Modules with integrable connections on analytic and algebraic complex varieties II: Regular connections & regular singular points in dimension 1.

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<u>Recap</u>: X $\mathbb{C}$-analytic manifold, E vector bundle over X
$\mathcal{E}$ sheaf of analytic section of E ← locally free sheaf $/\mathcal{O}_X$

A connection on $\mathcal{E}$ is $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ + Leibniz rule.

$$\nabla_\theta (f \cdot s) = f \cdot \nabla_\theta(s) + s \otimes df.$$

NOTATION: $\nabla_\theta(s) \cdot f + s \otimes df.$

On a trivializing open set, we have frames, $U \hookrightarrow X$

$$\underline{s} : \mathcal{O}_U^{\oplus e} \xrightarrow{\sim} \mathcal{E}|_U, \quad \Omega \in M_e(\Omega^1_X(U)), \quad \nabla \underline{s} = \underline{s} \cdot \Omega$$

$$\nabla(\underline{s} \cdot \underline{f}) = \sum_{\alpha=1}^{e} s$$

$$s \cdot (\text{"} d\underline{f} + \Omega \underline{f}).$$

Locally ∇ "=" $d + \Omega$.

We saw the extension $\nabla : \mathcal{E} \otimes \Omega^i_X \longrightarrow \mathcal{E} \otimes \Omega^{i+1}_X$ "Leibniz rule"

$\nabla^2 := \nabla^1 \circ \nabla^0 : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^2_X$ is $\mathcal{O}$ linear and

can be written $\nabla^2 s = R s$ $\quad$ R section of $\mathrm{End}(\mathcal{E}) \otimes \Omega^2_X$.

$$\begin{cases} R \text{ "="} \ d\Omega + \Omega^2 \\ \nabla^2(\underline{s}\underline{f}) = \underline{s} R \underline{f}. \end{cases}$$

Direct sum, tensor product and internal Hom w/ explicit formulas, we have a tensor category.

$\mathrm{Hom}(E, F) \rightsquigarrow$ flat sections of the internal Hom.

$f : X' \longrightarrow X \rightsquigarrow$ described inverse images.

MIC = modules with integrable connections means that R vanishes = curvature vanishes = integrability

Always satisfied in dim 1.

→ local systems of $\mathbb{C}$-vector spaces over X.

= locally constant sheaf $\mathbb{E}$ of finite dimensional $\mathbb{C}$-vector spaces.

= locally $\mathbb{E} \simeq \underline{\mathbb{C}}^e$

X connected, $x_0 \in X$, there is an equivalence of categories

$$\left\{ \begin{matrix} \text{local systems of} \\ \text{---- on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite dim. rep of} \\ \pi_1(X, x) \end{matrix} \right\}$$

"MONODROMY"

Loop at x_0,

$\mathbb{E}_{x_0} = \mathbb{E}_{\gamma(0)}$; $\gamma^* \mathbb{E} \simeq \mathbb{E}_{\gamma(0)} \times \underline{\mathbb{C}}_{[0,1]}$

$\downarrow$

$\mathbb{E}_{\gamma(1)} = \mathbb{E}_{x_0}$

$\mathbb{E}$ on X, $\rightsquigarrow (E, \nabla)$

$$E \cdots \mathcal{E} := \mathbb{E} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X$$

$$\nabla := \mathrm{Id} \otimes d$$

Thm: This construction is an equivalence of categories

$$\left\{ \begin{matrix} \text{local systems} \\ \text{on } X \end{matrix} \right\} \longleftrightarrow \mathrm{MIC}(X)$$

If E is a vector bundle on X, there is a bijection

$$\left\{ \begin{matrix} \text{locally constant} \\ \mathbb{E} \subset \mathcal{E} \\ \text{s.t. } \mathbb{E} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X \xrightarrow{\sim} \mathcal{E} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \\ \text{integrable connection} \end{matrix} \right\}$$

$$\mathbb{E} \longmapsto \cancel{\mathrm{Id} \otimes \mathcal{O}_X} (\mathbb{E} \otimes \mathcal{O}_X, \mathrm{Id} \otimes d)$$

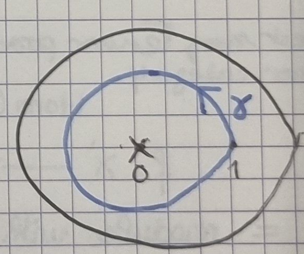
$$\mathcal{E}^{\nabla = 0} \longleftarrow (E, \nabla)$$

Adding the previous equivalence we obtain an eq. of categories.

$$\left\{ \begin{matrix} \text{f.d rep of} \\ \pi_1(X, x_0) \end{matrix} \right\} \longleftrightarrow \{ \mathrm{M.I.C} \}$$

Example of monodromy representation

① $X = \dot{D} := \dot{D}(0, r) \setminus \{0\}$ $\quad r > 1$, $x_0 = 1$



$$\pi_1(X, x_0) \xrightarrow{\sim} \mathbb{Z}$$
$$[\gamma_0] \longmapsto 1$$

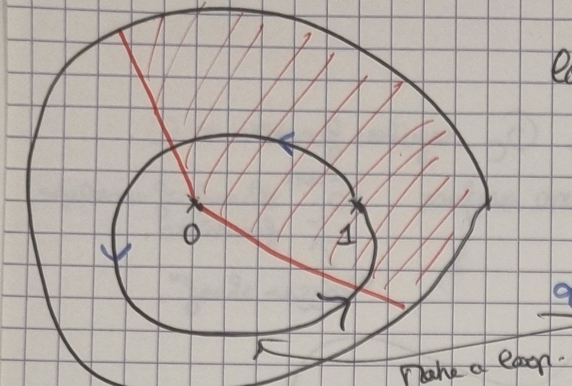
$e = 1$: $\mathrm{Hom}(\pi_1(X, x_0), \mathbb{C}^\times) = \mathbb{C}^\times$

$$\rho \longmapsto \lambda = \rho([\gamma_0]).$$

for $\lambda \in \mathbb{C}^\times$, we want to construct (E, ∇) in $\mathrm{MIC}(D)$ with monodromy λ.

$\mathcal{E} = \mathcal{O}_X$, $\quad \nabla = d + \underbrace{\Omega}_{\text{1-form}}$. Let $\ell \in \mathbb{C}$ such that $\exp(2\pi i \ell) = \lambda$

and let $\Omega = -\ell \frac{dz}{z}$, flat sections are $\mathcal{E}^{\nabla = 0} = \{ \nabla \varphi = 0 \}$



$\log : S \xrightarrow{\text{analytic}} \mathbb{C}$

$\nabla\varphi = 0 \iff d\varphi = \ell\, d(\log(z))\varphi$

$\iff \varphi = c\,\underbrace{\exp(\ell\log(z))}_{\text{“}z^{\ell}\text{”}}$

To find the monodromy.

Make a loop.

$\exp(\ell\log(z) + \ell 2\pi i) = \lambda z^{\ell}$

This can be generalized to higher dimensions.

$$\begin{aligned}\rho : \pi_1(X, x) &\longrightarrow GL_e(\mathbb{C})\\ [\gamma] &\longmapsto T\end{aligned}$$

So question becomes, with $T \in GL_e(\mathbb{C})$, how to construct (E, ∇) with monodromy T.

Write $T = DU$, D diagonal, $U = I_e + N$, $[D, N] = 0$.
N nilpotent

Pick a logarithm of the eigenvalues of D and construct a matrix "$C = \frac{1}{2i\pi}\log(D)$" diagonal in the same frame/base as D.

In particular, $[C, N] = 0$

Set $\log(I + N) = \sum_{k=1}^{e} \frac{(-1)^{k-1}}{k} N^k$

$[C, \log(I+N)] = 0$ so $\exp(2\pi i C + \log(I+N)) = DU = T$

Set $B = C + \frac{1}{2\pi i}\log(I+N)$. Now set $E = \mathcal{O}_D^{\oplus e}$, $\nabla = d + \Omega$.

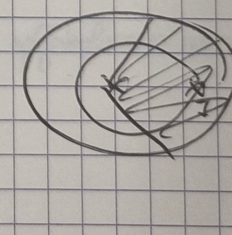
$\Omega = -B\frac{dz}{z}$ (automatically integrable.)

So we compute $\varphi \in E^{\nabla = 0}$ $\iff d\cdot\underline{\varphi} = B\frac{dz}{z}\underline{\varphi}$

"$\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_e\end{pmatrix}$"   constant $\begin{pmatrix}c_1\\ \vdots\\ c_e\end{pmatrix}$ "

$\iff \underline{\varphi} = \exp(\log(z)\cdot B)\cdot\underline{\varphi}_0$

$= \text{“}z^{B}\text{”}\,\varphi_0$ abuse of notations.



Follow the section on γ, $\log(z)$ becomes $\log(z) + 2\pi i B$.

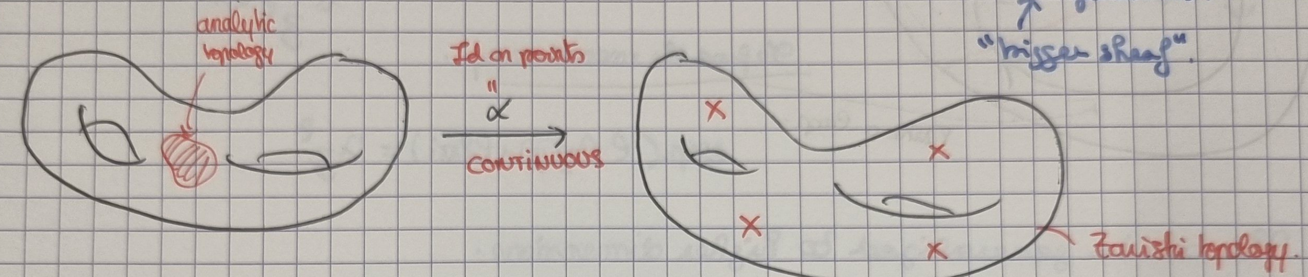
so it multiplies by $\exp(2\pi i B) = T$ ∎

Riemann-Hilbert on projective curves.

C smooth connected projective curve over $\mathbb{C}$. $\mathcal{O}_C$ regular functions on C.

"Analytify" $\rightsquigarrow$ C^{an} smooth compact connected Riemann surface. $\mathcal{O}_C^{an}$ sheaf of analytic functions.

"bigger sheaf".



$$(C^{an}, \mathcal{O}_C^{an}) \qquad\qquad (C, \mathcal{O}_C)$$

α is also a morphism of ringed spaces

$$\alpha^{-1}\mathcal{O}_C \longrightarrow \mathcal{O}_C^{an}$$

regular functions $\subset$ analytic functions

meromorphic.

Answer: $U \overset{Z}{\underset{o}{\subset}} C \qquad \mathcal{O}_C(U) = \mathcal{O}_C^{an}(U) \cap \mathcal{M}(C^{an})$.

E vector bundle on C, $\qquad E^{an} := \alpha^* E$

"analytic" vector bundle over C^{an}.

$f : E \longrightarrow F$ morphism of alg. varieties

$f^{an} : E^{an} \longrightarrow F^{an}$ $\qquad \Omega_C^{1\,an} \simeq \Omega^1_{C^{an}}$

We can analytify $\quad \nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$. "algebraic connection"

$$\alpha^{-1}\mathcal{E} \otimes_{\alpha^{-1}\mathcal{O}_C} \mathcal{O}_C^{an} \longrightarrow \alpha^{-1}(\mathcal{E} \otimes_{\mathcal{O}_C} \Omega^1_C) \otimes_{\alpha^{-1}\mathcal{O}_C} \mathcal{O}_C^{an}$$

$$\mathcal{E}^{an} \;\text{(}=\text{ the left side)} \qquad\qquad \cong \alpha^{-1}\mathcal{E} \otimes_{\mathcal{O}_C^{an}} \Omega^1_{C^{an}} = \mathcal{E}^{an} \otimes \Omega^1_{C^{an}}$$

~~$\nabla(sf) = \nabla s \cdot f + s\,df.$~~

$\nabla(s \otimes f) = \nabla s \otimes f + s\,df.$

We defined a way to attach, to any

$$(E,\nabla) \text{ in } \mathrm{MIC}(C) \xrightarrow[\sim]{\text{GAGA}} (E^{an},\nabla^{an}) \text{ in } \mathrm{MIC}(C^{an})$$

alg.   analytic

$$\downarrow \wr$$

$$\left\{ \begin{array}{c} \text{Finite dim. repr.} \\ \text{of } \pi_1(X,x_0) \end{array} \right\}$$

Riemann-Hilbert problem

(Generalizes to smooth projective algebraic varieties)

<u>pre-GAGA</u> (Puiseux - Riemann)

$$\left\{ \begin{array}{c} \text{smooth connected} \\ \text{projective curves} \\ /\mathbb{C} \end{array} \right\} \xrightarrow{an} \left\{ \begin{array}{c} \text{compact connected} \\ \text{Riemann surfaces} \end{array} \right\}$$

$$C \rightsquigarrow C^{an}$$

Specific to dimension 1

non constant morphism dominant / surj. of algebraic varieties

non constant $\mathbb{C}$-analytic.

$$C \mapsto \mathbb{C}(C)$$

$$\left\{ \begin{array}{c} \text{function fields over } \mathbb{C} \\ \text{finitely generated ext.} \\ K \text{ of } \mathbb{C}, \ \deg \mathrm{tr}_{\mathbb{C}}(K)=1 \end{array} \right\}$$

$$S \mapsto \mathcal{M}(S)$$

meromorphic

<u>GAGA</u> over $\mathbb{C}$

<u>comparison</u>: E vector bundle over C, $H^i(C,E) \xrightarrow{\sim} H^i(C^{an},E^{an})$, $i=0,1$.

<u>existence</u>: Every analytic vector bundle over C^{an}, is isomorphic to E^{an} for some vector bundle E over C

$$\left\{ \begin{array}{c} \text{vector bundles} \\ \text{over } C \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{analytic vector} \\ \text{bundles over } C^{an} \end{array} \right\}$$

use comparison in $\mathrm{Hom}_{\mathcal{O}_C^{an}}(E^{an},F^{an}) = H^0(C^{an},E^{\vee an}\otimes F^{an})$

Pb: to algebraize ΠIC on C^{an} : how to algebraize connections ? these are not $\mathcal{O}_{C^{an}}$-linear !!!.

Solution: use Atiyah extension.

X alg/an smooth complex

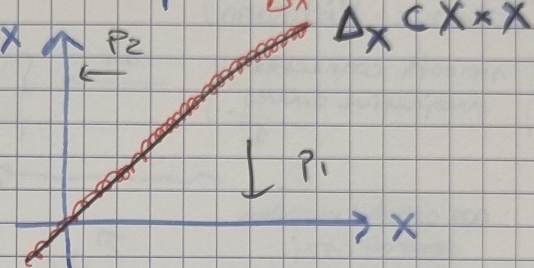
E vector bundle over X

Introduce the jet bundle of E over X

$$J^1E_x = \mathcal{E}_x / m_x^2 \mathcal{E}_x$$

(recall $E_x = \mathcal{E}_x / m_x \mathcal{E}_x$)

Consider the product



J_X ideal sheaf of Δ_X in $\mathcal{O}_{X \times X}$

Consider

Δ'_X subspace of $X \times X$ defined by J_X^2,

Set $J^1E := (p_2)_* (\mathcal{O}_{\Delta'_X} \otimes p_1^* E)$.

Atiyah short exact sequence of vector bundles

$$At(E) \qquad 0 \longrightarrow \Omega^1_X \otimes E \longrightarrow J^1E \underset{ev}{\overset{\sigma}{\rightleftarrows}} E \longrightarrow 0.$$

$$\begin{tikzcd}
0 \arrow[r] & (\Omega^1_X \otimes E)_x \arrow[r] \arrow[d, phantom, "\wr\wr"] & (J^1E)_x \arrow[r, "ev"'] \arrow[d, phantom, "\wr\wr"] & E_x \arrow[r] \arrow[d, phantom, "\wr\wr"] & 0 \\
& \Omega^1_{X,x} \otimes E_x \arrow[d, phantom, "\wr\wr"] & \mathcal{E}_x / m_x^2 \mathcal{E}_x & \mathcal{E}_x / m_x \mathcal{E} & \\
& m_x / m_x^2 \otimes \mathcal{E}_x / m_x & & &
\end{tikzcd}$$

Construction: A splitting σ of $At(E)$

$\rightsquigarrow$ connection ∇ on E, $s \in \mathcal{E}(U) \longmapsto j^1 s \in J^1E(U)$

$\sigma \cdot s \in J^1E(U)$

then $\nabla s = j^1 s - \sigma . s \in (\mathcal{E} \otimes \Omega^1_X)(U)$

$\downarrow ev$

0.

$\sigma : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X$

is a connection on $\mathcal{E}$.

And this defines a bijection.

So $\{\text{splittings of } At(E)\} \xrightarrow{\sim} \{\text{connections on } E\}$.

$$\sigma \longmapsto \nabla = j^1 - \sigma.$$

$$(j^1 s)(x) = [s_x] \qquad s_x \in E_x \longrightarrow E_x/m_x^2$$

$$J^1 \mathcal{O}_X = \mathcal{O}_X \oplus \Omega^1_X$$

$$\mathcal{D}iff^{\leq 1}_X \simeq \mathcal{O}_X \oplus \Theta_X.$$

E algebraic $\quad 0 \to \Omega^1_X \otimes E \hookrightarrow J^1 E \to E \to 0.$ $\qquad At(E)^{an}$

$\wr$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\mid$

E^{an} analytic $\quad 0 \to \Omega^1_X \otimes E \to J^1 E \to E \to 0$ $\qquad At(E^{an})$

σ

Start with E alg

analytic splitting of $At(E^{an}) \Leftrightarrow$ alg. splitting of $At(E)$

algebraic connection on E.