

Harmonic measure with Robin boundary conditions

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Similar results by Anna Rozanova-Pierrat. Help of Jill Pipher.

The team



Figure: Stefano Decio



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Figure: Svitlana Mayboroda



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First definition of the Robin boundary condition

Goal: study “harmonic measure” for the following Robin problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ Rob_a(u) := \frac{1}{a} \frac{\partial u}{\partial n} + u = f & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Here $\Omega \subset \mathbb{R}^n$ is bounded and $\frac{\partial u}{\partial n}$ is the outwards normal derivative. Locally, with u given on $\Omega \setminus B(x, r)$, often we just consider $f = 0$.

$a > 0$ is some constant (say);

$a = 0$ corresponds to Neumann boundary conditions;

$a = +\infty$ corresponds to Dirichlet boundary conditions.

Comment: we chose the outwards normal, so that typically $u > 0$ on $\partial\Omega$ and u is larger on Ω than on $\partial\Omega$.

Trivial examples, with $f = 0$: $u(x, t) = t + \frac{1}{a}$ on $\mathbb{R}_+^2$;

$u(x) = \frac{1}{a} - \log(|x|)$ on $\Omega = B(0, 1) \subset \mathbb{R}^2$.

Comment: $L = \operatorname{div} A \nabla$, A bounded elliptic, is allowed too.

Harmonic measure

To simplify the definitions assume $\partial\Omega$ is sufficiently smooth of co-dimension 1 so that everything is well defined.

Take a pole $X \in \Omega$. Define the **Robin** harmonic measure ω_{Rob}^X by $\omega_{Rob}^X(E) = u_E(X)$, where for $E \subset \partial\Omega$, u_E solves (at least formally)

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \text{Rob}_a(u) := \frac{1}{a} \frac{\partial u}{\partial n} + u = \mathbb{1}_E & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Or ω_{Rob}^X is the probability measure on $\partial\Omega$ such that the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \text{Rob}_a(u) = f & \text{on } \partial\Omega. \end{cases} \quad (3)$$

is given, for $f \in C(\partial\Omega)$, by

$$u(X) = \int_{\xi \in \partial\Omega} f(\xi) d\omega_{Rob}^X(\xi). \quad (4)$$

[Same as for the usual Dirichlet harmonic measure ω_{Dir}^X , where we would require $u = f$ on $\partial\Omega$. But we need to construct all this.]

Brownian interpretation

In the Dirichlet case, we think of $\omega_{Dir}^X(E)$ as the probability that a Brownian particle starting at X first exits Ω through a point of E . For $\omega_{Rob}^X(E)$, think of a Brownian particle that starts from X , and each time it hits $\partial\Omega$, has a certain “probability” (small if a is small) of being absorbed. And if not we start it again from where it is, and continue playing until the particle is absorbed.

In fact, easier to define discretely, with random walks; otherwise one would try to use local time in $\partial\Omega$.

Many applications because full absorption is rare.

Example: the deep lung.

But also heat in a room (and the Robin boundary condition is also called **Fourier boundary condition**).

Main question for today: Define the Robin problem and $\omega_{Rob}^X(E)$, and try to find out where on $\partial\Omega$ is $\omega_{Rob}^X(E)$ supported and how regular it is.

Lungs are fractal

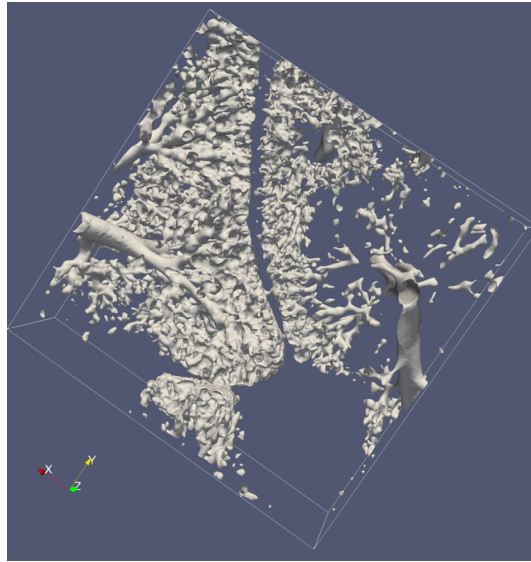
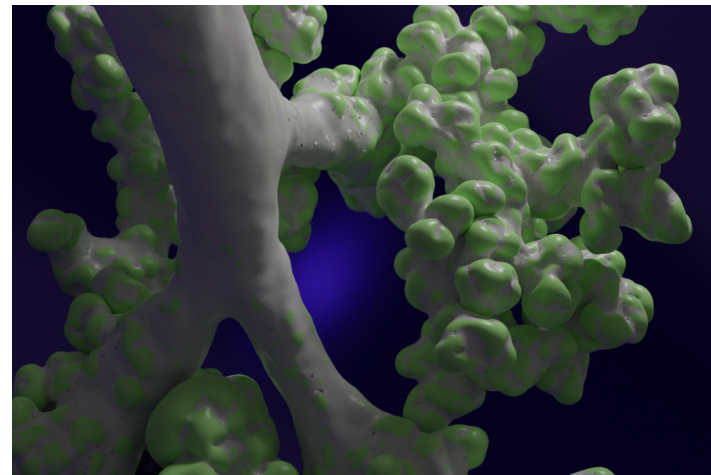
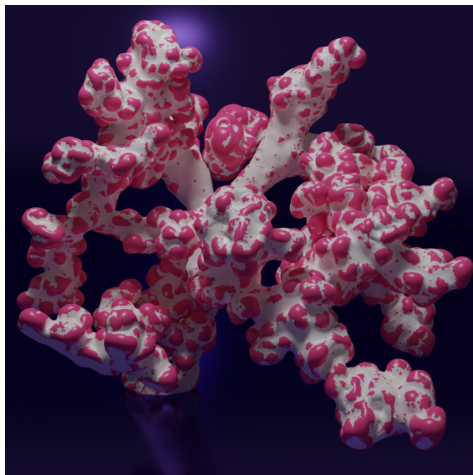


Figure: Pictures of rat lungs by tomography performed at the Grenoble Synchrotron. Credits: S. Bayat, H. Leclerc, S. Martin, B. Maury, B. Semin.



Dirichlet Harmonic measure and why we could be afraid

Rapid review of the situation of $\omega_{Dir}^X(E)$.

This is a long story starting, say, from Riesz and many authors (La, Ca, Ma, Jo, Wo, ... conformal mappings) when $n = 2$, and then more work from Dahlberg to HoLaMarMayMouToVo ($n \geq 3$).

For $\partial\Omega \subset \mathbb{R}^n$ of dimension $n - 1$, the (mutual) absolute continuity of ω_{Dir}^X with respect to the surface measure $\sigma = \mathcal{H}_{|\partial\Omega}^{n-1}$ is mostly a matter of connectedness for Ω (typically, NTA) and rectifiability for $\partial\Omega$ (typically, UR).

Some results when $\partial\Omega$ is of dimension $d \neq n - 1$, mostly proofs that $\omega_{Dir} \perp \mathcal{H}_{|\Omega}^d$ at least on fractals.

But there are amusing recent examples of $\partial\Omega$ Ahlfors regular of dimension d , $n - 2 < d < n - 1$, with mutual absolute continuity.

Dirichlet Harmonic measure (2) and our question

Also, one may consider solutions of $Lu = 0$, for elliptic operators $L = -\operatorname{div} A \nabla$ other than $-\Delta$.

Then positive results exist when L is sufficiently close to Δ (or a constant coefficient operator). In terms of Carleson measure estimates on the oscillation of A . And counterexample otherwise.

Return to $L = -\Delta$, and think of two main examples of co-dimension 1:

- Lipschitz graphs
- Cantor sets like the “Garnett Ivanov set” on the next page.

Do we expect the same thing to happen for ω_{Rob}^X ?

Maybe more smooth, but is there a phase transition when a varies?

But, because of the lung, we'll also be interested in Ahlfors regular boundaries of dimensions $d \in (n - 2, n)$, i.e., there is a measure μ on $\partial\Omega$ such that, for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$

$$C^{-1}r^d \leq \mu(\partial\Omega \cap B(x, r)) \leq Cr^d.$$

The Garnett-Ivanov 1-dimensional Cantor set

$K = \bigcap_{k \geq 0} K_k$, suggested by the picture.

K_k is composed of 4^k squares of size 4^{-k}

A natural measure μ on K gives the same mass 4^{-k} to each square of K_k . And then $\mu = cH^1|_K$.

K is totally unrectifiable: $\mu(E \cap \Gamma) = 0$ for every curve Γ with finite length

This set is one-dimensional Ahlfors regular, with a NTA complement.

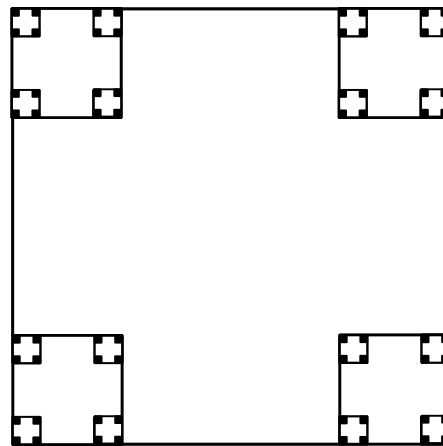


Figure: The set K_3 (three generations of the construction of K ; then just keep the dust at the limit)

Pictures for **Non-Tangentially-Accessible** domains (here the domain is outside)

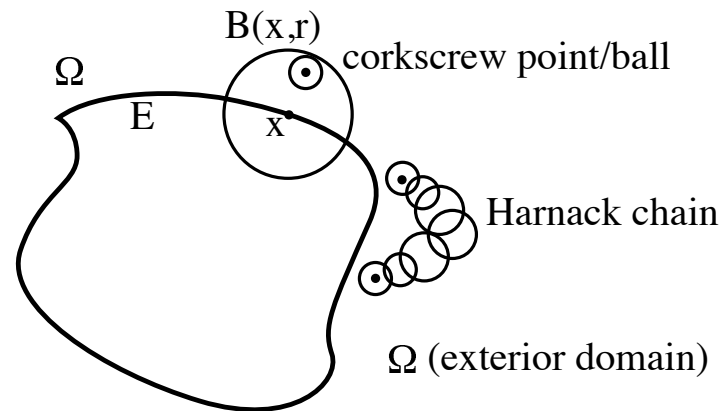


Figure: A corkscrew ball (top) and a Harnack chain between two points (right). The domain is outside.

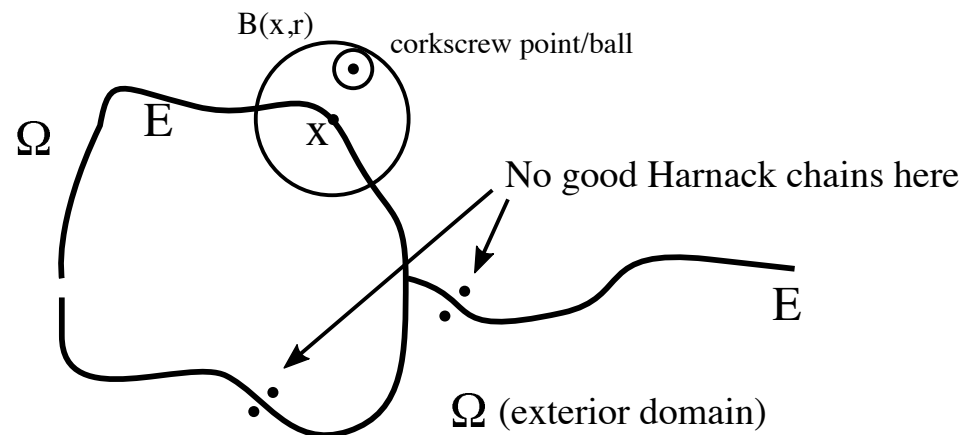


Figure: A situation with bad NTA constants

Typical best harmonic measure estimate for Dirichlet

THEOREM (Many Authors, and better variants exist)

For a domain Ω with NTA (non-tangential access) and an Ahlfors regular and uniformly rectifiable boundary of dimension $n - 1$, we have the following A_∞ estimate for ω_{Dir} : For $B = B(x, r)$ centered on $\partial\Omega$, $r < \text{diam}(\partial\Omega)$, $X \in B$ such that $\text{dist}(X, \partial\Omega) \geq C^{-1}r$, and $E \subset B \cap \partial\Omega$,

$$C^{-1} \left(\frac{\sigma(E)}{\sigma(D)} \right)^\alpha \leq \frac{\omega_{Dir}^X(E)}{\omega_{Dir}^X(D)} \leq C \left(\frac{\sigma(E)}{\sigma(D)} \right)^\beta. \quad (5)$$

for some $\alpha, \beta > 0$ (that depend on Ω).

Thus $\omega_{Dir}^X \ll \sigma \ll \omega_{Dir}^X$, with uniform estimates.

True for the Laplacian, but also for good enough elliptic operators.

For Lipschitz graphs, if I recall correctly we can take $\alpha = 2$ and $\beta = 1/2$. But for general UR domains I don't think we know α, β .

Comments about Neumann

We would expect the Robin situation to be somewhere “between” Dirichlet and Neumann.

- Calculus of variation likes the (vanishing) Neumann condition: if you minimize $\int_{\Omega \cap B(0,R)} |\nabla u|^2$ under the constraint $u_{\partial B(0,R) \cap \Omega} = u_0$, then automatically $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega \cap B(0, R)$.

In this case, $\frac{\partial u}{\partial n} = 0$ amounts to no boundary constraint on u . And, if we are lucky, regularity could be like interior regularity.

- We'll see that for simple bounded domains, $\omega_{Neu}^X(E) = \frac{\sigma(E)}{\sigma(\partial\Omega)}$. (uniform presence of the Brownian particle). So ω_{Neu}^X is tame!

Note: for the moment, think $\partial\Omega$ is $(n - 1)$ -dimensional, but not smooth, and $\sigma = \mathcal{H}_{|\partial\Omega}^{n-1}$. But recall we want more interesting cases.

Different scale invariance. And what is $\frac{\partial u}{\partial n}$?

Unpleasant, but we have to remember this: when u is a solution of $\text{Rob}_a(u) = 0$ on $\partial\Omega$, i.e., when $\frac{1}{a} \frac{\partial u}{\partial n} + u = 0$, then the function v given by $v(x) = u(\lambda^{-1}x)$ solves $\text{Rob}_{\lambda^{-1}a}(v) = 0$ on $\partial(\lambda\Omega)$,

And/or: the constant a scales like $\frac{1}{\text{length}}$.

Our results will have to acknowledge this. Said in other words, **at small scales we expect u to look more like a Neumann solution,** and at large scales like a Dirichlet solution.

Another issue: when $\partial\Omega$ is not smooth, and even more when its dimension is $\neq n - 1$, what is $\frac{\partial u}{\partial n}$?

For my fun: a variational definition of ω_{Rob}^X

We do a last computation in co-dimension 1 with a smooth boundary. Call σ the surface measure.

Let $E \subset \partial\Omega$. Minimize (for the given Robin constant $0 < a < +\infty$)

$$J(u) = \frac{1}{a} \mathcal{E}(u) + \int_{\partial\Omega} u^2 d\sigma - 2 \int_E u d\sigma, \quad \text{with } \mathcal{E}(u) = \int_{\Omega} |\nabla u|^2. \quad (6)$$

Not so hard to prove that a **unique minimizer** u_E exists, by **convexity, Poincaré, and the existence of a nice trace**.

Next, the minimizer $u = u_E$ is harmonic on Ω .

We can integrate by parts to compute that $\mathcal{E}(u) = \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\sigma$.

By Lagrange (i.e., expand $J(u + tv)$ and differentiate at $t = 0$), **$Rob(u) = \mathbb{1}_E$ on $\partial\Omega$** . So

$$\omega_{Rob}^X(E) = u_E(X).$$

A variational definition of ω_{Rob}^X (2)

Recall

$$\omega_{Rob}^X(E) = u_E(X), \quad (7)$$

where u_E minimizes

$$J(u) = \frac{1}{a} \int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} u^2 d\sigma - 2 \int_E u d\sigma. \quad (8)$$

For Dirichlet ($a = +\infty$), we would minimize the same $\mathcal{E}(u)$ with the constraint that $u = \mathbb{1}_E$ on $\partial\Omega$. But this is harder to manage.

For Neumann ($a = 0$), u must be a constant c , and a direct computation with (6) gives $u \equiv \frac{\sigma(E)}{\sigma(\partial\Omega)}$.

Return to (7). Notice that all this makes sense whenever σ is a measure on $\partial\Omega$ and the pair (Ω, σ) is regular enough for a trace $\text{Tr}(u)$ to exist on $\partial\Omega$ and lie in $L^2(\sigma)$. A good sign.

But in fact we proceed otherwise (weak definition of $\frac{\partial u}{\partial n}$ and estimates on weak solutions).

Assumptions and results

Here is the basic statement.

- Ω is assumed to be bounded (for simplicity);
- A is elliptic (above we even wrote $A = I$);
- Quantitative Connectedness: Ω is a one-sided NTA domain: it contains corkscrew balls and there are Harnack chains of balls in Ω .
- $\partial\Omega$ is Ahlfors regular of dimension $d \in (n - 2, n)$. That is, there exists $C \geq 1$ and a measure σ on $\partial\Omega$ such that

$$C^{-1}r^d \leq \sigma(\partial\Omega \cap B(x, r)) \leq Cr^d$$

for $x \in \partial\Omega$ and $0 < r \leq \text{diam}(\Omega)$.

In fact, still works when σ is a doubling measure on $\partial\Omega$ with a mixed dimensions condition (something that says that $d > n - 2$).

Existence and regularity results

THEOREM (D.-Decio-Engelstein-Mayboroda-Michetti)

Let Ω be as above, and $0 < a < +\infty$. Then the Robin problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ Rob_a(u) := \frac{1}{a} \frac{\partial u}{\partial n} + u = f & \text{on } \partial\Omega. \end{cases} \quad (9)$$

has a unique solution in Ω for every continuous f on $\partial\Omega$.

This solution u is given by a collection of probability measure ω_{Rob}^X .
If f is Hölder continuous on $\partial\Omega$, then u is also Hölder continuous on $\overline{\Omega}$.

Finally $\omega_{Rob}^X \ll \sigma \ll \omega_{Rob}^X$, with uniform estimates of A_∞ type.

More precisions below.

Notice Cantor sets and “bad” elliptic A are allowed!

a does not need to be constant : $a_0 \leq a(x) \leq Ca_0$ is allowed too.

About the regularity of solutions and existence

Main difference with the Dirichlet case: we also need to run the Moser argument for the Neumann problem.

At the center of the proof, show that for a (weak) solution u in $B(0, R)$, the oscillation of u in $B(0, 10^{-2}R)$ is at most $(1 - \eta)$ times the oscillation in $B(0, R)$.

For the rest: Lax-Milgram (weak solutions exist), Moser, the maximum principle, and the Hölder regularity and comparison principle (Hölder regularity at the boundary too).

Important new feature: a gives a scale, and the constants for estimates in $B(x, r)$ depend on ar^{2+d-n} (in fact, $a\sigma(B(x, r))r^{2-n}$).

Thus the mutual absolute continuity constants below will logically diverge when $ar^{2+d-n} \gg 1$, because we get close to Dirichlet.

Absolute continuity estimates for ω_{Rob}

Here is a precise version of the absolute continuity.

- ω_{Rob} is doubling: $\omega_{Rob}^X(B(x, 2r)) \leq C \omega_{Rob}^X(B(x, r))$ for $x \in \partial\Omega$ and $r > 0$, with C that does not depend on a or r .

- Mutual absolute continuity estimates with optimal exponent 1: Consider $E \subset B = B(x, r) \cap \partial\Omega$, and a pole $X \in B(x, r)$ at distance $C^{-1}r$.

Set $\sigma = \mathcal{H}_{|\partial\Omega}^d$. Then

$$C^{-1} \frac{\sigma(E)}{\sigma(B)} \leq \frac{\omega_{Rob}^X(E)}{\omega_{Rob}^X(B)} \leq C \frac{\sigma(E)}{\sigma(B)} \quad (10)$$

in the normalized “close-to-Neumann” case when $x \in \partial\Omega$, $\text{diam}(\partial\Omega) \geq Cr$, and $ar^{2+d-n} < 1$.

Otherwise, C depends also (like a power) on ar^{2+d-n} .

Comments: Optimal power 1, and no phase transition in a . Good for (D_p) . Thank you Jill Pipher. More estimates, on the size of the Green function $G_{Rob}(X, Y)$, are coming.

About the proof (if time allows)

To amuse you, an argument for a.c. by the calculus of variation (assuming symmetry of the coefficient matrix).

But the best proof (below) uses a Green function argument.

Recall that $\omega_{Rob}^X(E) = u_E(X)$ where $u = u_E$ minimizes

$$J(u) = \frac{1}{a} \mathcal{E}(u) + \int_{\partial\Omega} u^2 d\sigma - 2 \int_E u d\sigma, \quad \text{with } \mathcal{E}(u) = \int_{\Omega} |\nabla u|^2.$$

We want to show that $\omega^X(E) = 0$ if and only if $\sigma(E) = 0$.

- If $\sigma(E) = 0$, then $J \geq 0$, the minimum is for $u \equiv 0$, and then $\omega_{Rob}^X(E) = 0$.
- If $\omega_{Rob}^X(E) = 0$, then $u_E = 0$ everywhere on Ω by nonnegativity and Harnack, and so $J(u) \geq 0$ for all u . But if $\sigma(E) > 0$, taking $u = c$, where c is a very small constant, gives $J(u) < 0$.

This was easy! A clumsy (and not completely checked) proof with calculus of variation starts like this.

Use the Green function

But we can also use the Green function. This is a nonnegative function $G_{Rob}(X, Y)$, which satisfies the equation with

$$Rob_a(G_{Rob}(\cdot, Y)) = 0 \quad \text{on } \partial\Omega \quad (11)$$

and has a normalized singularity at Y .

Existence and some regularity for G_{Rob} a little bit as usual, once solutions are known to be regular.

Here we will use the fact that morally $\frac{\partial G}{\partial n} = -aG$ at the boundary while (traditionally) the density of ω^Y is $\frac{\partial G}{\partial n}$, here equal to $-aG$.

In fact we have the nice formula

$$\omega_{Rob}^Y(E) = c \int_E G_{Rob}(x, Y) d\sigma(x) \quad (12)$$

for $E \subset \partial\Omega$. And we can apply the comparison principle to the positive solution $x \mapsto G_{Rob}(x, Y)$ (far from the pole Y) to get that all the values of $G_{Rob}(x, Y)$, $x \in B = B(x, r)$, are comparable, whence the desired result (integrate on E or on B).

Comments. More on the Green function

After all, no “phase transition” when a varies.

We will need to check what happens for Ω unbounded.

We intend to do some optimization and “explain” the (dimension of the) lung.

Almost done: estimates on G_{Rob} that relate it to G_{Dir} (and the strange homogeneity above). Consider $G_{Rob}^X = G_{Rob}(X, Y)$ with pole X , and the Dirichlet Green function $G_{Rob}^X = G_{Rob}(X, Y)$. In the most standard situation where X is close to the center and Y close to the boundary, we get that

$$C^{-1} G_{Dir}^X(Y_y) \leq G_{Rob}^X(Y) \leq C G_{Dir}^X(Y^r)$$

where Y^r is a point of Ω chosen near Y , at distance comparable to $r > 0$ from both Y and ∂ , where r is chosen so that $ar^{2-n}\sigma(B(Y, r)) \sim 1$ (the transition radius).

Saved by weak conditions (and the calculus of variations?)

An important progress (I claim) was the setup, which allows Ω with a rough boundary of co-dimension $\neq 1$.

Maybe you noticed that on the Cantor set, $\frac{\partial u}{\partial n}$ is not well defined. And this is worse for fractal boundaries (such as other Cantor sets or snowflakes).

So we need to define $\frac{\partial u}{\partial n}$ and $Rob_a(u)$ weakly (see the next page).

Comment designed to help me: the functional

$$aJ(u) = \mathcal{E}(u) + a \int_{\partial\Omega} u^2 d\sigma - 2a \int_E u d\sigma$$

is well defined, essentially for any measure σ on $\partial\Omega$, as long as functions $u \in W^{1,2}(\Omega)$ (i.e., such that $\mathcal{E}(u) < +\infty$) have a nice trace in $L^2(\sigma)$. Here we are : all our results use the existence of nice traces!

Notice that the main actor is $a d\sigma$, so for instance different but equivalent σ allow to account for slightly variable coefficients a .

Official definition of a weak solution

Finally what we mean by a (weak) solution to the Robin problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{1}{a} \frac{\partial u}{\partial n} + u = f & \text{on } \partial\Omega. \end{cases} \quad (13)$$

It is a function $u \in W^{1,2}(\Omega)$ such that

$$\frac{1}{a} \int_{\Omega} \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} \text{Trace}(u) \varphi d\sigma = \int_{\partial\Omega} f \varphi d\sigma \quad (14)$$

for all test functions $\varphi \in C_c^1(\mathbb{R}^n)$.

[and to get that $\Delta u = 0$ on Ω , just consider $\varphi \in C_c^1(\Omega)$.]

Thanks

Thanks for the invitation and listening!

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