CHAPTER 6

Uniformization of surfaces of genus 0

The main goal of this chapter is to give a proof of the following result:

Theorem 6.1. *Let* (Σ^2, g_0) *be a closed Riemannian surface with positive Euler characteristic. Then the unique solution to the normalized Ricci flow starting from* g_0 *converges exponentially in any* C^k *norm to a smooth round metric* g_∞ *as* t *tends to* $+\infty$*.*

We present below a proof that combines Hamilton's original proof [**Ham88**] in the case of a initial metric g_0 with positive scalar curvature and Chow's proof [**Cho91**] that extends Hamilton's work for an arbitrary metric g_0. Notice that Hamilton's proof does invoke the uniformization theorem through the so called Kazdan-Warner identity. Chen-Lu-Tian [**CLT06**] managed to get rid of such identity by classifying shrinking gradient Ricci solitons on a 2-sphere independently as explained in Chapter 2.

1. Strategy of the proof

According to Chapter 5, there is a unique solution to the NRF coming out of a given smooth Riemannian metric on Σ that exists for all time: see Corollary 5.9. The main issue in case Σ is a sphere is the convergence of the flow to a metric with constant positive curvature. Indeed, observe that the bounds obtained so far in Chapter 5 are blowing up as t tends to $+\infty$. One of Hamilton's ideas was to consider a tensor that measures the defect of the flow to be a shrinking gradient Ricci soliton. Such a tensor called $M(t)$ has been introduced in Proposition 5.8 and has not been used so far. Let us compute its evolution equation along the NRF:

Proposition 6.2. *Along the NRF,*

$$\frac{\partial}{\partial t}M(t) = \Delta_{g(t)}M(t) + (r - 2\,\mathrm{R}_{g(t)})M(t), \quad t > 0.$$

In particular,

$$\frac{\partial}{\partial t}|M(t)|^2_{g(t)} = \Delta_{g(t)}|M(t)|^2_{g(t)} - 2|\nabla^{g(t)}M(t)|^2_{g(t)} - 2\,\mathrm{R}_{g(t)}\,|M(t)|^2_{g(t)}, \quad t > 0.$$

Before we proceed to the proof of Proposition 6.2, we explain what it suggests to prove next: *if* we knew that $\mathrm{R}_{g(t)} \geq c > 0$ for all time t sufficiently large then the maximum principle would imply that $|M(t)| \leq Ce^{-ct}$ for some positive constant C. As t tends to $+\infty$, it strongly suggests that the flow should converge to a metric whose associated M-tensor vanishes identically, i.e. it should converge to a shrinking gradient Ricci soliton! We will make these heuristics more precise later. Before proving Proposition 6.2, we need a general lemma that computes the evolution of a Lie derivative of a gradient vector field along an arbitrary flow of metrics:

Lemma 6.3. *For a one-parameter family of Riemannian metrics* $g(t)$ *and a one-parameter family of smooth functions* $f(t)$*:*

$$\nabla^{g(t),2}\left(\partial_t f - \Delta_{g(t)} f\right) = \left(\partial_t - \Delta_{L,g(t)}\right)\nabla^{g(t),2} f + T(\nabla^{g(t)} f),$$

where

$$T(\nabla^{g(t)}f)_{ij} := \frac{1}{2}g(t)^{kl}\Big(\nabla_i^{g(t)}(\partial_t g + 2\operatorname{Ric}(g(t))_{jk} + \nabla_j^{g(t)}(\partial_t g + 2\operatorname{Ric}(g(t))_{ik} \\ - \nabla_k^{g(t)}(\partial_t g + 2\operatorname{Ric}(g(t))_{ij}\Big)\nabla_l^{g(t)}f.$$

PROOF. Using commutation formulae, for a Riemannian metric g and a smooth function f:

$$\begin{aligned}(\nabla^{g,2}(\Delta_g f))_{ij} &= \nabla_i^g\nabla_j^g\nabla_k^g\nabla_k^g f = \nabla_i^g\nabla_k^g\nabla_k^g\nabla_j^g f - \nabla_i^g(\operatorname{Ric}(g)_j^l\nabla_l^g f)\\ &= \nabla_k^g\nabla_i^g\nabla_k^g\nabla_j^g f - \operatorname{Rm}(g)_{ikk}^l\nabla_l^g\nabla_j^g f - \operatorname{Rm}(g)_{ikj}^l\nabla_k^g\nabla_l^g f\\ &\quad - g^{kl}\nabla_i^g\operatorname{Ric}(g)_{jk}\nabla_l^g f - \operatorname{Ric}(g)_j^l\nabla_i^g\nabla_l^g f\\ &= (\Delta_g\nabla^{g,2}f)_{ij} - \nabla_k^g\left(\operatorname{Rm}(g)_{ikj}^l\nabla_l^g f\right) - \operatorname{Ric}(g)_i^l\nabla^{g,2}f_{lj} - \operatorname{Rm}(g)_{ikj}^l\nabla_k^g\nabla_l^g f\\ &\quad - g^{kl}\nabla_i^g\operatorname{Ric}(g)_{jk}\nabla_l^g f - g^{kl}\operatorname{Ric}(g)_{jk}\nabla^{g,2}f_{il}\\ &= (\Delta_{L,g}\nabla^{g,2}f)_{ij} - g^{kl}\left(\nabla_i^g\operatorname{Ric}(g)_{jk} + \nabla_j^g\operatorname{Ric}(g)_{ik} - \nabla_k^g\operatorname{Ric}(g)_{ij}\right)\nabla_l^g f,\end{aligned}$$

where we have used the second Bianchi identity $\nabla_k^g\operatorname{Rm}(g)_{ikj}^l = g^{kl}(\nabla_j^g\operatorname{Ric}(g)_{ik} - \nabla_k^g\operatorname{Ric}(g)_{ij})$ in the last line.

On the other hand, thanks to Lemma 1.13 and the definition of the Hessian of a function,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^{g(t),2}f_{ij} &= \nabla^{g(t),2}\left(\frac{\partial}{\partial t}f\right)_{ij} - \frac{\partial}{\partial t}\Gamma(g(t))_{ij}^l\nabla_l^{g(t)}f\\ &= \nabla^{g(t),2}\left(\frac{\partial}{\partial t}f\right)_{ij} - \frac{1}{2}g(t)^{kl}\left(\nabla_i^{g(t)}(\partial_t g)_{jk} + \nabla_j^{g(t)}(\partial_t g)_{ik} - \nabla_k^{g(t)}(\partial_t g)_{ij}\right)\nabla_l^{g(t)}f.\end{aligned}$$

This ends the proof of the lemma. □

We are now in a position to prove Proposition 6.2:

PROOF OF PROPOSITION 6.2. Lemma 6.3 applied to the NRF gives:

$$\left(\partial_t - \Delta_{L,g(t)}\right)\nabla^{g(t),2}f = \nabla^{g(t),2}\left(\partial_t f - \Delta_{g(t)}f\right),$$

since $\partial_t g + 2\operatorname{Ric}(g(t)) = (r - \mathrm{R}_{g(t)})g(t) + \mathrm{R}_{g(t)}\,g(t) = rg(t)$ is parallel with respect to $\nabla^{g(t)}$.

Now, in dimension 2,

$$\operatorname{Rm}(g(t))_{ijkl} = \frac{\mathrm{R}_{g(t)}}{2}\left(g(t)_{il}g(t)_{jk} - g(t)_{ik}g(t)_{jl}\right).$$

Therefore,

$$\Delta_{L,g(t)}\nabla^{g(t),2}f = \Delta_{g(t)}\nabla^{g(t),2}f + \mathrm{R}_{g(t)}(\Delta_{g(t)}f)g(t) - 2\,\mathrm{R}_{g(t)}\,\nabla^{g(t),2}f.$$

In particular,

$$\begin{aligned}\left(\partial_t - \Delta_{g(t)}\right)\nabla^{g(t),2}f &= \nabla^{g(t),2}\left(\partial_t f - \Delta_{g(t)}f\right) + \mathrm{R}_{g(t)}(\Delta_{g(t)}f)g(t) - 2\,\mathrm{R}_{g(t)}\,\nabla^{g(t),2}f\\ &= (r - 2\,\mathrm{R}_{g(t)})\nabla^{g(t),2}f + \mathrm{R}_{g(t)}(\Delta_{g(t)}f)g(t)\\ &= (r - 2\,\mathrm{R}_{g(t)})M(t) + r\frac{\Delta_{g(t)}f}{2}g(t).\end{aligned}$$

Since,

$$\begin{aligned}\left(\partial_t - \Delta_{g(t)}\right)\left(\frac{\Delta_{g(t)}f}{2}g(t)\right) &= \left[\left(\partial_t - \Delta_{g(t)}\right)\Delta_{g(t)}f\right]\frac{g(t)}{2} + \frac{\Delta_{g(t)}f}{2}\partial_t g\\ &= \frac{\mathrm{R}_{g(t)}}{2}(\Delta_{g(t)}f)g(t) + \frac{\Delta_{g(t)}f}{2}(r - \mathrm{R}_{g(t)})g(t)\\ &= r\frac{\Delta_{g(t)}f}{2}g(t),\end{aligned}$$

we finally obtain the desired evolution equation of the tensor $M(t)$.

Thanks to the previous computation, the definition of the norm of $M(t)$ with respect to $g(t)$, i.e. $|M(t)|^2_{g(t)} := g(t)^{ik}g(t)^{jl}M(t)_{ij}M(t)_{kl}$, implies:

$$\begin{aligned}\frac{\partial}{\partial t}|M(t)|^2_{g(t)} &= 2(\mathrm{R}_{g(t)} - r)|M(t)|^2_{g(t)} + 2\langle M(t), \Delta_{g(t)}M(t) + (r - 2\,\mathrm{R}_{g(t)})M(t)\rangle_{g(t)} \\ &= \Delta_{g(t)}|M(t)|^2_{g(t)} - 2|\nabla^{g(t)}M(t)|^2_{g(t)} - 2\,\mathrm{R}_{g(t)}\,|M(t)|^2_{g(t)},\end{aligned}$$

as expected. □

We end this section by summarizing the main steps leading to the proof of the uniform lower bound $\mathrm{R}_{g(t)} \geq c > 0$ along the NRF **in case the initial metric satisfies** $\mathrm{R}_{g_0} > 0$**.**

(i) Define an entropy for surfaces with positive curvature and show that it is strictly decreasing along the NRF unless the flow is static, i.e. the flow is a time-independent family consisting of an Einstein metric.
(ii) The aforementioned monotonicity will help us prove uniform bounds on the scalar curvature and its gradient. A uniform bound on the diameter will also be derived.
(iii) A suitable differential Harnack inequality for the scalar curvature will let us compare its values at different space-time points.
(iv) We will then be in a good position to prove the desired uniform lower bound on the scalar curvature along the NRF.
(v) We will show the convergence of the NRF to a shrinking gradient Ricci soliton up to suitable diffeomorphisms.
(vi) By invoking the classification of such solitons in Chapter 2, we will conclude the proof of Theorem 6.1.

2. Hamilton's entropy

Let us start with a formal definition:

Definition 6.4. Let (Σ^2, g) be a Riemannian manifold with positive scalar curvature $\mathrm{R}_g > 0$. The entropy of g, denoted by $N(g)$, is defined by:

$$N(g) := \int_\Sigma \mathrm{R}_g \log \mathrm{R}_g \; d\mu_g.$$

Recall the following straightforward fact that we leave as an exercise:

Lemma 6.5. *Along the NRF,*

$$\frac{\partial}{\partial t}\left(\mathrm{R}_{g(t)}\, d\mu_{g(t)}\right) = \Delta_{g(t)}\,\mathrm{R}_{g(t)}\, d\mu_{g(t)}.$$

We derive a first version of the first variation of the entropy:

Lemma 6.6. *Along the NRF starting from a metric g_0 with positive scalar curvature,*

$$\frac{\partial}{\partial t}N(g(t)) = -\int_\Sigma \frac{|\nabla^{g(t)}\,\mathrm{R}_{g(t)}\,|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} + \int_\Sigma (\mathrm{R}_{g(t)} - r)^2\, d\mu_{g(t)}.$$

The previous lemma has a main drawback, it does seem obvious to show that the entropy is monotonic along the NRF.

PROOF. It is a straightforward computation that uses the definition of the NRF and that of the entropy only:

$$\begin{aligned}\frac{\partial}{\partial t}N(g(t)) &= \int_\Sigma \left(\frac{\partial}{\partial t}\log \mathrm{R}_{g(t)}\right)\mathrm{R}_{g(t)}\ d\mu_{g(t)} + \int_\Sigma \log \mathrm{R}_{g(t)}\frac{\partial}{\partial t}\left(\mathrm{R}_{g(t)}\ d\mu_{g(t)}\right)\\ &= \int_\Sigma \Delta_{g(t)}\mathrm{R}_{g(t)} + \mathrm{R}_{g(t)}(\mathrm{R}_{g(t)} - r)\, d\mu_{g(t)} + \int_\Sigma \log \mathrm{R}_{g(t)}\Delta_{g(t)}\mathrm{R}_{g(t)}\ d\mu_{g(t)}\\ &= \int_\Sigma \mathrm{R}_{g(t)}(\mathrm{R}_{g(t)} - r)\, d\mu_{g(t)} - \int_\Sigma g(t)(\nabla^{g(t)}\log \mathrm{R}_{g(t)}, \nabla^{g(t)}\mathrm{R}_{g(t)})\, d\mu_{g(t)}\\ &= \int_\Sigma (\mathrm{R}_{g(t)} - r)^2\, d\mu_{g(t)} - \int_\Sigma g(t)(\nabla^{g(t)}\log \mathrm{R}_{g(t)}, \nabla^{g(t)}\mathrm{R}_{g(t)})\, d\mu_{g(t)},\end{aligned}$$

where we have used Lemma 6.5 in the second line. An integration by parts has been performed in the antepenultimate line together with the definition of r in the last line. This computation leads to the desired result. □

The next result is based on the previous computation and it shows the desired monotonic property of Hamilton's entropy:

Lemma 6.7. *Let $(\Sigma, g(t))_{t\in[0,T)}$ be a solution to NRF such that $\mathrm{R}_{g(0)} > 0$. Then,*

$$\frac{\partial}{\partial t}N(g(t)) = -\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)} - \mathrm{R}_{g(t)}\nabla^{g(t)}f(t)|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} - 2\int_\Sigma |M(t)|^2_{g(t)}\, d\mu_{g(t)} \leq 0,$$

with equality if and only if $(\Sigma, g(t))_{t\in[0,T)}$ is a shrinking round sphere.

PROOF. Let us start from the righthand side and let us expand it as follows:

$$\begin{aligned}&-\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)} - \mathrm{R}_{g(t)}\nabla^{g(t)}f(t)|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} =\\ &-\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)}|^2_{g(t)} - 2\,\mathrm{R}_{g(t)}\, g(t)(\nabla^{g(t)}\mathrm{R}_{g(t)}, \nabla^{g(t)}f(t)) + \mathrm{R}^2_{g(t)}|\nabla^{g(t)}f(t)|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} =\\ &-\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)}|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} - 2\int_\Sigma \mathrm{R}_{g(t)}\Delta_{g(t)}f(t)\, d\mu_{g(t)} - \int_\Sigma \mathrm{R}_{g(t)}|\nabla^{g(t)}f(t)|^2_{g(t)}\, d\mu_{g(t)} =\\ &-\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)}|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} + 2\int_\Sigma \mathrm{R}_{g(t)}(\mathrm{R}_{g(t)} - r)\, d\mu_{g(t)} - \int_\Sigma \mathrm{R}_{g(t)}|\nabla^{g(t)}f(t)|^2_{g(t)}\, d\mu_{g(t)} =\\ &-\int_\Sigma \frac{|\nabla^{g(t)}\mathrm{R}_{g(t)}|^2_{g(t)}}{\mathrm{R}_{g(t)}}\, d\mu_{g(t)} + 2\int_\Sigma (\mathrm{R}_{g(t)} - r)^2\, d\mu_{g(t)} - \int_\Sigma \mathrm{R}_{g(t)}|\nabla^{g(t)}f(t)|^2_{g(t)}\, d\mu_{g(t)}.\end{aligned} \tag{2.1}$$

Here we have used integration by parts in the second line together with the equation satisfied by $f(t)$ in the third line. Finally, we have used the fact that $\mathrm{R}_{g(t)} - r$ has zero mean value in the last line.

Now,

$$\begin{aligned}-2\int_\Sigma |M(t)|^2_{g(t)}\, d\mu_{g(t)} &= -2\int_\Sigma \left|\nabla^{g(t),2}f(t) - \frac{1}{2}\Delta_{g(t)}f(t)g(t)\right|^2_{g(t)}\\ &= -2\int_\Sigma \left|\nabla^{g(t),2}f(t)\right|^2_{g(t)} - \langle \nabla^{g(t),2}f(t), \Delta_{g(t)}f(t)g(t)\rangle_{g(t)} + \frac{1}{2}(\Delta_{g(t)}f(t))^2\, d\mu_{g(t)}\\ &= -2\int_\Sigma \left|\nabla^{g(t),2}f(t)\right|^2_{g(t)} - \frac{1}{2}(\Delta_{g(t)}f(t))^2\, d\mu_{g(t)}.\end{aligned} \tag{2.2}$$

The last step consists in linking the L^2 norm of $\mathrm{R}_{g(t)} - r$ with that of the Hessian $\nabla^{g(t),2} f(t)$ through the Bochner formula for functions:

$$\begin{aligned}\int_\Sigma (\mathrm{R}_{g(t)} - r)^2 \, d\mu_{g(t)} &= \int_\Sigma (\Delta_{g(t)} f(t))^2 \, d\mu_{g(t)} \\ &= -\int_\Sigma g(t)(\nabla^{g(t)} f(t), \nabla^{g(t)}(\Delta_{g(t)} f(t))) \, d\mu_{g(t)} \\ &= -\int_\Sigma g(t)(\nabla^{g(t)} f(t), \Delta_{g(t)}(\nabla^{g(t)} f(t))) - \mathrm{Ric}(g(t))(\nabla^{g(t)} f(t), \nabla^{g(t)} f(t)) \, d\mu_{g(t)} \\ &= \int_\Sigma |\nabla^{g(t),2} f(t)|^2_{g(t)} + \frac{1}{2} \mathrm{R}_{g(t)} |\nabla^{g(t)} f(t)|^2_{g(t)} \, d\mu_{g(t)}.\end{aligned} \tag{2.3}$$

Finally, we add (2.1) and (2.2) to get the expected monotonicity once we invoke together with Lemma 6.6. □

We conclude this section by stating the following straightforward but crucial consequence of Lemma 6.7:

Corollary 6.8. *Let $(\Sigma, g(t))_{t \geq 0}$ be a solution to NRF such that $\mathrm{R}_{g(0)} > 0$. Then Hamilton's entropy is uniformly bounded from above for all time. More precisely,*

$$N(g(t)) \leq N(g(0)), \quad t \geq 0.$$

3. Uniform curvature bounds and their consequences

Lemma 6.9. *Let $(\Sigma, g(t))_{t \geq 0}$ be a solution to NRF such that $\mathrm{R}_{g(0)} > 0$. Then for any $t \geq 0$,*

$$\max_\Sigma \mathrm{R}_{g(s)} \leq 2 \max_\Sigma \mathrm{R}_{g(t)}, \quad t \leq s \leq t + \frac{1}{2 \max_\Sigma \mathrm{R}_{g(t)}}.$$

In particular,

$$e^{-1} g(t) \leq g(s) \leq \sqrt{e} g(t), \quad t \leq s \leq t + \frac{1}{2 \max_\Sigma \mathrm{R}_{g(t)}}.$$

Proof. The evolution equation for the scalar curvature gives in particular:

$$\frac{\partial}{\partial t} \mathrm{R}_{g(t)} \leq \Delta_{g(t)} \mathrm{R}_{g(t)} + \mathrm{R}^2_{g(t)}.$$

The maximum principle implies that $\max_\Sigma \mathrm{R}_{g(s)} \leq y(s)$, where y solves $y'(s) = y(s)^2$ for $s \geq t$ and $y(t) := \max_\Sigma \mathrm{R}_{g(t)}$. Therefore,

$$\max_\Sigma \mathrm{R}_{g(s)} \leq \frac{1}{\frac{1}{\max_\Sigma \mathrm{R}_{g(t)}} + t - s} \leq 2 \max_\Sigma \mathrm{R}_{g(t)},$$

if $s \leq t + \frac{1}{2 \max_\Sigma \mathrm{R}_{g(t)}}$.

The bounds on the metric comes immediately from the integrated version of the normalized Ricci flow equation together with the previously established bound on the curvature:

$$g(s) = \exp\left\{ \int_t^s (r - \mathrm{R}_{g(s')}) \, ds' \right\} g(t), \quad s \geq t.$$

In particular, if $t \leq s \leq t + \frac{1}{2 \max_\Sigma \mathrm{R}_{g(t)}}$,

$$g(s) \leq \exp\{r(s-t)\} g(t) \leq \sqrt{e} g(t), \quad s \geq t.$$

The lower bound can be proved similarly. □

The next result uses the so called Bernstein-Shi technique that we already used in Chapter 3.

Lemma 6.10 (Uniform gradient bounds)**.** *There exists a universal positive constant C such that for any solution $(\Sigma, g(t))_{t\geq 0}$ to the normalized Ricci flow with $\mathrm{R}_{g(t)} \geq 0$. Then,*

$$\sup_\Sigma |\nabla^{g(t)} \mathrm{R}_{g(t)}|_{g(t)} \leq \frac{C}{\sqrt{t}} \sup_\Sigma |\mathrm{R}_{g(0)}|, \quad t \in \left(0, (8 \sup_\Sigma |\mathrm{R}_{g(0)}|)^{-1}\right].$$

Notice that we derived such bounds for the Ricci flow, not for the normalized Ricci flow.

Proof. Let us consider the following quantity: $\mathrm{R}^2_{g(t)} + t|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}$. Let us derive its evolution equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left(\mathrm{R}^2_{g(t)} + t|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}\right) &= -2|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)} + 2\,\mathrm{R}^2_{g(t)}(\mathrm{R}_{g(t)} - r)\\ &\quad - 2t|\nabla^{g(t),2} \mathrm{R}_{g(t)}|^2_{g(t)} + \left(t(4\,\mathrm{R}_{g(t)} - 3r) + 1\right)|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}.\end{aligned}$$

In particular, since $r \geq 0$,

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left(\mathrm{R}^2_{g(t)} + t|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}\right) \leq \left(4t\,\mathrm{R}_{g(t)} - 1\right)|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)} + 2\,\mathrm{R}^3_{g(t)}.$$

Now, $\partial_t \mathrm{R}_{g(t)} \leq \Delta_{g(t)} \mathrm{R}_{g(t)} + \mathrm{R}^2_{g(t)}$ on Σ which implies by the maximum principle $\mathrm{R}_{g(t)} \leq 2\max_\Sigma \mathrm{R}_{g(0)}$ for $0 \leq t \leq (2\max_\Sigma \mathrm{R}_{g(0)})^{-1}$. As an intermediate conclusion, $4t\,\mathrm{R}_{g(t)} \leq 1$ for $0 \leq t \leq (8\max_\Sigma \mathrm{R}_{g(0)})^{-1}$.

Inserting this curvature bound back to the previous differential inequality leads to:

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\left(\mathrm{R}^2_{g(t)} + t|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}\right) \leq 16(\max_\Sigma \mathrm{R}_{g(0)})^3, \quad 0 \leq t \leq (8\max_\Sigma \mathrm{R}_{g(0)})^{-1}.$$

Invoking the maximum principle once more gives us the expected result:

$$\mathrm{R}^2_{g(t)} + t|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)} \leq \max_\Sigma \mathrm{R}^2_{g(0)} + 16(\max_\Sigma \mathrm{R}_{g(0)})^3 t \leq 3\max_\Sigma \mathrm{R}^2_{g(0)},$$

as long as $0 \leq t \leq (8\max_\Sigma \mathrm{R}_{g(0)})^{-1}$. □

We now bound the curvature along the normalized Ricci flow uniformly in time on a 2-sphere endowed with an initial metric with positive scalar curvature:

Proposition 6.11. *Let $(\Sigma, g(t))_{t\geq 0}$ be a solution to the normalized Ricci flow with $\mathrm{R}_{g(t)} > 0$. Then the (scalar) curvature is uniformly bounded in time, i.e. there exists a uniform positive constant C such that:*

$$\sup_{\Sigma\times\mathbb{R}_+} \mathrm{R}_{g(t)} \leq C.$$

Before proving Proposition 6.11, we recall the following injectivity radius estimate:

Theorem 6.12 (Klingenberg's theorem)**.** *Let (M^{2n}, g) be an orientable manifold with positive sectional curvature, i.e. $K_g > 0$. Then*

$$\mathrm{inj}(M, g) \geq \frac{\pi}{\sqrt{\max_M K_g}}.$$

A proof is given in the list of exercises below.

Proof. Recall from Corollary 6.8 that Hamilton's entropy $N(g(t))$ is uniformly bounded from above for $t \geq 0$. In particular, if $r \leq \pi/\sqrt{\max_\Sigma K_{g(t)}} = \pi\sqrt{2}/\sqrt{\max_\Sigma \mathrm{R}_{g(t)}}$, Theorems 6.12 and A.4

ensure that:

$$\begin{aligned} C \geq N(g(t)) &= \int_\Sigma \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \, d\mu_{g(t)} \\ &= \int_{B_{g(t)}(x,r)} \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \, d\mu_{g(t)} + \int_{\Sigma \setminus B_{g(t)}(x,r)} \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \, d\mu_{g(t)} \\ &\geq \min_{B_{g(t)}(x,r)} \left\{ \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \right\} \mathrm{vol}_{g(t)} B_{g(t)}(x,r) - 2e^{-1} \mathrm{vol}_{g(t)}(\Sigma \setminus B_{g(t)}(x,r)) \\ &\geq \min_{B_{g(t)}(x,r)} \left\{ \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \right\} \mathrm{vol}_{\mathbb{S}^2(\max_\Sigma \mathrm{R}_{g(t)}/2)} B_{\mathbb{S}^2(\max_\Sigma \mathrm{R}_{g(t)}/2)}(r) - e^{-1} \mathrm{vol}_{g(t)}(\Sigma) \\ &\geq \min_{B_{g(t)}(x,r)} \left\{ \mathrm{R}_{g(t)} \log \mathrm{R}_{g(t)} \right\} (cr^2) - e^{-1} \mathrm{vol}_{g(0)}(\Sigma), \end{aligned}$$

for some universal positive constant c. Here we have used the fact that $\min_{x>0} x \log x = -e^{-1}$ in the third line together with the constancy of the volume in the last line.

The goal is then to find a radius r comparable to $(\max_\Sigma \mathrm{R}_{g(t)})^{-1/2}$ so that the minimum on a ball of radius r centered at a point to be defined is bounded from below by $\max_\Sigma \mathrm{R}_{g(t)}$. Indeed, the previous set of inequalities will show the uniform boundedness of $\max_\Sigma \mathrm{R}_{g(t)}$ thanks to the additional log term.

To make this reasoning formal, for $T \geq 0$, define $\kappa(T) := \max_{\Sigma\times[0,T]} \mathrm{R}_{g(t)}$. Assume that $\kappa(T) > \max\left\{\kappa(1), 4^{-1}\right\}$. Otherwise, there is nothing to prove. Notice in particular that $T > 1$. Pick a space-time point (x_1, t_1) in $(1, T] \times \Sigma$ such that $\mathrm{R}_{g(t_1)}(x_1) = \kappa(T)$. We want to show that on a ball $B_{g(t_1)}(x_1, \varepsilon/\sqrt{\kappa(T)})$ with $\varepsilon > 0$ to be chosen later, $\mathrm{R}_{g(t_1)}$ is comparable from below to $\kappa(T)$. Lemma 6.10 applied to an interval of the form $(t_1 - (4\kappa(T))^{-1}, t_1]$ (once the solution translated by time $t_1 - (4\kappa(T))^{-1}$) shows that

$$|\nabla^{g(t)} \mathrm{R}_{g(t)}|_{g(t)}\Big|_{t=t_1} \leq \frac{C}{\sqrt{t - ((t_1 - (4\kappa(T))^{-1})}} \kappa(T)\Big|_{t=t_1} = 2C\kappa(T)^{\frac{3}{2}}.$$

In particular, on a ball $B_{g(t_1)}(x_1, \varepsilon/\sqrt{\kappa(T)})$,

$$\mathrm{R}_{g(t_1)}(y) \geq \mathrm{R}_{g(t_1)}(x_1) - 2C\kappa(T)^{\frac{3}{2}} \cdot \frac{\varepsilon}{\sqrt{\kappa(T)}} = \kappa(T)(1 - 2\varepsilon C) \geq \frac{\kappa(T)}{2},$$

if ε is chosen to be small enough (uniformly in time). This concludes the proof. □

With such a uniform bound on curvature in hands provided by the previous proposition, we can furthermore bound the diameter uniformly in time:

Proposition 6.13. *Let $(\Sigma, g(t))_{t\geq 0}$ be a solution to the normalized Ricci flow with $\mathrm{R}_{g(t)} > 0$. Then the diameter is uniformly bounded in time, i.e. there exists a uniform positive constant C such that:*

$$\sup_{t\geq 0} \mathrm{diam}(g(t)) \leq C.$$

Remark 6.14. Perelman has generalized this fact for solutions to the Kähler-Ricci flow on a Fano manifold starting from a Kähler metric in the first Chern class: see [**ST08**] for a proof. Hamilton's entropy is replaced by Perelman's entropy.

Proof. The combination of Klingenberg's theorem 6.12 together with the uniform upper bound on curvature guaranteed by Proposition 6.11 leads to the existence of a positive constant C such that for all $t \geq 0$, $\mathrm{inj}(\Sigma, g(t)) \geq \iota_0 > 0$.

Now, if x an y are two points in Σ such that $\mathrm{diam}(g(t)) = d_{g(t)}(x, y)$ then let $(p_i)_{0\leq i\leq N}$ denote a sequence of points in Σ (along a minimizing geodesic connecting x and y for instance) such that $d_{g(t)}(p_i, p_{i+1}) \geq \iota_0$ for $i = 0, ..., N$ with $p_0 := x$ and $p_N := y$ and $\mathrm{diam}(g(t))/\iota_0 \leq N < \mathrm{diam}(g(t))/\iota_0 + 1$. Then the balls $B_{g(t)}(p_i, \iota_0/2)$ are pairwise disjoint and embedded in Σ. Moreover, Theorem A.4

gives a uniform lower bound, say v_0, on the volume of each such balls. Therefore, if $V_0 := \mathrm{vol}_{g(0)} \Sigma$,

$$\mathrm{diam}(g(t)) \leq N\iota_0,$$
$$(N+1)v_0 \leq \mathrm{vol}_{g(t)} \Sigma = V_0,$$

since the volume is constant in time by the very definition of the normalized Ricci flow. This ends the proof: $\mathrm{diam}(g(t)) \leq \iota_0 V_0/v_0$. □

4. Differential Harnack estimate

Theorem 6.15. *Let $(\Sigma, g(t))_{t\in[0,T)}$ be a solution to the normalized Ricci flow with $\mathrm{R}_{g(t)} > 0$ then there exists a constant $C(g_0) > 1$ such that for $t \geq 0$,*

$$\frac{\partial}{\partial t} \log \mathrm{R}_{g(t)} - |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} = \Delta_{g(t)} \log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} - r \geq -\frac{Cre^{rt}}{Ce^{rt} - 1}.$$

We only prove the second version, i.e. in the setting a normalized Ricci flow. We start with a lemma that computes the evolution equation of $\log \mathrm{R}_{g(t)}$:

Lemma 6.16. *Let $(\Sigma, g(t))_{t\in[0,T)}$ be a solution to the normalized Ricci flow with $\mathrm{R}_{g(t)} > 0$ then*

$$\frac{\partial}{\partial t} \log \mathrm{R}_{g(t)} = \Delta_{g(t)} \log \mathrm{R}_{g(t)} + |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r.$$

Proof. It is a straightforward computation based on the evolution equation satisfied by $\mathrm{R}_{g(t)}$:

$$\begin{aligned}\frac{\partial}{\partial t} \log \mathrm{R}_{g(t)} &= \frac{1}{\mathrm{R}_{g(t)}} \frac{\partial}{\partial t} \mathrm{R}_{g(t)} \\ &= \frac{1}{\mathrm{R}_{g(t)}} \left(\Delta_{g(t)} \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)}(\mathrm{R}_{g(t)} - r)\right) \\ &= \Delta_{g(t)} \log \mathrm{R}_{g(t)} + |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r,\end{aligned}$$

since for any sufficiently regular function u on Σ, $u \cdot \Delta_{g(t)} \log u = \Delta_{g(t)} u - u|\nabla^{g(t)} \log u|^2_{g(t)}$. □

From now on, we define the differential Harnack quantity:

$$Q(t) := \frac{\partial}{\partial t} \log \mathrm{R}_{g(t)} - |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)}, \tag{4.1}$$

which equals $\Delta_{g(t)} \log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} - r$ according to Lemma 6.16.

In order to prove Theorem 6.15, we derive the evolution equation of Q in the next lemma:

Lemma 6.17. *Let $(\Sigma, g(t))_{t\in[0,T)}$ be a solution to the normalized Ricci flow with $\mathrm{R}_{g(t)} > 0$ then,*

$$\begin{aligned}\frac{\partial}{\partial t} Q(t) &= \Delta_{g(t)} Q(t) + 2g(t)(\nabla^{g(t)} Q(t), \nabla^{g(t)} \log \mathrm{R}_{g(t)}) \\ &\quad + 2\left|\nabla^{g(t),2} \log \mathrm{R}_{g(t)} + \frac{1}{2}(\mathrm{R}_{g(t)} - r)g(t)\right|^2_{g(t)} + rQ(t).\end{aligned}$$

Proof. It is a brutal force computation. Let us start by differentiating in time the function Q based on Lemma 6.16:

$$\begin{aligned}\frac{\partial}{\partial t} Q(t) &= \frac{\partial}{\partial t} \left(\Delta_{g(t)} \log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} - r\right) \\ &= \Delta_{g(t)} \left(\frac{\partial}{\partial t} \log \mathrm{R}_{g(t)}\right) + (\mathrm{R}_{g(t)} - r)\Delta_{g(t)} \log \mathrm{R}_{g(t)} + \frac{\partial}{\partial t} \mathrm{R}_{g(t)} \\ &= \Delta_{g(t)} \left(\Delta_{g(t)} \log \mathrm{R}_{g(t)} + |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r\right) + (\mathrm{R}_{g(t)} - r)\Delta_{g(t)} \log \mathrm{R}_{g(t)} \\ &\quad + \mathrm{R}_{g(t)} \left(\Delta_{g(t)} \log \mathrm{R}_{g(t)} + |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r\right) \\ &= \Delta_{g(t)} Q(t) + \Delta_{g(t)} |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + (\mathrm{R}_{g(t)} - r)\Delta_{g(t)} \log \mathrm{R}_{g(t)} \\ &\quad + \mathrm{R}_{g(t)} \left(\Delta_{g(t)} \log \mathrm{R}_{g(t)} + |\nabla^{g(t)} \log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r\right).\end{aligned}$$

Now, thanks to the Bochner formula applied to the function $\log \mathrm{R}_{g(t)}$:

$$\begin{aligned}\frac{\partial}{\partial t}Q(t) &= \Delta_{g(t)}Q(t) + 2g(t)\left(\nabla^{g(t)}\Delta_{g(t)}\log \mathrm{R}_{g(t)}, \nabla^{g(t)}\log \mathrm{R}_{g(t)}\right) + \mathrm{R}_{g(t)}|\nabla^{g(t)}\log \mathrm{R}_{g(t)}|^2_{g(t)}\\
&\quad + 2|\nabla^{g(t),2}\log \mathrm{R}_{g(t)}|^2_{g(t)} + (\mathrm{R}_{g(t)} - r)\Delta_{g(t)}\log \mathrm{R}_{g(t)}\\
&\quad + \mathrm{R}_{g(t)}\left(\Delta_{g(t)}\log \mathrm{R}_{g(t)} + |\nabla^{g(t)}\log \mathrm{R}_{g(t)}|^2_{g(t)} + \mathrm{R}_{g(t)} - r\right)\\
&= \Delta_{g(t)}Q(t) + 2g(t)\left(\nabla^{g(t)}Q(t), \nabla^{g(t)}\log \mathrm{R}_{g(t)}\right)\\
&\quad + 2|\nabla^{g(t),2}\log \mathrm{R}_{g(t)}|^2_{g(t)} + (\mathrm{R}_{g(t)} - r)\left(2\Delta_{g(t)}\log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} - r\right) + rQ(t).\end{aligned}$$

Finally, notice that:

$$\begin{aligned}&2\left|\nabla^{g(t),2}\log \mathrm{R}_{g(t)} + \frac{1}{2}(\mathrm{R}_{g(t)} - r)g(t)\right|^2_{g(t)} =\\
&2|\nabla^{g(t),2}\log \mathrm{R}_{g(t)}|^2_{g(t)} + (\mathrm{R}_{g(t)} - r)^2 + 2(\mathrm{R}_{g(t)} - r)\Delta_{g(t)}\log \mathrm{R}_{g(t)} =\\
&2|\nabla^{g(t),2}\log \mathrm{R}_{g(t)}|^2_{g(t)} + (\mathrm{R}_{g(t)} - r)\left(2\Delta_{g(t)}\log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} - r\right),\end{aligned}$$

which leads to the expected result. □

We are in a good position to prove Theorem 6.15:

Proof of Theorem 6.15. Lemma 6.17 ensures that Q satisfies:

$$\begin{aligned}\frac{\partial}{\partial t}Q(t) &\geq \Delta_{g(t)}Q(t) + 2g(t)(\nabla^{g(t)}Q(t), \nabla^{g(t)}\log \mathrm{R}_{g(t)})\\
&\quad + \left|\Delta_{g(t)}\log \mathrm{R}_{g(t)} + (\mathrm{R}_{g(t)} - r)\right|^2 + rQ(t)\\
&\geq \Delta_{g(t)}Q(t) + 2g(t)(\nabla^{g(t)}Q(t), \nabla^{g(t)}\log \mathrm{R}_{g(t)}) + Q(t)^2 + rQ(t).\end{aligned}$$

Here we have used the elementary inequality for symmetric 2-tensors T on a Riemannian manifold (M^n, g):

$$n|T|^2_g \geq (\mathrm{tr}_g T)^2.$$

The minimum principle for functions requires to solve the ODE: $y'(t) = y^2(t) + ry(t)$, $y(0) := y_0$. Here we have no idea of the exact value of $\min_\Sigma Q(0)$. Therefore, we chose an initial condition q_0 so small that $\min\{\min_\Sigma Q(0), -r\} > q_0$. In particular,

$$\min_\Sigma Q(t) \geq q(t) = -\frac{rq_0e^{rt}}{q_0e^{rt} - q_0 - r} =: -\frac{Cre^{rt}}{Ce^{rt} - 1} > -\frac{re^{rt}}{e^{rt} - 1}, \quad t \geq 0,$$

where $C := \frac{q_0}{q_0 + r}$. This ends the proof of this theorem. □

As promised, we can compare values of the (scalar) curvature at different space-time points along any solution to the normalized Ricci flow with positive scalar curvature.

Corollary 6.18. *Let $(\Sigma, g(t))_{t\in[0,T)}$ be a solution to the Ricci flow with $\mathrm{R}_{g(t)} > 0$. Then there exists a constant $C > 0$ depending on g_0 only such that for all x_1, x_2 and $0 \leq t_1 < t_2$,*

$$\frac{\mathrm{R}_{g(t_2)}(x_2)}{\mathrm{R}_{g(t_1)}(x_1)} \geq \exp\left\{-\frac{1}{4}\inf_\gamma \int_{t_1}^{t_2} |\dot\gamma(t)|^2_{g(t)}\, dt\right\}\frac{Ce^{rt_1} - 1}{Ce^{rt_2} - 1},$$

where the infimum is taken over all regular paths $\gamma : [t_1, t_2] \to M$ such that $\gamma(t_i) = x_i$, $i = 1, 2$.

This estimate is called a parabolic Harnack inequality: it is the integrated version of the differential inequality obtained in Theorem 6.15.

PROOF. Let $\gamma : [t_1, t_2] \to M$ be a C^1 path connecting x_1 to x_2. Then:

$$\begin{aligned}
\log\left(\frac{\mathrm{R}_{g(t_2)}(x_2)}{\mathrm{R}_{g(t_1)}(x_1)}\right) &= \int_{t_1}^{t_2} \frac{d}{dt} \log \mathrm{R}_{g(t)}(\gamma(t))\, dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t} \mathrm{R}_{g(t)}(\gamma(t)) + g(t)\left(\nabla^{g(t)} \mathrm{R}_{g(t)}(\gamma(t)), \dot\gamma(t)\right)\right) dt \\
&\geq \int_{t_1}^{t_2} \left(|\nabla^{g(t)} \mathrm{R}_{g(t)}|^2_{g(t)}(\gamma(t)) - \frac{Cre^{rt}}{Ce^{rt}-1} - |\nabla^{g(t)} \mathrm{R}_{g(t)}|_{g(t)}(\gamma(t))|\dot\gamma(t)|_{g(t)}(\gamma(t))\right) dt \\
&\geq -\int_{t_1}^{t_2} \frac{Cre^{rt}}{Ce^{rt}-1} + \frac{1}{4}|\dot\gamma(t)|^2_{g(t)}(\gamma(t))\, dt \\
&= -\frac{1}{4}\int_{t_1}^{t_2} |\dot\gamma(t)|^2_{g(t)}(\gamma(t))\, dt - \log\left(\frac{Ce^{rt_2}-1}{Ce^{rt_1}-1}\right),
\end{aligned}$$

where we have used Theorem 6.15 together with Young's inequality $ab \leq a^2 + \frac{1}{4}b^2$ in the penultimate line. The result follows by minimizing over the set of such curves, the expected Harnack inequality follows. □

5. End of the proof of the main theorem

As explained in the introduction of this chapter, one crucial ingredient for the proof of Theorem 6.1 is a uniform *positive* lower bound on the scalar curvature. This is achieved in the next result:

Proposition 6.19. *Let $(\Sigma, g(t))_{t\in[0,+\infty)}$ be a solution to the Ricci flow with $\mathrm{R}_{g(0)} > 0$. Then there exists a uniform positive constant c such that for all $t \geq 0$:*

$$\mathrm{R}_{g(t)} \geq c > 0.$$

In particular, $\sup_\Sigma |M(t)|_{g(t)} \leq Ce^{-ct}$, $t \geq 0$ for some uniform positive constant C.

PROOF. By the minimum principle, Chapter ensures that $\min_\Sigma \mathrm{R}_{g(t)} > 0$ for all $t \geq 0$. In particular, it is enough to prove the desired estimate for large time, say $t \geq 1$. For that purpose, take x_1 such that $r \leq \mathrm{R}_{g(t-1)}(x_1)$. From Corollary 6.18, the Harnack inequality implies that for all $(x, t) \in \Sigma \times \{t\}$,

$$\begin{aligned}
\mathrm{R}_{g(t)}(x) &\geq r \exp\left\{-\frac{1}{4}\inf_\gamma \int_{t-1}^{t} |\dot\gamma(t)|^2_{g(t)}\, dt\right\} \frac{Ce^{r(t-1)}-1}{Ce^{rt}-1} \\
&\geq r\frac{C-1}{Ce^r-1} \exp\left\{-\frac{1}{4}\inf_\gamma \int_{t-1}^{t} |\dot\gamma(s)|^2_{g(s)}\, ds\right\}.
\end{aligned}$$

Here we have used the fact that the function $\frac{Ce^{r(t-1)}-1}{Ce^{rt}-1}$ is non-decreasing.

Therefore, we only need to bound the quantity $\inf_\gamma \int_{t-1}^t |\dot\gamma(t)|^2_{g(t)}$ from above uniformly in time. On the segment $[t-1, t]$, the integrated version of the normalized Ricci flow equation gives:

$$g(x,s) = \exp\left\{\int_{t-1}^{s} r - \mathrm{R}_{g(s')}(x)\, ds'\right\} g(x,t-1) \leq \exp\left\{\int_{t-1}^{t} r - \mathrm{R}_{g(s')}(x)\, ds'\right\} g(x,t-1) \leq e^r g(x,t-1),$$

since the scalar curvature is nonnegative for all time. In particular,

$$\inf_\gamma \int_{t-1}^{t} |\dot\gamma(s)|^2_{g(s)}\, ds \leq e^r \inf_\gamma \int_{t-1}^{t} |\dot\gamma(s)|^2_{g(t-1)}\, ds \leq e^{r+R} \inf_\gamma \int_{t-1}^{t} |\dot\gamma(s)|^2_{g(t)}\, ds,$$

where $R := \sup_{\Sigma\times\mathbb{R}_+} \mathrm{R}_{g(t)} < +\infty$ thanks to Proposition 6.11. To sum it up,

$$\begin{aligned}\mathrm{R}_{g(t)}(x) &\geq r\frac{C-1}{Ce^r-1}e^{-\frac{d_{g(t)}(x,x_1)^2}{4}}\\ &\geq r\frac{C-1}{Ce^r-1}e^{-\frac{(\mathrm{diam}_{g(t)}\Sigma)^2}{4}}\\ &\geq r\frac{C-1}{Ce^r-1}e^{-\frac{D^2}{4}}, \quad t\geq 1,\end{aligned}$$

where we have used the uniform upper bound on the diameter proved in Proposition 6.13.

The exponential decay on $M(t)$ follows from Proposition 6.2 and the discussion that follows. □

Before we end the proof of Theorem 6.1, we need a priori decay in time estimates on the covariant derivatives of the tensor $M(t)$:

Lemma 6.20. *Let $(\Sigma, g(t))_{t\in[0,+\infty)}$ be a solution to the Ricci flow with $\mathrm{R}_{g(0)} > 0$. Then for each $k \geq 0$, there exists positive constants C_k and c_k such that:*

$$\sup_{\Sigma} |\nabla^{g(t),k} M(t)|_{g(t)} \leq C_k e^{-c_k t}, \quad t\geq 0.$$

PROOF. The proof of this result is left as an exercise.

(i) Use the evolution equation for $M(t)$ from Proposition 6.2 to derive an evolution equation for $\nabla^{g(t),k}M(t)$ by invoking commutation formulae for each $k\geq 0$.

(ii) Use the same technique we used to derive Shi type estimates in Proposition 3.8 to derive the expected exponential decay estimate in time. Here we also need to invoke the boundedness of all the covariant derivatives of the scalar curvature obtained thanks to the previously mentioned proposition.

□

We are now in a good position to give a proof of the main Theorem 6.1:

PROOF OF THEOREM 6.1. Let (ψ_t) be the flow generated by the gradient $-\nabla^{g(t)}f(t)$ starting from the identity at $t=0$: $\partial_t\psi_t = -\nabla^{g(t)}f(t)\circ\psi_t$, $\psi_t|_{t=0} = \mathrm{Id}_\Sigma$. Then the metrics $\tilde{g}(t) := \psi_t^* g(t)$ satisfy:

$$\begin{aligned}\partial_t\tilde{g}(t) &= \psi_t^*\left(-\mathcal{L}_{\nabla^{g(t)}f(t)}(g(t)) + \partial_t g(t)\right)\\ &= \psi_t^*\left(-\mathcal{L}_{\nabla^{g(t)}f(t)}(g(t)) + \Delta_{g(t)}f(t)g(t)\right)\\ &= -2\psi_t^* M(t)\\ &=: -2\tilde{M}(t), \quad t\geq 0.\end{aligned}$$

In particular, since the maximum of a function is invariant by diffeomorphisms, Lemma 6.20 ensures that the covariant derivatives of the tensor $\tilde{M}(t)$ with respect to the metric $\tilde{g}(t)$ all decay exponentially fast as t tends to $+\infty$:

$$\sup_{\Sigma} |\nabla^{\tilde{g}(t),k}\tilde{M}(t)|_{\tilde{g}(t)} \leq C_k e^{-c_k t}, \quad t\geq 0.$$

Proposition 3.10 ensures the existence of a metric g_∞ on Σ such that $\tilde{g}(t)$ converges exponentially fast to g_∞ in the smooth topology. In particular, if one could show the convergence of the potentials $\tilde{f}(t)$ to a smooth function f_∞ then we could ensure that the limit metric satisfies:

$$\nabla^{g_\infty,2}f_\infty = \frac{\Delta_{g_\infty}f_\infty}{2}g_\infty, \quad \Delta_{g_\infty}f_\infty = r - \mathrm{R}_{g_\infty},$$

i.e. $(\Sigma, g_\infty, \nabla^{g_\infty}f_\infty)$ is a shrinking gradient Ricci soliton. Theorem 2.15 states that f_∞ is constant, i.e. g_∞ is a constant positive curvature metric: its value must be r by Gauss-Bonnet formula. Therefore, unravelling the definition of $\tilde{g}(t)$ with respect to $g(t)$ implies that: $\max_\Sigma |r - \mathrm{R}_{g(t)}| =$

$\max_\Sigma |r - \mathrm{R}_{\tilde{g}(t)}| = \max_\Sigma |\mathrm{R}_{g_\infty} - \mathrm{R}_{\tilde{g}(t)}| \leq Ce^{-ct}$ for all $t \geq 0$ for some positive constant c since the convergence of the flow $\tilde{g}(t)$ towards its limit metric g_∞ is exponential. The same argument shows that the covariant derivatives of $\mathrm{R}_{g(t)}$ with respect to $g(t)$ must converge to 0 at an exponential rate too. This ends the proof of the theorem. □

6. Exercises

Exercise 6.21. *If* $\frac{\partial}{\partial t} g(t) = -\varepsilon \, \mathrm{R}_{g(t)} \, g(t)$ *on* Σ^2 *with* $\mathrm{R}_{g(t)} > 0$ *and* $\varepsilon \geq 0$, *show that if*

$$\frac{\partial}{\partial t} u = \Delta_{g(t)} u + \varepsilon \, \mathrm{R}_{g(t)} \, u,$$

then

$$\frac{\partial}{\partial t} \log u - |\nabla^{g(t)} u|^2_{g(t)} + \frac{1}{t} = \Delta_{g(t)} \log u + \varepsilon \, \mathrm{R}_{g(t)} + \frac{1}{t} \geq 0.$$

Exercise 6.22. *Prove that if* $(\Sigma^2, g(t))_{t \in [0,T)}$ *is a solution to the Ricci flow with* $\mathrm{R}_{g(t)} > 0$ *then*

$$\frac{\partial}{\partial t} \log \mathrm{R}_{g(t)} - |\nabla^{g(t)} \mathrm{R}_{g(t)} |^2_{g(t)} + \frac{1}{t} = \Delta_{g(t)} \log \mathrm{R}_{g(t)} + \mathrm{R}_{g(t)} + \frac{1}{t} \geq 0.$$

Exercise 6.23. *Let* $(\Sigma^2, g(t))_{t \in [0,T)}$ *be a solution to the Ricci flow with* $\mathrm{R}_{g(t)} > 0$. *Prove that*

$$\frac{t_2 \, \mathrm{R}_{g(t_2)}(x_2)}{t_1 \, \mathrm{R}_{g(t_1)}(x_1)} \geq \exp\left\{ -\frac{1}{4} \inf_{\gamma} \int_{t_1}^{t_2} |\dot\gamma(t)|^2_{g(t)} \, dt \right\},$$

where the infimum is taken over all regular paths $\gamma : [t_1, t_2] \to M$ *such that* $\gamma(t_i) = x_i$, $i = 1, 2$. *In particular, show that*

$$\frac{t_2 \, \mathrm{R}_{g(t_2)}(x_2)}{t_1 \, \mathrm{R}_{g(t_1)}(x_1)} \geq \exp\left\{ -\frac{1}{4} \frac{d^2_{g(t_1)}(x_1, x_2)}{t_2 - t_1} \right\}.$$

Exercise 6.24. *For every sequence of times* $(t_i)_i$ *diverging to* $+\infty$, *prove that the family of potentials* $(\tilde f(t_i))_{i \geq 0}$ *normalized so that their mean values with respect to the metrics* $\tilde g(t_i)$ *are* 0, *defined in the proof of Theorem 6.1, admits a subsequence that converges smoothly to a function* f_∞ *on* Σ. *To do so:*

(i) *Check that the normalization* $\int_M \tilde f(t_i) \, d\mu_{\tilde g(t_i)} = 0$ *does not cause any trouble with the proof of Theorem 6.1 unlike that of Theorem 5.1.*

(ii) *Prove that* $(\tilde f(t_i))_{i \geq 0}$ *is a bounded sequence in* $W^{2,1}(M)$ *by invoking the Bochner formula and the Poincaré inequality* $\|\nabla^{\tilde g(t_i)} \tilde f(t_i)\|_{L^2} \geq \lambda \|\tilde f(t_i)\|_{L^2}$ *for some uniform positive* λ.

(iii) *Prove that for each* $k \geq 0$, $(\tilde f(t_i))_{i \geq 0}$ *is a bounded sequence in* $W^{k,2}(M)$ *and invoke Sobolev embeddings to ensure that* $(\tilde f(t_i))_{i \geq 0}$ *is a bounded sequence in* $C^l(M)$ *for every* $l \geq 0$.

(iv) *Conclude.*