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General introduction

The notion of uniform rectifiability probably appeared near 1990, in connection with the question of finding on which d-dimensional sets of $\mathbb{R}^n$ the natural Calderón-Zygmund kernels define bounded operators on L^2. It was suspected that rectifiability should play an important role in this question, and various quantitative rectifiability conditions were available. The phrase "uniform rectifiability" only came later (in [**DS4**]), and at that time it became more clear that uniform rectifiability should be the natural notion in many problems involving rectifiability in a quantitative and scale-invariant way. In particular, [**DS4**] contains loads of equivalent definitions and characterizations.

The main goal of these notes is to present a general introduction to uniform rectifiability, which would be slightly less intimidating than [**DS4**] even though the author's impression is that Part II of [**DS4**] is not so hard to read.

We shall try give an idea of some of the techniques involved in the proofs, which means that we shall single out some results that are both reasonably easy and significant in some way, and give a proof or a sketch of proof for them. We shall even try to present an argument based on the Corona construction. We shall also mention some domains where the notion was useful (analytic capacity, and the study of the Mumford-Shah functional and Quasiminimal sets).

Some of the results presented here are quite old; the reader is kindly asked to forgive this. The author wishes to thank the organizers of the Park City summer school for a very nice event, and Raanan Schul and Gilad Lerman for many suggestions and improvements.

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LECTURE 1

Definitions, singular integrals, and big pieces

In this first part we want to give initial (but strong) definitions of uniform rectifiability, and then rapidly discuss singular integrals on sets, because it was the initial motivation for introducing the notion. We shall talk a little more about "big pieces", because we think it is a nice example where some mild local control at all scales and locations implies much stronger conditions.

1. Ahlfors-regular sets

To make the exposition more pleasant, we shall mostly restrict to sets $E \subset \mathbb{R}^n$ that are Ahlfors-regular of some dimension $d < n$. This means that E is closed, not reduced to one point, and that there is a Borel measure μ supported on E such that

(1) $$C_0^{-1} r^d \leq \mu(B(x,r)) \leq C_0 r^d \text{ for all } x \in E \text{ and } 0 < r < \operatorname{diam}(E)$$

for some constant $C_0 \geq 1$. It is easy to check (with a small covering argument) that if this is the case, then the restriction to E of the d-dimensional Hausdorff measure H^d is equivalent to μ (in the sense that $C^{-1}\mu(A) \leq H^d(A) \leq C\mu(A)$ for every Borel set $A \subset E$). See Exercise 1. Thus we could take $\mu = H^d_{|E}$ in (1), and this would give an equivalent definition. The presentation above has the advantage that we shall not feel compelled to define H^d. Let us just say that the restriction of H^d to E coincides with surface measure when E is a smooth d-dimensional submanifold, and refer to [**Fa**], [**Fe**], or [**Ma3**] for details.

Note that Ahlfors-regularity is *not* a regularity property. It is just a strong scale-invariant way to demand that E be d-dimensional.

It is a very pleasant assumption to do, because it gives a clear relation between the size of sets and their diameters. Also, (1) implies that that μ is doubling (i.e., that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for $x \in E$ and $r > 0$). In other words, E is a space of homogeneous type, and many of the standard techniques of analysis in Euclidean space work just as well on E. But we should say that our Ahlfors-regularity assumption is often here only for comfort, and that there are very interesting results, even in the next lectures, where it is not needed.

2. Uniform rectifiability

We first define uniform rectifiability for Ahlfors-regular sets of dimension 1.

Definition 2. *An Ahlfors-regular curve is a set of the form $\Gamma = z(I)$, where $I \subset \mathbb{R}$ is a closed interval (not reduced to a point, but possibly unbounded) and $z : I \to \mathbb{R}^n$ is such that*

(3) $$|z(s) - z(t)| \leq C_1|s-t| \textit{ for } s,t \in I$$

and

$$\Big|\big\{s \in I\,;\, z(s) \in B(x,r)\big\}\Big| \leq C_1 r \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0 \tag{4}$$

for some constant $C_1 \geq 1$. We call Ahlfors-regular mapping a function z such that (3) *and* (4) *hold.*

Regular curves were introduced in [**Ah1**] as early as 1935, in a context that is fairly different from the one here. Note that (3) and (4) go in different directions; once we know (3), (4) says that $z(s)$ never stays too long in a given ball. Lipschitz graphs and chord-arc curves are Ahlfors-regular curves, but the latter may also have cusps and self-intersections (to a limited extent). That is, Ahlfors-regular parametrizations z are not necessarily injective, but it is easy to check that $z^{-1}(x)$ never has more than $1 + \frac{C_1^2}{2}$ points, for instance (see Exercise 2). In terms of smoothness, Ahlfors-regular curves are almost as good as Lipschitz curves.

Also, it is easy to check that Ahlfors-regular curves are connected Ahlfors-regular sets; we can take for μ the direct image by z of the Lebesgue measure on I. The converse is true for compact sets, i.e., every compact connected Ahlfors-regular set of dimension 1 is the image of an Ahlfors-regular curve; see Lemma 66. This fails slightly for unbounded sets; for instance the union of the two coordinate axes in the plane is not a regular curve, because any parametrization would have to go infinitely many times through the origin. But even in this case the difference is quite small: one can show that every connected Ahlfors-regular set is contained in an Ahlfors-regular curve.

Definition 5. *The one-dimensional Ahlfors-regular set E is said to be uniformly rectifiable when there is an Ahlfors-regular curve Γ that contains E.*

As we shall see, there are other equivalent definitions; this one is probably the simplest (and the most demanding in appearance).

Maybe this is the right time to say why this is stronger than rectifiability. Let us recall the definition directly in dimension d. A set $F \subset \mathbb{R}^n$, say, with sigma-finite Hausdorff measure H^d, is said to be rectifiable (some important references say "countably rectifiable", or even "regular") if

$$F \subset N \cup \Big(\bigcup_{i\in\mathbb{N}} \Gamma_i\Big) \tag{6}$$

for some negligible set N (i.e., such that $H^d(N) = 0$) and some countable family of C^1 embedded submanifolds Γ_i of dimension d. But we get an equivalent definition if we only require the sets Γ_i to be Lipschitz images of (subsets of) $\mathbb{R}^d$; this is fairly easy to check, because Lipschitz images of $\mathbb{R}^d$ can be covered, up to sets of H^d-measure zero, by submanifolds. [Think about Lusin's theorem.]

The difference with uniform rectifiability is that we only use one curve Γ (instead of countably many), and we have a uniform control on Γ at all scales and locations through the constant C_1 in (3) and (4). Incidentally, note that our definitions are invariant under translations and dilations. That is, the images of Γ and E by translations and dilations are (respectively) Ahlfors-regular and uniformly rectifiable with the same constants C_0 and C_1 as Γ and E.

Let us now give our first definition of uniform rectifiability in higher dimensions. We try to imitate the definition above, and say that uniformly rectifiable sets are Ahlfors-regular sets of dimension d that are contained in sets with a reasonable parametrization. But since it is somewhat harder to parametrize sets of dimensions larger than one, we shall have to content ourselves with parametrizations that are not quite Lipschitz. Instead they will be controlled by weights $\omega \in A_1$. Recall that $A_1(\mathbb{R}^d)$ is the class of positive, locally integrable functions ω such that for some constant $C_2 \geq 1$,

$$\frac{1}{|B|}\int_B \omega(y)dy \leq C_2 \operatorname*{ess\,inf}_B \omega \quad \text{for every ball } B \subset \mathbb{R}^d. \tag{7}$$

Definition 8. *Let $\omega \in A_1(\mathbb{R}^d)$ be given. An ω-regular mapping is a function $z : \mathbb{R}^d \to \mathbb{R}^n$ such that for some $C_3 \geq 1$,*

$$|z(s) - z(t)| \leq C_3 \Big\{ \int_{B_{s,t}} \omega(y)dy \Big\}^{1/d} \quad \textit{for } s, t \in \mathbb{R}^d, \tag{9}$$

where we set $B_{s,t} = B(\frac{s+t}{2}, |s-t|)$, say, and

$$\int \mathbf{1}_{\{s\in\mathbb{R}^d\,;\, z(s)\in B(x,r)\}}(s)\,\omega(s)ds \leq C_3 r^d \textit{ for all } x \in \mathbb{R}^n \textit{ and } r > 0. \tag{10}$$

The set $z(\mathbb{R}^d)$ will be called an ω-regular surface.

Notice the similarity with Definition 2, which corresponds to $d = 1$ and $\omega = 1$. We could also consider ω-regular mappings defined on a cube of $\mathbb{R}^d$ (with a similar definition), but this will not be necessary for the purpose of these lectures.

It may help to think of $\omega(B_{s,t})^{1/d}$ as a semidistance between s and t (only the triangle inequality fails, and not so much) governed by the weight ω; then (9) says that z is Lipschitz (on the "space of homogeneous type" $\mathbb{R}^d$, equipped with the measure $\omega(s)ds$ and this semidistance). Or you may just think that ω gives some bound on the Jacobian determinant of z. Anyway, once we know (9), the left-hand side of (10) gives an upper bound on the surface measure of $z(\mathbb{R}^d) \cap B(x, r)$.

It is fairly easy to check that $z(\mathbb{R}^d)$ is an Ahlfors-regular set of dimension d when z is an ω-regular mapping; one can use the direct image of $\omega(s)ds$ by z to prove (1).

The notion of ω-regular parametrization dates from the eighties, and the point was to give a class of sets as large as possible, where standard Calderón-Zygmund kernels would yield bounded singular integral operators (see [**Da2**]). For this purpose, it was even enough to assume that ω lies in the Muckenhoupt class A_∞. There is not much point to do so here, because we are happy to restrict to the smaller class of weights.

Definition 11. *Let $E \subset \mathbb{R}^n$ be an Ahlfors-regular set of dimension d. We say that E is uniformly rectifiable if we can find a weight $\omega \in A_1(\mathbb{R}^d)$ and an ω-regular mapping $z : \mathbb{R}^d \to \mathbb{R}^{n+1}$ such that $E \subset z(\mathbb{R}^d)$.*

The slightly annoying fact that we may have to use a set $z(\mathbb{R}^d)$ in $\mathbb{R}^{n+1}$ comes from the fact that it may be difficult to extend ω-regular mapping in $\mathbb{R}^n$, and the extra room in $\mathbb{R}^{n+1}$ is useful in some proofs. The issue does not arise when $n \geq 2d$, and we can take z with values in $\mathbb{R}^n$ in this case.

Also note that when $d = 1$, our two definitions of uniform rectifiability are equivalent, because one can reparametrize $z(\mathbb{R})$ by arclength, or just replace the ω-regular mapping z with $z \circ \varphi^{-1}$, where φ is an increasing mapping such that $\varphi'(s) = \omega(s)$ almost-everywhere and distribution wise. This is why weights are not needed when $d = 1$. A similar attempt to re-parametrize images of ω-regular mappings in higher dimensions leads to serious questions. S. Semmes [**Se7**] showed that there are strong A_∞ weights ω for which we cannot find such a change of variable, i.e, such that there is no quasiconformal change of variable whose Jacobian determinant is equivalent in size to ω. But as far as I know, there is no example known like this where there is an ω-regular mapping for such ω. Also, Semmes [**Se4**] showed that for every A_1 weight, there is an ω-regular mapping into some $\mathbb{R}^N$ (N possibly very large), but we do not know if every A_1 weight is equivalent in size to the Jacobian determinant of some quasiconformal change of variable (we suspect not).

Ahlfors-regular curves and ω-regular surfaces (and hence also uniformly rectifiable sets) have all sorts of good geometric and analytic properties. We shall spend some time on geometric properties; for the analytic ones, we shall say merely a few words about the L^2-boundedness of singular integral operators, and the reader may consult [**DS4**] for other ones. See Chapters III-2 and III-3 (and in particular (III.2.9) and Theorem III.3.14) for analogues of the square function estimates on Cauchy integrals of functions on E, and Proposition III.4.2 for the good approximation of Lipschitz functions on E by affine functions. These results have partial converses, which was one of the main points of the study in [**DS4**].

In most of the rest of this text, we take for granted that uniformly rectifiable sets are nice, and we shall look for sufficient conditions for uniform rectifiability, and examples. We shall even try to convince the reader that natural scale-invariant, quantitative conditions that imply rectifiability should often imply uniform rectifiability.

3. Singular integral operators

Since this was the original excuse to study uniform rectifiability, let us say a few words about singular integrals on E. We shall consider kernels $K(z)$ defined on $\mathbb{R}^n \setminus \{0\}$, such that

$$|\nabla^j K(z)| \leq C_4 |z|^{-d-j} \text{ for } z \in \mathbb{R}^n \setminus \{0\} \text{ and } j = 0, 1, 2, \dots, \tag{12}$$

and

$$K(-z) = -K(z) \quad \text{for } z \in \mathbb{R}^n \setminus \{0\}. \tag{13}$$

One could consider slightly weaker size conditions, i.e., require only a finite number of derivatives in (12) (depending on n), and different cancellation properties, but this is not the point here. The most typical examples are the Cauchy kernel $K(z) = \frac{1}{z}$ (with $d = 1$, $n = 2$, and where $\mathbb{R}^2$ is identified with $\mathbb{C}$), and the Riesz kernels $K_{n,d}(z) = \dfrac{z}{|z|^{d+1}}$, often with $d = n-1$. Since the integral $\int_E K(x-y) f(y) d\mu(y)$ rarely converges, the best way to define L^2-boundedness is to ask for uniform estimates on truncated integrals, as follows. If K satisfies (12) and (13) and μ is a locally finite Borel measure on $\mathbb{R}^n$, we can set

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y) f(y) d\mu(y) \tag{14}$$

for f continuous and compactly supported on $\mathbb{R}^n$ and $\varepsilon > 0$, and we say that K defines a bounded operator on $L^2(\mu)$ if there is a constant C_5 such that

$$\int |T_\varepsilon f(x)|^2 d\mu(x) \leq C_5 \int |f(y)|^2 d\mu(y) \quad \text{for all } f \text{ and } \varepsilon > 0. \tag{15}$$

If we are just given a set E, we take for μ the restriction to E of the Hausdorff measure H^d, and say that K defines a bounded operator on $L^2(E)$ if it defines a bounded operator on $L^2(\mu)$. But if E is Ahlfors-regular we could take any measure μ such that (1) holds, and this would yield an equivalent definition.

If E is uniformly rectifiable and μ is as in (1), then every kernel K such that (12) and (13) hold defines a bounded operator on $L^2(\mu)$ [**Da2**]. The main point is that $z(\mathbb{R}^d)$ contains big pieces of Lipschitz graphs when z is an ω-regular mapping, so that one can reduce to the theorem of Coifman, McIntosh, and Meyer that gives the result in the case of Lipschitz graphs; see Proposition 16 below. To be precise, the initial proof in [**Da2**] is a little less direct than this, but one could really proceed this way, because of [**Da3**] or [**Jo2**].

The situation for the converse is a little more complicated. First of all, if E is Ahlfors-regular, μ is as in (1), and all the kernels K such that (12) and (13) hold define bounded operators on $L^2(\mu)$, then E is uniformly rectifiable [**DS3**]. The proof also shows that if d is not an integer, then there is no Ahlfors-regular set E of dimension d such that every kernel K with (12) and (13) defines a bounded operator on $L^2(\mu)$.

But there are other, probably more natural questions. Mainly, is it true that if the Riesz kernel $K_{n,d}(z) = \frac{z}{|z|^{d+1}}$ defines a bounded operator on $L^2(\mu)$, then E is uniformly rectifiable? This is known only when $d = 1$. When $d = 1$ and $n = 2$, Mattila, Melnikov, and Verdera [**MMV**] showed that if the Cauchy kernel defines a bounded operator on L^2 on the Ahlfors-regular set E, then E is uniformly rectifiable. The proof uses the magic formula on the Menger curvature of triples that will be rapidly presented in Section 5. Then H. Farag [**Fr2**] showed that the (positivity for the) formula extends to $d = 1$ and $n > 2$ (and even with $K(x) = x\,\{\sum_{i=1}^{n} |x_i|^p\}^{-2/p}$), so that L^2 boundedness still implies uniform rectifiability in this case. He also showed that the positivity fails when $d > 1$ [**Fr1**].

In higher dimensions, much less is known. If E is Ahlfors-regular and all kernels of the form $K(z) = \varphi(|z|)K_{n,d}(z)$ that satisfy (12) and (13) define bounded operators on $L^2(\mu)$, then E is uniformly rectifiable. This relies on a density theorem of Mattila and Preiss (which they also used to get the result on principal values quoted below). In codimension 1, it is also enough to consider a class of radial kernels (with a different cancellation property than (13), though). See [**DS4**], pages 48-49 for details.

One cannot go too far in this direction. The case of the Riesz kernels is unknown, except for non-integer dimensions and Ahlfors-regular measures, for which M. Vihtilä [**Vh**] showed that they cannot define L^2-bounded operators. But there are nontrivial kernels K that define bounded operators on the Cantor set of dimension 1 in the plane known as Garnett-Ivanov's example [**Da9**]. Also see [**C**].

A slightly different issue is the existence of principal values $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ for almost every x, when E, K, and the T_ε are as above.

If K defines a bounded operator and principal values exist almost-everywhere for a dense class of functions f, then they also exist almost-everywhere for every $f \in L^2$; but for instance the L^2-boundedness alone for a given K does not imply the existence of principal values almost-everywhere (in the main example of [**Da9**], there is no principal value for $f = 1$). One can still hope that in the special case of the Riesz kernels $K_{n,d}$, the situation is different and one can deduce the existence of principal values from the L^2-boundedness. This would give a way to prove the uniform rectifiability of E when $K_{n,d}$ defines a bounded operator.

It is slightly easier to start from the existence of principal values almost-everywhere, because one can use it to find tangent measures to μ for which the operators T_ε vanish. Mattila and Preiss [**MP**] showed that if $E \subset \mathbb{R}^n$ is measurable for H^d and $H^d(E) < +\infty$, then E is rectifiable if and only if the lower density $\liminf_{r\to 0} r^{-d} H^d(E \cap B(x,r))$ is positive for H^d-almost every $x \in E$, and the principal value $\lim_{\varepsilon\to 0} \int_{E\setminus B(x,\varepsilon)} K_{n,d}(x-y)\, dH^d(y)$ exists for H^d-almost every $x \in E$. The case of the Cauchy kernel in the plane was known before [**Ma2**]. Also see [**Hu1**] for a similar result with slightly different kernels.

Here again, not every kernel works perfectly; P. Huovinen [**Hu2**] constructed Ahlfors-regular sets of dimension 1 in the plane that are totally non rectifiable (i.e., such that $H^1(E \cap \Gamma) = 0$ for every curve Γ of finite length), and yet for which certain kernels (like $K(z) = \mathrm{Re}(z)/|z|^2 - \mathrm{Re}(z)^3/|z|^4$) give principal values almost-everywhere on E. These sets E also have subsets of positive measure where the kernels K define bounded operators.

We should mention that even for the question of L^2-boundedness, Ahlfors-regularity is not a completely natural assumption. It is true that the upper bound in (1) follows from the boundedness on $L^2(d\mu)$ of the Riesz kernel (or any sufficiently nondegenerate collection of kernels); see for instance[**Da4**], page 56. But there is no reason a priori to require the lower bound in (1). X. Tolsa [**To1**], and then independently Nazarov, Treil, and Volberg [**NTV1**] characterized the measures μ on the plane such that the Cauchy kernel defines a bounded operator on $L^2(d\mu)$, in terms of total Menger curvature. Also see [**Ve2**] for a very nice proof.

We refer to the surveys [**Ma5, Ma6, Ma7**] for additional information on these issues. Also see [**MaV, NTV2, To5, To6, To7, Vo**] for more recent developments or surveys.

4. Big pieces of better sets

The notion of big pieces was initially used to extend results of L^2-boundedness from one class of sets to another. Let $\mathcal{F}$ be a class of Ahlfors-regular sets. A typical example would be the class of (images under rotations of $\mathbb{R}^n$) of Lipschitz graphs with a given constant. We say that the Ahlfors-regular set E has big pieces of sets in $\mathcal{F}$ when there is a constant $\theta > 0$ such that, for each $x \in E$ and each $r < \mathrm{diam}\,(E)$, we can find $F \in \mathcal{F}$ such that $H^d(E \cap F \cap B(x,r)) \geq \theta r^d$. [Here we decided to measure the size of $E \cap F \cap B(x,r)$ with Hausdorff measure, but we could also have used an Ahlfors-regular measure as in (1).]

Proposition 16. *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfy (12), and let $\mathcal{F}$ be a class of Ahlfors-regular sets. Suppose that there is a constant M such that for each $F \in \mathcal{F}$, there is a measure μ supported on F, such that (1) holds with $C_1 = M$ and (15) holds with $C_5 = M$. Then K defines a bounded operator on $L^2(E)$ for every Ahlfors-regular set E that has big pieces of sets in $\mathcal{F}$.*

Actually, one only needs $j = 0$ and $j = 1$ in (12); the cancellation property (13) is not needed in the statement, but it is often useful to verify that the measures μ satisfy the assumption (15). The proof of Proposition 16 also gives bounds on the constant C_5 in the analogue of (15) for the restriction of H^d to E, which depend only on n, d, C_4, M, the Ahlfors-regularity constant for E, and θ.

To the author's best knowledge, the first time an argument like Proposition 16 showed up was in a paper of Coifman and Meyer, where the authors deduced the L^2-boundedness of the Cauchy integral on chord-arc curves with small constants from the same result on small Lipschitz graphs [**CM**]. The method was systematized in [**Da1**] and further work. For a proof of Proposition 16, see Proposition 3.2 on page 60 of [**Da4**].

The proof of Proposition 16 uses standard real variable techniques, and is based on the "good λ inequalities" of Burkholder and Gundy. Even though this is now old and well known, let us state a good λ inequalities lemma, just to make sure that the reader knows about them.

Lemma 17. *Let $(X, \mathcal{A}, \mu)$ be a measured space, and let $1 \leq p < +\infty$ be given. Let $u : X \to \mathbb{R}^+$ be measurable, and assume that u coincides, out of a set of finite measure, with a function of $L^p(d\mu)$. [But require no bounds.] Also assume that we can find a nonnegative function $v \in L^p(d\mu)$ and constants $\nu \in (0,1)$, $\alpha \in (0,1)$, $\gamma > 0$, and $C_0 \geq 0$ such that $\alpha < (1-\nu)^{-1/p} - 1$ and*

$$\begin{aligned}&\mu(\{x \in X\,;\, u(x) > (1+\alpha)\lambda\}) \\ &\quad \leq (1-\nu)\,\mu(\{x \in X\,;\, u(x) > \lambda\}) + C_0\mu(\{x \in X\,;\, v(x) > \gamma\lambda\})\end{aligned} \tag{18}$$

for every $\lambda > 0$. Then $u \in L^p(d\mu)$, and

$$||u||_p \leq C(\nu, \alpha)\gamma^{-1} C_0^{1/p}\, ||v||_p. \tag{19}$$

The proof of Lemma 17 is straightforward. See for instance [**Da1**]. Notice that (18) is implied by

$$\begin{aligned}&\mu(\{x \in X\,;\, u(x) > (1+\alpha)\lambda \text{ and } v(x) \leq \gamma\lambda\}) \\ &\quad \leq (1-\nu)\mu(\{x \in X\,;\, u(x) > \lambda\}),\end{aligned} \tag{20}$$

which is often as easy to prove.

Most of the time (including for Proposition 16) Lemma 17 is applied in the following way. We want to control a function u, typically the maximal operator $T_* f(x) = \sup_{\varepsilon > 0} T_\varepsilon f(x)$ applied to a test function f (to ensure the qualitative assumption in the lemma), and we take for v a mixture of maximal functions of u that we control. We want to prove (20), so we fix $\lambda > 0$ and we cover the open set $\{u(x) > \lambda\}$ by essentially disjoint cubes or balls Q_j (typically, Whitney cubes). A small accounting argument shows that it is enough to prove that

$$\mu(\{x \in Q_j\,;\, u(x) > (1+\alpha)\lambda\}) \leq \tau\mu(Q_j) \tag{21}$$

for every j such that Q_j contains at least a point ξ such that $v(\xi) \leq \gamma\lambda$, and where $\tau < 1$ is a constant. Here it is interesting to take τ very close to 1; the price to pay is that we have to take ν in (18) and (20) very small and then α very small too, but this is usually not a problem because we can take γ as small as we wish, depending on all these constants.

Notice that (in our typical situation) there is a point $z \in CQ_j$ such that $u(z) \leq \lambda$, for instance because Q_j is a Whitney cube. Then we cut u in two, a faraway piece that varies little and a local piece. For instance, we can take $u_{far} = T_*(\mathbf{1}_{\mathbb{R}^n \setminus 2CQ_j} f)$ as the faraway piece, and show that $u_{far}(x) \leq u(z) + Cv(\xi) \leq \lambda + C\gamma\lambda$ for $x \in Q_j$.

Such estimates typically use the regularity of the kernel (i.e., (12)) and maximal functions.

Then we would show that the local part $u_{loc}(x)$ is less than $C\gamma\lambda$ on some small but substantial piece A_j of Q_j, so that $u(x) \leq u_{far}(x)+u_{loc}(x) \leq \lambda+C\gamma\lambda \leq (1+\alpha)\lambda$ on A_j (if γ is small enough), and hence (21) holds.

In the case of Proposition 16, for instance, our control on the local part would come from the assumption that there is a big piece of $E \cap Q_j$ in some nice set $F \in \mathcal{F}$, where we control the L^2 integral of the local part by definition of $\mathcal{F}$.

In the true proof of Proposition 16, we also use a small trick: we bilinearize the operators. That is, we define our operators as integrating against a first measure μ, but we compute L^2 norms with a second measure $\widetilde{\mu}$. This makes things easier, because we can modify μ or $\widetilde{\mu}$ alone.

We do not wish to elaborate too much on that proof, but let us still say that it is one more example of situations where asking for some apparently small control on a set or a function, but at all scales and locations, yields a much better control than expected. A very well known example of this principle is the theorem of John and Nirenberg, where we take a BMO function with small norm (a priori, only integrable on each interval) and show that its exponential is locally integrable. Lemma 17 is a very nice tool for this sort of argument.

Nothing prevents us from applying Proposition 16 many times, to get larger and larger classes of sets where singular integral operators are bounded on L^2. This is what was originally done to prove that kernels K as in (12) and (13) define bounded operators on $L^2(E)$ when E is an ω-regular surface, by showing that E contains big pieces of (sets that contain big pieces of)$^{d-1}$ Lipschitz graphs. Amusingly, we now know that the process stops after two iterations. Indeed, it happens that ω-regular surfaces contain big pieces of Lipschitz graphs (see [**Da3**] or [**Jo2**]), so uniformly rectifiable sets contain big pieces of sets (namely, ω-regular surfaces) that contain big pieces of Lipschitz graphs. We do not need to iterate more, because Ahlfors-regular sets that contain big pieces of uniformly rectifiable sets are themselves uniformly rectifiable. One can use Proposition 16 and the characterization of L^2-boundedness of singular integrals in [**DS3**] to prove this, but there is also a geometric proof with corona decompositions in [**DS4**], Theorem IV.2.29.

Also, the two iterations are needed; already in dimension 1, Tomasz Hrycak found uniformly rectifiable sets that do not contain big pieces of Lipschitz graphs. In fact, if E is uniformly rectifiable of dimension d, E contains big pieces of Lipschitz graphs if and only if it has big projections [**DS2**]. This last means that there is a constant $\theta > 0$ such that, if $x \in E$ and $0 < r < \mathrm{diam}\,(E)$, we can find a d-plane P such that $H^d(\pi_P(E \cap B(x,r))) \geq \theta r^d$, where π_P is the orthogonal projection onto P. And Hrycak constructed uniformly rectifiable sets of dimension 1 (Ahlfors-regular sets contained in Ahlfors-regular curves) in the plane, which do not have big projections. His construction is based on venetian blinds; since it does not seem to be published anywhere, we give a rapid description in Exercise 9.

We complete this section with two results about big pieces. First we prove that Ahlfors-regular curves contain big pieces of Lipschitz graphs. Thus Proposition 16 allows us to deduce the L^2-boundedness of singular integral operators on Ahlfors-regular curves (and hence, trivially, on uniformly rectifiable sets) from the same result on Lipschitz graphs [**CMM**]. Then we shall (almost) show that Ahlfors-regular sets that contain big pieces of connected sets are uniformly rectifiable.

Proposition 22. *For each constant $C_1 \geq 1$, we can find $C_6 \geq 1$, with the following property. Let $z : I \to \mathbb{R}^n$ be a regular mapping, with constant C_1, and set $E = z(I)$. Then for each $x \in I$ and $0 < r < |I|$, we can find a C_6-Lipschitz graph Γ such that*

$$|z^{-1}(E \cap \Gamma \cap B(z(x), r))| \geq C_6^{-1} r. \tag{23}$$

Here we used the parametrization z to express everything, but since it is fairly easy to show that $|z^{-1}(A)|$ is equivalent to $H^1(A)$ for $A \subset E$ (see Exercise 5), our statement says that E contains big pieces of Lipschitz graphs. [By C_6-Lipschitz graph we mean the image under a rotation of $\mathbb{R}^n$ of the graph of a C_6-Lipschitz function $f : \mathbb{R} \to \mathbb{R}^{n-1}$.]

Let us prove the proposition. Let I, z, x, and r be given, and let J be an interval of length $l = (2C_1)^{-1} r$ starting or ending at x and contained in I. We shall actually find a Lipschitz graph Γ such that

$$|J \cap z^{-1}(E \cap \Gamma)| \geq C_6^{-1} r, \tag{24}$$

and this is a little stronger than (23) because (3) says that $z(J) \subset B(z(x), r)$.

First observe that $z(J)$ cannot be contained in $B(z(x), (2C_1)^{-1} l)$, because this would contradict (4). So we can find $x_1 \in J$ such that $|z(x_1) - z(x)| \geq (2C_1)^{-1} l$. Without loss of generality, we can assume that $x = 0$, $J = (0, l)$, $z(x) = 0$ and

$$z_1(x_1) \geq z_1(x) + (2C_1)^{-1} l = (2C_1)^{-1} l = (2C_1)^{-2} r, \tag{25}$$

where z_1 denotes the first coordinate of z. Set $J_1 = [0, x_1)$, $\tau = \frac{z_1(x_1)}{2x_1} \geq \frac{z_1(x_1)}{2l} \geq (4C_1)^{-1}$, and then

$$g(y) = z_1(y) - \tau y \quad \text{and} \quad h(y) = \sup_{0 \leq t \leq y} g(t) \quad \text{for } y \in [0, x_1]. \tag{26}$$

Thus h is the smallest nondecreasing function such that $g(y) \leq h(y)$ on $[0, x_1]$. The following description is known as the Rising Sun Lemma of F. Riesz; see for instance [Zy] for a proof, but hopefully a picture is already quite convincing.

First, g and h are both Lipschitz, and $0 \leq h'(y) \leq C_1 - \tau$ almost-everywhere. Set $H = \{y \in J_1 \,;\, h(y) = g(y)\}$. Then H is closed J_1, and $J_1 \setminus H$ is an at most countable union of disjoint open intervals $L_j = (a_j, b_j)$. Moreover, h is constant (and equal to $g(a_j)$) on each L_j. We want a lower bound on the size of H, so we write that

$$g(x_1) \leq h(x_1) = \int_0^{x_1} h'(y) dy = \int_H h'(y) dy \leq C_1 |H| \tag{27}$$

because $h(0) = 0$, $\int_{L_j} h'(y) dy = 0$ for every j, and $h'(y) \leq C_1$ almost-everywhere. Thus

$$|H| \geq C_1^{-1} g(x_1) = C_1^{-1}[z_1(x_1) - \tau x_1] = C_1^{-1} z_1(x_1)/2 \geq (2C_1)^{-3} r, \tag{28}$$

by definition of τ and (25). Now set $\varphi(y) = h(y) + \tau y$ on J_1; then φ is Lipschitz, with $\tau \leq \varphi'(y) \leq C_1$ almost-everywhere. We want to use this to check that $z(H)$ lies on a Lipschitz graph.

Set $J_2 = \varphi(J_1)$; thus φ is a biLipschitz mapping from J_1 to J_2. We define $f : J_2 \to \mathbb{R}^{n-1}$ by $f(t) = z_*(\varphi^{-1}(t))$, where z_* is composed of all the coordinates of z, except the first one. Obviously, f is Lipschitz (with constant $\tau^{-1} C_1 \leq 4C_1^2$), and we claim that $z(H) \subset \Gamma$, where Γ is the graph of f. This will prove (24) (with $C_6 = 8C_1^3$) because of (28), and Proposition 22 will follow.

So let $y \in H$ be given. Then $h(y) = g(y)$, hence $\varphi(y) = z_1(y)$ (by (26)) and $z(y) = (z_1(y), z_*(y)) = (z_1(y), f(z_1(y)))$ by definitions. The claim, and then Proposition 22, follow. We get $C_6 = 8C_1^3$, but we would get a cleaner control on the constants by demanding that z be 1-Lipschitz instead of (3), parametrizing $z(E)$ by arclength, and getting rid of some of the powers of C_1 above. □

If z is ω-regular, then $z(\mathbb{R}^d)$ contains big pieces of sets that contain big pieces of sets ... that contain big pieces of Lipschitz graphs, with at most d iterations. This fact is technically more delicate to prove when $d > 1$, but the proof is based on the same principle (and a version of the Rising Sun Lemma with $d-1$ parameters).

But we also know that $z(\mathbb{R}^d)$ directly contains big pieces of Lipschitz graphs. To get this, one extracts big pieces of biLipschitz mappings from given Lipschitz mappings with big images, and the argument is different. See [**Da3**] or [**Jo2**].

Proposition 29. *Let E be an Ahlfors-regular set of dimension 1 in $\mathbb{R}^n$, which contains big pieces of connected sets. That is, there is a constant C_7 such that, for each $x \in E$ and $0 < r < \mathrm{diam}\,(E)$, we can find a connected set $\Gamma_{x,r}$ such that $H^1(\Gamma_{x,r}) \leq C_7 r$ and $H^1(\Gamma_{x,r}) \cap E \cap B(x,r) \geq C_7^{-1} r$. Then E is uniformly rectifiable.*

If I recall well, Proposition 29 was first known with a fairly indirect combination of results. Then P. Jones suggested that there should be a direct construction of the Ahlfors-regular curve that contains E, but apparently he did not publish his proof. The proof that is outlined below first appeared as Proposition 3.35 in [**DS7**] (an unexpected place, since the rest of the paper deals with higher dimensional sets). And the second part of the argument follows a suggestion of J.-M. Morel and S. Solimini (See [**MoS**], Proposition 16.25 p.210 for a similar argument).

We shall only give an outline of the proof of Proposition 29 (and even, only for bounded sets); for additional details, we refer to [**DS7**] and to Chapter 31 of [**Da11**].

The proof of Proposition 29 decomposes neatly into two parts. Here is the first one.

Lemma 30. *If E is as in the proposition, then for $x_0 \in E$ and $0 < r_0 < \mathrm{diam}\,(E)$, we can find a connected set $\Gamma = \Gamma(x_0, r_0)$ such that $E \cap B(x_0, r_0) \subset \Gamma$ and $H^1(\Gamma) \leq C_8 r_0$.*

The proof of Lemma 30 is the same as in [**DS7**] and Chapter 31 of [**Da11**]. As we shall see, the point is to add more and more little connected sets $\Gamma_{x,r}$ until we cover $E \cap B(x_0, r_0)$. Let $x_0 \in E$ and $r_0 < \mathrm{diam}\,(E)$ be given. We shall construct a nondecreasing sequence of compact connected sets Γ_j that contain more and more of the set

$$E_0 = E \cap B(x_0, r_0). \tag{31}$$

Set $\Gamma_0 = \Gamma_{x_0, r_0}$. Thus

$$H^1(\Gamma_0) \leq C_7\, r_0 \tag{32}$$

and

$$H^1(E \cap \Gamma_0 \cap B(x_0, r_0)) \geq C_7^{-1}\, r_0. \tag{33}$$

We can modify Γ_0 (without changing (32) and (33)), so that $\Gamma_0 \subset \overline{B}(x_0, r_0)$. Indeed, let $\pi : \mathbb{R}^n \to \overline{B}(x_0, r_0)$ be defined by

$$\begin{cases} \pi(z) = z, & \text{for} z \in \overline{B}(x_0, r_0) \\ \pi(z) = x_0 + r_0 \frac{z - x_0}{|z - x_0|}, & \text{for} z \in \mathbb{R}^n \setminus \overline{B}(x_0, r_0). \end{cases} \tag{34}$$

It is easy to see that π is $1-$Lipschitz, and so $\pi(\Gamma_0)$ still satisfies (32). Of course $\pi(\Gamma_0)$ is still connected, (33) stays the same, and now $\pi(\Gamma_0) \subset \overline{B}(x_0, r_0)$.

We want to continue our construction by induction, so let us assume that $j \geq 0$ is given and that we already have a nondecreasing (finite) sequence of compact connected sets $\Gamma_0, \ldots, \Gamma_j$, and let us define Γ_{j+1}. Set

$$U_j = E_0 \setminus \Gamma_j = E \cap B(x_0, r_0) \setminus \Gamma_j \tag{35}$$

and, for each $x \in U_j$,

$$\delta_j(x) = \operatorname{dist}(x, \Gamma_j) \quad \text{and} \quad B_j(x) = B(x, \delta_j(x)). \tag{36}$$

Note that

$$\delta_j(x) \leq 2r_0 < 2 \operatorname{diam}(E) \tag{37}$$

because $x \in B(x_0, r_0)$, $\Gamma_0 \cap B(x_0, r_0) \neq \emptyset$ by (33), and hence $\Gamma_j \cap B(x_0, r_0) \neq \emptyset$ too.

By the usual "5-covering argument" (see the first pages of [**St**]), we can find a finite or countable set $A_j \subset U_j$ such that

$$\text{the balls } \overline{B}_j(x), \ \ x \in A_j, \ \text{ are disjoint} \tag{38}$$

and

$$U_j \subset \bigcup_{x \in A_j} \overline{B}(x, 5\delta_j(x)). \tag{39}$$

In particular,

$$H^1(U_j) \leq \sum_{x \in A_j} H^1(E \cap \overline{B}(x, 5\delta_j(x))) \leq C \sum_{x \in A_j} \delta_j(x), \tag{40}$$

because E is Ahlfors-regular. Since $\delta_j(x) < 2 \operatorname{diam}(E)$, we can apply the hypothesis of Proposition 29 to each each $x \in A_j$, with the radius $\frac{1}{2}\,\delta_j(x)$. We get a compact connected set $\gamma_j(x) = \Gamma_{x, \delta_j(x)/2}$ such that

$$H^1(\gamma_j(x)) \leq \frac{1}{2}\, C_7\, \delta_j(x) \tag{41}$$

and

$$H^1(E \cap \gamma_j(x) \cap B_j(x)) \geq \frac{1}{2}\, C_7^{-1}\, \delta_j(x). \tag{42}$$

Also, by the same argument as for Γ_0, we can always replace $\gamma_j(x)$ with its image by a contraction onto $\overline{B}\big(x, \frac{1}{2}\,\delta_j(x)\big)$ (like π in (34)), and so we may assume that $\gamma_j(x) \subset B_j(x)$.

Call $z_j(x)$ a point of $\Gamma_j \cap \partial B_j(x)$. Such a point exists, by (36). Add to $\gamma_j(x)$ a line segment that connects it to $z_j(x)$. We get a new compact connected set $\alpha_j(x)$ such that

$$H^1(\alpha_j(x)) \leq C\, \delta_j(x), \tag{43}$$

$$H^1(E \cap \alpha_j(x)) \geq (2C_7)^{-1}\, \delta_j(x), \tag{44}$$

and

$$\alpha_j(x) \subset B_j(x), \text{ except for the point } z_j(x) \in \Gamma_j \cap \partial B_j(x). \tag{45}$$

Note that

$$\text{the sets } \alpha_j(x)\,,\ x \in A_j, \text{ are disjoint,} \tag{46}$$

by (45) and (38). Also, if we set

$$V_j = E \cap B(x_0, 3r_0) \setminus \Gamma_j, \tag{47}$$

then

$$E \cap \alpha_j(x) \subset V_j \cup \{z_j(x)\}. \tag{48}$$

This comes from (45) and (36), plus (37) and the fact that $x \in A_j \subset U_j \subset E_0 \subset B(x_0, r_0)$ for the part about $B(x_0, 3r_0)$. Now

$$\sum_{x \in A_j} \delta_j(x) \leq 2C_7 \sum_{x \in A_j} H^1(E \cap \alpha_j(x)) \leq 2C_7\, H^1(V_j) < +\infty \tag{49}$$

by (44), (46), (48), and the Ahlfors-regularity of E.

Since $\sum_{x \in A_j} \delta_j(x) < +\infty$, we can find a finite set $A'_j \subset A_j$ such that

$$\sum_{x \in A'_j} \delta_j(x) \geq \frac{1}{2} \sum_{x \in A_j} \delta_j(x). \tag{50}$$

Set

$$\Gamma_{j+1} = \Gamma_j \cup \Big(\bigcup_{x \in A'_j} \alpha_j(x) \Big). \tag{51}$$

By construction, Γ_{j+1} is still compact (because A'_j is finite) and connected (because each $\alpha_j(x)$ is connected and touches the connected set Γ_j).

Our construction of sets Γ_j is now finished. We want to take

$$\Gamma = \Big\{ \bigcup_{j \geq 0} \Gamma_j \Big\}^- \tag{52}$$

but let us first see why the process has a chance of converging. Note that for $x \in A'_j$, $\alpha_j(x)$ is contained in Γ_{j+1} (by (51)). Hence it does not meet V_{j+1} (see (47)), and so

$$E \cap \alpha_j(x) \subset (V_j \setminus V_{j+1}) \cup \{z_j(x)\}, \tag{53}$$

by (48). Since all these sets are disjoint by (46), we get that

$$H^1(V_j \setminus V_{j+1}) \geq \sum_{x \in A'_j} H^1(E \cap \alpha_j(x)) \geq (2C_7)^{-1} \sum_{x \in A'_j} \delta_j(x), \tag{54}$$

by (44). Note that

$$\sum_{j \geq 0} H^1(V_j \setminus V_{j+1}) \leq H^1(V_0) \leq H^1(E \cap B(x_0, 3r_0)) \leq Cr_0 \tag{55}$$

by (47) and because E is Ahlfors-regular. Hence

$$\sum_{j \geq 0} \sum_{x \in A'_j} \delta_j(x) \leq Cr_0 < +\infty, \tag{56}$$

by (54) and (55). Now we can verify that

$$E_0 = E \cap B(x_0, r_0) \text{ is contained in } \Gamma, \tag{57}$$

where Γ is defined by (52).

Let $z \in E_0$ be given. If $z \in \Gamma_j$ for some $j \geq 0$, there is nothing to prove (because $\Gamma_j \subset \Gamma$). So we can assume that $z \in U_j = E_0 \setminus \Gamma_j$ for each j. (See the definition (35).) Then $z \in \overline{B}(x, 5\delta_j(x))$ for some $x \in A_j$, by (39), and so

$$\operatorname{dist}(z, \Gamma) \leq \operatorname{dist}(z, \Gamma_j) \leq |z - x| + \operatorname{dist}(x, \Gamma_j) \leq |z - x| + \delta_j(x) \leq 6\delta_j(x), \tag{58}$$

by (36). Note also that

$$\delta_j(x) \leq \sum_{y \in A_j} \delta_j(y) \leq 2 \sum_{y \in A'_j} \delta_j(y) \tag{59}$$

for $x \in A_j$, by (50). The right-hand side of (59) tends to 0 when j tends to $+\infty$, by (56). Hence $\operatorname{dist}(z, \Gamma) = 0$, by (58) and (59), and $z \in \Gamma$ (which is closed by definition). This proves (57).

Next we want to check that

$$H^1(\Gamma) \leq C\, r_0. \tag{60}$$

Set $\Gamma^* = \bigcup_{j \geq 0} \Gamma_j$. We already know that

$$H^1(\Gamma^*) \leq H^1(\Gamma_0) + \sum_{j \geq 0} \sum_{x \in A'_j} H^1(\alpha_j(x)) \leq C\, r_0 + C \sum_{j \geq 0} \sum_{x \in A'_j} \delta_j(x) \leq C\, r_0 \tag{61}$$

by (51), (32), (43) and (56). So it is enough to control $\Gamma \setminus \Gamma^*$.

Let $z \in \Gamma \setminus \Gamma^*$ be given, and let $\{z_\ell\}$ be a sequence in Γ^* which converges to z. If infinitely many z_ℓ lie in a same Γ_j, then $z \in \Gamma_j$ as well. This is impossible, since $z \notin \Gamma^*$. Thus for ℓ large enough, $z_\ell \in \alpha_{j(\ell)}(x_\ell)$ for some index $j(\ell)$ that tends to $+\infty$ and some $x_\ell \in A_{j(\ell)}$. By construction, $x_\ell \in A_{j(\ell)} \subset U_{j(\ell)} \subset E_0$ (see (35) and the discussion that follows it). Since $\alpha_{j(\ell)}(x_\ell) \subset \overline{B}_{j(\ell)}(x_\ell)$ by (45), we get that

$$\operatorname{dist}(z_\ell, E_0) \leq \operatorname{dist}(z_\ell, x_\ell) \leq \delta_{j(\ell)}(x_\ell) \leq 2 \sum_{x \in A'_{j(\ell)}} \delta_{j(\ell)}(x), \tag{62}$$

by (50). The right-hand side of (62) tends to 0 by (56), hence $\operatorname{dist}(z_\ell, E_0)$ tends to 0 and $z \in E \cap \overline{B}(x_0, r_0)$. Thus $\Gamma \setminus \Gamma^* \subset E \cap \overline{B}(x_0, r_0)$, and

$$H^1(\Gamma) \leq H^1(\Gamma^*) + H^1(E \cap \overline{B}(x_0, r_0)) \leq C\, r_0, \tag{63}$$

by (61) and because E is Ahlfors-regular.

Our compact set Γ is connected, because each Γ_j is. Let us rapidly check this. Let $\Gamma = F_1 \cup F_2$ be a decomposition of Γ into disjoint closed subsets. Then for each j, $\Gamma_j = (\Gamma_j \cap F_1) \cup (\Gamma_j \cap F_2)$ is a similar decomposition of Γ_j. Since Γ_j is connected, $\Gamma_j \cap F_1 = \emptyset$ or $\Gamma_j \cap F_2 = \emptyset$. If $\Gamma_j \cap F_1 = \emptyset$ for all j, then $F_1 = \Gamma \cap F_1$ is also empty, because all the Γ_j lie in F_2, which lies at positive distance from F_1. If $\Gamma_{j_0} \cap F_1 \neq \emptyset$ for some j_0, then $\Gamma_j \cap F_1 \neq \emptyset$ for all $j \geq j_0$, and hence $\Gamma_j \cap F_2 = \emptyset$ for all $j \geq j_0$. In this case F_2 is empty (by the same argument as above). Our partition of Γ was trivial, and Γ is connected. With (57) and (60), this completes our proof of Lemma 30. □

For the second part of the proof of Proposition 29, we start with the case when E is bounded, and we have two main options. The first one is to follow the proof of Lemma 30 (on a ball that contains E) carefully, and build our set Γ so that it is also Ahlfors-regular. The point is that when we construct the $\alpha_j(x)$, we should make sure that we do not add lots of curves or line segments that go trough a given little ball. That is, when we feel like adding lots of curves that go through a ball, we should group them into a single one, either by hand or by forcing the curves to lie on a given network of edges of dyadic cubes. This is probably what P. Jones suggested, and the argument would be similar to the proof of Theorem 3.14 below (that is, Proposition II.1.3 in [**DS4**]). To my knowledge, this is not written anywhere.

Here we shall describe an automatic way to improve the connected set given by Lemma 30, which is based on a variational argument and was suggested by J.-M. Morel and S. Solimini. To say the truth, the argument is probably a little longer than the direct route altogether, but it is a pleasant illustration of a general principle that can be useful: a nice way to produce a set with good properties at all scales and locations is to minimize a functional.

Actually, we shall give two versions of the same argument. In both cases, we first assume that E is bounded.

We start with a proof that works nicely when $n = 2$, but uses Golab's theorem. Golab's theorem says that if Γ_k is a sequence of compact connected sets in $\mathbb{R}^n$ that converges to a compact set Γ (for the Hausdorff distance), then $H^1(\Gamma) \leq \liminf_{k\to+\infty} H^1(\Gamma_k)$. See for instance [**MoS**] or [**Da11**] for a proof. Note that if we forget the connectedness assumption, the result fails miserably [see Exercise 11]. But there is a very nice sufficient condition that can replace connectedness and also works for higher dimensional Hausdorff measure: the "uniform concentration property" introduced in [**DMS**]. See also [**MoS**], or even [**Da11**] and [**Da10**].

Now let E be bounded and satisfy the hypotheses of Proposition 29. Apply Lemma 30 with any $x_0 \in E$ and $r_0 = 2\,\mathrm{diam}\,(E)$; the proof still works in this case (even though r_0 is a little larger than $\mathrm{diam}\,(E)$), and anyway you could also do a trivial covering with a bounded number of pieces). This gives a compact connected set Γ that contains E and such that $H^1(\Gamma) \leq C\,\mathrm{diam}\,(E)$.

Now we can choose Γ so that $H^1(\Gamma)$ is minimal. Indeed, let Γ_k be a minimizing sequence. That is, Γ_k is connected, $E \subset \Gamma_k$ and $H^1(\Gamma_k)$ tends to

$$m = \inf\{H^1(\Gamma)\,;\, \Gamma \text{ is compact, connected, and contains } E\}. \tag{64}$$

Then we can extract a subsequence for which Γ_k converges to a limit Γ, and this limit is connected. One has to check this (see Exercise 12), but the verification is not too hard; the simplest way for us to do it would be to use (uniformly) Lipschitz parametrizations of the Γ_k by a fixed interval (see below the proof of Lemma 66), and extract a convergent subsequence of the parametrization. Now Γ contains E because $E \subset \Gamma_k$ for each k, and $H^1(\Gamma) = m$, by Golab's theorem and as needed.

We claim that Γ is automatically an Ahlfors-regular set. If not, there is a ball $B_0 = B(x_0, r_0)$ such that $H^1(\Gamma \cap B_0) \geq C_9 r_0$, where the constant C_9 will be chosen soon. Now Lemma 30 (applied to any point $x \in E \cap B_0$, if any, and the radius $3r_0$) says that there is a compact connected set G that contains $E \cap B_0$, and such that $H^1(G) \leq 2C_8 r_0$. Set $\Gamma' = [\Gamma \setminus B_0] \cup \partial B_0 \cup G \cup L$, where L is a closed line segment that connects G to ∂B_0. Then Γ' is compact and connected, and it contains E because G covers $E \cap B_0$ just as well as $\Gamma \cap B_0$ did. Moreover,

$$\begin{aligned} H^1(\Gamma') &\leq H^1(\Gamma) - H^1(\Gamma \cap B_0) + 2\pi r_0 + H^1(G) \\ &\leq H^1(\Gamma) - C_9 r_0 + 2\pi r_0 + 2C_8 r_0 \\ &< H^1(\Gamma) \end{aligned} \tag{65}$$

if we take C_9 large enough. This is impossible because $H^1(\Gamma)$ is minimal, and hence Γ is Ahlfors-regular.

To complete our proof of Proposition 29 when $n = 2$ and E is compact, we just need the following.

Lemma 66. *If Γ is a compact connected Ahlfors-regular set of dimension 1, we can find a compact interval I and an Ahlfors-regular mapping $z : I \to \mathbb{R}^n$ such that $z(I) = \Gamma$.*

We only sketch the proof; more details on the (standard) graph construction can be found in Section 30 of [**Da11**].

Let Γ be as in the lemma, and let $\delta < \operatorname{diam}(\Gamma)$ be given. Let A_δ be a maximal set of points of Γ, subject to the constraint that $|x-y| \geq \delta$ for $x, y \in A_\delta$, $x \neq y$. We construct a graph G_δ by connecting all the pairs x, y of A_δ such that $|x-y| \leq 3\delta$.

First observe that G_δ is connected, because otherwise we can find a partition $A_\delta = B_1 \cup B_2$ of A_δ into nontrivial subsets, with $\operatorname{dist}(B_1, B_2) \geq 3\delta$. Since every point of Γ lies within δ of A_δ (by maximality of A_δ), we get a partition of Γ into closed sets $\Gamma_j = \{x \in \Gamma\,;\, \operatorname{dist}(x, B_j) \leq \delta\}$, $j = 1, 2$, and this contradicts the connectedness of Γ.

Then notice that $H^1(\Gamma \cap B(y, \delta/2)) \geq \delta/2$ for $y \in A_\delta$. Indeed, call π the radial projection $x \to |y-x|$. Then $\pi(\Gamma)$ contains 0 (because $y \in \Gamma$) and some points larger than $\delta/2$ (because $\delta < \operatorname{diam}(E)$); hence $\pi(\Gamma)$ contains $[0, \delta/2]$, because Γ is connected. The conclusion follows, because $H^1(\Gamma \cap B(y, \delta/2)) \geq H^1(\pi(\Gamma \cap B(y, \delta/2))) = \delta/2$ (since π is 1-Lipschitz). Now, if $x \in \mathbb{R}^n$ and $r > \delta$,

$$
\begin{aligned}
\sharp(A_\delta \cap B(x,r)) &\leq \sum_{y \in A_\delta \cap B(x,r)} \frac{2}{\delta}\, H^1(\Gamma \cap B(y, \delta/2)) \\
&\leq \frac{2}{\delta}\, H^1(\Gamma \cap B(x, r+\delta)) \qquad (67) \\
&\leq \frac{C}{\delta} r
\end{aligned}
$$

because the $B(y, \delta/2)$ are disjoint (by definition of A_δ) and Γ is Ahlfors-regular.

The advantage of using graphs is that they are easier to parametrize. Indeed, it is classical and easy that there is way to find a path that runs along each of the intervals that compose G_δ exactly twice, once in each direction. Let us parametrize this path by arclength; this gives a function $z_\delta : I_\delta \to G_\delta$ that is onto and 2-to-1 (except at edges and points where two intervals or more meet). It is easy to see that z_δ is Ahlfors-regular, with estimates that do not depend on δ. Indeed the Lipschitz condition is obvious, and we can estimate $|z_\delta^{-1}(B(x,r)|$ using (67) and the fact that there are never more than C edges of G_δ that meet a ball of radius δ.

Finally observe that $C^{-1} \operatorname{diam}(E) \leq |I_\delta| = 2\text{length}(G_\delta) \leq C \operatorname{diam}(E)$, by (67) and because $\operatorname{diam}(G_\delta) \geq \operatorname{diam}(A_\delta) \geq \operatorname{diam}(E)/2$ (if $\delta < \operatorname{diam}(E)/2$, say). Thus we can replace the z_δ with other Ahlfors-regular parametrizations z_δ^* defined on a fixed interval I of length diam E. And then we can find a sequence $\{\delta_k\}$ that tends to 0 such that the $z_{\delta_k}^*$ converge to a limit z, uniformly on I. It is easy to check that z is Ahlfors-regular, and that $z(I) = \Gamma$. This proves the lemma. □

So we have a proof of Proposition 29 when $n = 2$ and E is bounded. But the reader may now be upset because we used Golab's theorem, and we had to parametrize sets anyway. The first objection is not that serious, because we could use the argument above in a more constructive way. That is, we could start from a curve Γ_0 that contains E, then look for a ball B_0 such that $H^1(\Gamma_0) - C_9\, r(B_0)$ is (positive and) largest, do the same surgery as above to replace Γ_0 with a shorter curve Γ_1 that contains E, and iterate. The process converges (because we win at least r_k in length whenever we do a modification in a ball of radius r_k), and the limit is the desired Ahlfors-regular curve that contains E.

But anyway, let us try to work directly with parametrizations and avoid Golab's theorem. Our argument will be even more sketchy than before, and we refer to Section 31 in [**Da11**] for details about the proof.

We start the argument as before. That is, we assume that E is bounded, apply Lemma 30 to any $x_0 \in E$ and $r_0 = 2\,\mathrm{diam}\,(E)$, and get a compact connected set Γ that contains E and such that $H^1(\Gamma) \leq C\,\mathrm{diam}\,(E)$. But this time we take a parametrization of Γ immediately. A small modification of the proof of Lemma 66 gives a Lipschitz function $f : [0,1] \to \mathbb{R}^n$ such that $f([0,1])$ contains E, and with Lipschitz constant at most $C\,\mathrm{diam}\,(E)$. See Section 30 of [**Da11**], for instance.

Since the set of functions on $[0,1]$ with a Lipschitz constant at most M and $f(0) = 0$ is compact (for the topology of uniform convergence), we can even find f such that $f([0,1])$ contains E and the Lipschitz constant of f is minimal. Another way to say the same thing is that the length of the path defined by f is minimal. [Compare with Exercise 4.]

The rest of the proof consists in checking that f is an Ahlfors-regular parametrization. Here is the idea. Suppose not. Then there is a ball $B_0 = B(x_0, r_0)$, centered on $f([0,1])$, and such that $|f^{-1}(B_0)| \geq C_9 r_0$, with C_9 very large. On the other hand, Lemma 30 says that we can cover $E \cap B(x_0, 2r_0)$ with a connected set G of length at most Cr_0. The idea is to replace the curves that compose $f([0,1]) \cap B(x_0, 2r_0)$ with another set of curves, with strictly smaller total length, and that contains G (so that we no longer need $f([0,1]) \cap B(x_0, 2r_0)$ to cover. Once we do this, we get the desired contradiction with the definition of f.

The construction is a little painful, because $f([0,1]) \cap B(x_0, 2r_0)$ may be composed of a very large number of small curves and we have to match ends rather carefully to get a single curve at the end. The general idea is to leave alone the curves that do not go as far in as B_0, and to remove the other ones. This way we know that we remove a total length of at least Nr_0, where N is the number of curves. But we have a collection of free ends, and we have to connect them to each other (by pairs), to get a single curve. One of the points of the argument is that if N is large (the most interesting case), then we can use the fact that there are often lots of ends that are close to each other, and if we do the connections efficiently, this will cost much less than Nr_0 in length. Hopefully the reader will find it more pleasant to believe that we can construct a curve that is better than $f([0,1])$, instead of checking all details.

Our sketch of proof is over for bounded sets E. If E is unbounded, the argument sketched above says that if $x_0 \in E$, then for each $k \geq 0$, we can find an Ahlfors-regular mapping $f_k : [0, 2^k] \to \mathbb{R}^n$, with uniformly bounded Ahlfors-regularity constants, such that $E \cap \overline{B}(x_0, 2^k) \subset f_k([0, 2^k])$. Then it is not too hard to find a single Ahlfors-regular curve that covers E. Think about what you would do in the case when E is the union of the two axes in $\mathbb{R}^2$, or see [**DS4**] for the details. □

LECTURE 2

Approximation by d-planes, geometric lemmas

Here we shall discuss ways to measure how rectifiable a set is, by way of functions defined on the set of balls. To the author's knowledge, the first time such a thing was done was when P. Jones proved the "geometric lemma" for Lipschitz graphs, and the point was to get a new proof of L^2-boundedness for the Cauchy integral on those graphs. Since then, many variants of the geometric lemma have been proved, with applications.

The most standard way to measure rectifiability is in terms of closeness to affine d-planes, as follows. Let the integer dimension $d < n$ be fixed, and let E be a closed set in $\mathbb{R}^n$. Call $\mathcal{P}_d$ the set of affine planes of dimension d and set

$$\beta_E(x,r) = \frac{1}{r} \inf_{P \in \mathcal{P}_d} \big\{ \sup \{ \mathrm{dist}\,(y,P)\,;\, y \in E \cap B(x,r) \} \big\} \tag{1}$$

for $x \in \mathbb{R}^n$ and $r > 0$. When $E \cap B(x,r)$ is empty, we set $\beta_E(x,r) = 0$.

We divided by r to make $\beta_E(x,r)$ dimensionless. Note that $\beta_{\lambda E}(x,r) = \beta_E(x/\lambda, r/\lambda)$ and $\beta_E(x,r) \leq 1$ in all cases. In some cases it is more convenient to replace the *sup* with an L^q norm and set

$$\beta_{E,q}(x,r) = \frac{1}{r} \inf_{P \in \mathcal{P}_d} \Big\{ r^{-d} \int_{E \cap B(x,r)} \mathrm{dist}\,(y,P)^q \, d\mu(y) \Big\}^{\frac{1}{q}} \tag{2}$$

for $x \in \mathbb{R}^n$ and $r > 0$, and where (unless otherwise specified) μ is the restriction of H^d to E. As we shall see, there are interesting connections between geometric or analytic properties of E and the size of the $\beta_{E,q}(x,r)$.

The geometric lemma for Ahlfors-regular sets

1. The geometric lemma for Ahlfors-regular sets

We start with the slightly simpler case of d-dimensional Ahlfors-regular sets. In this case it will be convenient to restrict our attention to $x \in E$ in (1) and (2); also note that then the integral in (2) is essentially an L^q-average; thus $\beta_{E,p}(x,r) \leq C\beta_{E,q}(x,r)$ for $p \leq q \leq +\infty$, and where we naturally set $\beta_{E,\infty}(x,r) = \beta_E(x,r)$. We start with a regularity result for one-dimensional uniformly rectifiable sets.

Theorem 3 (**[Jo3, Ok]**). *If E is an Ahlfors-regular curve in $\mathbb{R}^n$, there is a constant C such that*

$$\int_{y \in E \cap B(x,r)} \int_{0 < t \leq r} \beta_E(y,t)^2 \, \frac{dH^1(y)\, dt}{t} \leq Cr \quad \text{for } x \in E \text{ and } r > 0. \tag{4}$$

The reader may be more familiar with a discretized version of (4), where one sums quantities like $\beta_E(Q)^2 \, \mathrm{diam}\,(Q)$ over dyadic cubes. Here we prefer to stress the importance of the almost-invariant measure $\frac{dH^1(y)\,dt}{t}$, possibly at the expanse of computability. Translations of estimates like (4) from the discrete version to the continuous one (and back) are fairly easy to do anyway.

In fact, P. Jones first proved Theorem 3 for Lipschitz graphs, and used it to get another proof of L^2-boundedness for the Cauchy kernel [**Jo1**]. The case of Ahlfors-regular curves is more complicated; the difficulty is that we cannot immediately reduce to a more standard problem of approximation of Lipschitz functions by affine functions, because of self-intersections. Actually, Jones (when $n = 2$) and Okikiolu (for any n) directly worked for general sets and proved Theorem 9 below; we stated Theorem 3 mostly to prepare the reader to the next result.

Observe that we also have the analogue of (4) with the numbers $\beta_{E,q}(x,r)^2$, since $\beta_{E,q}(x,r) \leq C\beta_E(x,r)$. Also, (4) stays true for uniformly rectifiable sets (and even for any subset of an Ahlfors-regular curve), because $\beta_E(x,r) \leq \beta_F(x,r)$ when $E \subset F$.

Theorem 3 generalizes to higher dimensions, but we have to be slightly careful with the Sobolev exponents, and the $\beta_{E,q}(x,r)$ are more appropriate.

Theorem 5 ([**DS3**], page 12)**.** *Let $E \subset \mathbb{R}^n$ be uniformly rectifiable, and let $1 \leq q < \frac{2d}{d-2}$ be given. Then there is a constant C such that*

$$(6) \qquad \int_{y\in E\cap B(x,r)} \int_{0<t\leq r} \beta_{E,q}(y,t)^2 \, \frac{dH^d(y)dt}{t} \leq Cr^d \quad \text{for } x \in E \text{ and } r > 0.$$

When $d = 1$, we can take $q = +\infty$ (i.e., use β_E), by Theorem 3. [But the proof is different.] Note that for Ahlfors-regular sets, $\beta_{E,p}(y,t) \leq C\beta_{E,q}(y,t)$ for $p \leq q$, by Hölder, so (6) is stronger when q is large.

Xiang Fang (probably in [**Fg**]) showed that (6) may fail for $q \geq \frac{2d}{d-2}$, and the point is that when there is no Sobolev inequality, one may construct counterexamples with lots of little spikes.

The proof of (6) is based on a corona decomposition of E (see Section 4), which allows one to reduce to the case of Lipschitz graphs with small constant. For Lipschitz graphs, (6) is a result of approximation of Lipschitz functions by affine functions, which is due to Dorronsoro [**Do**]. It is not so easy to prove, but at least it is directly in the spirit of Littlewood-Paley theory. We shall rapidly return to this result of Dorronsoro near the end of Section 3 (when we discuss its extension to Lipschitz functions on uniformly rectifiable sets, see near (3.22).

When E is a Lipschitz graph and $q = 2$, (6) follows from a rather easy computation based on Plancherel. We shall not do this computation here, but it is similar to the derivation of upper bounds for triple integrals of Menger curvature, in Lemma 5.9.

In this context, the square power in (4) and (6) comes naturally from orthogonality in Plancherel; in geometric proofs it comes directly from Pythagoras. Note that (6) says that $J_{E,q}(y) = \int_0^1 \beta_{E,q}(y,t)^2 \frac{dt}{t} < +\infty$ for almost every y, but even when $q = +\infty$, we cannot say that E has a tangent d-plane at y as soon as $J_{E,q}(y) < +\infty$. [This would rather require the convergence of $\int_0^1 \beta_{E,q}(y,t) \frac{dt}{t}$; the square power allows for slowly turning spirals centered at y.] Nevertheless, we shall see soon that (6) implies uniform rectifiability (and hence the existence of a tangent d-plane almost-everywhere).

The conditions (4) and (6) can be seen as a Carleson measure estimate on the $\beta_{E,p}$. In fact, it really helps to see the results of this section as analogues for sets of some aspects of Littlewood-Paley theory for functions. [I think S. Semmes was the first one to take this point of view seriously.] We all know that it is sometimes useful, in order to study functions f on $\mathbb{R}^d$, to consider extensions of f to the upper half-space $\mathbb{R}^{d+1}_+$ (for instance, the harmonic extension, but not always). And we want to do something like this for sets.

Here the appropriate analogue of $\mathbb{R}^{d+1}_+$ is the set of balls

$$\mathcal{A} = \{(x,r)\,;\, x \in E \text{ and } 0 < r < \operatorname{diam} E\}. \tag{7}$$

The natural measure on $\mathcal{A}$ is the essentially scale-invariant (hence infinite) measure $dH^d(x)\frac{dr}{r}$; thus (4) and (6) really say that the $\beta_{E,q}(x,r)$ are small most of the time.

We call Carleson measure on $\mathcal{A}$ a measure ν such that $\nu(B(x,r)\times(0,r)) \leq Cr^d$ for $x \in E$ and $0 < r < \operatorname{diam}(E)$. Thus (6) says that $\beta_{E,q}(x,r)^2\,\frac{dH^d(x)dr}{r}$ is a Carleson measure, and similarly for (4).

Perhaps we should say that there is an additional difficulty with Littlewood-Paley theory with sets: it is not so easy to decompose sets into standard little pieces. The reason why we like so much the corona construction (rapidly described below), is that at least it provides a way to linearize some questions, and reduce to the case of small Lipschitz graphs.

Here is a converse to Theorems 3 and 5. We shall see in Section 3 that much more can be done in this direction.

Theorem 8 ([**DS3**]). *Let $E \subset \mathbb{R}^n$ be Ahlfors-regular of dimension $d \geq 1$, and suppose that there are constants $q \geq 1$ and $C \geq 0$ such that (6) holds. Then E is uniformly rectifiable.*

There are lots of other ways to measure how rectifiable E is; one can ask whether it is almost symmetric with respect to most of its points, or almost convex, and so on. See Section 3 for a little more about this, and [**DS4**] for a lot more.

2. The traveling salesman

We start with one-dimensional sets, and the following question. Given a bounded set $E \subset \mathbb{R}^n$, when is it possible to find a curve with finite length that contains E?

Theorem 9 ([**Jo2, Ok**]). *Set*

$$\beta_{tot}(E) = \int_{x\in\mathbb{R}^n}\int_0^\infty \beta_E(x,r)^2\,\frac{dx\,dr}{r^n}, \tag{10}$$

for $E \subset \mathbb{R}^n$. Then there is a curve Γ of finite length that contains E (in its image) if and only if E is bounded and $\beta_{tot}(E) < +\infty$. Moreover, there is a constant $C = C(n)$ such that

$$\begin{aligned} C^{-1}[\operatorname{diam}(E) + \beta_{tot}(E)] &\leq \inf\big\{\text{length}(\Gamma)\,;\,\Gamma \textit{ is a curve that contains } E\big\} \\ &\leq C[\operatorname{diam}(E) + \beta_{tot}(E)] \end{aligned} \tag{11}$$

for all Borel sets $E \subset \mathbb{R}^n$.

P. Jones proved the existence of a curve Γ when $\beta_{tot}(E)$ is finite, but was only able to establish the converse (i.e., estimates on $\beta_{tot}(E)$ when E is a rectifiable curve) when $n = 2$; the proof even used conformal mappings! The extension to higher dimensions (with a more natural proof) is due to K. Okikiolu.

The reader should not worry about the contribution of radii $r > \operatorname{diam}(E)$ to the integral in (10); it is finite because $\beta_E(x,r) = 0$ when $E \cap B(x,r) = \emptyset$, so we only need to care about x when $\operatorname{dist}(x,E) \leq r$, and then $\beta_E(x,r) \leq r^{-1}\operatorname{diam}(E)$.

It is fairly easy to check that Theorem 9 implies Theorem 3. The different powers of r come from the fact that we now integrate on $\mathbb{R}^n$, while in Theorem 3 we integrated on E only. In the case of Ahlfors-regular sets (as in Theorem 3), the two are equivalent.

Theorem 9 has surprising applications. For instance, Jones and Bishop were able to use it to prove conjectures about harmonic measure in simply connected domains [**BJ1, BJ2**]! Also see [**BJ3**] for a use of $\beta_E(x,r)$ in the context of limit sets of groups.

Let us discuss a few variants or improvements of Theorem 9. We start with a localization. For $E \subset \mathbb{R}^n$ and $x \in E$, set

$$J_E(x) = \int_0^1 \beta_E(x,t)^2 \frac{dt}{t}. \tag{12}$$

Theorem 13 ([**BJ2**])**.** *There is a constant C such that if E is a compact set of diameter 1 and $J_E(x) \leq M$ for every $x \in E$, then there is a curve of length at most Ce^{CM} that contains E in its image.*

Theorem 13 is only stated for subsets of the plane, but the proof works in any dimension n, with $C = C(n)$ (or else I missed something important). The exponential growth is optimal.

Theorem 9 is not so easy to extend to higher dimensions. There is an extension to $d = 2$ in [**Pa1, Pa2**], where fairly reasonable sufficient conditions are given, in terms of the $\beta_E(x,r)$ and densities, which allow to say that E is contained in a set with a nice parametrization. The restriction to dimension 2 is due to the explicit way surfaces are constructed (with triangles), and some estimates get bad when the density of E is too small, because even if the $\beta_E(x,r)$ are very small, we do not get a good control on the way the best d-planes depend on x and r.

Then there are recent results by Lerman [**Le2**] (in dimension 1), and then Jones and Lerman (all dimensions) that allow to say that some substantial part of a set is contained in a uniformly rectifiable set. The advantage is that the result works in all dimensions, and with measures (with potentially low densities); the price to pay is that only a part of the support is controlled.

Let us try to say a little more. We need some notation. Let the integer dimensions d and n be given, and let μ be a locally finite Borel measure on $\mathbb{R}^n$. For each cube $Q \subset \mathbb{R}^n$ (with sides parallel to the axes), set

$$\beta(Q) = \frac{1}{\operatorname{diam}(Q)} \inf_{P \in \mathcal{P}_d} \left\{ \int_Q \operatorname{dist}(y,P)^2 \frac{d\mu(y)}{\mu(Q)} \right\}^{1/2}, \tag{14}$$

where the infimum is still taken over all affine d-planes P, and then

$$J(x) = \sum_{k \in \mathbb{Z}} \sup \big\{ \beta(Q)^2 \,;\, Q \text{ is a cube of side length } 2^{-k} \text{ that contains } x \big\}. \tag{15}$$

To be exact, Jones and Lerman use less cubes than this (only dyadic cubes, and a small number of translations of dyadic cubes) and this probably makes a big difference in computational applications, but we let us not bother here.

Also call $J_{Q_0}(x)$, where Q_0 is a given cube (with sides parallel to the axes) the analogue of $J(x)$, but where we only consider cubes $Q \subset Q_0$.

Theorem 16 ([**JL**]). *There exist constants C_1 (an absolute constant) and C_2 (that depends only on n and d) such that if μ is a locally finite Borel measure, Q_0 is a cube, and if*

$$\int_{C_1 Q_0} e^{C_2 J_{Q_0}(x)} d\mu(x) \leq A\mu(Q_0) \tag{17}$$

(for some $A > 0$), then there is an ω-regular surface Γ, with constants less than $C(A, n, d)$ such that

$$\mu(\Gamma) \geq C_2^{-1} A^{-1} \mu(Q_0). \tag{18}$$

Even though this only gives a big piece of uniformly rectifiable set in the support of μ, it is surprising that one can give such a precise result. Here the difficulty was not so much about finding parametrizations of sets, but really constructing a coherent set Γ from μ, especially at places where μ is so tenuous (or placed near a low-dimensional subset) that it does not give good hints on the direction of the tangent plane to Γ. The general idea of the construction is still to proceed from large scales down, like in the corona constructions, but there are lots of tricks (and the choice of L^2 norms in (14) helps).

Hopefully this result will be useful, for instance in data analysis (where it could help approximate and describe large sets of points which tend to accumulate on a d-dimensional subset; note that for such applications, using measures instead of sets makes sense). Already see Le1.

As was mentioned above, the numbers $\beta_E(x, r)$ are very convenient, but they are not the only ones that can give a good description of the rectifiability properties of E. See the next section for some possibilities, and Section 5 for Menger curvature.

See [**Ha1, Ha2, Sc1, Sc2**] for more recent versions of the Jones-Okikiolu theorem in the context of metric and Hilbert spaces.

LECTURE 3

Weaker characterizations of uniform rectifiability

In practice it may be hard to check that a given set E is contained in an ω-regular surface, and the verification of the Carleson condition (2.4) on the $\beta_E(x,r)^2$ is even worse. In this section we shall mention a few apparently weaker conditions that imply uniform rectifiability.

We start with two or three conditions relative to big pieces of sets, but we shall rapidly concentrate on weak variants of the geometric lemma. Then we shall give slightly more exotic conditions.

1. More big pieces of sets

We already saw that Ahlfors-regular sets of dimension 1 that contain big pieces of connected sets are uniformly rectifiable. In fact, we have the following necessary and sufficient conditions in any dimension.

Theorem 1 ([**DS3**]). *Let E be an Ahlfors-regular set of dimension d in $\mathbb{R}^n$. The following properties are equivalent:*

(2) *E is uniformly rectifiable (i.e., is contained in an ω-regular surface),*

(3) *E contains big pieces of Lipschitz images of subsets of $\mathbb{R}^d$,*

(4) *E contains very big pieces of biLipschitz images of $\mathbb{R}^d$ inside $\mathbb{R}^{\mathrm{Max}\,(n,2d+1)}$.*

Here (3) means that there exist constants $\theta > 0$ and $M \geq 1$ such that for $x \in E$ and $0 < r < \mathrm{diam}\,(E)$ we can find an M-Lipschitz mapping f defined on the ball $B(0,r)$ in $\mathbb{R}^d$, with values in $\mathbb{R}^n$, and such that $H^d(E \cap f(B(0,r))) \geq \theta r^d$.

The stronger condition (4) means that given $\varepsilon > 0$ we can find $M \geq 1$ such that for $x \in E$ and $0 < r < \mathrm{diam}\,(E)$ there is a M-biLipschitz mapping $\varphi : \mathbb{R}^d \to \mathbb{R}^{\mathrm{Max}\,(n,2d+1)}$ such that $H^d(E \cap B(x,r) \setminus \varphi(\mathbb{R}^d)) \leq \varepsilon r^d$. When $d = 1$, $\varphi(\mathbb{R}^d)$ is a chord-arc curve, and (4) says that E contains very big pieces of chord-arc curves.

2. The bilateral weak geometric lemma and variants

We now turn to "weak conditions", where we find some way to say that E is nice in a given ball $B(x,r)$, and require that this property holds for "Carleson-almost" every $B(x,r)$.

Definition 5. *Let E be an Ahlfors-regular set of dimension d. A Carleson set of balls is a set $\mathcal{B}$ of pairs (y,t), with $y \in E$ and $0 < t < \mathrm{diam}\,(E)$, such that*

$$\int_{y \in E \cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}}(y,t)\, dH^d(y) \frac{dt}{t} \leq C r^d \tag{6}$$

for $x \in E$ and $0 < r < \mathrm{diam}\,(E)$.

In other words, $\mathbf{1}_{\mathcal{B}}(y,t)dH^d(y)\frac{dt}{t}$ is a Carleson measure on $E \times (0, \mathrm{diam}\,(E))$. We could indifferently have used a measure μ such that (1) holds, instead of the restriction of H^d to E.

Here is a first example of a weak condition. We say that the Ahlfors-regular set E satisfies the weak geometric lemma (in short, $E \in WGL$) if for each $\varepsilon > 0$, the set $\mathcal{B}(\varepsilon) = \{(y,t)\,;\, y \in E, 0 < t < \mathrm{diam}\,(E), \text{ and } \beta_E(y,t) > \varepsilon\}$ is a Carleson set.

If E satisfies the geometric lemma (2.6) for any $q \geq 1$, then $E \in WGL$. For $q = +\infty$, this comes directly from Chebyshev. For other q, this uses the fact that $\beta_E(y,t) \leq C\beta_{E,q}(y,2t)^{q/d+q}$, which is fairly easy to establish. See for instance [**DS4**], page 27, or Exercise 15. Now $\mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t) \leq C\varepsilon^{-2(d+q)/q}\beta_{E,q}(y,2t)^2$, and (6) follows from (2.6).

"Obviously", the WGL does dot imply (2.6), even when $d = 1$; see Section 20 of [**DS3**] for a counterexample. Let us now introduce stronger variants of the WGL. We start with the following bilateral variant of the β numbers. Set

$$(7) \qquad b\beta_E(x,r) = \frac{1}{r}\inf_{P \in \mathcal{P}_d}\left\{\sup_{y \in E \cap B(x,r)}\{\mathrm{dist}\,(y,P)\} + \sup_{y \in P \cap B(x,r)}\{\mathrm{dist}\,(y,E)\}\right\}$$

for $x \in E$ and $0 < r < \mathrm{diam}\,(E)$. Thus we care about how flat E is in $B(x,r)$ but also whether it has significant holes there.

The numbers $b\beta_E(x,r)$ are not really new, since they appear in Reifenberg's theorem [**Re**] that says that under suitable conditions, the set E is a topological disk of dimension d, and even has a bi-hölder parametrization, provided that the $b\beta_E(x,r)$ stay uniformly small.

The estimate (2.4) also holds with β_E replaced with $b\beta_E$, and similarly (2.6) would hold with appropriate bilateral L^q variants of the $\beta_{E,q}$. If the author recalls correctly, the proof of (2.6) just extends.

We say that E satisfies the bilateral weak geometric lemma (in short, $E \in BWGL$) if for each $\varepsilon > 0$,

$$b\mathcal{B}(\varepsilon) = \{(y,t)\,;\, y \in E, 0 < t < \mathrm{diam}\,(E), \text{ and } b\beta_E(y,t) > \varepsilon\}$$

is a Carleson set.

But there are other ways to measure how close to a d-plane E is locally. For instance, we could measure whether $E \cap B(x,r)$ is approximately convex, with

$$(8) \qquad cv(x,r) = \frac{1}{r}\sup\Big\{\mathrm{dist}\,(\frac{y+z}{2}, E)\,;\, y,z \in E \cap B(x,r)\Big\},$$

or approximately symmetric, with

$$(9) \qquad sym(x,r) = \frac{1}{r}\sup\Big\{\mathrm{dist}\,(2y-z, E)\,;\, y,z \in E \cap B(x,r)\Big\},$$

where $x \in E$ and $0 < r < \mathrm{diam}\,(E)$ are given. As before, when E is uniformly rectifiable one can prove analogues of (2.6) for the L^q versions of $cv(x,r)$ and $sym(x,r)$.

And we say that E is locally convex (in short, $E \in LCV$) if

$$(10) \qquad \Big\{(y,t)\,;\, y \in E, 0 < t < \mathrm{diam}\,(E), \text{ and } cv(y,t) > \varepsilon\Big\}$$

is a Carleson set for each $\varepsilon > 0$, and similarly that E is locally symmetric ($E \in LS$) if

$$(11) \qquad \Big\{(y,t)\,;\, y \in E, 0 < t < \mathrm{diam}\,(E), \text{ and } sym(y,t) > \varepsilon\Big\}$$

is a Carleson set for every $\varepsilon > 0$.

Theorem 12 ([**DS4**])**.** *The Ahlfors-regular set E is uniformly rectifiable if and only if $E \in BWGL$, if and only if $E \in LCV$, and if and only if $E \in LS$.*

The equivalence between $BWGL$, LCV, and LS is fairly easy; it is barely harder than the verification that an Ahlfors-regular set of dimension d that is convex, or symmetric with respect to each of its points, is a d-plane; besides this, the proof is a convincing illustration of the flexibility of our weak conditions. The fact that uniformly rectifiable sets (or sets for which singular integrals define bounded operators, as in Section 1) satisfy LS, for instance, is relatively easy. The surprising fact, in principle, is that our weak conditions (say, the $BWGL$) imply uniform rectifiability. And again the proof relies on a corona construction.

Let us also add that in the proof that $BWGL$, LCV, or LS implies uniform rectifiability, we do not really need to know that for every ε, $b\mathcal{B}(\varepsilon)$ or its analogue in (10) or (11) is a Carleson set. We just need one value of ε, which can be computed from n, d, and the Ahlfors-regularity constant for E (but which is so small that we may as well prove the result for all ε). The same thing happens with many of the weak characterizations mentioned below.

3. Weak connectedness in dimension 1

When $d = 1$, thanks to the special role of connectedness, the proof is somewhat easier. In fact, we can use the following property of weak connectedness.

Definition 13. *Let E be a one-dimensional Ahlfors-regular set in $\mathbb{R}^n$. For each $0 < \alpha < 1$ and $A \geq 1$, call $\mathcal{G}(\alpha, A)$ the set of pairs $\{(x,r) \in E \times (0, \operatorname{diam}(E))\}$ such that for all choices of y, $z \in E \cap B(x,r)$ such that $|y-z| \geq 10^{-1}r$, we can find a chain of points $v_j \in E \cap B(x, Ar)$, $0 \leq j \leq m = m(y,z)$, with $v_0 = y$, $v_m = z$, and $|v_j - v_{j-1}| \leq \alpha|y-z|$ for $1 \leq j \leq m$. We say that E is weakly connected (in short, $E \in WC$) is we can find $\alpha < 1$ and $A \geq 1$ such that $[E \times (0, \operatorname{diam}(E))] \setminus \mathcal{G}(\alpha, A)$ is a Carleson set.*

Theorem 14 ([**DS4**], page 70)**.** *Every weakly connected Ahlfors-regular set of dimension 1 is uniformly rectifiable.*

The converse is true, but not so interesting; notice that the bilateral weak geometric lemma, or the local convexity condition, clearly imply WC. Theorem 14 is so easy to prove that we can present an argument here. Note the similarity with Proposition 2.29 on big pieces of connected sets.

So let us see how to prove Theorem 14. As for Proposition 2.29, it will be enough to show that for $x \in E$ and $0 < r < \operatorname{diam}(E)$, we can find a connected set Γ such that $H^1(\Gamma) \leq Cr$ and $E \cap B(x,r) \subset \Gamma$. Without loss of generality, we may assume that $x = 0$ and $r = 1$.

It will be more convenient to use a discretized version of $[E \times (0, \operatorname{diam}(E))] \setminus \mathcal{G}(\alpha, A)$. Call $A^* \geq A$ a reasonably large constant (to be chosen soon). For $k \geq 1$, call X_k the set of $x \in E \cap B(0, A^*)$ such that we can find y, $z \in E \cap B(x, 2^{-k})$ such that $|y - z| \geq 5^{-1}2^{-k}$, for which there is no chain $\{v_j\}$, $0 \leq j \leq m$, in $E \cap B(x, A2^{-k+1})$ with $v_0 = y$, $v_m = z$, and $|v_j - v_{j-1}| \leq \alpha|y-z|$ for $1 \leq j \leq m$. Further choose inside X_k a maximal set Y_k of points with mutual distances at least 2^{-k-1}.

Note that if $x \in X_k$, then all the pairs (w,t) with $w \in E \cap B(x, 2^{-k})$ and $2^{-k} \leq t < 2^{-k+1}$ lie in $[E \times (0, \operatorname{diam}(E))] \setminus \mathcal{G}(\alpha, A)$. Then

$$
\begin{aligned}
&\sum_{k\geq 1} 2^{-k}\, \sharp(Y_k) \\
&\quad \leq C \sum_{k\geq 1} \sum_{x\in Y_k} H^1(E\cap B(x,2^{-k-2})) \\
&\quad \leq C \sum_{k\geq 1} \sum_{x\in Y_k} \int_{w\in E\cap B(x,2^{-k-2})} \int_{2^{-k}\leq t\leq 2^{-k+1}} dH^1(w) \frac{dt}{t} \\
&\quad \leq \int_{w\in E\cap B(0,2A^*)} \int_{0<t<1} \mathbf{1}_{[E\times(0,\operatorname{diam}(E))]\setminus\mathcal{G}(\alpha,A)}(w,t)\, dH^1(w) \frac{dt}{t} \\
&\quad \leq C
\end{aligned}
\tag{15}
$$

because all these sets are disjoint and $[E \times (0, \operatorname{diam}(E))] \setminus \mathcal{G}(\alpha, A)$ is a Carleson set.

For each $x \in Y_k$, call $G(x)$ a compact connected set of length $C2^{-k}$ that contains a collection of points of $E \cap B(x, 2^{-k+2})$ that is 2^{-k-3}-dense in $E \cap B(x, 2^{-k+2})$. Then set $G = \cup_{k\geq 1} \cup_{x\in Y_k} G(x)$, and $\Gamma = G \cup [E \cap \overline{B}(0, A^*)]$. Obviously

$$
\begin{aligned}
H^1(\Gamma) &\leq H^1(E\cap \overline{B}(0,A^*)) + H^1(G) \leq C + \sum_{k\geq 1}\sum_{x\in Y_k} H^1(G(x)) \\
&\leq C + C\sum_{k\geq 1} 2^{-k}\, \sharp(Y_k) \leq C,
\end{aligned}
\tag{16}
$$

by (15). Also, Γ is closed, because $E\cap\overline{B}(0, A^*)$ and each $G(x)$ is compact, and then if z is the limit of a sequence of points $\{y_j\}$, with $y_j \in G(x_j)$ for some $x_j \in Y_{k(j)}$ and with $k(j)$ unbounded, then $\operatorname{dist}(z, E \cap \overline{B}(0, A^*)) \leq \liminf_{j\to+\infty} \operatorname{dist}(y_j, E \cap \overline{B}(0, A^*)) \leq \liminf_{j\to+\infty} |y_j - x_j| = 0$, because $k(j)$ is unbounded, so that $z \in E \cap \overline{B}(0, A^*)$.

So it is enough to check that the connected component of 0 in Γ contains $E\cap B(0, 1)$. Or that given a point $z \in E\cap B(0, 1)$ and an $\varepsilon > 0$, we can find a finite chain $\{y_j\}$ in Γ, with $y_0 = 0$ and $y_m = z$, so that $|y_j - y_{j-1}| \leq \varepsilon$ for $1 \leq j \leq m$. In fact, we shall directly get a combination of ε-chains and curves contained in G, which is just as good. And the construction gives a sufficient control on the combination in question to allow us to extract a curve in Γ from 0 to z, by Montel.

We get our combinations of chains and curves recursively, starting from the large scales and going to smaller ones little by little. The point is the following. Let z_1 and z_2 be two points of $E\cap B(0, A^*)$; these could be successive points of one of our chains, and we want to find a finer combination of chains and curves from z_1 to z_2. One possibility is that we can find a sequence $\{y_j\}$ in E, as in the definition of WC, and in particular so that $|y_j - y_{j-1}| \leq \alpha|z_1 - z_2|$. In this case we are happy, we can replace the link from z_1 to z_2 by the finer chain given by the $\{y_j\}$.

If we cannot find the sequence $\{y_j\}$, we choose k so that $2^{-k} \leq |z_1 - z_2| \leq 2^{-k+1}$, and then we know that $z_1 \in X_k$. Thus can find $x \in Y_k$ such that z_1 and z_2 both lie in $B(x, 2^{-k+1})$. In this case, we can connect z_1 to z_2 by a small jump from z_1 to some point of $E \cap G(x)$, followed by an arc on $G(x)$ (which we can actually keep unchanged till the end of the construction), and then another small jump from some other point of $E \cap G(x)$ to z_2. When we follow the construction suggested here, we never go further than $CA(1+\alpha+\alpha^2+\dots)$, which is all right if we choose A^* large enough.

The reader may consult [**DS4**] for more details about the construction. The proof is slightly different there because we already had constructed dyadic cubes on E (so we used them), and also because one constructed an Ahlfors-regular curve directly (by being more careful about how we chose the various $G(x)$ so that not so many of them go through the same ball), rather than merely constructing a connected set as we did, and relying on a variational argument to choose the best one. □

4. Weak approximation by some classes of sets: other variants of the BWGL

There are other ways to measure how nice the set E is, at different scales and locations, and these ways sometimes lead to characterizations of uniform rectifiability. We do not want to bore the reader with a full list, so we shall only give a few hints about what is possible, and refer to [**DS4**] for further information.

First there are modifications of the $BWGL$ in the following vein. Consider a class $\mathcal{A}$ of closed sets in $\mathbb{R}^n$. For the $BWGL$, we would take the class of affine d-planes. Then define the analogue for $\mathcal{A}$ of the numbers $b\beta_E(x,r)$ by

$$(17)\qquad ap_{\mathcal{A}}(x,r) = \frac{1}{r}\inf_{A\in\mathcal{A}}\left\{\sup_{y\in E\cap B(x,r)}\{\mathrm{dist}\,(y,A)\} + \sup_{y\in A\cap B(x,r)}\{\mathrm{dist}\,(y,E)\}\right\}$$

for $x\in E$ and $0<r<\mathrm{diam}\,(E)$.

Let us call $Approx(\mathcal{A})$ the set of Ahlfors-regular sets E such that for each $\varepsilon>0$,

$$(18)\qquad \mathcal{B}_A(\varepsilon) = \big\{(x,r)\in E\times(0,\mathrm{diam}\,(E))\,;\, ap_{\mathcal{A}}(x,r)>\varepsilon\big\}$$

is a Carleson set. Thus $BWGL$ is the same as $Approx(\mathcal{A})$ when $\mathcal{A}$ is the class of affine d-planes, while WGL corresponds to the class of (closed) subsets of affine d-planes, and it is fairly easy to check that $LCV = Approx(\mathcal{A})$ when $\mathcal{A}$ is the class of convex closed sets.

When we take for $\mathcal{A}$ the class of closed sets that contain an affine d-planes, $Approx(\mathcal{A})$ is a class called $OUWLG$ (for other unilateral weak geometric lemma) in [**DS4**]. And every set in $Approx(\mathcal{A})$ is uniformly rectifiable. [See Propositions II.3.17 and 18 in [**DS4**].]

When $\mathcal{A}$ is the class of images under isometries of $\mathbb{R}^n$ of d-dimensional Lipschitz graphs with constant less than M (M given in advance), we get a class called $BALG$. The author's impression is that E is uniformly rectifiable when $E\in BALG$. This is Corollary II.4.10 in [**DS4**] when $d=n-1$, and I think this was supposed to follow from the results in [**DS10**] in higher codimensions, but unfortunately I do not recall how and this was probably never written. Please ask S. Semmes for details.

When $d=n-1$ and $\mathcal{A}$ is the class of closed sets $A\subset\mathbb{R}^n$ such that every connected component of $\mathbb{R}^n\setminus A$ is convex, we get a condition called weak exterior convexity (WEC), and which is also equivalent to uniform rectifiability (see Theorem I.2.18 in [**DS4**]). Also look for the $GWEC$ for a generalization of this condition to higher codimensions.

5. Characterizations by analysis on E ; approximation by affine functions

It is also tempting to try to characterize uniform rectifiability in terms of density properties of a measure on E. For instance, we could consider the quantities

$$(19) \qquad \delta_\mu(x,r) = \inf_{a>0} \sup \big\{ |\mu(B(y,t)) - at^d| \,;\, y \in E \cap B(x,r) \text{ and } 0 < t \le r \big\},$$

where $x \in E$, $0 < r < \mathrm{diam}\,(E)$, and μ is a measure on E such that (1.1) holds.

If $E = \mathbb{R}^d$ and $d\mu = b(x)dx$ for some bounded density b,

$$(20) \qquad \mathcal{B}_d(\varepsilon) = \big\{ (x,r) \in E \times (0, \mathrm{diam}\,(E)) \,;\, \delta_\mu(x,r) > \varepsilon \big\}$$

is a Carleson set for every $\varepsilon > 0$. This is reasonably easy to prove, for instance by Plancherel, and one can even get stronger square function estimates. This is still true if E is uniformly rectifiable and μ satisfies (1.1) (see Section 6 in [**DS3**] or Theorem 2.52 in [**DS4**]), but the converse is less clear.

We shall say that E satisfies the "slightly stronger weak density condition" if there is a measure μ supported on E, such that (1.1) holds and $\mathcal{B}_d(\varepsilon)$ is a Carleson set for every $\varepsilon > 0$. The condition is slightly stronger than the WCD in [**DS4**], where one is allowed to choose a different measure $\mu_{x,r}$ for each choice of x and r, provided that (1.1) stays true with a fixed constant C_0. The WCD implies uniform rectifiability when $d = 1$ or $n-1$, where it follows from a result of D. Preiss [**Pr**]; see Theorem I.2.56 in [**DS4**]. This result about WCD is related to the discussion about singular integrals that we had in Section 1, and in codimension 1 one can reduce slightly the class of kernels K that satisfy (1.12) and (1.13), and for which the L^2-boundedness of (every) associated singular integral on E implies the uniform rectifiability of E. See [**MP**] and Theorem I.2.59 in [**DS4**].

It is amusing to try to measure the regularity of E more indirectly, by looking at whether some standard theorems of analysis on a d-plane still hold on E. For instance, one could ask whether some square function estimates (out of E) on the Cauchy or Cauchy-Clifford integrals of functions defined of E still hold, as it happens for Lipschitz graphs. See Part III of [**DS4**] for some results of this type.

We want to say a few words about affine approximation of Lipschitz functions on E, because the statements are amusing (if rare) and there is a potential for application to (Ahlfors-regular) metric spaces.

First consider functions f from $\mathbb{R}^d$ to $\mathbb{R}$. For $0 \le M \le +\infty$, call $A(M)$ the set of affine functions a on $\mathbb{R}^d$ such that $|\nabla a| \le M$, and then set

$$(21) \qquad \gamma_{q,M}(x,r) = r^{-1} \inf_{a \in A(M)} \Big\{ r^{-d} \int_{B(x,r)} |f(y) - a(y)|^q dy \Big\}^{1/q}$$

for $x \in \mathbb{R}^d$, $r > 0$, $1 \le q \le +\infty$, and $0 \le M \le +\infty$ (with the obvious adaptation when $q = +\infty$).

A theorem of Dorronsoro [**Do**] says that if f is locally integrable on $\mathbb{R}^d$ and if

$$(22) \qquad 1 \le q < \frac{2d}{d-2} \text{ when } d > 1; \; 1 \le q \le +\infty \text{ when } d = 1,$$

the distributional gradient of f lies in L^2 if and only if

$$(23) \qquad \int_{\mathbb{R}^d} \int_0^1 \gamma_{q,+\infty}(x,r)^2 \, \frac{dxdt}{t} < +\infty.$$

Recall that this result is somewhat easier to prove when $q = 2$; in particular the direct estimate eventually reduces to a Fourier computation with Plancherel. [See Exercise 17.]

By an easy localization argument, we can deduce from this that if $f : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz and (22) holds, then

$$\int_{y \in B(x,r)} \int_{t \in (0,r)} \gamma_{q,+\infty}(y,t)^2 \, \frac{dydt}{t} \leq C r^d \, ||f||_{Lip}^2 \tag{24}$$

for $x \in \mathbb{R}^d$ and $r > 0$. We even have the same result with $\gamma_{q,+\infty}$ replaced with $\gamma_{q,M}$, where M is any bound for $||f||_{Lip}$.

Of course it is not a coincidence that the authorized exponents in (22) are the same as in the geometric lemma above, and indeed if f is a Lipschitz function, the numbers $\beta q, \Gamma$ for the graph Γ of f are equivalent to the $\gamma_{q,+\infty}$ for f. This is the easy part of the analogy between some of our results on uniform rectifiability for sets and Littlewood-Paley theory for functions.

Now consider Lipschitz functions on the uniformly rectifiable set E. It turns out that in this case the Carleson estimate (24) is still valid, and even

$$\int_{y \in E \cap B(x,r)} \int_{t \in (0,r)} \gamma_{q,M}(y,t)^2 \, \frac{dydt}{t} \leq C r^d \, ||f||_{Lip}^2 \,, \tag{25}$$

with a similar definition of $\gamma_{q,M}(x,r)$ (except that we integrate on $E \cap B(x,r)$ and with respect to H^d in (21)), and as long as we take $M \geq ||f||_{Lip}$. See Proposition 4.2 in [**DS4**].

Now we want to see whether such a result is characteristic of uniformly rectifiable sets. As often, let us consider a weaker property. We say that the d-dimensional Ahlfors-regular set satisfies the property of <u>weak approximation by Lipschitz functions</u> on E <u>by affine functions</u>, or more shortly that $E \in WALA$, if there is a constant $M \geq 1$ such that, for every Lipschitz function $f : E \to \mathbb{R}$ with $||f||_{Lip} \leq 1$, and every $\varepsilon > 0$,

$$\big\{(x,r) \in E \times (0, \mathrm{diam}\,(E))\,;\, \gamma_{+\infty,M}(x,r) > \varepsilon\big\} \text{ is a Carleson set.} \tag{26}$$

It is fairly easy to see that we could replace $\gamma_{+\infty,M}$ in (26) with any $\gamma_{q,M}$, because $\gamma_{+\infty,M}$ is largest, and dominated by a power of each given $\gamma_{q,M}$, so that changing the power q in (26) can be compensated by replacing ε with a power of ε. Thus (26) is weaker than (25). The situation in this respect is quite similar to what happened with the Geometric Lemma (2.6) and its weak counterpart, the WGL defined below (6).

The discussion concerning M is a little more subtle. We do not think that it would be reasonable to take $M = +\infty$ here. If E is a d-plane, then it is true that we can easily restrict to approximations of $f : E \to \mathbb{R}$ by affine functions on E with Lipschitz constants at most $||f||_{Lip}$. And these functions can be extended to functions that are defined and Lipschitz on $\mathbb{R}^n$ with the same constant. But in general, we do not have any reason to think that if $f : E \to \mathbb{R}$ is very close to an affine function in a ball, then this function is Lipschitz with constant less than $C||f||_{Lip}$, or even that it can be replaced with another affine function with similar bounds. And when this does not happen it seems legitimate to consider that f is not well approximated by affine functions (even though it is very close to one).

We would like to say that E is rectifiable as soon as $E \in WALA$, but unfortunately we can only prove it in dimension $d = 1$. See Theorem 2.49 in [**DS4**].

Maybe we should say that the $WALA$ is a nice way to say that it is not so easy to construct Lipschitz functions (because they must have all the rigidity hidden in the fact that they are so close to affine functions). In contrast, it is very easy to construct Lipschitz functions on a Cantor set, say. The point is that you can essentially decide independently about what the functions do at each scale, which gives loads of "fractal" Lipschitz functions. See Exercise 20.

The $WALA$ also has a very nice generalization, which we shall call the $GWALA$. Let E be Ahlfors-regular, and give yourself, for each $x \in E$ and $0 < r < \operatorname{diam}(E)$, a finite dimensional vector space $A_{x,r}$ of functions $a : E \to \mathbb{R}$. But assume that $N = \sup_{x,r} dim(A_{x,r})$ is finite.

For each Lipschitz function f, we can define the $\gamma_{q,M}(x,r)$ as we did in (1), except that we replace $A(M)$ with the set of functions $a \in A_{x,M}$ such that $||a||_{lip} \leq M$.

And we say that E satisfies the $GWALA$ (generalized $WALA$) when we can find M, N, and a collection of vector spaces $A_{x,r}$ of dimensions at most N, such that the analogue of (26) holds for all Lipschitz functions f with norm less than 1 and all $\varepsilon > 0$. [To be precise, the definition in [**DS4**] is a little different, because we wanted to avoid issues of measurability and allow other values of q, but the reader will probably be happy to ignore the difference for the moment.] It is still true that the $GWALA$ implies uniform rectifiability in dimension $d = 1$; see Theorem III.4.14 in [**DS4**]. And it would be nice to know what happens in general, in particular because the $GWALA$ can be stated nicely for an abstract Ahlfors-regular metric space (while the notion of affine functions depends on an embedding in $\mathbb{R}^n$). If the finite-dimensional vector space $A_{x,t}$ does not depend on x and r, we already get a nice condition, and perhaps the functions in A can be used as a special set of coordinates on E.

See [**DS4**], and in particular the table of acronyms for additional and even more exotic conditions.

LECTURE 4

Corona decompositions

The goal of this section is to give an idea of what typical arguments using the corona construction look like. Perhaps let us first say where the name comes from. All the constructions described below are strongly inspired of the stopping time argument that was invented by L. Carleson ([**Ca**], see also [**Ga**]) to prove the corona conjecture for bounded holomorphic functions in the disk.

We shall try to give an idea of the proof of uniform rectifiability for Ahlfors-regular sets that satisfy a geometric lemma (as in (2.6)), or a bilateral weak geometric lemma (as in Section 3, near (3.7)). But there are other arguments that use the same general scheme.

Since such arguments are based on stopping times, it is easier to use analogues on E of dyadic cubes. We shall do this, in particular because we shall not have to construct the cubes, but it should be said that there are other ways to do stopping time arguments when E is not Ahlfors-regular, either by working on standard dyadic cubes (not especially adapted to E), as in [**Le1, Le2**] and [**JL**] or [**To4**], or just on the set of pairs (x, t), as in [**Leg**].

1. Dyadic cubes on E

Let E be an Ahlfors-regular set of dimension d in $\mathbb{R}^n$, and let us first assume that E is unbounded. Then there is a family Δ of "dyadic cubes" on E, with the following properties. First, Δ is the disjoint union of subsets Δ_k, $k \in \mathbb{Z}$, and for each k

$$E \text{ is the disjoint union of the sets } Q, Q \in \Delta_k. \tag{1}$$

The cubes are correctly embedded, i.e.,

$$Q \subset R \text{ whenever } Q \in \Delta_k,\ R \in \Delta_l,\ k \leq l, \text{ and } Q \cap R \neq \emptyset. \tag{2}$$

Then there is the size condition

$$C^{-1} 2^k \leq \operatorname{diam}(Q) \leq C 2^k \text{ for } Q \in \Delta_k. \tag{3}$$

Also, each cube Q comes with a "center" c_Q such that

$$\operatorname{dist}(c_Q, E \setminus Q) \geq C^{-1} \operatorname{diam}(Q). \tag{4}$$

Finally, we have the "small boundary condition"

$$\begin{aligned} H^d(\{x \in Q; \operatorname{dist}(x, E \setminus Q) < \tau 2^k\}) + H^d(\{x \in E \setminus Q; \operatorname{dist}(x, Q) < \tau 2^k\}) \\ \leq C \tau^{1/C} 2^{kd} \end{aligned} \tag{5}$$

whenever $Q \in \Delta_k$ and $0 < \tau < 1$. The constant C in (3)-(5) depends on n, d, and the regularity constant for E.

When E is bounded, we have a family $\Delta = \cup_k \Delta_k$ as above, except that we restrict to $k \leq k_0$, where k_0 is such that $\operatorname{diam} E \leq 2^{k_0} \leq 2 \operatorname{diam} E$. We can also choose Δ so that Δ_{k_0} has only one element (i.e., $Q = E$).

There is a minor difference with the usual dyadic cubes in $\mathbb{R}^d$, because the number of children of a given cube $Q \in \Delta_k$ (i.e., of cubes $R \in \Delta_{k-1}$ that are contained in Q) may depend on Q. This number may even be equal to 1 in some cases. But it is always less than C, and altogether our family Δ is often as easy to use as dyadic cubes in $\mathbb{R}^d$.

See [**Da4**], page 86 and Appendix 1, for a construction of Δ, [**Ch2**] for an extension to spaces of homogeneous type, and even [**DM**] for a (slightly more complicated) variant where the measure is not doubling.

2. Construction of a corona region

Let E be Ahlfors-regular, and let us assume (for instance) that E satisfies a geometric lemma, as in (2.6), or a bilateral weak geometric lemma ($E \in BWGL$), as defined below (3.7). We want to prove that E is uniformly rectifiable.

Let us choose a set Δ of dyadic cubes on E, as above. The proof will use what is called a coronization in [**DS4**], i.e., a decomposition of Δ into stopping time regions. We shall not give a formal definition of coronizations here, but just show how to construct some.

Pick any cube $Q_0 \in \Delta$, and let us first see how to construct a "corona region" $\mathcal{S}$ under Q_0. First, we should give ourselves a good set of cubes $\mathcal{G}$, and its complement $\mathcal{B} = \Delta \setminus \mathcal{G}$. For the present argument, we choose a very small constant $\varepsilon > 0$ and a reasonably large geometric constant C_0, and call $\mathcal{G}$ the set of cubes $Q \in \Delta$ for which there is an affine d-plane P_Q such that

$$\text{dist}\,(x, P_Q) \le \varepsilon\,\text{diam}\,(Q) \text{ for } x \in E \cap B(c_Q, C_0\,\text{diam}\,(Q)), \tag{6}$$

where c_Q is the center of Q (but here any other point of Q would do). Of course, in this sort of argument, we should not choose $\mathcal{G}$ and $\mathcal{B}$ completely at random, because we shall need later a control on the size of $\mathcal{B}$ for the Carleson packing condition (15). If E satisfies the geometric lemma (2.6) (say, with $q = 1$), and for the choice of $\mathcal{B}$ above, this control will come from the fact that $E \in WGL$ (see below (3.6)); if $E \in BWGL$, it will follow more directly from the definitions. But let us not worry about this for the moment.

Actually, if $E \in BWGL$, let us even require that in addition to (6),

$$\text{dist}\,(x, E) \le \varepsilon\,\text{diam}\,(Q) \text{ for } x \in P_Q \cap B(c_Q, C_0\,\text{diam}\,(Q)), \tag{7}$$

i.e., we have a bilateral approximation of E by P_Q near Q.

In the construction below, we shall stop whenever we hit a cube $Q \in \mathcal{B}$. But we may also decide to stop for other reasons, that may depend on Q_0. For instance, if E satisfies the geometric lemma (2.6), let us decide to stop when Q and Q_0 lie in $\mathcal{G}$, but

$$\text{Angle}(P_Q, P_{Q_0}) \ge \delta \tag{8}$$

for some other small constant $\delta > \varepsilon$. [The precise definition of angle does not really matter.] In other arguments, other stopping reasons exist.

To construct $\mathcal{S} = \mathcal{S}(Q_0)$, we start from Q_0. If $Q_0 \in \mathcal{B}$, we stop and set $\mathcal{S} = \{Q_0\}$. Otherwise, we still put Q_0 in $\mathcal{S}$, but we continue and look at the children of Q_0. For those which lie in $\mathcal{B}$, we stop; this means that we include them in $\mathcal{S}$, but will not consider any of their descendants. If E satisfies the geometric lemma (2.6), we also stop for the cubes which lie in $\mathcal{G}$, but satisfy (8). We are left with some good children R of Q_0. For each R, we consider the children Q of R, put them all in $\mathcal{S}$, but stop at those which lie in $\mathcal{B}$ or satisfy (8) (if E satisfies (2.6)), and continue with the others. This means that we consider the children of these others, and proceed with them as before. Eventually, we get a description of the stopping-time region $\mathcal{S}$ by induction.

Let us call $Stop(\mathcal{S})$ the collection of cubes where we stopped, and $End(\mathcal{S})$ the set of points $x \in Q_0$ such that all cubes Q such that $x \in Q \subset Q_0$ lie in $\mathcal{S}$. Then

$$Q_0 = End(\mathcal{S}) \cup \Big(\bigcup_{Q \in Stop(\mathcal{S})} Q \Big), \tag{9}$$

and this union is disjoint.

So far we only played with definitions, but if we did things right we can already build something that will be useful. For instance, consider the case when E satisfies the geometric lemma (2.6), and we kept cubes that satisfy (6) and not (8). [The situation for the $BWGL$ is a little more complicated, and will be addressed later.] We claim that we can find Lipschitz graph $\Gamma = \Gamma(\mathcal{S})$ which approximates E fairly well at all the scale between $\mathrm{diam}\,(Q_0)$ and the size of the cubes of $Stop(\mathcal{S})$. That is, Γ is the graph of some $C\delta$-Lipschitz function $f : P_{Q_0} \to P_{Q_0}^{\perp}$, and

$$\mathrm{dist}\,(x, \Gamma) \leq \eta\, \mathrm{diam}\,(Q) \quad \text{whenever } Q \in \mathcal{S} \text{ and } x \in E \cap B(c_Q, 2\,\mathrm{diam}\,(Q)), \tag{10}$$

where η is any small constant given in advance. [But we need to take ε small enough, depending on η.] Also, we have the analogous estimate for $End(\mathcal{S})$, i.e.,

$$End(\mathcal{S}) \subset \Gamma \tag{11}$$

(which actually follows from (10)). The construction of Γ takes some time, but is not especially subtle. The point is to use the good d-planes P_Q that satisfy (6), plus the fact that they vary fairly slowly with Q (a reasonably easy consequence of (6) and the Ahlfors-regularity of E), and some argument with partitions of unity at the scale of $Stop(\mathcal{S})$. See Section 7 of [**DS3**].

The Lipschitz graph Γ will be useful in more than one way. At the end of the argument, we shall use the various sets Γ that correspond to selected top cubes Q_0 to glue them and find an ω-regular surface that contains E, but even before that it will be useful to be able to go back and forth between E and Γ to code information about β-numbers. Of course Γ will be more useful if $End(\mathcal{S})$ is large, or if $\mathcal{S}$ at least contains a lot of generations of cubes. For instance, in the perfect case where $End(\mathcal{S}) = Q_0$, (11) says that $Q_0 \subset \Gamma$. But even if $End(\mathcal{S})$ is empty but most of $Stop(\mathcal{S})$ is composed of very small cubes, (10) will often say that E stays very close to Γ (compared to the initial scale $\mathrm{diam}\,(Q_0)$. We expect this to be the case on average, but this will have to be proved.

3. The corona condition

Once we know how to define a stopping time region $\mathcal{S} = \mathcal{S}(Q_0)$ associated to a given cube, we can decompose our set Δ (or large regions of Δ if E is unbounded) into stopping time regions.

Let us fix some cube $Q_0 \in \Delta$; if E is bounded, we may as well take $Q_0 = E$. Call $\Delta(Q_0)$ the set of descendants of Q_0 (grosso modo, the set of cubes $Q \in \Delta$ that are contained in Q, but for a formal definition we should be a little more careful, since a given set can correspond to cubes of a few different generations). We want to decompose $\Delta(Q_0)$ into a family $\mathcal{F}$ of stopping-time regions $\mathcal{S}$.

We first apply the construction above to Q_0 and get a stopping-time region $\mathcal{S}(Q_0)$. We also apply the construction above to every child Q of a cube $Q' \in Stop(\mathcal{S}(Q_0))$ and get various regions $\mathcal{S}(Q)$. Then we apply the construction to the children of cubes of all the $Stop(\mathcal{S}(Q))$ and get new regions. We continue like this, unless we eventually come to a situation where there is no cube in any $Stop(\mathcal{S}(Q))$ of some generation (and then we are finished). Our family $\mathcal{F}$ is the collection of all the stopping time regions that we get this way. By construction, $\Delta(Q_0)$ is the disjoint union of the regions $\mathcal{S}$, $\mathcal{S} \in \mathcal{F}$.

Our construction will not lead us too far if we do not control the number of stopping time regions. [Think about the amount of information that we would have if all the regions were reduced to one cube!] Call $Q(\mathcal{S})$ the top cube of the region $\mathcal{S}$ (i.e., in our construction, the cube Q such that $\mathcal{S} = \mathcal{S}(Q)$). The estimate we need is the following "Carleson packing condition": there is a constant C such that

$$\sum_{\mathcal{S}\in\mathcal{F};\, Q(\mathcal{S})\subset B(x,r)} H^d(Q(\mathcal{S})) \leq Cr^d \tag{12}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. In other words, since almost all top cubes are children of stopped cubes, we did not stop too often.

Let us concentrate on the case when E satisfies the geometric lemma (2.6), with $q = 1$ (we lose no generality because $q = 1$ gives the weaker form of (2.6)). Then we used the conditions (6) and (8) to define the stopping time regions.

If (12) holds, we shall say that E has a corona decomposition. Actually, there is a more abstract and general definition of the locution "E has a corona decomposition", where we allow coverings of Δ that may have been obtained from different algorithms. Nonetheless, the covering regions $\mathcal{S}$ are required to satisfy some coherence conditions (i.e., each $\mathcal{S}$ really looks like a stopping time region constructed from a top cube), we require the existence of Lipschitz graphs $\Gamma = \Gamma_{\mathcal{S}}$ (with uniform bounds on the Lipschitz constants) such that (10) and (11) hold, and the Carleson packing condition (12) should hold too. See Section 2 of [**DS3**] or [**DS4**] for definitions, but note that in [**DS3**] there is a small imprecision in the definition of coherence, which is corrected in [**DS4**].

The notion was introduced and used before by Semmes [**Se3**], where the point was to get a corona decomposition and use it to prove estimates on singular integrals and harmonic measure on some class of Ahlfors-regular sets of codimension 1 (the sets with "Condition B" of [**Se1**]). See Exercise 14 for a definition of Condition B.

So there are two things to do in our argument: first prove (12), and then show that (12) (or the existence of a corona decomposition) implies uniform rectifiability.

4. The Carleson packing estimate when E satisfies a geometric lemma

So how do we prove a Carleson packing estimate like (12)? Let us distinguish between different sorts of regions $\mathcal{S}$. Call $\mathcal{F}_1$ the set of regions $\mathcal{S} \in \mathcal{F}$ such that

$$H^d(End(\mathcal{S})) \geq 10^{-1} H^d(Q(\mathcal{S})). \tag{13}$$

Notice that when $x \in End(\mathcal{S})$ for some $\mathcal{S}$, then it does not lie in any of the stopped cubes $Q \in Stop(\mathcal{S})$, nor of course in their descendants. Also, $End(\mathcal{S}) \subset Q(S)$.

Then the sets $End(\mathcal{S})$, $\mathcal{S} \in \mathcal{F}$ are disjoint, and

$$\begin{aligned}\sum_{\mathcal{S}\in\mathcal{F}_1;\, Q(\mathcal{S})\subset B(x,r)} H^d(Q(\mathcal{S})) &\leq 10 \sum_{\mathcal{S}\in\mathcal{F}_1;\, Q(\mathcal{S})\subset B(x,r)} H^d(End(\mathcal{S})) \\ &\leq 10 H^d(E\cap B(x,r)) \leq Cr^d.\end{aligned} \tag{14}$$

This takes care of $\mathcal{F}_1$. Next consider

$$\mathcal{F}_2 = \Big\{\mathcal{S}\in\mathcal{F}\,;\, \sum_{Q\in Stop(\mathcal{S})\cap\mathcal{B}} H^d(Q) \geq 10^{-1} H^d(Q(\mathcal{S}))\Big\}. \tag{15}$$

Then

$$\begin{aligned}&\sum_{\mathcal{S}\in\mathcal{F}_2;\, Q(\mathcal{S})\subset B(x,r)} H^d(Q(\mathcal{S})) \\ &\qquad\leq 10 \sum_{\mathcal{S}\in\mathcal{F}_2;\, Q(\mathcal{S})\subset B(x,r)} \ \sum_{Q\in Stop(\mathcal{S})\cap\mathcal{B}} H^d(Q) \\ &\qquad\leq 10 \sum_{Q\in\mathcal{B};\, Q\subset B(x,r)} H^d(Q) \\ &\qquad\leq Cr^d,\end{aligned} \tag{16}$$

by (15), because our stopping time regions are disjoint, and (for the last inequality) because E satisfies a weak geometric lemma. The verification is straightforward; one has to check quietly that a Carleson measure estimate on a bad set of balls (the Carleson set $\mathcal{B}(\varepsilon/C_1)$, where $\mathcal{B}(\varepsilon)$ is defined a little below (3.6) and C_1 is somewhat larger than C_0) transforms into a Carleson packing estimate on our set $\mathcal{B}$ of bad cubes.

If we were trying to prove (12) in the case when bad cubes are the only reason for stopping (which is the case for the proof with the $BWGL$), we would be finished because (9) would say that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. In the present situation, we still have to control $\mathcal{F}_3 = \mathcal{F} \setminus \mathcal{F}_1 \cup \mathcal{F}_2$.

Call $Stop_2(\mathcal{S})$ the set of cubes $Q \in Stop(\mathcal{S})$ that do not lie in $\mathcal{B}$. Theses are cubes where we stopped because of (8). By definitions (and (9)),

$$\sum_{Q\in Stop_2(\mathcal{S})\cap\mathcal{B}} H^d(Q) \geq \frac{8}{10}\, H^d(Q(\mathcal{S})) \quad \text{for } \mathcal{S}\in\mathcal{F}_3. \tag{17}$$

Now we want to control $\sum_{\mathcal{S}\in\mathcal{F}_3;\, Q(\mathcal{S})\subset B(x,r)} H^d(Q(\mathcal{S}))$. To do so, we prove that for each $\mathcal{S} \in \mathcal{F}_3$, we can associate a region $\mathcal{H} = \mathcal{H}(\mathcal{S})$ of

$$[E\cap B(c_{Q(\mathcal{S})}, C\,\mathrm{diam}\,(Q(\mathcal{S}))] \times (0, C\,\mathrm{diam}\,(Q(\mathcal{S})))$$

such that

$$\int_{\mathcal{H}} \beta_{E,1}(y,t)^2\, \frac{dH^d(y)dt}{t} \geq C(\varepsilon,\delta,n,d)^{-1} H^d(Q(\mathcal{S})), \tag{18}$$

and also such that the regions $\mathcal{H}(\mathcal{S})$, $\mathcal{S} \in \mathcal{H}$, have bounded overlap (that is, no point of $E \times (0, +\infty)$ lies in more than C regions $\mathcal{H}(\mathcal{S})$). Once we do this, we get that

$$
\begin{aligned}
&\sum_{\mathcal{S}\in\mathcal{F}_3;\,Q(\mathcal{S})\subset B(x,r)} H^d(Q(\mathcal{S}))\\
&\qquad \le C(\varepsilon,\delta,n,d) \sum_{\mathcal{S}\in\mathcal{F}_3;\,Q(\mathcal{S})\subset B(x,r)} \int_{\mathcal{H}} \beta_{E,1}(y,t)^2\,\frac{dH^d(y)dt}{t}\\
&\qquad \le C'(\varepsilon,\delta,n,d) \int_{y\in B(x,Cr)} \int_{t=0}^{Cr} \beta_{E,1}(y,t)^2\,\frac{dH^d(y)dt}{t}\\
&\qquad \le C''(\varepsilon,\delta,n,d)\, r^d,
\end{aligned}
\tag{19}
$$

by (17), because the $\mathcal{H}(\mathcal{S})$ have bounded overlap and are contained in $B(x,Cr)\times(0,Cr)$, and by the geometric lemma (2.6). [Recall that we took $q=1$ in (2.6), but anyway (2.6) is stronger when $q>1$, by Hölder.]

The construction of $\mathcal{H}(\mathcal{S})$ and the verification of (18) is a little technical, so we only give an idea of the proof. Grosso modo, $\mathcal{H}$ is the set of pairs (y,t) such we can find $Q\in\mathcal{S}$ with $\operatorname{dist}(y,Q)\le\operatorname{diam}(Q)$ and $C^{-1}\operatorname{diam}(Q)\le t\le C\operatorname{diam}(Q)$. In particular, $(y,t)\in\mathcal{H}$ for t small if $y\in End(\mathcal{S})$. Then the sets $\mathcal{H}(\mathcal{S})$, $\mathcal{S}\in\mathcal{H}$, have bounded overlap essentially by construction, and the important part is (18).

Recall that there is a Lipschitz graph $\Gamma=\Gamma_{\mathcal{S}}$ associated to $\mathcal{S}$. Call $f=f_{\mathcal{S}}:P_{\mathcal{S}}\to P_{\mathcal{S}}^{\perp}$ the Lipschitz function whose graph is Γ (in the appropriate set of coordinates). Because of (10) and (11), we can go back and forth between estimates on the $\beta_{E,1}(y,t)$ and the numbers

$$
\gamma_{f,1}(\xi,s)=s^{-1}\inf_{a\text{ affine}}\left\{\frac{1}{|B(\xi,s)|}\int_{B(\xi,s)}|f(\zeta)-a(\zeta)|\,d\zeta\right\}
\tag{20}
$$

That is, we introduce a region $\mathcal{H}^*$ in $P_{\mathcal{S}}\times(0,C\operatorname{diam}(Q(\mathcal{S})))$ that corresponds to $\mathcal{H}$ (just a little bit smaller), and we prove that (18) holds as soon as

$$
\int_{\mathcal{H}^*}\gamma_{f,1}(\xi,s)^2\,\frac{d\xi ds}{s}\ge C(\varepsilon,\delta,n,d)^{-1}\operatorname{diam}(Q(\mathcal{S}))^d.
\tag{21}
$$

In the computations, we have to estimate the differences between the numbers $\beta_{E,1}(y,t)$ and $\gamma_{f,1}(\xi,s)$, but the errors turn out to be summable because they are of a definite (small) size at the scale of cubes of $Stop(\mathcal{S})$, and then decrease geometrically at larger scales, because (10) (applied to the same cubes of $Stop(\mathcal{S})$) gets comparatively better.

Why should (21) hold when $\mathcal{S}\in\mathcal{F}_3$? Recall that $Stop_2(\mathcal{S})$ is the set of cubes $Q\in Stop(\mathcal{S})$ where we stopped because of (8). Thus, near such a cube, E is close to a d-plane P_Q that makes an angle at least δ with $P_{\mathcal{S}}$. Because of this, $|\nabla f|\ge\delta/2$ near the projection of Q onto $P_{\mathcal{S}}$. By (17), the union of these neighborhoods of projections has a measure at least $C^{-1}H^d(Q(\mathcal{S}))$, and hence

$$
\int_{P_{\mathcal{S}}}|\nabla f|^2\ge C^{-1}\delta^2 H^d(Q(\mathcal{S})).
\tag{22}
$$

This gives a lower bound on $\int_{P_{\mathcal{S}}\times(0,+\infty)}\gamma_{f,1}(\xi,s)^2\,\frac{d\xi ds}{s}$, (compare with (3.23)) and since we can also control $\int_{P_{\mathcal{S}}\times(0,+\infty)\setminus\mathcal{H}^*}\gamma_{f,1}(\xi,s)^2\,\frac{d\xi ds}{s}$ quite well (by construction of Γ and if ε is small enough), one eventually gets (21).

This completes our sketch of proof of (12) when E satisfies the geometric lemma (2.6).

5. Construction of an ω-regular surface

We continue with the case when E satisfies (2.6), and show how the existence of a corona decomposition leads to uniform rectifiability. As before, the argument below is just a sketch, and the details (or necessary modifications) can be found in Section 18 of [**DS3**]. Let us just try to find an ω-regular surface that contains a given cube Q_0 in E; thus for an unbounded set E there would be an additional gluing argument like the one we already did not do for connected Ahlfors-regular sets of dimension 1.

So let $Q_0 \subset E$ be given. We construct a family $\mathcal{F}$ of stopping time regions $\mathcal{S}$, each with a Lipschitz graph $\Gamma_{\mathcal{S}}$ associated to it, and the Carleson condition (12) says that there are not too many regions.

Notice that by (12) and Fubini,

$$\int_{Q_0} \sum_{\mathcal{S}\in\mathcal{F}} \mathbf{1}_{Q(\mathcal{S})}(x) \leq \sum_{\mathcal{S}\in\mathcal{F}} H^d(Q(\mathcal{S})) \leq C\,\mathrm{diam}\,(Q_0)^d, \tag{23}$$

so that for H^d-almost every $x \in Q_0$, x only lies in finitely many cubes $Q(\mathcal{S})$. Then $x \in End(\mathcal{S})$ for some $\mathcal{S}$ (namely, the region whose top cube is the smallest $Q(\mathcal{S})$ that contains x). Hence

$$Q_0 \subset \Big(\bigcup_{\mathcal{S}\in\mathcal{F}} End(\mathcal{S})\Big)^- \subset \Big(\bigcup_{\mathcal{S}\in\mathcal{F}} \Gamma_{\mathcal{S}}\Big)^- \tag{24}$$

(where the bars mean "closure"), and it is enough to construct an ω-regular surface that contains all the $\Gamma_{\mathcal{S}}$, or even the sets $End(\mathcal{S}) \subset \Gamma_{\mathcal{S}}$.

Let us first choose the pieces of $\Gamma_{\mathcal{S}}$ that we want to keep and glue to each other. Fix $\mathcal{S}$, call $P_{\mathcal{S}}$ the good d-plane associated to the top cube $Q(\mathcal{S})$, and $\pi_{\mathcal{S}}$ the orthogonal projection onto $P_{\mathcal{S}}$. Select a point $c_{\mathcal{S}}$ in $Q(\mathcal{S})$, and set $D^+(\mathcal{S}) = \pi_{\mathcal{S}}^{-1}(B(\pi_{\mathcal{S}}(c_{\mathcal{S}}), C_1\,\mathrm{diam}\,(Q(\mathcal{S})))$; the constant C_1 will be chosen soon.

Also call $Chop(\mathcal{S})$ the set of children of cubes of $Stop(\mathcal{S})$. We care about the cubes of $Chop(\mathcal{S})$ because they are the top cubes of the regions $\mathcal{S}'$ that lie directly "under" $\mathcal{S}$ in our family $\mathcal{F}$. For each $Q \in Chop(\mathcal{S})$, call $c_Q \in Q$ the center of Q (as in (4)). This time we shall really use the fact that $\mathrm{dist}\,(c_Q, E\setminus Q) \geq C^{-1}\,\mathrm{diam}\,(Q)$.

Set $D(Q) = \pi_{\mathcal{S}}^{-1}B(\pi_{\mathcal{S}}(c_Q), C_1^{-1}\,\mathrm{diam}\,(Q))$ for every $Q \in Chop(\mathcal{S})$. If we choose C_1 large enough, then $2D(Q) \subset D^+(\mathcal{S})$ for $Q \in Chop(\mathcal{S})$, and

$$\mathrm{dist}\,(D(Q), D(R)) \geq C^{-1}[\mathrm{diam}\,(Q) + \mathrm{diam}\,(R)] \quad \text{for } Q, R \in Chop(\mathcal{S}), Q \neq R, \tag{25}$$

essentially by (4) and if δ is small enough. Now set

$$G(\mathcal{S}) = [\Gamma_{\mathcal{S}} \cap D^+_{\mathcal{S}}] \setminus \big[\bigcup_{Q\in Chop(\mathcal{S})} D(Q)\big]; \tag{26}$$

this is the piece of $\Gamma_{\mathcal{S}}$ that we want to keep. It looks like a big disk, with lots of little holes in it, one for each cube of $Chop(\mathcal{S})$. It is easy to check that $End(\mathcal{S}) \subset G(\mathcal{S})$, again by (25) and if δ is small enough. Thus it is enough to find an ω-regular surface that contains all the $G(\mathcal{S})$.

Now we connect all the $G(\mathcal{S})$. That is, we connect each $\Gamma_{\mathcal{S}}$ to the various $G\mathcal{S}')$, where $\mathcal{S}'$ is a region such that $Q(\mathcal{S}') \in Chop(\mathcal{S})$. For this we just use a tube $T(\mathcal{S}, \mathcal{S}')$ from the $(d-1)$-sphere $\Gamma_{\mathcal{S}} \cap \partial D(Q)$ to the (probably somewhat larger) $(d-1)$-sphere $\Gamma_{\mathcal{S}'} \cap \partial D^+(\mathcal{S}')$. Once we do this for all the regions $\mathcal{S}$ and take the closure, we get a connected surface Σ; and then we have to prove that Σ is an ω-regular surface.

First, we have to verify that Σ is an Ahlfors-regular set. It is true that our various sets $\Gamma_{\mathcal{S}}$ may meet each other, but it is possible to control $H^d[(\bigcup \Gamma_{\mathcal{S}}) \cap B(x,r)]$ because the $\Gamma_{\mathcal{S}}$ are Lipschitz graphs that stay close to E, and E is Ahlfors-regular. We have to be a little more careful with the tubes $T(\mathcal{S},\mathcal{S}')$, because we could inadvertently build lots of tubes of different scales that pass through a given tiny ball. To avoid this, we decide to use an extra dimension and construct our tubes in $\mathbb{R}^{n+1}$. The extra dimension allows us to make sure that the last coordinate of a point of $T(\mathcal{S},\mathcal{S}')$ is essentially bounded below by its distance to E, which diminishes the chances of fortuitous intersections. This little complication is the reason why we decided to allow ω-regular mappings with values in $\mathbb{R}^{n+1}$ in Definition 1.11 (of uniform rectifiability). When $2d \leq n$, there is enough room in $\mathbb{R}^n$ to construct tubes $T(\mathcal{S},\mathcal{S}')$ that almost avoid each other, and we do not need the extra dimension. [But the argument is a little painful.] At any rate, it is possible to prove that Σ is an Ahlfors-regular set.

We also need to parametrize Σ by a disk D_0 of radius $\operatorname{diam}(Q_0)$ in $\mathbb{R}^d$, but there is no serious difficulty. Of course it is very easy to parametrize the top graph $\Gamma_{\mathcal{S}_0}$ with D_0, where $\mathcal{S}_0$ is the region such that $Q(\mathcal{S}_0) = Q_0$. The smaller set $G(\mathcal{S}_0)$ corresponds to D_0, minus a collection of small disjoint disks D_j, that correspond to cubes $Q_j \in Chop(\mathcal{S}_0)$. We leave the annuli $D_j \setminus \frac{1}{2}D_j$ for a parametrization of the tubes $T(\mathcal{S}_0,\mathcal{S}_j)$ (where $\mathcal{S}_j$ is the region whose top cube is Q_j), and we are ready to continue the argument. That is, we parametrize the graph $\Gamma_{\mathcal{S}_j}$ with D_j, and remove the part of the D_j that corresponds to cubes of the $Chop(\mathcal{S}_j)$. This leaves a collection of very small disks D'_j; we keep the annuli $D'_j \setminus \frac{1}{2}D'_j$ to parametrize tubes, and we are left with smaller disks $\frac{1}{2}D'_j$ that we use to parametrize all the remaining sets $G(\mathcal{S})$. Eventually, we get the desired parametrization z of Σ.

Of course our estimates for z deteriorate a little each time we go through a generation of stopping time regions. For instance, note that the disks D_j of the first generation are already about C_1 times smaller than the corresponding cubes Q_j. This seems to be needed in the argument, because we want the sets D_j to be disjoint and reasonably far from each other, so it is more convenient to make them small. Of course the $\frac{1}{2}D_j$ are even a little bit smaller. Thus we are forced to parametrize the $\Gamma_{\mathcal{S}_j}$ about C_1 times faster than would be natural. And the parametrization of graphs of the k^{th} generation will be about C_1^k times faster.

The result of this is that z is not Lipschitz, there are places where its derivative is very large. But these places are not too numerous, because of the Carleson packing condition (12) and the fact that Γ is Ahlfors-regular. And eventually one proves that z is ω-regular for some A_1 weight ω. But the reader easily imagines that it would be much harder to find a 1-regular parametrization of Σ.

We did not give details here, but the complete proof of the fact that the existence of a corona decomposition of E implies its uniform rectifiability is not too bad. It takes 18 pages in [**DS3**]; the reader should see this as an indication of the strength of corona decompositions. In [**Se3**], such decompositions are also used to prove estimates on singular integrals on E, and even harmonic measure estimates for domains bounded by E (when E satisfies Condition B, see Exercise 14). The point is that these results are known for Lipschitz graphs, and then (12) allows one to control errors.

6. The case of $BWGL$

Let us also say a few words about the case when $E \in BWGL$, and we want to prove uniform rectifiability. Here also we shall merely sketch the proof, and even tell a small pedagogical lie if the occasion arises. In the present case we decided to stop only when we meet a bad cube, so at least (12) is easy to get (see (14) and (16)). But of course we do not have a Lipschitz graph Γ as in (10) and (11), because the approximating planes P_Q are allowed to turn slowly when Q becomes smaller.

So instead of building a Lipschitz graph, we construct a surface $E(\mathcal{S})$. We use a building block Σ_0, which is the intersection of the unit ball with a finite union of d-planes, which we take large enough to make sure that the projection of Σ_0 on every d-plane is reasonably large. For each cube Q, we pick a point c_Q such that (25) holds, and set $\Sigma(Q) = c_Q + (2C)^{-1}\Sigma_0$, with C as in (25). Thus $\Sigma(Q)$ stays reasonably far from $E \setminus Q$. Finally set

$$E(\mathcal{S}) = \Big[End(\mathcal{S}) \cup \bigcup_{Q \in Stop(\mathcal{S})} \Sigma(Q)\Big]^{-}. \tag{27}$$

The main point of the argument is that $E(\mathcal{S})$ is a nice set. First, it is Ahlfors-regular. This comes from the Ahlfors-regularity of E, plus the fact that $E(\mathcal{S})$ always stays close to E; the details are tedious, but easy.

Next, E satisfies the weak geometric lemma; again this comes from the same property for E. Alternatively, if you pick a small constant ε_0 (depending on the Ahlfors-regularity constant) and just want a control on the single bad set $\mathcal{B}(\varepsilon_0) = \big\{(x,t) \in E(\mathcal{S}) \times (0, \mathrm{diam}\,(Q_0))\,;\, \beta_{E(\mathcal{S})} > \varepsilon_0\big\}$, I think (but did not check thoroughly) that you can get it for free (i.e., without using our assumption that $E \in BWGL$) by definition of $\mathcal{S}$, and if ε is chosen sufficiently small compared to ε_0.

Then we can show that $E(\mathcal{S})$ has big projections. Recall that this means that there is a constant $\theta > 0$ such that, if $x \in E(\mathcal{S})$ and $0 < r < \mathrm{diam}\,(Q_0)$, we can find a d-plane P such that $H^d(\pi_P(E \cap B(x,r))) \geq \theta r^d$, where π_P is the orthogonal projection onto P. At scales r smaller than the diameters of the cubes $Q \in Stop(\mathcal{S})$, this comes from our choice of Σ_0. At larger scales, we have to use the definition of $\mathcal{S}$, and in particular the fact that we decided to stop also when Q is a cube with a good approximating plane P_Q as in (6), but (7) does not hold, i.e., when there is an apparent hole in Q of size $\geq \varepsilon\,\mathrm{diam}\,(Q)$.

The argument is a little easier to imagine in codimension 1. The idea is that at all the large scales, and until we hit a bad cube, E looks like a hyperplane. This allows us to cut the space near E roughly into two components. Because we can do this for all the cubes until we stop, there is even a coherent way to do the rough splitting. That is, we can define something like an orientation of E near Q for every $Q \in \mathcal{S}$. With a little more effort, we can prove the big projection property; the argument is a little technical, but hopefully the reader imagines that if you start from a cube $Q \in \mathcal{S}$ and pick two balls of radius $\mathrm{diam}\,(Q)/10$ near Q, lying on different sides of E, then many of the line segments from one ball to the other one will have to cross $E(\mathcal{S})$. Clearly, such segments must pass at distance $\leq 2\varepsilon\,\mathrm{diam}\,(Q)$ from E, by (7), and the idea of the proof is to look more carefully at what happens at smaller scales, when you pass near E and go from one local side of E to the other.

The case of higher codimension also uses the same sort of idea (starting from the construction a local orientation), but is a little more algebraic. We leave the details; see Chapter 2.2 in [**DS4**].

Notice that all our estimates on $E(\mathcal{S})$ get better when we take ε smaller; the only thing that gets worse is the constant in the Carleson packing estimate (12). The conclusion of all this is that since $E(\mathcal{S})$ is Ahlfors-regular that satisfies a weak geometric lemma and has big projections, the main result in [**DS2**] says that $E(\mathcal{S})$ is uniformly rectifiable, and even contains big pieces of Lipschitz graphs. Since for this result one only needs to control the bad set $\mathcal{B}(\varepsilon_0)$ above for a single (but small) ε_0, the verification of the weak geometric lemma should be slightly easier than what we did in [**DS4**].

The proof is not finished yet. Now we have a decomposition similar to what we had in the last subsection, when E satisfied a geometric lemma. The difference is that we now have uniformly rectifiable sets $E(\mathcal{S})$ with big pieces of Lipschitz graphs, instead of the Lipschitz graphs $\Gamma_{\mathcal{S}}$. With the vocabulary of [**DS4**], E has a generalized corona decomposition, in terms of uniformly rectifiable sets. And there are various proofs of uniform rectifiability in that case; for instance, one could use the fact that each $E(\mathcal{S})$ satisfies a geometric lemma, with uniform estimates, to show that E itself satisfies a geometric lemma (see [**DS4**]). Or we could use the L^2-boundedness of singular integrals on the $E(\mathcal{S})$. But apparently there is no direct argument.

7. Last comments

We like the corona construction and its geometric variants for various reasons. First, it has a strong algorithmic flavor. Once we give ourselves the stopping rules for the construction of regions $\mathcal{S}$, we get an automatic decomposition of $\Delta(Q_0)$ (our set of cubes) into regions $\mathcal{S}$, and we usually can associate to each region a nice set $E(\mathcal{S})$. This is almost the same as decomposing our initial set E into nicer sets $E(\mathcal{S})$ (a missing piece in the analogy with Littlewood-Paley theory on sets). It was proposed to try to study data sets (or large clouds of points) this way; the result would be something like an automatic cartography of E.

A second important feature of the construction is that we can do estimates on the various regions, or the sets $E(\mathcal{S})$, without really knowing the good properties of E yet. These properties show up only when we try to prove the Carleson packing estimate (12). In some cases (like in our proof of uniform rectifiability when we have a geometric lemma), we can already use estimates on the approximating sets $E(\mathcal{S})$ to prove (12). In other cases, like when one tries to prove results about operators on E, the corona construction can be used a little bit like a linearization: we prove that E has a corona decomposition into better sets (like Lipschitz graphs with small constants), and then we can reduce to proving the desired result on the better class.

The sort of construction that we presented here, but often more elaborate, has been used in a range of papers. See other results in [**DS4, JL, Leg, Le2, To4**], and [**To5**].

LECTURE 5

Menger curvature, other applications

Menger curvature is at the center of a very impressive series of recent results, and even though it is not directly related to uniform rectifiability a priori, it seems a good idea to spend some time discussing it. More details on many results of this section will be found in the recent book [**Pa3**].

Recall that the Menger curvature of a triple (x, y, z) is the inverse of the radius of the circle that goes through these three points. We shall denote it by $c(x, y, z)$. We set $c(x, y, z) = 0$ when the three points lie on a line.

Menger curvature can be seen as another way to measure the rectifiability (or flatness) of a set, a little bit like the Jones numbers $\beta_E(x, r)$ in Section 2. And in this respect it is easy to invent higher-dimensional analogues, with the same sort of rectifiability results. But since Menger curvature is a little harder to manipulate (for instance, because it gives rise to triple integrals), its true justification in Euclidean spaces is its close relation with the Cauchy integral. Menger curvature can also be used in metric spaces as a means to control the geometry, as was originally planned by Menger. See for instance [**Ha2**].

1. The Cauchy kernel

The main reason why the Menger curvature is so useful is through the following magic formula, discovered by M. Melnikov and rapidly used by Mattila, Melnikov, and Verdera in connection with the Cauchy operator (see for instance [**MMV**]): if z_1, z_2, and z_3 are three distinct points of the complex plane, then

(1) $$\sum_{\sigma \in \Sigma_3} \frac{1}{(z_{\sigma(1)} - z_{\sigma(2)})\overline{(z_{\sigma(1)} - z_{\sigma(3)})}} = c(z_1, z_2, z_3)^2,$$

where Σ_3 denotes the set of permutations of the set $\{1, 2, 3\}$. The verification is of course straightforward, but the miracle is that the right-hand side is nonnegative and has a geometric meaning.

Because of (1), there is an obvious relation between the Cauchy kernel and Menger curvature. Let us say how it could be used. Let μ be a locally finite positive Borel measure on the plane, without atoms, and set

(2) $$T_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)d\mu(y)}{x - y}$$

for f continuous and compactly supported, say, and $\varepsilon > 0$. Suppose in addition that the $T_{\mu,\varepsilon}$, $\varepsilon > 0$, are uniformly bounded operators on $L^2(d\mu)$. It is not too hard to show that μ has linear growth, i.e., that there is a constant $C \geq 0$ such that

(3) $$\mu(B(x, r)) \leq Cr \text{ for } x \in \mathbb{C} \text{ and } r > 0.$$

See for instance [**Da4**], page 56. But we seek additional geometric information. Let $B = B(x,r)$ be any disk in the plane. Notice that

$$||T_{\mu,\varepsilon}\mathbf{1}_B||_2^2 \leq C\,||\mathbf{1}_B||_2^2 = C\mu(B) \leq Cr. \tag{4}$$

On the other hand,

$$||\mathbf{1}_B T_{\mu,\varepsilon}\mathbf{1}_B||_2^2 = \int\int\int_{\Delta} \frac{1}{(x-y)\overline{(x-z)}}\, d\mu(y)d\mu(z)d\mu(x), \tag{5}$$

where $\Delta = \big\{(x,y,z) \in B^3\,;\, |x-y| > \varepsilon \text{ and } |x-z| > \varepsilon\big\}$. Now let us try to replace Δ with the more symmetric domain $\Delta' = \big\{(x,y,z) \in B^3\,;\, |x-y| > \varepsilon, |x-z| > \varepsilon, \text{ and } |y-z| > \varepsilon\big\}$. A fairly straightforward estimate shows that

$$\int\int\int_{\Delta\setminus\Delta'} |(x-y)(x-z)|^{-1} d\mu(x)d\mu(y)d\mu(z) \leq C\mu(B), \tag{6}$$

because of (3); see for instance [**Ve2**], page 187. And then

$$\begin{aligned}&\int\int\int_{\Delta'} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)\\ &\qquad = \frac{1}{6}\int\int\int_{\Delta'} \frac{1}{(x-y)\overline{(x-z)}}\, d\mu(x)d\mu(y)d\mu(z) \leq C\mu(B),\end{aligned} \tag{7}$$

because we can symmetrize the integral in Δ', then use (1), (6), (5), and (4). Since the estimates are uniform, we can let ε tend to 0 and we get that

$$\int\int\int_{B^3} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \leq C\mu(B) \quad \text{for every ball } B. \tag{8}$$

Thus we get some geometric information on the support of μ. This was in particular used in [**MMV**] to prove that if $E \subset \mathbb{C}$ is Ahlfors-regular, μ is the restriction of H^1 to E (or just any Ahlfors-regular measure on E) and if the $T_{\mu,\varepsilon}$ are uniformly bounded on $L^2(d\mu)$, then E is uniformly rectifiable. Once they have (8), they do not need to use the Cauchy kernel any more, and most of the proof consists in going from estimates on the Menger curvature to Carleson measure estimates on the squares of the Jones numbers $\beta_{E,2}(x,r)$.

After this, Tolsa [**To1**] and Nazarov, Treil, Volberg [**NTV1**] completed the characterization of uniform boundedness of the $T_{\mu,\varepsilon}$, by showing that (3) and (8) are also sufficient.

The converse to [**MMV**], i.e., the fact that the $T_{\mu,\varepsilon}$ are uniformly bounded when μ is the restriction of H^1 to an Ahlfors-regular curve (or a uniformly rectifiable set) was already known before, but even before [**MMV**], Verdera noticed that (1) could give very elegant proofs of L^2-boundedness results. That is, let μ be the restriction of H^1 to an Ahlfors-regular set E (of dimension 1), and suppose that you can prove that (8) holds. Then a minor variant of the $T(1)$-theorem shows that the $T_{\mu,\varepsilon}$ are uniformly bounded. The proof of $T(1)$ can even be simplified in this special case, and the argument in [**Ve2**] avoid using the $T(1)$-theorem.

So Melnikov and Verdera [**MeV**] got a new proof of L^2-boundedness for the Cauchy integral on Lipschitz graphs, where they just needed to show that (8) holds for Lipschitz graphs. This turns out to follow from a simple computation based on Plancherel. Let us rapidly see how.

Let $A : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and set $z(x) = x + iA(x)$. Thus z gives a parametrization of the graph of A, and (8) will follow at once from the next lemma.

Lemma 9. *There is a universal constant C such that*

$$\int\int\int_{I^3} c(z(x),z(y),z(w))^2 dxdydw \leq C||A'||_\infty^2 |I| \tag{10}$$

for every Lipschitz function A and every interval $I \subset \mathbb{R}$.

Of course we can restrict to $x < y < w$, and a simple geometric estimate shows that in this case

$$c(z(x),z(y),z(w))^2 \leq \frac{C}{(w-x)^2}\,\Delta_A(x,y,w)^2, \tag{11}$$

where we set $\Delta_A(x,y,w) = \frac{A(y)-A(x)}{y-x} - \frac{A(w)-A(y)}{w-y}$. It may help the reader to think of $\Delta_A(x,y,w)$ as the difference between the slopes of the segments $[z(x),z(y)]$ and $[z(y),z(w)]$.

So we want to find upper bounds for triple integrals of the right-hand side of (11). Let us start with an L^2 estimate, and then we shall localize to get (10). We claim that if F is a compactly supported Lipschitz function, then

$$\int\int\int_{x<y<w} \frac{1}{(w-x)^2}\,\Delta_F(x,y,w)^2 dxdydw \leq C||F'||_2^2\,. \tag{12}$$

Fix x, and write $w = x+t$ for some $t > 0$ and $y = x+st$ for some $s \in (0,1)$. Also set $f = F'$, and note that the slopes $\frac{F(y)-F(x)}{y-x}$ and $\frac{F(w)-F(y)}{w-y}$ are the means of f on the intervals $[x,y]$ and $[y,w]$ respectively, so that

$$\Delta_F(x,y,w) = (\psi_{s,t} * f)(x), \tag{13}$$

where $\psi_{s,t}(\xi) = \frac{1}{t}\,\psi_s(\xi/t)$ and $\psi_s = \frac{1}{s}\mathbf{1}_{[0,s]} - \frac{1}{1-s}\,\mathbf{1}_{[s,1]}$. The reader may think of s as being a parameter, and then ψ_s is a function with a fairly mild singularity (which will go away when we integrate on s) and integral zero.

Call J the triple integral in (12). Then

$$\begin{aligned} J &= \int_{s\in(0,1)}\int_{t>0}\int_{x\in\mathbb{R}} |\psi_{s,t} * f|^2(x)\,\frac{dxdtds}{t} \\ &= \int_{s\in(0,1)}\int_{t>0}\int_{\xi\in\mathbb{R}} |\widehat{\psi_{s,t}}(\xi)\widehat{f}(\xi)|^2\,\frac{d\xi dtds}{t} \\ &= \int_{s\in(0,1)}\int_{t>0}\int_{\xi\in\mathbb{R}} |\widehat{\psi}_s(t\xi)|^2|\widehat{f}(\xi)|^2\,\frac{d\xi dtds}{t}, \end{aligned} \tag{14}$$

by Plancherel. For each given ξ, set $\varepsilon = \xi/|\xi|$, and observe that

$$\int_{t>0} |\widehat{\psi}_s(t\xi)|^2\frac{dt}{t} = \int_{u>0} |\widehat{\psi}_s(u\varepsilon)|^2\frac{du}{u} = \int_{u>0} |\widehat{\psi}_s(u)|^2\frac{du}{u} \tag{15}$$

(set $u = |\xi|t$ and use the fact that ψ_s is real). Then (14) yields

$$J = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \int_{s\in(0,1)}\int_{u>0} |\widehat{\psi}_s(u)|^2\frac{duds}{u}d\xi = T||f||_2^2, \tag{16}$$

with $T = \int_{s\in(0,1)}\int_{u>0} |\widehat{\psi}_s(u)|^2\frac{duds}{u}$. And (12) will follow as soon as we prove that $T < +\infty$.

The estimate for T is of course straightforward; essentially we just have to make sure that the singularity in s is small enough. Let us do the computation, but neglect the factors $\pm 2\pi$. First, $\frac{1}{s}\widehat{\mathbf{1}_{[0,s]}}(u) = \frac{1}{s}\int_0^s e^{iux}dx = \frac{1}{ius}(e^{ius}-1)$, and similarly $\frac{1}{1-s}\widehat{\mathbf{1}_{[s,1]}}(u) = \frac{1}{iu(1-s)}(e^{iu}-e^{ius})$. So

$$\widehat{\psi}_s(u) = \frac{1}{ius}(e^{ius}-1) - \frac{1}{iu(1-s)}(e^{iu}-e^{ius}). \tag{17}$$

We shall only estimate the contribution of $s \leq 1/2$; the other case would work the same way by symmetry.

When $u > 1/s$, we say that $|\widehat{\psi}_s(u)| \leq \frac{4}{us}$.

When $1 \leq u \leq 1/s$, we simply say that $|\widehat{\psi}_s(u)| \leq 2$.

When $u < 1$, we have to use the fact that $\int \psi_s(x)dx = 0$, so we do an expansion in (17) near $u = 0$ and find that $\widehat{\psi}_s(u) = \frac{1}{ius}[ius + O(u^2s^2)] + \frac{1}{iu(1-s)}[iu - ius + O(u^2)] = O(u)$. Finally, $\int_0^\infty |\widehat{\psi}_s(u)|^2 \frac{du}{u} \leq C\int_0^1 udu + 4\int_1^{1/s}\frac{du}{u} + 16\int_{1/s}^\infty \frac{du}{s^2u^3} \leq C + 4Log(1/s) + 8$, and then $T \leq 2\int_0^{1/2}[C + 4Log(1/s) + 8]ds < +\infty$; (12) follows.

Return to the Lipschitz function of Lemma 9, let the interval I be given, and define an auxiliary function F as follows. Write $I = [a,b]$ and set $F(x) = 0$ for $x < a$ and $x > 2b-a$, $F(x) = A(x) - A(a)$ on I, and $F(x) = A(2b-x) - A(a)$ for $b \leq x \leq 2b-a$. Note that F is Lipschitz and compactly supported. Then

$$\begin{aligned} &\int\int\int_{I^3} c(z(x),z(y),z(w))^2 dxdydw \\ &\leq C\int\int\int_{I^3} \frac{1}{(w-x)^2}\,\Delta_A(x,y,w)^2 dxdydw \\ &= C\int\int\int_{I^3} \frac{1}{(w-x)^2}\,\Delta_F(x,y,w)^2 dxdydw \\ &\leq C||F'||_2^2 = C\int_I |A'(x)|^2 dx \leq C|I|\,||A'||_\infty^2 \end{aligned} \tag{18}$$

by (11) and (12). This completes our proof of Lemma 9. □

So (8) holds for Lipschitz graphs Γ, and we have a rapid proof of boundedness for the Cauchy operator on Γ. The proof even gives fairly good estimates: it shows that the kernel $\frac{1}{z(x)-z(y)}$ defines an operator on $L^2(\mathbb{R})$, with norm less than $C||A'||_\infty$. Recall that the optimal estimate in that case is by $C||A'||_\infty^{1/2}$; see [**Mu**].

A modification of the argument above even gives the best known result for the k^{th} Calderón commutator. Recall that it is the operator with kernel $K(x,y) = \frac{[A(x)-A(y)]^k}{(x-y)^{k+1}}$, where A is a (real or complex valued) Lipschitz function on $\mathbb{R}$. Mateu and Verdera [**MV**] observed that when we symmetrize $K(x,y)$ as we did for the Cauchy kernel, simplifications occur and one is left with terms that can be estimated. We actually just did the necessary verification for the first commutator; the other ones lead to more complicated computations, but the end result is that one can show that $K(x,y)$ defines an operator with norm less than $C(1+k)||A'||_\infty^k$. This is a little shocking; the best result so far was $C_\varepsilon(1+k)^{1+\varepsilon}||A'||_\infty^k$, and was obtained by Christ and Journé [**ChJ**] with impressive multilinear integral expansions.

2. Analytic capacity

The most important recent use of Menger curvature is with analytic capacity. Recall that the analytic capacity of a compact set $K \subset \mathbb{C}$ is defined by

$$\gamma(K) = \sup\left\{|f'(\infty)|\,;\, f \text{ is a bounded analytic function on } \mathbb{C}\setminus K, \text{ with } ||f||_\infty \leq 1\right\} \tag{19}$$

[notice that since f is bounded and analytic in a pointed neighborhood of ∞, it has a removable singularity there, and a Laurent expansion $f(z) = f(\infty) + f'(\infty)/z + \ldots$]. This quantity was introduced by Ahlfors [**Ah2**], and measures how many bounded analytic functions live on $\mathbb{C}\setminus K$. Ahlfors also showed that $\gamma(K) = 0$ if and only if K is removable for bounded analytic functions, which means that if Ω is a neighborhood of K and f is a bounded analytic function on $\Omega \setminus K$, then f extends to a bounded analytic function on Ω.

X. Tolsa [**To4**] recently showed that analytic capacity is semiadditive; there is even an absolute constant C such that $\gamma(K_1 \cup K_2 \ldots \cup K_n) \leq C[\gamma(K_1) + \gamma(K_2) + \ldots \gamma(K_n)]$ for all choices of compact sets $K_1, \ldots K_n$. He also showed that $\gamma(K) > 0$ if and only if there is a positive measure μ supported on K, with linear growth (i.e., (3)), and such that $\mu(K) > 0$ and the total Menger curvature $c^2(\mu) = \int\int\int_{K^3} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)$ is finite. [There is also a quantitative statement that we do not copy down here.]

The main point was to prove the equivalence of $\gamma(K)$ with its variant $\gamma_+(K)$, which is defined as in (19), but with the additional constraint that f should be the Cauchy integral of some positive measure on K. [Thus $\gamma_+(K) \leq \gamma(K)$ trivially; the other inequality is nontrivial.] This required a lot of machinery (such as $T(b)$-theorems for non doubling measures), as well as a clever bootstrap argument. The connection between $\gamma_+(K)$ and the Menger curvature was known a little before: see [**Me, To2, To3**].

Let us just say a few words about the special case when $0 < H^1(K) < +\infty$. This is not such a weird assumption to make, because Painlevé showed that K is removable when $H^1(K) = 0$, and a simple argument with the Cauchy integral of a Frostman measure shows that $\gamma(K) > 0$ when the Hausdorff dimension of K is strictly larger than 1. In this case Tolsa's theorem was proved a little before (in [**DM**] and [**Da6**]), and can be stated in simpler terms: $\gamma(K) = 0$ if and only if K is totally unrectifiable (that is, $H^1(K \cap \Gamma) = 0$ for every curve Γ with finite length).

This is a little more directly in the subject of these lectures, because one of the important ingredients of the proof is a result of [**Leg**], which says in particular that if K is compact, $H^1(K) < +\infty$, and μ is a positive measure on K, with linear growth and such that $\mu(K) > 0$ and $c^2(\mu) < +\infty$, then K has a nontrivial rectifiable piece, i.e., there is a curve Γ with finite length such that $H^1(K \cap \Gamma) > 0$. And of course the proof in [**Leg**] is based on a corona construction.

See [**Da7**] for a vague description of [**Leg**], [**Da8**], [**Ma6, Ma7**] and [**Ve3**] for surveys on the results on analytic capacity prior to [**To4**], and [**Pa3**] for more details on the same subject.

The question arose as to whether the characterization above by the existence of a positive measure with linear growth and finite Menger curvature was really a good geometric characterization. [The proof did not really construct a measure.] At least, Tolsa showed that this characterization is invariant under biLipschitz mappings [**To5**]. Once again, the argument will make a lover of elaborate stopping time arguments particularly happy!

See [**NTV2, Pa1, Pa2, To6, To7, Vo**] for surveys or more recent results.

LECTURE 6

Other applications

Here we just want to report very rapidly on two other domains where uniform rectifiability has played a role. This subsection will look more like an advertisement than real mathematics.

The first domain is the study of minimizers for the Mumford-Shah functional. The functional is given by the formula

$$J(u,K) = H^{n-1}(K) + \int_{\Omega\setminus K} |\nabla u|^2 + \int_{\Omega\setminus K} |u-g|^2, \tag{1}$$

where Ω is a given simple domain in $\mathbb{R}^n$, g is a given function in $L^\infty(\Omega)$, K is required to be a closed subset of Ω, H^{n-1} denotes the Hausdorff (surface) measure of codimension 1, and u is defined on $\Omega \setminus K$, with a derivative in $L^2(\Omega \setminus K)$.

It was originally proposed to use J for image segmentation. Here Ω would be a screen, g would represent an image (i.e., a grey level at each point), and we would be looking for a simplified, cartoon-like approximation u of g. In particular, u would be required to be reasonably smooth, except that we would allow jumps on a singular set K. The three terms of the functional reflect these two constraints, plus of course a last one that $u-g$ be reasonably small, and we expect minimizers of J to give a decent compromise between the constraints, and hence an acceptable segmentation. There are a few variants of this, but mathematically they are not so far from J.

Even though the existence of minimizers for J was fairly hard to establish, it is true, and the main concern is the regularity of the minimizing pairs (u,K). In fact, u is fairly easy to describe once we know K, so we can concentrate on the regularity of K.

When $n=2$, Mumford and Shah conjectured that modulo a set of vanishing H^1 measure that does not matter, K is a finite union of C^1 curves, and such curves may only meet by sets of three and with 120° angles. In higher dimensions, we do not have such a precise conjecture, but it is probably safe to conjecture that K is a finite union of C^1 hypersurfaces, that meet on a reasonably nice set of dimension $n-2$.

At some point of time, it looked reasonable to try to prove that K is locally uniformly rectifiable, because it was known to be rectifiable almost for abstract reasons (it is the singular set of some function of bounded variation) and locally Ahlfors-regular. (see [**DMS**] when $n=2$ and [**CL**] in larger dimensions). And indeed it is the case [**DS5**], [**DS6**]. However it is not true that uniform rectifiability is the right notion here, because we expect K to be somewhat more regular than that, and actually there are already results where it is shown that there are lots of balls B centered on K such that $K \cap B$ is a C^1 hypersurface [**AFP**], [**Da5**]. But some of the standard techniques of these lectures, including a modest use of

Carleson measures and big pieces, were quite useful in the subject. We refer the reader to the beginning of [**Da11**] for a longer introduction, and to [**MoS**] for a discussion of functionals in the context of image processing.

Our second application concerns quasiminimal sets. It is perhaps a little more serious, in the sense that the class of objects under scrutiny is invariant under biLipschitz mappings, so we cannot expect more smoothness than uniform rectifiability.

What we shall call quasiminimal sets here is a class of subsets of $\mathbb{R}^n$ introduced and studied by F. Almgren [**Al**], under the name of "restricted sets". A slightly long (but clever) definition is needed.

Let $\Omega \subset \mathbb{R}^n$ be open, and let $E \subset \Omega$ be closed in Ω. The quasiminimality of E will be determined by comparing E with $f(E)$, when f is a suitable deformation. But let us first say what we mean by that.

We only consider Lipschitz mappings f from Ω to itself, but we never require any bound on the Lipschitz constant. For such f, we set

$$W_f = \{x \in \Omega\,;\, f(x) \neq x\}, \tag{2}$$

and demand that

$$W_f \cup f(W) \text{ be relatively compact in } \Omega, \tag{3}$$

(which means that its closure is compact and contained in Ω) and

$$\operatorname{diam}(W \cup f(W)) < \delta, \tag{4}$$

where $\delta \in (0, +\infty]$ is some constant.

The condition (3) acts like a Dirichlet constraint on $\partial\Omega$ (we are not allowed to modify E on $\partial\Omega$), while (4) is a way to make our definition of quasiminimal sets local. We can take $\Omega = \mathbb{R}^n$, and then (3) simply says that $f(x) = x$ out of a compact set. We can also take $\delta = +\infty$, and then (4) is void.

The statement below is still valid if we add the constraint that f is the endpoint of a one-parameter family $\{f_t\}$ of deformations, with conditions like (3) and (4) for the union of the $W_{f_t} \cup f_t(W_{f_t})$, but let us refer to [**DS10**] for details.

Note that our deformations f do not need to be bijective; we also allow Lipschitz functions f that crush parts of E.

Let $d < n$ be an integer, and $M \geq 1$ another constant. We say that E is an (Ω, M, δ)-quasiminimal set of dimension d when

$$H^d(E \cap B) < +\infty \text{ for every closed ball } B \subset \Omega \tag{5}$$

and

$$H^d(E \cap W_f) \leq M H^d(f(E \cap W_f)) \tag{6}$$

for every Lipschitz mapping $f : \Omega \to \Omega$ such that (3) and (4) hold.

Thus, when we do the comparison, we only count what has moved.

When $M = 1$, we get minimizers, i.e., sets E for which $H^d(E)$ cannot be made strictly smaller when we replace E with $f(E)$, f as above, but allowing M to be large can be very useful. The conditions (3) and (4) can be useful too. For instance, the line segment (a, b) is $(\Omega, 1, +\infty)$-quasiminimal in $\Omega = \mathbb{R}^n \setminus \{a, b\}$, and a circle is $(\mathbb{R}^n, M, \delta)$-quasiminimal for any given $M > 1$, if we take δ small enough (depending on M and the radius).

Here we want to work with δ and M fixed, and talk about uniform rectifiability, but often one considers smaller classes of sets E such that E is $(\Omega, M(\delta), \delta)$-quasiminimal for all small δ, with a function $M(\delta)$ that tends to 1 when δ tends to 0. Then E is asymptotically minimal at small scales, and there are more precise regularity results in this case [**Al**].

It is fairly easy to see that the image of a quasiminimal set by a biLipschitz mapping is a quasiminimal set (with probably worse constants). Thus Lipschitz graphs are quasiminimal, for instance, and so we cannot hope for great smoothness results.

A typical way in which quasiminimal sets may arise is when you try to minimize a functional like $J(E) = \int_E h(x) dH^d(x)$ under some topological constraint, with a function h on Ω such that $1 \leq h(x) \leq M$ everywhere. Then (5) may come directly from minimality (if the topological constraints are sufficiently stable), and if h is complicated it may be hard to use more precise information than that. A typical situation like this already occurs in [**DS9**], where $h = \mathbf{1}_F + M\mathbf{1}_{\mathbb{R}^n \setminus F}$ and we do not have so much control on F.

So the reader is asked to believe that quasiminimal sets show up. And the main point here is that, modulo a set of vanishing Hausdorff measure, they are locally Ahlfors-regular and uniformly rectifiable. More precisely, call

$$E^* = \{x \in E\,; H^d(E \cap B(x,r)) > 0 \text{ for all } r > 0\} \tag{7}$$

the closed support of the restriction of H^d to E. It is easy to see that E^* also is closed in Ω, and

$$H^d(E \setminus E^*) = 0. \tag{8}$$

Theorem 9. *For each choice of n, d, and $M \geq 1$, there are constants C_0 and C_1 with the following properties. Suppose $\Omega \subset \mathbb{R}^n$ is open, E is an (Ω, M, δ)-quasiminimal set of dimension d, and let E^* be as in (7). Then for all choices of $x \in E^*$ and $0 < r < \delta$ such that $B(x, 3r) \subset \Omega$, we can find a set F such that*

$$E^* \cap B(x,r) \subset F \subset E^* \cap B(x, 2r), \tag{10}$$

and F is Ahlfors-regular with constant C_0 and contains big pieces of Lipschitz graphs with constant C_1.

In particular, F is uniformly rectifiable. The local Ahlfors-regularity and plain rectifiability were proved in [**Al**], and the existence of big pieces of Lipschitz graphs was proved in [**DS10**]. See [**Da12**] for another application of Theorem 9.

Hopefully other examples of uniformly rectifiable sets will show up; we would like to say that this will often happen in situations where there is a sufficient amount of invariance, and rectifiability holds. And that uniform rectifiability is substantially easier to use.

Exercises

Exercise 1. We want to prove that if μ is a measure on E that satisfies (1.1), then there is a constant $C = C(C_0, n, d)$ such that

$$C^{-1}\mu(A) \leq H^d(E \cap A) \leq C\mu(A) \text{ for every Borel set } A. \tag{$*$}$$

(1) Check the first inequality. You will need to know the definition of Hausdorff measure.

(2) Prove that $H^d(E \cap B(x,r)) \leq Cr^d$ for $x \in \mathbb{R}^n$ and $r > 0$. (Cover $E \cap B(x,r)$ by balls of radius ε for each small $\varepsilon > 0$. You will need less than $C\varepsilon^{-d}r^d$ of them, if you take their centers at mutual distances $\geq \frac{\varepsilon}{2}$).
(3) Let K be a compact subset of E. Check that for each $\delta > 0$ we can find $\varepsilon > 0$ such that $\mu(K^\varepsilon) \leq \mu(K) + \delta$, where K^ε is an $\varepsilon-$neighborhood of K.
(4) Show that $H^d(K) \leq C\mu(K)$ for all compact subsets of E. (Cover K by balls of radius ε with centers on K and at mutual distances $\geq \frac{\varepsilon}{2}$).
(5) Complete the proof of $(*)$. (Use the regularity of $H^d_{|E}$).

Exercise 2.
(1) Check that in Definition 1.2, it is enough to verify (4) for $x \in z(I)$ and $0 < r < \operatorname{diam}(E)$.
(2) Show that if $z : I \to \mathbb{R}^n$ is an Ahlfors-regular mapping with constant C_1, $z^{-1}(x)$ never has more than $\frac{C_1^2}{2} + 1$ points. [Consider little intervals near these points.]

Exercise 3. Let Γ be a simple curve in $\mathbb{R}^n$, which we also assume to be locally of finite length. If Γ is open, we say that Γ is chord-arc if there exists a constant $C \geq 1$ such that y, z in Γ, the length of the arc of Γ from x to y is at most $C|x-y|$. If Γ is a loop, take the same definition, but only consider the shortest arc of Γ from x to y. Show that chord-arc curves are Ahlfors-regular curves.

Exercise 4. Let I be an interval in $\mathbb{R}$, choose an origin x_0 in I, and call $\mathcal{Z}(C_1)$ the set of mappings $z : I \to \mathbb{R}^n$ that are Ahlfors-regular with a given constant C_1 and for which $z(x_0) = 0$. Show that $\mathcal{Z}(C_1)$ is sequentially compact, in the sense that if $\{z_k\}$ is a sequence in $\mathcal{Z}(C_1)$, we can extract a subsequence that converges uniformly on bounded subsets of I, and the limit lies in $\mathcal{Z}(C_1)$.

Exercise 5.
(1) Check that $z(I)$ is an Ahlfors-regular set when $z : I \to \mathbb{R}^n$ is an Ahlfors-regular mapping. [Hint: take the image of dx.] Same question for an ω-regular mapping defined on (a cube of) $\mathbb{R}^d$.
(2) Show that $H^d(A)$ is equivalent to the Lebesgue measure of $z^{-1}(A)$ for $A \subset z(I)$ measurable. [Use a previous exercise.]

Exercise 6. Let E be the union of the two axes in the plane. Check that E is not an Ahlfors-regular curve, but find an Ahlfors-regular mapping z from $\mathbb{R}$ to $\mathbb{R}^2$ such that $E \subset z(\mathbb{R})$.

Exercise 7. Check that the two definitions of uniform rectifiability in dimension 1 are equivalent. That is, if $z : I \to \mathbb{R}^n$ is ω-regular for some weight $\omega \in A_1$, then there is an Ahlfors-regular mapping $z^* : I^* \to \mathbb{R}^n$ such that $z^*(I^*) = z(I)$.

Exercise 8. [Not every A_∞ weight gives ω-regular mappings.]
(1) Let ω be a continuous weight on $\mathbb{R}^d$. Show that if $z : \mathbb{R}^d \to \mathbb{R}^n$ satisfies (1.9), then z is locally Lipschitz and $|\nabla z(x)| \leq \omega(x)^{1/d}$ almost-everywhere.
(2) Set $\omega(x,y) = x^\alpha$ for $(x,y) \in \mathbb{R}^2$. This is a weight in A_∞. Show that if z satisfies (1.9), then z is constant on the vertical axis. Show that this that there is no ω-regular mapping for this weight. In fact, a necessary condition is the "strong A_∞ condition" from [**DS1**]. But Semmes [**Se4**]

showed that at least for each weight $\omega \in A^1(\mathbb{R}^d)$, then for n large enough there exist ω-regular mappings $z : \mathbb{R}^d \to \mathbb{R}^n$.

Exercise 9. [T. Hrycak's construction with venetian blinds.] Given a compact interval I in the plane, a small angle α, and a big integer N, call $F_{\alpha,N}(I)$ the closure of the set obtained from I by cutting it in N equal intervals and then replacing each of the N pieces with its image by a rotation of angle α centered at the center of the piece.

To construct our basic set E, we choose a very small α, a very large N, and start with a unit interval I_0. We replace I_0 with $F_{\alpha,N}(I_0)$, then replace each of the N intervals I that compose $F_{\alpha,N}(I_0)$ with $F_{\alpha,N}(I)$, then iterate. We iterate M times, where M is a little larger than α^{-1}. For definiteness, take for M the integer part of $1 + \alpha^{-1}$. The final set E is thus composed of about N^M intervals of length N^{-M}. For the rest of the exercise, we assume that α is small enough, and then N is small enough (depending on α).

(1) Show that E is Ahlfors-regular and contained in an Ahlfors-regular curve, with constants that do not depend on α and N.
(2) Call π the orthogonal projection onto any line L. Show that $H^1(\pi(E)) \leq C\alpha$. [Observe that there is a stage of the construction where all the segments I make angles less than α with L; if N is large enough, the small projection at this stage will not be disturbed by the rest of the construction.]
(3) Find an Ahlfors-regular set that is contained in an Ahlfors-regular curve, but does not have big projections. [Glue different sets E.]

Exercise 10. Check the description of the Rising Sun Lemma (see below (1.26)).

Exercise 11. Find a sequence of compact sets $E_k \subset [0,1]$ that converge to $[0,1]$, but such that $H^1(E_k)$ tends to any given number $\alpha \in [0,1)$.

Exercise 12. Find a sequence of compact curves in the plane, each with finite length, that converge to a set E which is not connected. Hint: $E = (-\infty, 0] \cup [1, +\infty)$.

Exercise 13. A logical attempt to deduce Proposition 1.29 from the case when E is bounded is the following. Suppose $0 \in E$, and suppose to simplify that for each $k \geq 0$ there is an Ahlfors-regular set E_k such that $E \cap B(0, 2^k) \subset E_k \subset B(0, 2^{k+1})$. The special case when E is bounded gives an Ahlfors-regular mapping $z_k : I_k \to E_k$ such that $z_k(I_k) = E_k$. We can even assume that $I_k = [0, 2^k]$ and $z_k(0) = 0$. Then we can extract a subsequence for which z_k converges to some z. What goes wrong? Check what happens with the union of two lines.

Exercise 14. An Ahlfors-regular set of dimension $n-1$ in $\mathbb{R}^n$ is said to satisfy condition B if there is a constant $C > 1$ such that, for $x \in E$ and $0 < r < \operatorname{diam}(E)$, we can find two balls B_1 and B_2 of radius $C^{-1}r$ that are contained in $B(x,r) \setminus E$, and that lie in different connected components of $\mathbb{R}^n \setminus E$. Show that if E satisfies Condition B, then it has big projections. That is, there is a constant $\theta > 0$ such that if $x \in E$ and $0 < r < \operatorname{diam}(E)$, there is a hyperplane P such that $H^{n-1}(\pi_P(E \cap B(x,r))) \geq \theta\, r^{n-1}$, where π_P denotes the orthonormal projection onto P. [Actually, E contains big pieces of Lipschitz graphs (see [**Da3**] or [**DJ**]), hence it is uniformly rectifiable. Previously Semmes had showed that singular integrals define bounded operators on E.

Exercise 15. Let $E \subset \mathbb{R}^n$ be Ahlfors-regular of dimension d, and define the numbers $\beta_E(x,r)$ and $\beta_{E,q}(x,r)$ as in (2.1) and (2.2). Show that $\beta_E(x,r) \leq C\beta_{E,q}(x,2r)^{q/d+q}$ for $x \in E$ and $0 < r < \operatorname{diam}(E)$. Hint: pick an almost optimal d-plane P in the definition of $\beta_{E,q}(x,2r)$, suppose $y \in E \cap B(y,r)$ lies at distance d from P, and estimate the contribution of $E \cap B(y,d/2)$ to $\beta_{E,q}(x,2r)$.

Exercise 16. Check that Theorem 2.3 follows from Theorem 2.9.

Exercise 17. Here we want to prove the estimate (3.23) on the numbers $\gamma_{2,+\infty}(x,t)$. To simplify things, we shall restrict to one dimension. Let φ be a smooth function on $\mathbb{R}$, with integral 1 and support in $[-1,1]$, and set $\psi = \varphi'$. Also set $\varphi_t(x) = \frac{1}{t}\varphi(x/t)$ and $\psi_t(x) = \frac{1}{t}\psi(x/t)$ for $t > 0$.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth and compactly supported. For $x \in \mathbb{R}$ and $t > 0$, set $a_{x,t} = (\frac{1}{t}\psi_t * f)(x) = (\varphi_t * f')(x)$ and $b_{x,t} = (\varphi_t * f)(x)$. Then set $A_{x,t}(y) = a_{x,t}(y-x) + b_{x,t}$ for $y \in \mathbb{R}$.

(1) Show that $\displaystyle\int_x \int_t \int_{y \in B(x,t)} |f(y) - A_{x,t}(y)|^2 \frac{dxdydt}{t^{n+3}} \leq C||f'||^2$. Check that (3.23) follows. [This may be long; see Lemma 5.9 for a hint.]
(2) Deduce (3.24) from (3.23). Do it also in $\mathbb{R}^d$.

Exercise 18. State analogues of the LCV or LS conditions for Lipschitz functions. A little longer: prove them. It could be easier to prove directly a (slightly stronger) L^2 version.

Exercise 19. Show that if E is Ahlfors-regular of dimension d and convex (or symmetric with respect to each of its points), then it is a d-plane.

Exercise 20. Set $H = \{0,1\}^{\mathbb{N}}$ and, for $\varepsilon = \{\varepsilon_k\} \in H$, $\varphi(\varepsilon) = \sum_k \varepsilon_k 4^{-k}$.

(1) Show that $E = \varphi(H)$ is Ahlfors-regular of dimension 1/2. [Hint: there is a natural measure on H.] Comment: thus E is a "middle half Cantor set". The set $E \times E \subset \mathbb{R}^2$ is our favorite non-rectifiable Ahlfors-regular set of dimension 1.
(2) Suppose we are given, for each finite sequence $(\varepsilon_0, \dots, \varepsilon_k)$, a number $a(\varepsilon_0, \dots, \varepsilon_k) \in [-1,1]$. Consider the function g defined on H by $g(\varepsilon) = \sum_k a(\varepsilon_0, \dots, \varepsilon_k) 4^{-k-1} \varepsilon_{k+1}$. Show that $g \circ \varphi^{-1}$ is Lipschitz on E. The point of this example is to show that there are lot of Lipschitz functions on E, much more than we would expect on a rectifiable set. In particular, the analogue of Rademacher's theorem on the differentiability almost-everywhere of Lipschitz functions fails brutally.
(3) Show that E does not satisfy the WALA, or the GWALA (see the end of Section 3). The same thing would be true on $E \times E$.

Exercise 21. Let E be Ahlfors-regular of dimension d, and let Δ be a collection of dyadic cubes as in (4.1)-(4.5).

(1) Show that for each $Q \in \Delta_k$, we can find a "center" $c_Q \in Q$ such that $\operatorname{dist}(c_Q, E \setminus Q) \geq C^{-1}2^k$.
(2) Show that if Q, $R \in \Delta_k$, $\int_Q \int_R |x-y|^{-d} dH^d(x) dH^d(y) \leq C2^{kd}$. This is the typical computation that makes (4.5) useful.

Exercise 22. Use the computations of Lemma 5.9 (and the $T(1)$-theorem) to show that for $A : \mathbb{R} \to \mathbb{R}$ Lipschitz, the kernel $\frac{A(x)-A(y)}{(x-y)^2}$ defines a bounded operator on

$L^2(\mathbb{R})$ (the first Calderón commutator). A little more complicated: do the same thing with the kernel $\frac{(A(x)-A(y))^2}{(x-y)^3}$.

Exercise 23. Show that chord-arc curves (see Exercise 3) are quasiminimal sets (with appropriate constants).

Exercise 24. Check (5.26).

Bibliography

[Ah1] L. Ahlfors, Zur Theorie der Überlagerungsfläschen, Acta Math. 65 (1935), 157-194.

[Ah2] L. Ahlfors, Bounded analytic functions, Duke Math. J. 14 (1947), 1-11.

[Al] F. J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, Memoirs of the Amer. Math. Soc. 165, volume 4 (1976), i-199.

[Am] L. Ambrosio, Existence theory for a new class of variational problems, Arch. Rational Mech. Anal. 111 (1990), 291-322.

[AFP] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Clarendon Press, Oxford 2000.

[BJ1] C. Bishop and P. Jones, Harmonic measure and arclength. Ann. of Math. (2) 132 (1990), no. 3, 511–547.

[BJ2] C. Bishop and P. Jones, Harmonic measure, L^2 estimates and the Schwarzian derivative, J. Anal. Math. 62 (1994), 77–113.

[BJ3] C. Bishop and P. Jones, Wiggly sets and limit sets, Ark. Mat. 35 (1997), no. 2, 201–224.

[C] V. Chousionis, Singular integrals on Sierpinski gaskets, Publ. Mat. 53 (2009), no. 1, 245–256.

[Ca] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. (2) 76 (1962), 547-559.

[CL] M. Carriero and A. Leaci, Existence theorem for a Dirichlet problem with free discontinuity set, Nonlinear Anal. 15 (1990), 661-677.

[Ch1] M. Christ, Lectures on Singular Integral Operators, Regional Conference series in Mathematics 77, AMS 1990.

[Ch2] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990), 1367-1381.

[ChJ] M. Christ and and J-L. Journé, Polynomial growth estimates for multilinear singular integral operators, Acta Math. 159 (1987), 51-80.

[CM] R. R. Coifman and Y. Meyer, Une généralisation du théorème de Calderón sur l'intégrale de Cauchy, Proceedings of the seminar on Fourier analysis, El Escorial 1979, edited by M. De Guzmán and I. Peral, pp. 87–116, Asoc. Mat. Espa nola, 1, Asociation Matematica Española 1980.

[CMM] R. R. Coifman, A. McIntosh and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, Ann. of Math. 116 (1982), 361-387.

[DMS] G. Dal Maso, J.-M. Morel, and S. Solimini, A variational method in image segmentation: Existence and approximation results, Acta Math. 168 (1992), 89-151.

[Da1] G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complexe, Ann. Sci. Ec. Norm. Sup. 17 (1984), 157-189.

[Da2] G. David, Opérateurs d'intégrale singulière sur les surfaces régulières, Ann. Sci. Ec. Norm. Sup., série 4, t.21 (1988), 225-258.

[Da3] G. David, Morceaux de graphes lipschitziens et intégrales singulières sur une surface, Revista Matematica Iberoamericana, vol. 4, 1 (1988), 73-114.

[Da4] G. David, Wavelets and singular integrals on curves and surfaces, Lecture Notes in Math. 1465, Springer-Verlag 1991.

[Da5] G. David, C-1 arcs for minimizers of the Mumford-Shah functional, SIAM Journal of Appl. Math. Vol. 56, N 3 (1996), 783-888.

[Da6] G. David, Unrectifiable 1-sets have vanishing analytic capacity, Revista Matematica Iberoamericana 14, 2 (1998), 369-479.

[Da7] G. David, Analytic capacity, Cauchy kernel, Menger curvature, and rectifiability, Harmonic Analysis and Partial Differential Equations, Essays in Honor of Alberto P. Calderón, M. Christ, C. Kenig, and C. Sadosky ed., The University of Chicago Press 1999, 183-197.

[Da8] G. David, Analytic capacity, Calderón-Zygmund operators, and rectifiability, Publicacions Matemàtiques 43 (1999), 3-25.

[Da9] G. David, Des intégrales singulières bornées sur un ensemble de Cantor, C. R. Acad. Sci. Paris, t. 332, Série I, p.1-6, 2001.

[Da10] G. David, Limits of Almgren quasiminimal sets, Proceedings of the conference on Harmonic Analysis, Mount Holyoke, A.M.S. Contemporary Mathematics series, Vol. 320 (2003), 119-145.

[Da11] G. David, Singular sets of minimizers for the Mumford-Shah functional, Progress in Mathematics 233, Birkhäuser Verlag, Basel, 2005. xiv+581 pp. ISBN: 978-3-7643-7182-1.

[Da12] G. David, Hausdorff dimension of uniformly non flat sets with topology, Publ. Mat. 48 (2004), no. 1, 187-225.

[DJ] G. David and D. Jerison, Lipschitz approximations to hypersurfaces, harmonic measure, and singular integrals, Indiana U. Math. Journal. 39, 3 (1990), 831-845.

[DM] G. David and P. Mattila, Removable sets for Lipschitz harmonic functions in the plane, Revista Matematica Iberoamericana 16, N1, 2000, 137-215.

[DS1] G. David and S. Semmes, Strong A_∞ weights, Sobolev inequalities, and quasi-conformal mappings, Analysis and partial differential equations, A collection of papers dedicated to Mischa Cotlar, ed. Sadosky, L.N. in Pure and Applied Math., Marcel Dekker 1989, 101-112.

[DS2] G. David and S. Semmes, Quantitative rectifiability and Lipschitz mappings, Transactions A.M.S. 337 (1993), 855-889.

[DS3] G. David and S. Semmes, Singular integrals and rectifiable sets in $\mathbb{R}^n$: au-delà des graphes lipschitziens, Astérisque 193, Société Mathématique de France 1991.

[DS4] G. David and S. Semmes, Analysis of and on uniformly rectifiable sets, A.M.S. series of Mathematical surveys and monographs, Volume 38, 1993.

[DS5] G. David and S. Semmes, On the singular sets of minimizers of the Mumford-Shah functional, Journal de Math. Pures et Appl. 75 (1996), 299-342.

[DS6] G. David and S. Semmes, On a variational problem from image processing, proceedings of the conference in honor of J.-P. Kahane, special issue of the Journal of Fourier Analysis and Applications, 1995, 161-187.

[DS7] G. David and S. Semmes, Uniform rectifiability and Singular sets, Annales de l'Inst. Henri Poincaré, Analyse non linéaire, Vol 13, N 4 (1996), p. 383-443.

[DS8] G. David and S. Semmes, Fractured fractals and broken dreams : Self-similar geometry through metric and measure, Oxford Lecture series in Mathematics and its applications 7, Oxford University Press 1997.

[DS9] G. David and S. Semmes, Quasiminimal surfaces of codimension 1 and John domains, Pacific Journal of Mathematics Vol. 183, N 2 (1998), 213-277.

[DS10] G. David and S. Semmes, Uniform rectifiability and quasiminimizing sets of arbitrary codimension, Memoirs of the A.M.S. Number 687, volume 144, 2000.

[DCL] E. De Giorgi, M. Carriero, and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, Arch. Rational Mech. Anal. 108 (1989), 195-218.

[Do] J. R. Dorronsoro, A characterization of potential spaces, Proc. A.M.S. 95 (1985), 21-31.

[Fa] K. Falconer, Fractal geometry, Mathematical foundation and applications, John Wiley and Sons, 1990.

[Fg] Xiang Fang, The Cauchy integral, analytic capacity and subsets of quasicircles, PhD. Thesis, Yale university.

[Fr1] H. Farag, The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature, Publ. Mat. 43 (1999), no. 1, 251-260.

[Fr2] H. Farag, Curvatures of the Melnikov type, Hausdorff dimension, rectifiability, and singular integrals on $\mathbb{R}^n$, Pacific J. Math. 196 (2000), no. 2, 317-339.

[Fe] H. Federer, Geometric measure theory, Grundlehren der Mathematishen Wissenschaften 153, Springer Verlag 1969.

[Ga] J. Garnett, Bounded analytic functions, Academic Press 1981.

[GaV] J. Garnett and J. Verdera, Analytic capacity, bilipschitz maps and Cantor sets, Math. Res. Lett. 10 (2003), no. 4, 515–522.

[Hu1] P. Huovinen, Singular integrals and rectifiability of measures in the plane, Dissertation, University of Jyväskylä, Jyväskylä, 1997, Ann. Acad. Sci. Fenn. Math. Diss. No. 109, (1997), 63 pp.

[Ha1] I. Hahlomaa, Curvature integral and Lipschitz parametrization in 1-regular metric spaces, Ann. Acad. Sci. Fenn. Math. 32 (2007), no. 1, 99–123.

[Ha2] I. Hahlomaa, Menger curvature and rectifiability in metric spaces, Adv. Math. 219 (2008), no. 6, 1894–1915.

[Hu2] P. Huovinen, A nicely behaved singular integral on a purely unrectifiable set. Proc. Amer. Math. Soc. 129 (2001), no. 11, 3345–3351.

[Jo1] P. Jones, Square functions, Cauchy integrals, analytic capacity, and harmonic measure, Proc. Conf. on Harmonic Analysis and Partial Differential Equations, El Escorial 1987 (ed. J. García-Cuerva), p. 24-68, Lecture Notes in Math. 1384, Springer-Verlag 1989.

[Jo2] P. Jones, Lipschitz and bi-lipschitz functions, Revista Matematica Iberoamericana, vol. 4, 1 (1988), 115-122.

[Jo3] P. Jones, Rectifiable sets and the traveling salesman problem, Inventiones Mathematicae 102, 1 (1990), 1-16.

[JKV] P. Jones, N. Katz, and A. Vargas, Checkerboards, Lipschitz functions and uniform rectifiability, Revista Matemetica Iberoamericana 13 (1997) 189-210.

[JL] P. Jones and G. Lerman, Manifold-like structures of measures via multiscale analysis, in preparation.

[Leg] J.-C. Léger, Menger curvature and rectifiability, Ann. of Math. (2) 149 (1999), no. 3, 831–869.

[Le1] G. Lerman, Geometric transcriptions of sets and their applications to data analysis, PhD thesis, Yale university.

[Le2] G. Lerman, Quantifying curvelike structures of measures by using L_2 Jones quantities, Comm. Pure Appl. Math. 56 (2003), no. 9, 1294–1365.

[MaV] P. Mattila and J. Verdera, Convergence of singular integrals with general measures, J. Eur. Math. Soc. 11 (2009) 257-271.

[MTV1] J. Mateu, X. Tolsa, and J. Verdera, On the semiadditivity of analytic capacity and planar Cantor sets, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 259–278, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.

[MTV2] J. Mateu, X. Tolsa, and J. Verdera, The planar Cantor sets of zero analytic capacity and the local $T(b)$-theorem, J. Amer. Math. Soc. 16 (2003), no. 1, 19–28.

[MV] J. Mateu and J. Verdera, personal communication.

[Ma1] P. Mattila, Smooth maps, null-sets for integralgeometric measure and analytic capacity, Ann. of Math. 123 (1986), 303-309.

[Ma2] P. Mattila, Cauchy singular integrals and rectifiability of measures in the plane, advances in math. 115, 1 (1995), 1-34.

[Ma3] P. Mattila, Geometry of sets and measures in Euclidean space, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press l995.

[Ma4] P. Mattila, On the analytic capacity and curvature of some Cantor sets with non-s-finite length, Publ. Mat. 40 (1996), 127-136.

[Ma5] P. Mattila, Singular integrals, analytic capacity and rectifiability, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. 3 (1997), Special Issue, 797–812.

[Ma6] P. Mattila, Rectifiability, analytic capacity, and singular integrals, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 657–664 (electronic).

[Ma7] P. Mattila, Removable sets for bounded analytic functions, Complex analysis and differential equations (Uppsala, 1997), 242–248, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 64, Uppsala Univ., Uppsala, 1999.

[MMV] P. Mattila, M. Melnikov, and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. 144, 1 (1996), 127-136.

[MP] P. Mattila and D. Preiss, Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals, J. London Math. Soc. (2) 52 (1995), no. 3, 482–496.

[Me] M. Melnikov, Analytic capacity : discrete approach and curvature of measure, Sbornik: Mathematics 186:6 (1995), 827-846.

[MeV] M. Melnikov and J. Verdera, A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs, International Math. Research Notices 7 (1995), 325-331.

[MoS] J.-M. Morel and S. Solimini, Variational methods in image segmentation, Progress in nonlinear differential equations and their applications 14, Birkhäuser 1995.

[Mo] F. Morgan, Geometric measure theory, A beginner's guide, Academic Press 1988.

[MuS1] D. Mumford and J. Shah, Boundary detection by minimizing functionals, IEEE Conference on computer vision and pattern recognition, San Francisco 1985.

[MuS2] D. Mumford and J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Comm. Pure Appl. Math. 42 (1989), 577-685.

[Mu] T. Murai, Boundedness of singular integral operators of Calderón type VI, Nagoya Math. Journal 102 (1986), 127-133.

[NTV1] F. Nazarov, S. Treil, and A. Volberg, Cauchy integral and Calderón-Zygmund operators on non homogeneous spaces, International Math. Res. Notices 1997, 15, 703-726.

[NTV2] F. Nazarov, S. Treil, and A. Volberg, F. Nazarov, S. Treil, and A. Volberg, The Tb-theorem on non-homogeneous spaces, Acta Math. 190 (2003), no. 2, 151–239.

[Ok] K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbb{R}^n$, J. of the London Math. Soc. 46 (1992), 336-348.

[Pa1] H. Pajot, Un théorème géométrique du "voyageur de commerce" en dimension 2, C. R. Acad Sci. Paris, t. 323, Série I (1996), 13-16.

[Pa2] H. Pajot, Conditions quantitatives de rectifiabilité, Bulletin de la Société Mathématique de France, Vol. 125 (1997), 15-53.

[Pa3] H. Pajot, Analytic capacity, rectifiability, Menger curvature and the Cauchy integral, L.N. in Math. 1799, Springer-Verlag 2002.

[Pra] L. Prat, Potential theory of signed Riesz kernels: capacity and Hausdorff measure, Int. Math. Res. Not. 2004, no. 19, 937–981.

[Pr] D. Preiss, Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities, Ann. of Math. 125 (1987), 537-643.

[Re] E. R. Reifenberg, Solution of the Plateau problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960), 1-92.

[Sc1] R. Schul, Analyst's traveling salesman theorems, A survey. In the tradition of Ahlfors-Bers. IV, 209–220, Contemp. Math., 432, Amer. Math. Soc., Providence, RI, 2007.

[Sc2] R. Schul, Subsets of rectifiable curves in Hilbert space—the analyst's TSP, J. Anal. Math. 103 (2007), 331–375

[Se1] S. Semmes, A criterion for the boundedness of singular integrals on hypersurfaces, Trans. Amer. Math. Soc. 311 (1989), no. 2, 501–513.

[Se2] S. Semmes, Differentiable function theory on hypersurfaces in $\mathbb{R}^n$ (without bounds on their smoothness), Indiana Univ. Math. Journal 39 (1990), 985-1004.

[Se3] S. Semmes, Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbb{R}^n$, Indiana Univ. Math. Journal 39 (1990), 1005-1036.

[Se4] S. Semmes, Bilipschitz mappings and strong A_∞ weights, Ann. Acad. Sci. Fenn. Ser. A I Math. 18 (1993), 211-248.

[Se5] S. Semmes, Finding structure in sets with little smoothness, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 875–885, Birkhäuser, Basel, 1995.

[Se6] S. Semmes, Good metric spaces without good parameterizations, Rev. Mat. Iberoamericana 12 (1996), 187-275.

[Se7] S. Semmes, On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. Rev. Mat. Iberoamericana 12 (1996), no. 2, 337-410.

[St] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton university press 1970.

[To1] X. Tolsa, L^2-boundedness of the Cauchy integral operator for continuous measures. Duke Math. J. 98 (1999), no. 2, 269–304.

[To2] X. Tolsa, Principal values for the Cauchy integral and rectifiability, Proc. Amer. Math. Soc. 128 (2000), no. 7, 2111–2119.

[To3] X. Tolsa, On the analytic capacity γ_+, Indiana Univ. Math. J. 51 (2002), no. 2, 317-343.

[To4] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. 190 (2003), no. 1, 105–149.

[To5] X. Tolsa, Analytic capacity, rectifiability, and the Cauchy integral, International Congress of Mathematicians Vol. II, 1505–1527, Eur. Math. Soc., Zrich, 2006.

[To6] X. Tolsa, Analytic capacity, rectifiability, and the Cauchy integral, International Congress of Mathematicians Vol. II, 1505–1527, Eur. Math. Soc., Zrich, 2006.

[To7] X. Tolsa, Principal values for Riesz transforms and rectifiability, J. Funct. Anal. 254 (2008), no. 7, 1811–1863.

[Ve1] J. Verdera, Removability, capacity and approximation, in Complex Potential Theory, (Montreal, PQ, 1993), 419-473, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 439, Kluwer Acad. Publ., Dordrecht, 1994.

[Ve2] J. Verdera, On the T(1)-Theorem for the Cauchy integral, Ark. Mat. 38 (2000), no. 1, 183–199.

[Ve3] J. Verdera, The fall of the doubling condition in Calderón-Zygmund theory, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. 2002, Vol. Extra, 275–292.

[Ve4] J. Verdera, L^2 boundedness of the Cauchy integral and Menger curvature, Harmonic analysis and boundary value problems (Fayetteville, AR, 2000), 139–158, Contemp. Math., 277, Amer. Math. Soc., Providence, RI, 2001.

[Vh] M. Vihtilä, The boundedness of Riesz s-transforms of measures in $\mathbb{R}^n$, Proc. Amer. Math. Soc. 124 (1996), no. 12, 3797–3804.

[Vi] A. G. Vitushkin, The analytic capacity of sets in problems of approximation theory, Uspekhi Mat. Nauk 22:6 (1967), 141-199 ; English transl. in Russian Math. Surveys 22.

[Vo] A. Volberg, Calderón-Zygmund capacities and operators on nonhomogeneous spaces, CBMS Regional Conference Series in Mathematics 100, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003. iv+167 pp. ISBN: 0-8218-3252-2

[Zy] A. Zygmund, Trigonometric series, Cambridge University Press 1968.