Some features of the typical Poisson-Delaunay/ Poisson-Voronoi cell

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Part 1 - Poisson-Delaunay

Poisson-Delaunay tessellation and the typical cell

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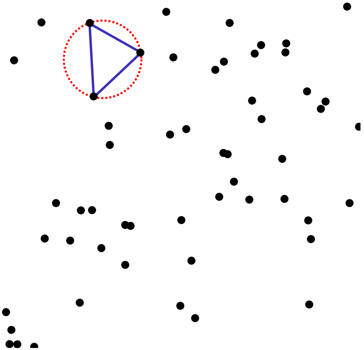

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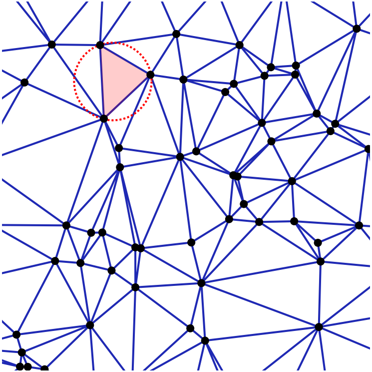
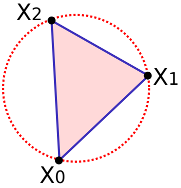


Typical cell: a cell chosen uniformly at random in a large window. Denote $X_0, \ldots, X_d$ its vertices.

$$\mathcal{C}_{\text{typ}}^{\text{Del}} = \text{Conv}\left(X_0, \ldots, X_d\right)$$



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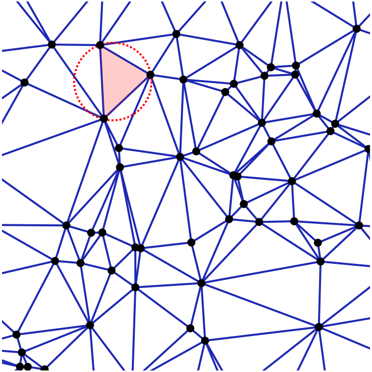

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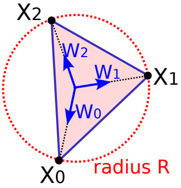


Law of the typical cell

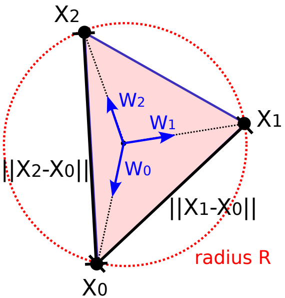
$$(\mathrm{X}_0, ..., \mathrm{X}_d) \overset{\mathrm{law}}{=} \mathrm{R}(\mathrm{W}_0, ..., \mathrm{W}_d)$$

where R has a known generalized Gamma distribution, $(\mathrm{W}_0, ..., \mathrm{W}_d)$ are points on the unit sphere with density proportional to $\mathrm{Vol}_d(\mathrm{Conv}(\mathrm{W}_0, ..., \mathrm{W}_d))$.

Some edge lengths of the typical cell

Question: what is the law of

$$(\|X_1 - X_0\|, ..., \|X_d - X_0\|) \ ?$$



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$$\mathbb{E}\left[\|X_1 - X_0\|^{\alpha_1} \cdots \|X_d - X_0\|^{\alpha_d}\right]$$

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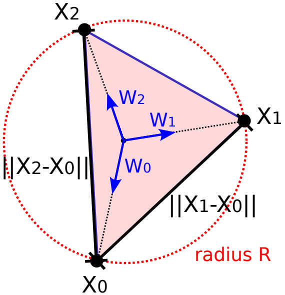


We want to compute the **Mellin transform**, that is

$$\mathbb{E}\left[\|X_1 - X_0\|^{\alpha_1} \cdots \|X_d - X_0\|^{\alpha_d}\right] = \mathbb{E}\left[R^{\alpha_1+\ldots+\alpha_d}\|W_1 - W_0\|^{\alpha_1} \cdots \|W_d - W_0\|^{\alpha_d}\right]$$

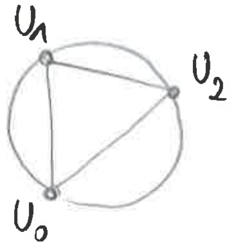
$$= \underbrace{\frac{\mathbb{E}\left[R^{\alpha_1+\ldots+\alpha_d}\right]}{\mathbb{E}\left[\mathrm{Vol}_d(\mathrm{Conv}(U_0, ..., U_d))\right]}}_{\text{found by [1]}} \mathbb{E}\left[\|U_1 - U_0\|^{\alpha_1} \cdots \|U_d - U_0\|^{\alpha_d} \mathrm{Vol}_d(\mathrm{Conv}(U_0, ..., U_d))\right]$$

where $U_0, ..., U_d$ are i.i.d. uniform on the unit sphere.

[1] R.E. Miles. Isotropic random simplices. *Adv. in App. Prob.* **Vol. 3, No. 2**, 353-382 (1971)

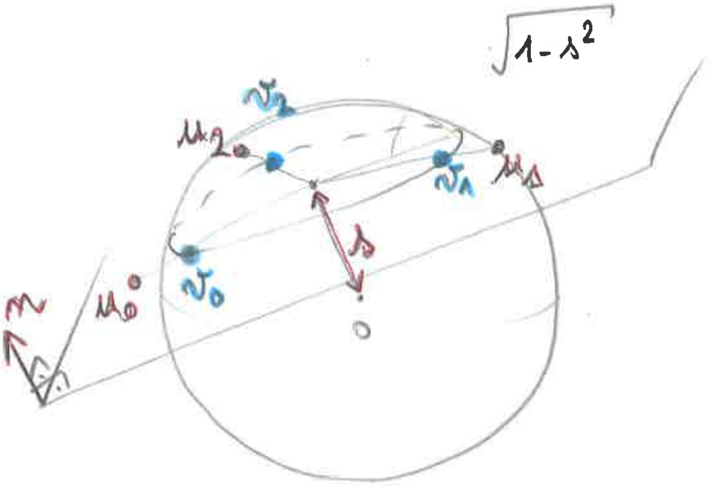
Miles $d = 2$

$$\mathbb{E}\left[\mathrm{Vol}\left(\mathrm{Conv}(U_0, U_1, U_2) \right) \right] = ?$$



iid $\mathrm{Unif}(\mathcal{S}_1)$.

miles starts from a different space ... a 2-dim sphere !



choose

$v_0, v_1, v_2 \in \mathcal{S}_2$ $\equiv$

choose

$n \in \mathcal{S}_2$

$s \in [0,1]$

$u_0, u_1, u_2 \in \mathcal{S}_1$

Choosing 3 points on $\mathcal{S}_2$ is easy ... we know how to do it.
However, we can do this choice in a much more complicated way, that will give us access to $\mathbb{E}\left[\mathrm{Vol}(\mathrm{Conv}(U_0, U_1, U_2))\right]$.

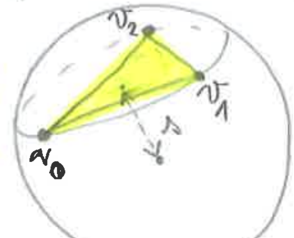
" v_0, v_1, v_2 live on a circle,

this circle lays on an affine plane

- this plane is orthogonal to vector $n \in \mathcal{S}_2$.
- this plane has distance $s \in [0,1]$ to the origin (this also fixes the radius of the circle : $\sqrt{1-s^2}$)
- finally, we need local coordinates within the affine plane : $u_0, u_1, u_2 \in \mathcal{S}_1$. "

Let's see how the volume unit changes between the two sets of coordinates

$$\underbrace{d\sigma_2(v_0)\, d\sigma_2(v_1)\, d\sigma_2(v_2)}_{\text{surface measure on } S_2} = 2\, \underbrace{d\sigma_2(n)}_{\text{surface measure on } S_2}\, \underbrace{ds}_{\text{Lebesgue}}\, (1-s^2)\, \underbrace{d\sigma_1(u_0)\, d\sigma_1(u_1)\, d\sigma_1(u_2)}_{\text{surface meas. on } S_1}\, Vol(Conv(u_0,u_1,u_2))$$

$(1-s^2)\cdot Vol(Conv(u_0,u_1,u_2)) = Vol\Bigg($  $\Bigg)$

We can integrate this expression and obtain

$$\int d\sigma_2(v_0)\int d\sigma_2(v_1)\int d\sigma_2(v_2) = 2\int d\sigma_2(n)\int ds\,(1-s^2)\int d\sigma_1(u_0)\int d\sigma_1(u_1)\int d\sigma_1(u_2)\, Vol(Conv(u_0,u_1,u_2))$$

$$(4\cancel{\pi})^3 = \cancel{2}\cdot 4\pi \cdot \frac{2}{3}\cdot (2\cancel{\pi})^3\, \mathbb{E}\left[Vol(Conv(U_0,U_1,U_2))\right]$$

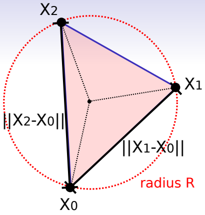
$$\Rightarrow \mathbb{E}\left[Vol(Conv(U_0,U_1,U_2))\right] = \frac{3}{2\pi}$$

What we were interested in was $\mathbb{E}\left[Vol(Conv(U_0,U_1,U_2))\,\|U_1-U_0\|^{\alpha_1}\|U_2-U_0\|^{\alpha_2}\right]$...

we find that we can insert these distances on either side of Miles' relation!

$$\underbrace{\overbrace{\int d\sigma_2(v_0)\int d\sigma_2(v_1)\int d\sigma_2(v_2)}^{\|v_1-v_0\|^{\alpha_1}\|v_2-v_0\|^{\alpha_2}}}_{\substack{\text{distance between indep.}\\ \text{points, that we can}\\ \text{compute} \to \text{Beta functions}}} = 2\underbrace{\int d\sigma_2(n)}_{\text{constant}}\underbrace{\overbrace{\int ds\,(1-s^2)}^{\text{renormalization: } (\sqrt{1-s^2})^{\alpha_1+\alpha_2}}}_{\text{Beta function}}\underbrace{\overbrace{\int d\sigma_1(u_0)\int d\sigma_1(u_1)\int d\sigma_1(u_2)\, Vol\, Conv(u_0,u_1,u_2)}^{\|u_1-u_0\|^{\alpha_1}\|u_2-u_0\|^{\alpha_2}}}_{=(2\pi)^3\,\mathbb{E}\left[Vol\cdot\|U_1-U_0\|^{\alpha_1}\|U_2-U_0\|^{\alpha_2}\right]}$$

We can solve the equation for and recognize the law ...



Miles "2.0"

Our problem: For $\mathrm{U}_0, \ldots, \mathrm{U}_d$ i.i.d. uniform on the unit sphere:

$$\mathbb{E}\left[\|\mathrm{U}_1 - \mathrm{U}_0\|^{\alpha_1} \cdots \|\mathrm{U}_d - \mathrm{U}_0\|^{\alpha_d} \mathrm{Vol}_d(\mathrm{Conv}(\mathrm{U}_0, \ldots, \mathrm{U}_d))\right] =? \qquad (1)$$

Re-purposing the methods used by Miles [1] to find $\mathbb{E}\left[\mathrm{Vol}_d(\mathrm{Conv}(\mathrm{U}_0, \ldots, \mathrm{U}_d))\right]$, we can compute (1) and recover a representation of the law of $(\|\mathrm{X}_1 - \mathrm{X}_0\|, \ldots, \|\mathrm{X}_d - \mathrm{X}_0\|)$.

Theorem.

$$(\|X_1 - X_0\|^2, \ldots, \|X_d - X_0\|^2) \overset{\text{law}}{=} R^2(L_{\sigma(1)}, \ldots, L_{\sigma(d)})$$

where R has density proportional to $r \mapsto r^{d^2-1} e^{-\kappa_d r^d}$, σ is a uniform random permutation and

$$(L_1, \ldots, L_d) \overset{\text{law}}{=} (CB_1D_1, \ldots, CB_{d-1}D_{d-1}, CB_d)$$

where the random variables $B_1, \ldots, B_d, D_1, \ldots, D_{d-1}, C$ are all independent and Beta distributed with the following parameters: $C \sim \mathrm{Beta}\left(\frac{d^2+1}{2}, \frac{1}{2}\right)$, $B_1, \ldots, B_d \sim \mathrm{Beta}\left(\frac{d}{2}+1, \frac{d}{2}-1\right)$ and $D_1, \ldots, D_{d-1} \sim \mathrm{Beta}\left(\frac{d}{2}, 1\right)$.

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Part 2 - Poisson-Voronoi

Poisson-Voronoi tessellation, the typical cell and its circumradius

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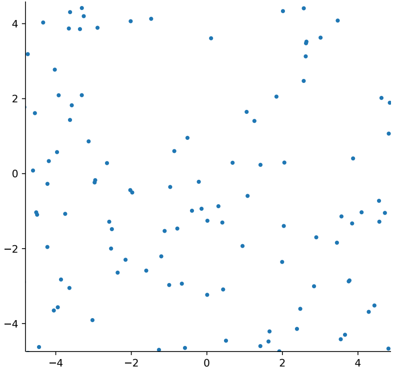

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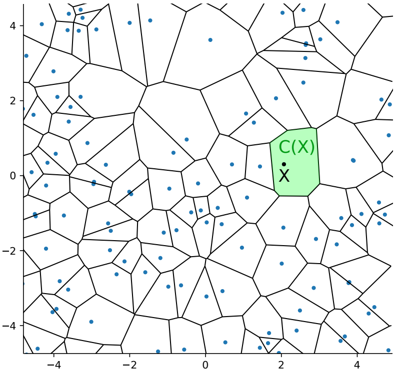
$$\mathcal{C}(\mathrm{X}) = \{y \in \mathbb{R}^d : \|y - \mathrm{X}\| \leq \|y - \mathrm{X}'\|, \mathrm{X}' \in \mathrm{PPP}\}$$



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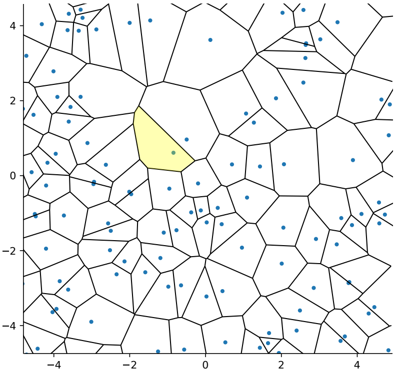
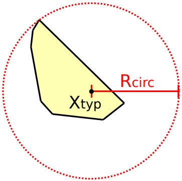


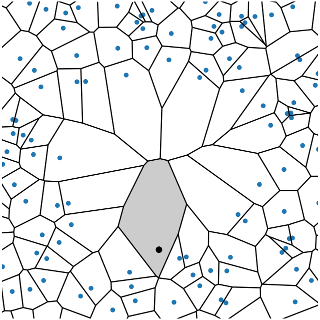
Typical cell $\mathcal{C}_{\mathrm{typ}} = \mathcal{C}(\mathrm{X}_{\mathrm{typ}})$: a cell chosen uniformly at random in a large window.



Circumradius $\mathrm{R}_{\mathrm{circ}}$ of the typical cell:

$$\mathrm{R}_{\mathrm{circ}} = \min(\mathrm{r} \geq 0 : \mathcal{C}_{\mathrm{typ}} \subseteq \mathcal{B}_r(\mathrm{X}_{\mathrm{typ}}))$$

Previous results in dimension 2



Calka, in [2], finds a double bound for $d = 2$. For t large enough,

$$2\pi t^2 e^{-\pi t^2} \leq \mathbb{P}\left(R_{\mathrm{circ}} \geq t\right) \leq 4\pi t^2 e^{-\pi t^2}.$$

Question: $\mathbb{P}\left(\mathrm{R}_{\mathrm{circ}} \geq t\right) \underset{t\to\infty}{\sim} ?$

[2] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *Adv. in Appl. Probab.* **34**, 702-717 (2002)

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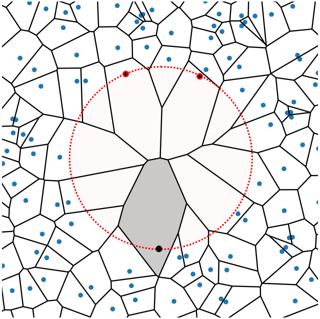

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factor t^2 : positioning of ◯ around X_{typ}

factor $e^{-\pi t^2}$:

$$e^{-\pi t^2} = e^{-\mathrm{Vol}_2(\bigcirc)}$$
$$= \mathbb{P}(\mathrm{PPP} \cap \bigcirc = \emptyset)$$

Guess:

$$\mathbb{P}(R_{\text{circ}} \geq t) \overset{t\to\infty}{\sim} C_d t^{n_d} e^{-\kappa_d t^d}$$

where κ_d is the volume of the unit ball,
C_d is a constant to be found,
n_d is a power to be found.

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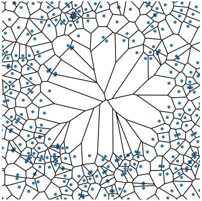
Theorem and implications regarding the extremal index

Theorem 1 (Distribution tail of the circumradius of the typical cell).

$$\mathbb{P}(R_{\text{circ}} \geq t) \overset{t\to\infty}{\sim} C_d t^{d(d-1)} e^{-\kappa_d t^d}$$

where κ_d is the volume of the unit ball and C_d is an explicit constant with $C_2 = 4\pi$.

Extremal index $= \dfrac{1}{\mathbb{E}[\#\text{ exceedances in a cluster of exceedances }]}$



Limit law of the max circumradius in an increasing window (properly rescaled)

[3] P. Calka & N. Chenavier. Extreme values for characteristic radii of a Poisson-Voronoi tessellation. *Extremes* **17**, 359-385 (2014)

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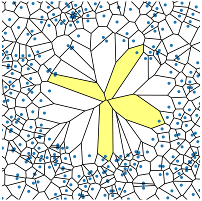

Theorem 2. Extremal index $= \frac{1}{2d}$.
For $d = 2$, extremal index $= \frac{1}{4}$

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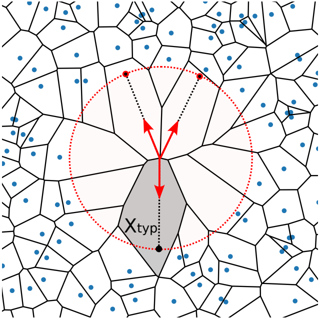
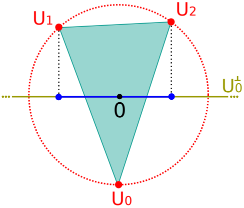
Random simplices in the unit sphere - value of the constant C_d

Theorem 3. Let $U_0, \ldots, U_d$ i.i.d. uniform on the $(d-1)$-dim unit sphere and denote κ_d the volume of the unit ball.

$$\frac{C_d}{(d\kappa_d)^d} = \mathbb{E}\left[\mathrm{Vol}_d\left(\mathrm{Conv}\left(U_0, \ldots, U_d\right)\right) \mathbb{1}_{0 \in \mathrm{Conv}\left(\mathrm{Proj}_{U_0^\perp}(U_1), \ldots, \mathrm{Proj}_{U_0^\perp}(U_d)\right)}\right]$$

$$= \frac{1}{2^{d-1}\sqrt{\pi}(d-1)!} \frac{\Gamma\left(\frac{d}{2}\right)^d}{\Gamma\left(\frac{d+1}{2}\right)^{d-1}}$$





$\left\{0 \in \mathrm{Conv}\left(\mathrm{Proj}_{U_0^\perp}(U_i)\right)\right\}$: describes locally the extremal shape of the cell.

$\mathrm{Vol}_d\left(\mathrm{Conv}\left(U_0, \ldots, U_d\right)\right)$: arises from a polar change of coordinates.

Thank you for your attention!