CHAPTER 4

Preserved curvature conditions

1. The strong maximum principle for functions

What happens if the scalar curvature attains its minimum at an interior space-time point along the Ricci flow ? The answer is obtained through the use of the strong maximum/minimum principle.

Lemma 4.1. *Let $(M, g(t))_{t\in[0,T)}$ be a smooth one-parameter family of metrics on a connected manifold M without boundary and let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function that satisfies*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u.$$

If $u \geq 0$ on $M \times [0, T)$ and $u(x_0, t_0) = 0$ for some $x_0 \in M$ and $t_0 > 0$ then $u(x, t) = 0$ for all $x \in M$ and $t \in [0, t_0)$.

Proof. There are many proofs of this result. We present two of them and we invite the reader to consult the reference [**PW84**] for instance. The order we choose reflects the amount of additional assumptions we make on top of those of Lemma 4.1.

First proof: Assume M to be compact. By the minimum principle Lemma 3.1, $u(x, t) \geq v(x, t)$ where v solves $\frac{\partial}{\partial t} v = \Delta_{g(t)} v$ on $M \times (0, T)$ and $v(x, 0) := u(x, 0)$ for $x \in M$. Then take for granted that there exists a kernel called the heat kernel, denoted by $K(x, t, y, s)$ for two space-time points (x, t) and (y, s) in $M \times [0, T)$ with $s < t$ such that the following representation formula holds true:

$$v(x,t) = \int_M K(x,t,y,s)v(y,s)\, d\mu_{g(s)}(y), \quad t > s \geq 0,\, x \in M. \tag{1.1}$$

It can be shown that $K(x, t, y, s) > 0$ for all such space-time points. In particular, if $u(x_0, t_0) = 0$ for some point $(x_0, t_0) \in M \times (0, T)$ then $v(x_0, t_0) = 0$ which forces the vanishing of $v(\cdot, s)$ for all $s \leq t_0$ thanks to (1.1). This implies in particular that $u(\cdot, 0) = v(\cdot, 0) = 0$ identically on M. The previous reasoning can be applied to any time $s \in [0, t_0)$.

One drawback of this proof is that it uses the notion of heat kernels whose existence in this setting is not straightforward and whose proof of its positivity relies on... the minimum principle! The major drawback however is that the strong maximum principle is a local statement whereas the previous proof gives the feeling that it is global.

Second proof:

If Ω is an open subset of M with smooth boundary that contains x_0 then let us consider the auxiliary solution to $\frac{\partial}{\partial t} v = \Delta_{g(t)} v$ on $\Omega \times (0, T)$ and $v(\cdot, t) := u(\cdot, t)$ on $\overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T)$. Then (an easy adaptation of) the minimum principle Lemma 3.1 ensures that $u(x, t) \geq v(x, t) \geq 0$ for $(x, t) \in \Omega \times [0, T)$. Now we invoke the parabolic Harnack inequality for nonnegative solutions to parabolic PDEs: for all $\Omega' \subset\subset \Omega$ and each $0 < s < t < T$, there exists C depending on Ω', s, t and the metrics $g(\cdot)$ such that:

$$\sup_{\Omega'} v(\cdot, s) \leq C \inf_{\Omega'} v(\cdot, t).$$

If we apply this inequality to $t = t_0$ and $s < t_0$, we get $v(\cdot, s) = 0$ since $v(x_0, t_0) = 0 = \min_{\Omega\times[0,T)} v$ if $u(x_0, t_0) = 0$. In particular, we have proved that $u(\cdot, s) = v(\cdot, s) = 0$ on $\partial\Omega$ for all $s \in [0, t_0)$ and all such sets Ω. This ends the proof.

The advantage of this proof modulo that of the parabolic Harnack inequalities is that it is purely local. The proof of the parabolic Harnack inequalities can be found for instance in [**Eva10**, Chapter 7, Theorem 10] in the case of smooth coefficients and we present it below in our Riemannian setting. $\square$

Proposition 4.2. *Ancient solutions to the Ricci flow $(M, g(t))_{t\in(-\infty,T)}$ on a closed manifold have nonnegative scalar curvature. More precisely, they are either Ricci flat or have positive scalar curvature, i.e. either $\mathrm{Ric}(g(t)) = 0$ on M for all $t < T$ or $\mathrm{R}_{g(t)} > 0$ for all $t < T$.*

Proof. Let us show first that ancient solutions to the Ricci flow have nonnegative scalar curvature. According to Proposition 3.4, $\mathrm{R}_{g(t)} \geq -\frac{n}{2t}$ for all $t \in [0, T)$. This lower bound does not depend on T. Since the solution is assumed to live on $(-\infty, T)$, we apply the previous reasoning to the translation of this solution by some time $T' > 0$: $\mathrm{R}_{g(t)} \geq -\frac{n}{2(t+T')}$ for $t \in [-T', T]$. Fix some time t in this interval and let T' go to $+\infty$ to reach the desired estimate.

The strong minimum principle Lemma 4.1 applied to the evolution equation satisfied by $\mathrm{R}_{g(t)}$ implies that either $\mathrm{R}_{g(t)} > 0$ for all $t \in (-\infty, T)$ or $\mathrm{R}_{g(t)} \equiv 0$ for all $t \in (-\infty, T)$. Now Proposition 1.23 also implies that $|\mathrm{Ric}(g(t))|^2_{g(t)} = 0$ for all $t \in (-\infty, T)$. This ends the proof.

$\square$

2. Baby Harnack differential inequality

In this section, we give a proof of a local version of the so called Harnack differential inequality for positive solutions to the heat equation:

Recall that on Euclidean space $(\mathbb{R}^n, \mathrm{eucl})$, the heat kernel

$$k(x, t, y, s) := (4\pi(t-s))^{-\frac{n}{2}} \exp\left\{-\frac{|x-y|^2}{t-s}\right\},$$

for x, y in $\mathbb{R}^n$ and $s < t$. Observe that:

$$\Delta_x \log k(\cdot, t, y, s) + \frac{n}{2(t-s)} = \partial_t \log k(\cdot, t, y, s) - |\nabla \log k|^2(\cdot, t, y, s) = 0.$$

Theorem 4.3. *Let (M^n, g) be a complete Riemannian manifold and let u be a positive solution to the heat equation: $\partial_t u = \Delta_g u$. Then, for all $\Omega' \subset\subset \Omega$, Ω a bounded domain of M, and for each $0 < s < t$, there exists C depending on Ω, s, t and the metric g such that:*

$$\sup_{\Omega'} u(\cdot, s) \leq C \inf_{\Omega'} u(\cdot, t).$$

Remark 4.4. Theorem 4.3 is stated for a time-independent metric g but the proof is robust enough to handle smooth 1-parameter families of metrics.

Proof. By covering Ω' by balls whose radii are smaller than the injectivity radius of g on Ω, it is enough to prove this inequality on such balls. We assume u to be positive, otherwise, we apply our reasoning to $u + \varepsilon$ for $\varepsilon > 0$.

Let $v := \log u$ and let us compute the evolution equation satisfied by $w := \Delta_g v$:

$$\begin{aligned}
\partial_t v &= \frac{\Delta_g u}{u} = \Delta_g v + |\nabla^g v|^2_g = w + |\nabla^g v|^2_g, \\
\partial_t w &= \Delta_g (\partial_t v) = \Delta_g w + 2|\nabla^{g,2} v|^2_g + 2g(\Delta_g \nabla^g v, \nabla^g v) \\
&= \Delta_g w + 2|\nabla^{g,2} v|^2_g + 2\,\mathrm{Ric}(g)(\nabla^g v, \nabla^g v) + 2g(\nabla^g w, \nabla^g v),
\end{aligned}$$

where we have used the Bochner formula in the last line. Since the Ricci curvature is bounded from below on Ω,

$$\partial_t w \geq \Delta_g w + 2|\nabla^{g,2} v|^2_g - C|\nabla^g v|^2_g + 2g(\nabla^g w, \nabla^g v).$$

Let us use $|\nabla^g v|_g^2$ as a barrier:

$$\begin{aligned}\partial_t|\nabla^g v|_g^2 &= 2g(\nabla^g(\Delta_g v+|\nabla^g v|_g^2),\nabla^g v)\\ &= 2g(\Delta_g\nabla^g v,\nabla^g v)-2\operatorname{Ric}(g)(\nabla^g v,\nabla^g v)+2g(\nabla^g|\nabla^g v|_g^2,\nabla^g v)\\ &= \Delta_g|\nabla^g v|_g^2-2|\nabla^{g,2}v|_g^2-2\operatorname{Ric}(g)(\nabla^g v,\nabla^g v)+2g(\nabla^g|\nabla^g v|_g^2,\nabla^g v)\\ &\geq \Delta_g|\nabla^g v|_g^2-2|\nabla^{g,2}v|_g^2-C|\nabla^g v|_g^2+2g(\nabla^g|\nabla^g v|_g^2,\nabla^g v),\end{aligned}$$

since the Ricci curvature is bounded from above on Ω.

Therefore, the function $\tilde{w}:=w+A|\nabla^g v|_g^2$ for $A>0$ satisfies:

$$\begin{aligned}(\partial_t-\Delta_g-2g(\nabla^g v,\nabla^g\cdot))\,\tilde{w} &\geq 2(1-A)|\nabla^{g,2}v|_g^2-C(1+A)|\nabla^g v|_g^2\\ &\geq |\nabla^{g,2}v|_g^2-C'|\nabla^g v|_g^2,\end{aligned}$$

if $A\in(0,1/2]$ and where $C'=C'(\Omega',n,A)$.

In order to localize the previous differential inequality satisfied by $\tilde{w}$, let us consider a ball $B_g(x,r)$ with $x\in\Omega'$ and $r<\operatorname{inj}_g(x)$ and a smooth cut-off function φ defined on $M\times(0,T)$ so that $\varphi\equiv 1$ on $B_g(x,r/2)\times[s,t]$ and $\varphi\equiv 0$ on $\partial B_g(x,r)\times(0,T)\cup\overline{B_g(x,r)}\times\{0\}$.

Then:

$$\begin{aligned}(\partial_t-\Delta_g-2g(\nabla^g v,\nabla^g\cdot))\left(\varphi^2\tilde{w}\right) \geq{}& \tilde{\omega}\,(\partial_t-\Delta_g-2g(\nabla^g v,\nabla^g\cdot))\,\varphi^2-2g(\nabla^g\varphi^2,\nabla^g\tilde{\omega})\\ &+\varphi^2|\nabla^{g,2}v|_g^2-C\varphi^2|\nabla^g v|_g^2\\ \geq{}& \varphi^2|\nabla^{g,2}v|_g^2-C\varphi\tilde{w}|\nabla^g v|_g-2\varphi^2 g(\nabla^g\varphi^2,\nabla^g\tilde{\omega})-C\varphi^2|\nabla^g v|_g^2-C\tilde{w},\end{aligned}$$

where C depends on r and the derivatives of φ. Now, the Cauchy-Schwarz inequality gives $n|\nabla^{g,2}v|_g^2\geq(\Delta_g v)^2$ which implies after multiplying the previous inequality across by φ^2:

$$\begin{aligned}\varphi^2\left(\partial_t-\Delta_g-2g(\nabla^g v,\nabla^g\cdot)\right)\left(\varphi^2\tilde{w}\right) \geq{}& \frac{1}{n}\varphi^4|\Delta_g v|^2-C\varphi^2|\tilde{w}||\nabla^g v|_g-C\varphi^4|\nabla^g v|_g^2\\ &-C\varphi^2|\tilde{w}|-2\varphi^2 g(\nabla^g\varphi^2,\nabla^g\tilde{\omega}).\end{aligned}$$

If there exists $(x_0,t_0)\in M\times(0,T)$ such that $\varphi^2\tilde{w}(x_0,t_0)<0$ then $t_0>0$, $\varphi^2\nabla^g\tilde{w}+w\nabla^g\varphi^2=0$ at that point and:

$$\begin{aligned}0 &\geq \frac{1}{n}\varphi^4|\Delta_g v|^2-C\varphi^3|\tilde{w}||\nabla^g v|_g-C\varphi^4|\nabla^g v|_g^2-C\varphi^2|\tilde{w}|-2\varphi^2 g(\nabla^g\varphi^2,\nabla^g\tilde{\omega})\\ &\geq \frac{1}{n}\varphi^4|\Delta_g v|^2-C\varphi^3|\tilde{w}||\nabla^g v|_g-C\varphi^4|\nabla^g v|_g^2-C\varphi^2|\tilde{w}|-2|\tilde{w}||\nabla^g\varphi^2|_g^2\\ &\geq \frac{1}{n}\varphi^4|\Delta_g v|^2-C\varphi^3|\tilde{w}||\nabla^g v|_g-C\varphi^4|\nabla^g v|_g^2-C\varphi^2|\tilde{w}|,\end{aligned}$$

at that point. Since $\tilde{w}(x_0,t_0)<0$, this implies $|\nabla^g v|_g^2<A^{-1}(-\Delta_g v)\leq A^{-1}|\Delta_g v|$ and $|\tilde{w}|=-\tilde{w}\leq|\Delta_g v|$ at that point so that:

$$\begin{aligned}0 &\geq \frac{1}{n}|\varphi^2\Delta_g v|^2-C\varphi^3|\tilde{w}||\Delta_g v|^{\frac{1}{2}}-C\varphi^4|\Delta_g v|-C\varphi^2|\tilde{w}|\\ &\geq \frac{1}{n}|\varphi^2\Delta_g v|^2-C(\varphi^2|\Delta_g v|)^{\frac{3}{2}}-C\varphi^2|\Delta_g v|-C\varphi^2|\Delta_g v|\end{aligned}$$

where C is a positive constant that may vary from line to line and where we have used the fact that $0\leq\varphi\leq 1$. Young's inequality used appropriately implies that there exists a positive constant C such that $\varphi^2\tilde{w}(x_0,t_0)\geq -C$.

This means that on $B_g(x,r/2)$, $\tilde{w}=\Delta_g\log u+A|\nabla^g\log u|_g^2\geq -C$. This in turn implies that

$$\partial_t\log u\geq(1-A)|\nabla^g\log u|_g^2-C\geq\frac{1}{2}|\nabla^g\log u|_g^2-C.$$

If x and y are points in $B_g(x,r/2)$, then choose a minimizing geodesic $\gamma(\tau)$ connecting $x=\gamma(0)$ to $y=\gamma(1)$ and consider the function $\log u(\gamma(\tau),(1-\tau)s+\tau t)$ as a function of τ:

$$\begin{aligned}\log u(x,t)-\log u(y,s) &= \int_0^1 \frac{d}{d\tau}\log u\,d\tau\\ &= \int_0^1 g(\nabla^g \log u,\dot\gamma)+(t-s)\partial_t \log u\,d\tau\\ &\geq \int_0^1 -|\nabla^g\log u|_g|\dot\gamma|_g+(t-s)\left(\frac{1}{2}|\nabla^g\log u|_g^2-C\right)\,d\tau\\ &\geq -C',\end{aligned}$$

where C' depends on $d_g(x,y)<r$ and $t-s$ and the geometry of Ω'.

This implies the desired result.

□

Remark 4.5. Notice that we have not been too subtle on the sign of the Ricci curvature in the previous proof. See exercices.

3. Maximum principles for symmetric 2-tensors

This section is devoted to the proof of the following maximum principle for symmetric 2-tensors:

Lemma 4.6. *Let $(M^n,g(t))_{t\in[0,T]}$ be a smooth one-parameter family of metrics on a closed manifold. Let $T(t)$ be a one-parameter family of symmetric 2-tensors satisfying on $M\times(0,T)$:*

$$\frac{\partial}{\partial t}T(t)\geq \Delta_{g(t)}T(t)+\nabla^{g(t)}_{X(t)}T(t)+S_T(t).$$

Here $S_T(t)=S(T(t),g(t))$ is a one-parameter family of symmetric 2-tensors satisfying

(i) *the null-eigenvector assumption: if $T(p,t)(v,\cdot)=0$ for some $v\in T_pM$ then $S_T(p,t)(v,v)\geq 0$;*
(ii) *a Lipschitz condition: if $T_1(t)$ and $T_2(t)$ are symmetric 2-tensors, then*

$$|S_{T_2}(t)-S_{T_1}(t)|_{g(t)}\leq C|T_2(t)-T_1(t)|_{g(t)},$$

for some constant $C=C\left(\sup_{M\times[0,T]}\left(|T_1(t)|_{g(t)}+|T_2(t)|_{g(t)}\right)\right)$.

If $T(0)\geq 0$ then $T(t)\geq 0$ for $t\in(0,T)$.

Proof. The idea consists in using the maximum principle for functions as stated in Lemma 3.1. Let us assume by contradiction that $T(t)$ is not non-negative for some time t which is necessarily positive. Then there must be a first time t_1 and a point x_1 together with a tangent vector $v_1\in T_{x_1}M$ such that $T(x_1,t_1)(v_1,\cdot)=0$. In particular, by definition of such space-time point, $T(x,s)(w,w)\geq 0$ for all $x\in M$, $s\in[0,t_1]$ and $w\in T_xM$. Extend the vector v_1 to a vector field defined on a neighborhood of (x_1,t_1) in a time-independent way as follows: on a ball of radius small enough (less than half the injectivity radius of M say), define V_1 by parallel transport along geodesics starting at x_1 so that $\nabla^{g(t_1)}V_1=0$ at x_1.

The function $T(t)(V_1,V_1)$ defined on a neighborhood of (x_1,t_1) satisfies the following differential inequality at the point (x_1,t_1):

$$\frac{\partial}{\partial t}T(t_1)(V_1,V_1)\geq \Delta_{g(t_1)}(T(t_1)(V_1,V_1))+X(t_1)\cdot T(t_1)(V_1,V_1),$$

because of the null-eigenvector assumption on $S(t)$. Moreover, the righthand side is nonnegative and the lefthand side is nonpositive because of the definition of t_1. This does not give a contradiction yet. To do so, we apply the above reasoning to the tensor $\tilde T(t):=T(t)+\varepsilon(\delta+t)g(t)$ where $\varepsilon>0$ and $\delta>0$ to be chosen later. This is reminiscent of the trick in the proof of the weak maximum principle.

Observe that if $t \in [0,\delta]$:

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{T}(t) &= \frac{\partial}{\partial t}T(t) + \varepsilon\left(g(t) - 2(\delta+t)\operatorname{Ric}(g(t))\right) \\
&\geq \Delta_{g(t)}T(t) + \nabla^{g(t)}_{X(t)}T(t) + S_T(t) + \varepsilon\left(1 - 4\delta\sup_{M\times[0,T]}|\operatorname{Ric}(g(t))|_{g(t)}\right)g(t) \\
&= \Delta_{g(t)}\tilde{T}(t) + \nabla^{g(t)}_{X(t)}\tilde{T}(t) + S_{\tilde{T}}(t) + \frac{\varepsilon}{2}g(t) - C|\tilde{T}(t) - T(t)|_{g(t)}g(t) \\
&> \Delta_{g(t)}\tilde{T}(t) + \nabla^{g(t)}_{X(t)}\tilde{T}(t) + S_{\tilde{T}}(t),
\end{aligned}$$

provided $t \leq \delta \leq (8\sup_{M\times[0,T]}|\operatorname{Ric}(g(t))|_{g(t)})^{-1}$ and $4\delta nC < 1$. We are then in a position to apply the first reasoning to $\tilde{T}$ to derive a contradiction so that $\tilde{T}(t) \geq 0$ on $[0,\delta]$. This lets us send ε to 0 and we deduce that $T(t) \geq 0$ on $[0,\delta]$. Repeating this reasoning a finite number of time ends the proof of the lemma. □

As a consequence of Lemma 4.6, we are in a good position to prove that nonnegativity of Ricci curvature in dimension 3 is preserved along the Ricci flow:

Proposition 4.7. *Let $(M^3, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow on a 3-dimensional closed Riemannian manifold.*

If $\operatorname{Ric}(g(0)) \geq 0$ *then* $\operatorname{Ric}(g(t)) \geq 0$ *for* $t \in [0,T]$.

Proof. Recall the evolution equation of the Ricci tensor derived in Proposition 1.22:

$$\frac{\partial}{\partial t}\operatorname{Ric}(g(t)) = \Delta_{g(t)}\operatorname{Ric}(g(t)) + \underbrace{2\overset{\circ}{\operatorname{Rm}}(\operatorname{Ric}(g(t))) - 2\operatorname{Ric}(g(t))\circ\operatorname{Ric}(g(t))}_{=:S_{\operatorname{Ric}}(t)}.$$

Now, all we need to check is whether the tensor S_{Ric} satisfies the so called null-eigenvector condition. This fact is proved by recalling the very special algebraic structure of the Riemann tensor in terms of the Ricci tensor in dimension 3:

$$S_{\operatorname{Ric}} = 3\operatorname{R}_g\operatorname{Ric}(g) - 6\operatorname{Ric}(g)\circ\operatorname{Ric}(g) + (2|\operatorname{Ric}(g)|^2_g - \operatorname{R}^2_g)g.$$

If $\operatorname{Ric}(g)(v,\cdot) = 0$ for some non zero vector $v \in T_pM$ then $S_{\operatorname{Ric}}(v,v) = (2|\operatorname{Ric}(g)|^2_g - \operatorname{R}^2_g)g(v,v)$. By diagonalizing $\operatorname{Ric}(g)$ at a point $p \in M$, it is not difficult to check that $2|\operatorname{Ric}(g)|^2_g - \operatorname{R}^2_g \geq 0$ at that point since 0 is an eigenvalue. This ends the proof of the proposition. □

4. Exercises

Exercise 4.8. *The purpose of this exercise is to show a integral version of the maximum principle on non-compact manifolds. Let (M^n, g) be a complete Riemannian manifold and let u be a Lipschitz weak subsolution of the heat equation $\partial_t u \leq \Delta_g u$ on $M \times [0,T]$ such that $u(\cdot, 0) \leq 0$ on M. Define $u_+ := \max\{u, 0\}$ and assume that there exists $\alpha > 0$ such that $e^{-\alpha d_g(p,\cdot)^2} u_+ \in L^2(M \times [0,T])$ for some point $p \in M$.*

(i) *Consider the weight function $w(x,t) := -\frac{d_g(p,x)^2}{4(t_0-t)}$ for $x \in M$ and $t \in [0, t_0)$. Prove that $\partial_t w + |\nabla^g w|_g^2 \leq 0$ almost everywhere.*

(ii) *Let φ_R be a cut-off function with values into $[0,1]$ such that $\varphi_R = 1$ on $B_g(p,R)$ and $\varphi_R = 0$ outside $B_g(p, R+1)$ with bounded gradient on M.*
- *Justify the existence of such a cut-off function.*
- *Show that there exists a universal positive constant C such that for all $R > 0$:*

$$\int_M u_+^2 \varphi_R^2 \, e^w \, d\mu_g \Big|_{t=t_0/2} \leq C \int_0^{t_0/2} \int_M u_+^2 |\nabla^g \varphi_R|_g^2 \, e^w \, d\mu_g dt.$$

- *Conclude by proving the existence of t_0 sufficiently small such that $u_+(\cdot,t) = 0$ for $t \in [0, t_0/2]$.*

(iii) *Adapt the above reasoning to show that a Lipschitz weak subsolution u to the heat equation along the Ricci flow with bounded curvature on compact time intervals that is initially nonpositive remains so for later times.*

Exercise 4.9. *Prove that Proposition 4.2 also holds for complete solutions to the Ricci flow on a non-compact manifold with bounded curvature on compact time intervals.*

Exercise 4.10. *Assume that (M^n, g) is a closed Riemannian manifold with nonnegative Ricci curvature. By inspecting the proof of Theorem 4.3, prove the so called Li-Yau Harnack differential inequality: for all $(x,t) \in M \times (0, \infty)$,*

$$\Delta_g \log u(x,t) + \frac{n}{2t} = \partial_t \log u(x,t) - |\nabla^g \log u|_g^2(x,t) + \frac{n}{2t} \geq 0.$$

(The same inequality holds for a complete non-compact metric with non-negative Ricci curvature but the localisation is more subtle than in the proof of Theorem 4.3 since the desired estimate requires to be independent of the cut-off function.)

Show that for all x, y in M and $0 < s < t$,

$$\frac{u(x,t)}{u(y,s)} \geq \left(\frac{t}{s}\right)^{-\frac{n}{2}} \exp\left(-\frac{d_g(x,y)^2}{4(t-s)}\right).$$

Exercise 4.11. *Deduce Yau's Liouville theorem from the previous exercise that a positive harmonic function on a complete Riemannian manifold with non-negative Ricci curvature is constant.*