

(1) remove 4 slits from the boundary of the disk to the 4 preimages of $-1/72$ in the horoball near -1
(2) a lune near R to remove the intersection of the horoball near 1 with the disk.

Remark 3. *The Riemann mapping theorem guarantees that such a domain exists, but we have to give explicit formulas to make computations. Therefore, the final choice will be an approximation of this domain, obtained by composing the operations Slits and Lunes that we define now.*

Definition 3. *(Lune) Fix $c > 1$. Let $L(c) := \mathbb{D} \setminus \mathbb{D} \cap D\left(-\frac{c^2+1}{c^2-1}, \frac{2c}{c^2-1}\right)$ be the lune of parameter c. The application*

$$Lune(z, c) := \frac{z(1+c^2) - 1 - c^2 + \sqrt{(1+c^2)^2(1+z^2)^2 - 16c^2 z}}{2(c^2-1)}$$

is a conformal isomorphism from the disk $\mathbb{D}$ to the lune $L(c)$ of conformal radius $\frac{c^2-1}{c^2+1}$.

Definition 4. *(Slit) For $0 < r < 1$, we have a conformal isomorphism between the disk $\mathbb{D}$ and the Slit $\mathbb{D}\setminus\,]-1; -r]$ given by*

$$Slit(z, r) := \frac{(r+1)^2 - 2(r-1)^2 z + (r+1)^2 z^2}{8rz} + \frac{(1+r)(-1+z)\sqrt{(1+r)^2 - 2(1-6r+r^2)z + (1+r)^2 z^2}}{8rz} .$$

The conformal radius of the Slit is $\frac{4r}{(1+r)^2}$.

Define the map from $\mathbb{D}$ to Ω as

$$G(z) := -R \cdot Lune(z_1 \cdot Slit(z_2 \cdot Slit(z_3 \cdot Slit(z_4 \cdot Slit(z, r_1), r_2), r_3), r_4), c)$$

where $R = 77/100$, the parameters z_i are in T to rotate the circle to remove the slits, the radius r_i are chosen so that $-r_i$ is near the i^{th} preimage, and the lune parameter c is chosen to remove the horoball near 1.

As it is not possible to determine with perfect precision the 4 preimages of $-1/72$, we define

$$\psi(z) := G\left(\frac{995}{1000} \cdot z\right)$$

as the final choice of contour so that this dilatation transforms the slits into "open" slits.

Proposition 2. *The constructed map $\varphi := h \circ \psi$ satisfy the conditions* (1) *and* (2) *of the introduction and we have the estimate with this choice of φ:*

$$\frac{\iint_{T^2} \log|\varphi(z) - \varphi(w)|\ \mu(z)\mu(w)}{\log|256\psi'(0)| - \tau} = 13,9938... < 14 .$$

References

[1] J.-B. Bost, F. Charles, *Quasi-projective and formal-analytic arithmetic surfaces*, arXiv:2206.14242 (2022), to appear in Annals of Mathematics Studies.

[2] F. Calegari, V. Dimitrov, Y. Tang, *The linear independence of* 1, $\zeta(2)$, *and* $L(2, \chi_{-3})$, arXiv:2408.15403 (2024).