Variational methods applied to discrete and evolutionary brittle damage models

Méthodes variationnelles appliquées à l'étude de modèles discrets et dynamiques en endommagement brutal

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Titre : Méthodes variationnelles appliquées à l'étude de modèles discrets et dynamiques en endommagement brutal

Mots clés : Calcul des variations, théorie géométrique de la mesure, Γ-convergence, relaxation, homogénéisation, mécanique des milieux continus dissipatifs, élasticité linéaire, rupture, plasticité, endommagement

Résumé : L'objectif de cette thèse consiste à étudier un modèle de la mécanique de l'endommagement brutal, dans différents régimes où la zone endommagée se concentre sur des ensembles de mesure nulle, et à identifier les modèles limites effectifs obtenus par analyse asymptotique basée sur la Γ-convergence des énergies totales. Nous commencerons par présenter différents modèles de la mécanique de l'endommagement et rappellerons quelques résultats fondateurs du cadre mathématique variationnel considéré par la suite. Ensuite, nous étudierons le comportement asymptotique d'une famille d'énergies d'endommagement brutal dans un cadre statique et discret en espace, où les énergies considérées agissent seulement sur des déplacements continus et affines par morceaux. Nous montrerons l'existence de différents modèles limites selon les lois d'échelle des régimes associés. En particulier, nous prouverons un résultat d'approximation discrète par éléments finis adaptatifs de la fonctionnelle de Griffith isotrope bidimensionnelle (en termes de Γ-convergence) par une suite d'énergies discrètes d'endommagement brutal définies pour des déplacements affines par morceaux et continus. Enfin, nous nous attarderons sur une interaction surprenante entre relaxation et irréversibilité au cours d'une évolution quasi-statique en mécanique de l'endommagement, en montrant que l'évolution quasi-statique d'un matériau élastique subissant un processus de déformation plastique indépendant de la vitesse de charge ne peut pas être obtenue de façon systématique comme modèle limite d'évolutions quasi-statiques d'endommagement brutal.

Title : Variational methods applied to discrete and evolutionary brittle damage models

Keywords : Calculus of variations, geometric measure theory, Γ-convergence, relaxation, homogenization, continuum mechanics with dissipative phenomena, linear elasticity, fracture, plasticity, damage

Abstract : The purpose of this thesis consists in studying brittle damage models, in different regimes where the damaged zone concentrates on vanishingly small sets, and in identifying the nature of the effective limit models obtained by means of an asymptotic analysis based on the Γ-convergence of the total energies. First, we introduce several damage mechanical models and recall some seminal results of the variational mathematical framework considered hereafter. Then, we address the question of the asymptotic analysis of brittle damage energies in the discrete and static setting, where the energies are restricted to piecewise affine continuous displacements. We exhibit different effective limit models according to the related regimes' scaling laws. In particular, we prove a discrete adaptive finite element approximation (in terms of Γ-convergence) of the isotropic two-dimensional Griffith functional, by a sequence of discrete brittle damage energy functionals defined on continuous piecewise affine displacements. Finally, we focus on a peculiar interplay between relaxation and irreversibility throughout a quasi-static evolution in damage mechanics, by showing that the quasi-static evolution of an elastic material undergoing a rate-independent process of plastic deformation cannot be systematically derived as the limit model of a sequence of quasi-static brittle damage evolutions.

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"La patience est l'art d'espérer."

Luc de Clapiers, marquis de Vauvenargues

"If you were right, I would agree with you."

Robin Williams, *Awakenings*

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1 - Introduction Générale

Les travaux présentés dans cette thèse portent sur l'analyse mathématique de modèles dissipatifs en mécanique des milieux continus, abordés du point de vue du calcul des variations et de la théorie géométrique de la mesure. Nous nous placerons dans le cadre de l'élasticité linéaire, sous l'hypothèse des petites déformations, et porterons une attention particulière sur certains phénomènes dissipatifs tels que l'endommagement (brutal ou progressif), la plasticité parfaite et la rupture fragile. Plus précisément, cette thèse a pour but d'étudier des modèles d'endommagement brutal sous différentes lois d'échelle et d'établir, par le biais d'une analyse asymptotique menée par Γ-convergence, la nature des modèles effectifs obtenus dans ces régimes respectifs. Cette question sera d'abord abordée dans un cadre statique aux chapitres 2 et 3, puis quasi-statique au chapitre 4.

Nous commençons ce chapitre introductif en rappelant succinctement quelques notions de base en thermomécanique des milieux continus et discuterons de la modélisation des milieux dissipatifs qui en découle, afin d'introduire et de justifier les modèles dissipatifs variationnels évoqués dans la suite de ce mémoire. Nous présenterons ensuite les objectifs de cette thèse, notamment en motivant l'articulation que l'on entend entre modèle d'endommagement et lois d'échelle. Nous décrirons et analyserons enfin les résultats obtenus au cours de cette thèse, avant de conclure cette introduction générale en introduisant les notations et outils de base empruntés ici.

1.1 . Mécanique des milieux continus dissipatifs

Ce paragraphe, à vocation heuristique, se base sur le livre de Gurtin, Fried et Anand [67], auquel nous renvoyons le lecteur pour une étude approfondie et rigoureuse de la mécanique et de la thermodynamique des milieux continus.

Nous considérons un matériau continu dont la configuration au repos, *i.e.* de référence, est un ouvert borné et régulier $\Omega \subset \mathbb{R}^N$. Au cours d'un intervalle de temps $[0, T]$, le matériau se déforme sous l'effet du chargement auquel il est soumis. Nous désignons par $\phi(t, \Omega)$ la configuration déformée qu'il occupe au temps $t \in [0, T]$, où nous supposerons que le champ des déformations

$$\phi : [0, T] \times \Omega \to \mathbb{R}^N$$

est régulier, afin de simplifier la suite des raisonnements de ce paragraphe. La position d'un point $x \in \Omega$ dans la configuration déformée au temps t est ainsi donnée par $\phi(t, x)$. Afin d'assurer les conditions de non-interpénétration de la matière et la préservation de l'orientation de l'espace, nous faisons l'hypothèse (classique) que $\phi(t, \cdot)$ est injective et que le jacobien du gradient (en espace) des déformations $\nabla\phi(t, \cdot)$ satisfait $\det \nabla\phi(t, \cdot) > 0$ dans Ω, en tout temps t. Toute courbe dans la configuration d'origine subit alors un changement de longueur mesuré à l'aide du tenseur de Cauchy-Green

$$\nabla\phi(t, \cdot)^T \nabla\phi(t, \cdot)$$

à l'instant t. Au cours du mouvement, les interactions entre les parties du milieu ou entre le milieu et son environnement extérieur résultent de trois types d'efforts :

(i) des forces de contact T (par unité de surface) à l'interface entre des parties internes adjacentes du corps,

(ii) des forces de contact g (par unité de surface) exercées par l'environnement extérieur sur le bord du milieu,

(iii) des forces volumiques f (par unité de volume) exercées par l'environnement extérieur à l'intérieur du milieu.

Un des axiomes fondamentaux de la mécanique des milieux continus concerne les forces de contact : si S est une surface régulière dans la configuration déformée $\phi(t,\Omega)$ séparant le matériau en deux parties P_1 et P_2, l'hypothèse de Cauchy suppose que P_2 exerce sur P_1 une force par unité de surface $T(\nu(y), y, t)$ en tout point $y \in S$, ne dépendant que de la normale ν à S orientée de P_1 vers P_2.

Conservation de la masse Les milieux continus possèdent une masse qui doit être conservée au cours du mouvement. Autrement dit, en notant $\varrho(t,\cdot)$ la masse volumique du milieu à l'instant t, le principe de conservation de la masse assure que pour toute sous-partie $\omega \subset \Omega$,

$$\int_{\phi(t,\omega)} \varrho(t,x)\,dx = \int_\omega \varrho(0,x)\,dx = \int_\omega \varrho(t,\phi(t,x))\mathrm{det}\big(\nabla\phi(t,x)\big)\,dx$$

où nous avons utilisé la formule de changement de variables. En particulier, il vient que

$$\varrho(t,\phi(t,x))\mathrm{det}\big(\nabla\phi(t,x)\big) = \varrho(0,x) \quad \text{pour tout } x \in \Omega \text{ et en tout temps.}$$

En introduisant la vitesse dans la configuration déformée $v(t,\phi(t,x)) := \dot\phi(t,x)$, on en déduit la version locale de la conservation de la masse :

$$\dot\varrho(t,y) + \mathrm{div}\,(\varrho v)(t,y) = 0 \quad \text{pour tout } t \in [0,T] \text{ et tout } y \in \phi(t,\Omega).$$

Théorème de Cauchy Les bilans de la quantité de mouvement et du moment cinétique expriment le fait que la variation temporelle de la quantité de mouvement (respectivement, du moment cinétique) est donnée par la somme des forces extérieures (respectivement, des moments des forces extérieures) appliquées au système. Autrement dit, on a pour toute sous-partie $\omega \subset \Omega$ et en tout temps :

$$\frac{d}{dt}\int_{\phi(t,\omega)} \varrho(t,y)v(t,y)\,dy = \int_{\phi(t,\omega)} f(t,y)\,dy + \int_{\partial\phi(t,\omega)} T(\nu,y,t)\,d\mathcal{H}^{N-1}(y) \tag{1.1.1}$$

et

$$\frac{d}{dt}\int_{\phi(t,\omega)} \varrho(t,y)v(t,y)\wedge y\,dy = \int_\omega f(t,y)\wedge y\,dy + \int_{\partial\phi(t,\omega)} T(\nu,y,t)\wedge y\,d\mathcal{H}^{N-1}(y)$$

où $\wedge$ désigne le produit vectoriel (ici, $N = 2$ ou 3). Il en découle le Théorème de Cauchy, qui assure que l'application $\nu \mapsto T(\nu,y,t)$ est symétrique et linéaire, d'où l'existence du tenseur des contraintes de Cauchy, $\sigma(t,y) \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$, tel que

$$T(\nu,y,t) = \sigma(t,y)\nu \quad \text{pour tout } y \in \phi(t,\Omega).$$

Hypothèse des petites transformations Dans ce mémoire, nous nous plaçons sous l'hypothèse des petites déformations, où la différence $\nabla\phi(t,\cdot) - I_N$ entre le gradient des déformations et l'identité est négligeable. Il est alors naturel d'introduire, à chaque instant t, le déplacement $u(t,x) := \phi(t,x) - x$ d'un point $x \in \Omega$ ainsi que le tenseur des déformations (de Green-St. Venant)

$$E_t := \frac{1}{2}\left(\nabla\phi(t,\cdot)^T\nabla\phi(t,\cdot) - I_N\right).$$

Le gradient et le tenseur des déformations sont ainsi liés au gradient du déplacement par les relations

$$\nabla\phi(t,\cdot) = I_N + \nabla u(t,\cdot) \quad \text{et} \quad E_t = \frac{1}{2}\left(\nabla u(t,\cdot) + \nabla u(t,\cdot)^T + \nabla u(t,\cdot)^T\nabla u(t,\cdot)\right).$$

Puisque le gradient des déplacements reste petit sous l'hypothèse des petites déformations, nous pouvons raisonnablement approcher le tenseur des déformations E_t par le tenseur dit des déformations linéarisées

$$e(t,u) := \frac{1}{2}\left(\nabla u(t,\cdot) + \nabla u(t,\cdot)^T\right)$$

où l'on a négligé le terme d'erreur quadratique dans l'expression de E_t. Nous pouvons également raisonnablement supposer que la configuration déformée du matériau coïncide en tout temps avec sa configuration au repos, à un déplacement homogène en espace près. En particulier, le principe de conservation de la masse assure que la masse volumique du corps $\varrho(t,x) =: \varrho(x)$ est indépendante du temps, de sorte que le bilan de la quantité de mouvement (1.1.1) conduit au *principe fondamental de la dynamique* :

$$\varrho(x)\dot{v}(t,x) = f(t,x) + \operatorname{div}\sigma(t,x) \quad \text{pour tout } x \in \Omega \text{ et en tout temps } t \in [0,T].$$

En imposant des conditions aux limites de type Neumann $g : [0,T] \times \Gamma_N \to \mathbb{R}^N$ et de Dirichlet $w : [0,T] \times \Gamma_D \to \mathbb{R}^N$, avec Γ_D et Γ_N deux surfaces telles que $\partial\Omega = \Gamma_D \sqcup \Gamma_N$, les équations du mouvement s'écrivent alors

$$\begin{cases} \varrho\ddot{u} = f + \operatorname{div}\sigma & \text{dans } [0,T] \times \Omega, \\ \sigma^T = \sigma & \text{dans } [0,T] \times \Omega, \\ \sigma\nu = g & \text{sur } [0,T] \times \Gamma_N, \\ u = w & \text{sur } [0,T] \times \Gamma_D. \end{cases}$$

Lorsque le mouvement est suffisamment lent, on peut négliger les effets d'inertie et les équations d'équilibre s'obtiennent alors à partir des équations du mouvement en négligeant le terme d'accélération :

$$f + \operatorname{div}\sigma = 0 \quad \text{dans } [0,T] \times \Omega.$$

Nous avons alors affaire à une évolution quasi-statique où le milieu cherche en chaque instant un état d'équilibre.
Dans la suite de ce mémoire, les équations du mouvement serons réécrites sous une formulation faible en terme d'énergie, permettant de s'affranchir des hypothèses de régularité forte des solutions.

Modélisation des milieux dissipatifs à l'aide d'une variable interne À ce stade de la modélisation, les équations du mouvement ne forment pas un système fermé. Il manque des relations liant le tenseur des contraintes et les déformations : c'est la loi de comportement, traduisant les propriétés physiques du matériau. Prenons par exemple le cas d'un solide purement élastique, caractérisé par la réversibilité de son comportement : il revient précisément à sa configuration de référence après un processus de charge-décharge, sans affaiblissement de ses propriétés élastiques. La modélisation des milieux continus élastiques (non dissipatifs) en petites déformations repose sur l'hypothèse que le tenseur des contraintes dérive d'un potentiel. En d'autres termes, nous supposons l'existence d'une densité volumique d'énergie élastique de classe C^1

$$W : \mathbb{M}^{N\times N}_{\text{sym}} \to \mathbb{R}^+$$

dépendant uniquement du tenseur des déformations $e(u) \in \mathbb{M}^{N\times N}_{\text{sym}}$, telle que la loi de comportement s'écrive

$$\sigma = \frac{\partial W}{\partial e(u)}(e(u)).$$

Dans la suite de ce mémoire, nous nous intéresserons principalement à des milieux **dissipatifs** (non purement élastiques) pour lesquels un affaiblissement des propriétés élastiques et des déformations permanentes (irréversibles) peuvent subvenir. Pour modéliser de tels matériaux (toujours sous l'hypothèse des **petites déformations**), nous supposons dorénavant que la densité volumique d'énergie élastique du système

$$W : \mathbb{M}^{N\times N}_{\text{sym}} \times \mathbb{R}^d \to \mathbb{R}^+$$

est une fonction de classe C^1 dépendant du tenseur des déformations $e(u) \in \mathbb{M}^{N\times N}_{\text{sym}}$ et d'une autre **variable interne**, notée $\alpha \in \mathbb{R}^d$, qui rendra compte de la partie irréversible de la déformation du matériau. Il est alors naturel d'introduire les tenseurs de contraintes *réversible*

$$\sigma^r = \frac{\partial W}{\partial e(u)}(e(u))$$

et *irréversible*

$$\sigma^i = \sigma - \sigma^r.$$

La déformation d'un milieu continu est une transformation thermodynamique particulière qui doit s'opérer en accord avec les deux principes de la thermodynamique. Le premier affirme qu'au cours du mouvement, la variation en temps de l'énergie totale (interne et cinétique) du milieu est égale à la quantité d'énergie échangée avec l'environnement extérieur, par transferts thermique (chaleur) et mécanique (travail des forces extérieures). Le deuxième principe assure que l'entropie, qui mesure le degré de désordre du milieu à l'échelle moléculaire, ne peut que croître au cours de la transformation. Dans cette thèse, nous considérerons toujours des évolutions **isothermes** (température constante en temps et en espace). Les effets de chaleur ne dissipent alors pas d'énergie, de sorte que les deux principes de la thermodynamique résultent en particulier en la validité de l'inégalité de dissipation mécanique :

$$\Delta := \sigma : e(\dot{u}) - \dot{W} \geq 0.$$

En définissant la force thermodynamique

$$A = -\frac{\partial W}{\partial \alpha}(e(u), \alpha)$$

associée à la variable interne α, nous avons alors que $\dot{W} = \sigma^r : e(\dot{u}) - A \cdot \dot{\alpha}$ et l'inégalité de dissipation mécanique s'écrit

$$\Delta = \sigma^i : e(\dot{u}) + A \cdot \dot{\alpha} \geq 0.$$

Afin d'obtenir les lois de comportement des milieux continus dissipatifs, il nous manque encore des lois complémentaires liant les vitesses $(e(\dot{u}), \dot{\sigma})$ et les forces (σ^i, A). À cet effet, nous nous restreignons à l'étude des **matériaux standards généralisés** (voir [73]) pour lesquels les forces thermodynamiques appartiennent au sous-différentiel d'un potentiel de dissipation évalué en les vitesses. Autrement dit, nous supposons l'existence d'un potentiel de dissipation $D : \mathbb{M}^{N\times N}_{\text{sym}} \times \mathbb{R}^d \to \mathbb{R}^+$ convexe, minimal et nul en 0, tel que

$$(\sigma^i, A) \in \partial D(e(\dot{u}), \dot{\alpha}).$$

En introduisant D^* la conjuguée convexe du potentiel, on vérifie alors que

$$D^*(\sigma^i, A) = \sup_{(\xi,z)\in\mathbb{M}^{N\times N}_{\text{sym}}\times\mathbb{R}^d} \left\{\sigma^i : \xi + A\cdot z - D(\xi, z)\right\} = \sigma^i : e(\dot{u}) + A\cdot\dot{\alpha} - D(e(\dot{u}), \dot{\alpha})$$

de sorte que l'inégalité de dissipation devient

$$\Delta = D(e(\dot{u}), \dot{\alpha}) + D^*(\sigma^i, A) \geq 0.$$

Remarquons que cette description englobe le cas d'un matériau élastique, où la densité d'énergie élastique $W(e(u), \alpha) = W(e(u))$ dépend uniquement du tenseur des déformations. Il n'y a donc pas de variable interne supplémentaire, de sorte que $D = 0$ et par conséquent la dissipation mécanique $\Delta = 0$, ce qui traduit bien le fait que le modèle est non dissipatif.

Il n'aura pas échappé au lecteur que certaines classes de matériaux et de comportements mécaniques n'entrent pas dans le formalisme présenté ci-dessus. Un certain manque de rigueur quant au caractère bien posé des lois précédentes est ainsi parfois inévitable, du fait de la trop grande rigidité sur les conditions de régularité imposées au matériau. L'approche par formulation variationnelle présente alors un intérêt certain de par sa plus grande flexibilité, en s'affranchissant des contraintes de dérivabilité classiques, et par conséquent de par son plus large cadre d'application. La pertinence mécanique des modèles variationnels introduits reste néanmoins sujet à discussion, tant de par leur non-unicité que de par l'incontournable nécessité de relaxer les formulations énergétiques afin de garantir des résultats d'existence.

Nous sommes désormais en mesure d'introduire les modèles variationnels des différents milieux dissipatifs dont il sera question dans la suite de ce mémoire. Ces modèles (à l'exception de d'élasticité linéaire et de la plasticité parfaite), ne sont pas équivalents au formalisme des lois de comportement précédent, bien que motivés par ce dernier.

1.1.1 . Élasticité linéaire

L'élasticité linéaire est un cas particulier du modèle d'élasticité en petites déformations, où l'on suppose que le tenseur des déformations est proportionnel à la sollicitation. Autrement dit, la déformation élastique du matériau et le tenseur des contraintes sont liés via la *Loi de Hooke* (loi d'élasticité)

$$\sigma = \mathbf{A}e(u)$$

où $\mathbf{A}$ est le tenseur de Hooke du matériau, un tenseur d'ordre 4 symétrique et coercif représentant les coefficients d'élasticité du matériau. En reprenant les notations précédentes, ce modèle correspond donc au choix de la densité d'énergie élastique volumique

$$W(e(u)) = \frac{1}{2}\mathbf{A}e(u) : e(u)$$

et du potentiel de dissipation nul $D = 0$ de sorte que $\sigma = \mathbf{A}e(u)$. La formulation variationnelle de ce modèle consiste alors seulement à résoudre la formulation faible des équations du mouvement avec conditions aux limites, également interprétée comme le principe des puissances virtuelles. Autrement dit,

$$u \in L^2([0,T]; H^1(\Omega;\mathbb{R}^N)) \cap W_0^{2,1}([0,T]; L^2(\Omega;\mathbb{R}^N))$$

doit satisfaire $u(t) = w(t)$ sur Γ_D en tout temps $t \in [0,T]$ et, pour tout champ de vitesse virtuel $v \in H^1(\Omega;\mathbb{R}^N)$ tel que $v = 0$ sur Γ_D,

$$\int_\Omega \mathbf{A}e(u(t)) : e(v)\,dx = \int_\Omega f(t)\cdot v\,dx + \int_{\Gamma_N} g(t)\cdot v\,d\mathcal{H}^{N-1} - \int_\Omega \varrho\ddot{u}(t)\cdot v\,dx,$$

où

$$\mathcal{P}_a(v) := \int_\Omega \varrho\ddot{u}(t)\cdot v\,dx$$

est la puissance virtuelle des forces d'inertie et

$$\mathcal{P}_i(v) := -\int_\Omega \mathbf{A}e(u(t)) : e(v)\,dx \quad \text{et} \quad \mathcal{P}_e(v) := \int_\Omega f(t)\cdot v\,dx + \int_{\Gamma_N} g(t)\cdot v\,d\mathcal{H}^{N-1}$$

désignent la puissance virtuelle des efforts internes et externes respectivement. En introduisant l'énergie totale au temps $t \in [0,T]$

$$\mathcal{E}(t) := \frac{1}{2}\|\sqrt{\varrho}\dot{u}(t)\|^2_{L^2(\Omega;\mathbb{R}^N)} + \frac{1}{2}\int_\Omega \mathbf{A}e(u(t)) : e(u(t))\,dx - \int_\Omega f(t)\cdot u(t)\,dx - \int_{\Gamma_N} g(t)\cdot u(t)\,d\mathcal{H}^{N-1},$$

la formulation variationnelle du deuxième principe de la thermodynamique $\Delta = \sigma : e(\dot{u}) - \dot{W} = 0$ intégré en temps sur $[0,t]$ et en espace correspond au bilan d'énergie :

$$\begin{aligned}\mathcal{E}(t) = \mathcal{E}(0) &- \int_0^t\int_\Omega \dot{f}(s)\cdot u(s)\,dxds - \int_0^t\int_{\Gamma_N} \dot{g}(s)\cdot u(s)\,d\mathcal{H}^{N-1}ds + \int_0^t\int_\Omega \varrho\ddot{u}\cdot\dot{w}\,dxds \\ &+ \underbrace{\int_0^t\int_\Omega \sigma(s) : e(\dot{w}(s))\,dxds - \int_0^t\int_\Omega f(s)\cdot\dot{w}(s)\,dxds - \int_0^t\int_{\Gamma_N} g(s)\cdot\dot{w}(s)\,d\mathcal{H}^{N-1}ds.}_{=\int_0^t\int_{\Gamma_D}\sigma\nu\cdot\dot{w}\,d\mathcal{H}^{N-1}ds}\end{aligned}$$

1.1.2 . Plasticité parfaite

La plasticité parfaite est un phénomène typiquement anélastique où des déformations permanentes surviennent lorsque la contrainte atteint un certain seuil. Plus précisément, le comportement du matériau est décrit par le biais de quatre quantités de la façon suivante : d'une part, le déplacement $u : [0,T]\times\Omega \to \mathbb{R}^N$ est tel que son tenseur des déformations $e(u) = e+p$ est additivement décomposé en la somme d'un tenseur des déformations élastiques $e : [0,T] \times \Omega \to \mathbb{M}^{N\times N}_{\rm sym}$ et d'un tenseur des déformations plastiques $p : [0,T] \times \Omega \to \mathbb{M}^{N\times N}_{\rm sym}$, représentant respectivement les parties élastique (réversible) et permanente (irréversible) de la déformation. D'autre part, le tenseur des contraintes ne dépend (linéairement) que de la partie élastique de la déformation et est astreint à demeurer dans un convexe fermé contenant l'origine, $K \subset \mathbb{M}^{N\times N}_{\rm sym}$, tout en respectant le principe du travail maximal de Hill selon lequel les déformations plastiques ne progressent que lorsque la contrainte atteint le bord ∂K. L'intérieur de K correspond ainsi au domaine des déformations élastiques. Autrement dit, l'évolution plastique du matériau est caractérisée par les lois de comportement

$$\sigma(t,x) = \mathbf{A}(x)e(t,x), \quad \sigma(t,x) \in K, \quad \dot p(t,x) : \sigma(t,x) = \max_{\tau \in K} \dot p(t,x) : \tau$$

en tout temps $t \in [0,T]$ et pour tout $x \in \Omega$, où $\mathbf{A}$ est le tenseur de Hooke du milieu, et par les équations du mouvement (au sens faible)

$$\begin{cases} \mathrm{div}\sigma(t) + f(t) = \varrho \ddot u(t) & \text{dans } \Omega, \\ p(t) = (w(t) - u(t)) \odot \nu & \text{sur } \Gamma_D, \\ \sigma(t)\nu = g(t) & \text{sur } \Gamma_N, \end{cases}$$

où $\nu \in \mathbb{S}^{N-1}$ désigne la normale extérieure à $\partial\Omega$. La condition de Dirichlet est relaxée dans ce modèle, car les observations physiques montrent que des déformations plastiques peuvent apparaître jusque sur le bord du matériau. Avec le formalisme précédent, on vérifie que ce modèle correspond au choix de la densité volumique d'énergie élastique

$$W(e(u),p) = \frac{1}{2}\mathbf{A}(e(u)-p) : (e(u)-p),$$

de sorte que la variable interne naturellement considérée ici est le tenseur des déformations plastiques p, tandis que le tenseur des contraintes irréversibles est nul ($\sigma = \sigma^r$, $\sigma^i = 0$), ce qui correspond à la première loi de comportement. En particulier, la force thermodynamique associée à p est donnée par $A = \mathbf{A}(e(u)-p) = \sigma$. En prenant pour potentiel de dissipation la fonction d'appui du convexe K

$$D = I_K^* : \xi \in \mathbb{M}^{N\times N}_{\rm sym} \mapsto \max_{\tau \in K} \xi : \tau$$

où $I_K = 0$ dans K et $+\infty$ sinon, on vérifie alors que $A \in \partial D(\dot p)$ si et seulement si $\sigma \in K$ et $\sigma : \dot p = \sup_{\tau \in K} \tau : \dot p$, ce qui correspond bien aux deux dernières lois de comportement mentionnées ci-dessus. Finalement, la dissipation mécanique est donnée par l'expression

$$\Delta = \sigma : \dot p \geq 0.$$

Notons que cette inégalité est automatiquement satisfaite grâce au principe du travail maximal de Hill dès lors qu'il existe une constante $r > 0$ telle que

$$B_r(0) \subset K, \tag{1.1.2}$$

puisque dans ce cas nous vérifions que $I_K^* \geq r\,|\cdot|$.

Plus récemment, le modèle de plasticité a été reformulé dans un cadre variationnel ([92, 93, 94]), donnant lieu à plusieurs résultats d'existence dans le cadre statique, quasi-statique pour des processus indépendants des vitesses, et dynamique (voir [83], [49, 13], [16, 9] par exemple). En plasticité parfaite, la notion de solution variationnelle coïncide parfaitement avec la notion précédente de solution, ce qui n'est pas le cas pour un grand nombre de phénomènes mécaniques dissipatifs. Nous décrivons ici le cas d'une évolution quasi-statique (*i.e.* l'accélération est négligée) et indépendante des vitesses. D'après [49, Definition 4.2], un triplet

$$(u, e, p) : [0,T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{N\times N}_{\text{sym}}) \times \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{N\times N}_{\text{sym}})$$

est une évolution variationnelle quasi-statique de plasticité parfaite si les propriétés suivantes sont satisfaites, pour tout $t \in [0,T]$:

(i) $Eu(t) = e(t)\mathcal{L}^N \llcorner \Omega + p(t) \llcorner \Omega$ dans $\mathcal{M}(\Omega; \mathbb{M}^{N\times N}_{\text{sym}})$;

(ii) $\sigma(t) = \mathbf{A}e(t) \in K$ presque partout dans Ω ;

(iii) $\operatorname{div}\sigma(t) + f(t) = 0$ dans $H^{-1}(\Omega; \mathbb{R}^N)$ et $\sigma(t)\nu = g(t)$ sur Γ_N au sens faible, *i.e.* pour toute fonction test $\psi \in H^1(\Omega; \mathbb{R}^N)$ telle que $\psi = 0$ sur Γ_D :

$$\int_\Omega \sigma(t) : e(\psi)\, dx = \int_\Omega f(t) \cdot \psi \, dx + \int_{\Gamma_N} g(t) \cdot \psi \, d\mathcal{H}^{N-1};$$

(iv) $p(t) \llcorner \Gamma_D = (w(t) - u(t)) \odot \nu \mathcal{H}^{N-1} \llcorner \Gamma_D$ dans $\mathcal{M}(\Gamma_D; \mathbb{M}^{N\times N}_{\text{sym}})$;

(v) $p : [0,T] \to \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{N\times N}_{\text{sym}})$ est à variation bornée.

Enfin, en introduisant le coût dissipatif cumulé dû aux déformations plastiques p au court de l'intervalle de temps $[0,t]$

$$\text{Diss}_K(p; 0, t) := \sup \left\{ \sum_{i=1}^n \int_{\Omega\cup\Gamma_D} I_K^* \left(\frac{d(p(s_i) - p(s_{i-1}))}{d\,|p(s_i) - p(s_{i-1})|} \right) d\,|p(s_i) - p(s_{i-1})| \ : \ n \in \mathbb{N},\ 0 = s_0 \le s_1 \le \ldots \le s_n = t \right\},$$

l'énergie totale

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega \mathbf{A}e(t) : e(t)\, dx + \text{Diss}_K(p; 0, t) - \int_\Omega f(t) \cdot u(t)\, dx - \int_{\Gamma_N} g(t) \cdot u(t)\, d\mathcal{H}^{N-1}$$

doit satisfaire le bilan d'énergie :

(vi) $$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega \sigma : e(\dot{w})\, dxds - \int_0^t \int_\Omega \left(\dot{f} \cdot u + f \cdot \dot{w}\right) dxds - \int_0^t \int_{\Gamma_N} (\dot{g} \cdot u + g \cdot \dot{w})\ d\mathcal{H}^{N-1} ds.$$

Remarquons que l'inégalité de dissipation mécanique $\Delta = \sigma : e(\dot{u}) - \dot{W} \ge 0$ est toujours satisfaite, puisqu'en intégrant Δ en temps sur $[0,t]$ et en espace sur Ω donne ici, formellement

$$\int_0^t \int_\Omega \sigma : e(\dot{u})\, dxds - \int_\Omega \frac{1}{2}\sigma(t) : e(t)\, dx + \int_\Omega \frac{1}{2}\sigma(0) : e(0)\, dx$$
$$= \text{Diss}_K(p; 0, t) = \int_0^t \int_{\Omega\cup\Gamma_D} \sigma : \dot{p}\, dxds = \int_0^t \int_{\Omega\cup\Gamma_D} I_K^*(\dot{p})\, dxds \ge 0$$

par (1.1.2).

L'équivalence de ces deux formulations provient du fait que toute solution variationnelle est absolument continue en temps (d'où l'existence de dérivées temporelles faibles en $\mathcal{L}^1$-presque tout temps, voir [49, Theorem 5.2]) et de l'équivalence entre le bilan d'énergie et la validité de la loi d'écoulement (elle-même équivalente au principe du travail maximal de Hill, voir [49, Theorem 6.1]) :

$$\sigma(t) : \dot{p}(t)\big(\Omega \cup \Gamma_D\big) = \int_{\Omega \cup \Gamma_D} I_K^* \left(\frac{d\dot{p}(t)}{d\,|\dot{p}(t)|} \right) d\,|\dot{p}(t)| \quad \text{pour } \mathcal{L}^1\text{-presque tout } t \in [0,T].$$

1.1.3 . Endommagement

L'endommagement est un processus de déformation au cours duquel les propriétés élastiques du milieu s'affaiblissent de façon irréversible sous l'effet du chargement auquel il est soumis. La modélisation d'un tel phénomène se base sur le postulat que le tenseur de rigidité $\mathbf{A}(t,x)$ (module de Young) du matériau est une fonction d'une variable interne d'endommagement $\chi(t,x)$:

$$\mathbf{A}(t,x) = \hat{\mathbf{A}}(\chi(t,x))$$

où χ varie dans l'intervalle $[0,1]$, les cas $\chi = 1$ correspondant à l'état d'endommagement maximal et $\chi = 0$ à l'état sain. Un choix simple et naturel définissant la dépendance du tenseur de rigidité par rapport à la variable d'endommagement consiste à prendre

$$\hat{\mathbf{A}}(\chi) = \chi \mathbf{A}_0 + (1-\chi)\mathbf{A}_1,$$

où $\mathbf{A}_0$ et $\mathbf{A}_1$ sont deux tenseurs d'ordre 4 symétriques et coercifs, représentant les propriétés élastiques d'une phase endommagée et saine respectivement, ce qui est traduit par la propriété d'ordre

$$\mathbf{A}_0 < \mathbf{A}_1$$

en tant que formes quadratiques agissant sur $\mathbb{M}^{N\times N}_{\mathrm{sym}}$. La densité volumique d'énergie élastique du matériau est alors donnée par

$$W(e(u),\chi) = \frac{1}{2}\left(\chi \mathbf{A}_0 + (1-\chi)\mathbf{A}_1\right) e(u) : e(u) + I_{[0,1]}(\chi).$$

Le cas où χ prend uniquement les valeurs 0 et 1 correspond à de l'endommagement brutal. Sinon, l'endommagement est dit progressif. Nous supposons ici que $\mathbf{A}_0$ est défini positif, traduisant un endommagement seulement partiel. Dans le cas où $\mathbf{A}_0$ est nul, l'endommagement est dit total. L'irréversibilité du processus est modélisé par la croissance en temps de la variable interne

$$\dot{\chi} \geq 0.$$

En endommagement brutal, la transformation du milieu est caractérisée par le biais d'un critère de contrainte maximale de type Griffith, selon lequel les déformations élastiques cèdent le pas à l'endommagement dès lors que le tenseur des contraintes excède un certain seuil. En d'autres termes, la rigidité du matériau passera de celle de l'état sain $\mathbf{A}_1$ à celle de l'état endommagé $\mathbf{A}_0$ dès lors que

$$(\mathbf{A}_1 - \mathbf{A}_0)\, e(u) : e(u) > 2\kappa,$$

où la ténacité $\kappa > 0$ est une constante intrinsèque du matériau représentant la densité d'énergie dépensée par unité de volume lors de la transition de l'état sain à l'état endommagé. Dans le cas général (progressif ou brutal), les lois de comportement du modèle d'endommagement sont données par le système

$$\begin{cases} 0 \leq \chi \leq 1, \quad \dot{\chi} \geq 0, \\ (\mathbf{A}_1 - \mathbf{A}_0)\, e(u) : e(u) \leq 2\kappa \quad \text{si } 0 \leq \chi < 1, \\ \big((\mathbf{A}_1 - \mathbf{A}_0) e(u) : e(u) - 2\kappa\big)\, \dot{\chi} = 0 \quad \text{si } 0 < \chi \leq 1. \end{cases}$$

Avec le formalisme ultérieur, cela correspond au choix du potentiel de dissipation homogène en contrainte

$$D(\chi) = \kappa\chi + I_{[0,+\infty)}(\chi)$$

de sorte que

$$\sigma^r = (\chi\mathbf{A}_0 + (1-\chi)\mathbf{A}_1)\, e(u) = \sigma \quad \text{et} \quad \sigma^i = 0.$$

La correspondance exacte avec le formalisme précédent est mise en difficulté lors de la détermination de la force thermodynamique associée à χ, le potentiel de dissipation n'étant pas suffisamment régulier. La détermination de la loi d'évolution pour la variable interne d'endommagement est alors mal posée et nécessite un choix (non unique) quant à la force thermodynamique A, qui doit satisfaire

$$A \in (\mathbf{A}_1 - \mathbf{A}_0)\, e(u) - \partial I_{[0,1]}(\chi), \quad A \in \partial D(\dot{\chi}).$$

Une fois un choix fixé pour A, l'inégalité de dissipation s'écrit alors

$$\Delta = A\dot{\chi} = \kappa\dot{\chi} \geq 0.$$

Ainsi, le recours à l'approche variationnelle pour la description des processus d'endommagement s'avère pertinent et légitime, sinon nécessaire, comme expliqué dans [64] dans le cas quasi-statique. L'idée proposée par Francfort et Marigo consiste à compléter le modèle avec un critère de stabilité (globale, conduisant à des débats quant à la pertinence mécanique du modèle, mais nécessaire pour le caractère bien posé du modèle mathématique) selon lequel toute solution variationnelle (u, χ) : $[0, T] \times \Omega \to \mathbb{R}^N \times \{0, 1\}$ doit minimiser la somme des énergies potentielles et dissipatives parmi tous les déplacements et réarrangements admissibles du matériau (*i.e.* tenant compte de l'historique passé du chargement et de l'irréversibilité du processus d'endommagement). Plus précisément, étant donnée une subdivision de $[0, T]$

$$0 = t_0 < \dots < t_I = T,$$

Francfort et Marigo introduisent l'énergie totale au temps t_i associée à un couple $(u, \chi) \in H^1(\Omega; \mathbb{R}^N) \times L^\infty(\Omega; \{0, 1\})$:

$$\mathcal{E}_i(u, \chi) := \int_\Omega \frac{1}{2}\, (\chi\mathbf{A}_0 + (1-\chi)\mathbf{A}_1)\, e(u) : e(u)\, dx + \kappa \int_\Omega \chi\, dx - \int_\Omega f(t_i) \cdot u\, dx - \int_{\Gamma_N} g(t_i) \cdot u\, d\mathcal{H}^{N-1}.$$

Le problème consiste alors à trouver les couples $(u_i, \chi_i) \in H^1(\Omega; \mathbb{R}^N) \times L^\infty(\Omega; \{0, 1\})$ qui minimisent

$$\mathcal{E}_i(u, \chi) \tag{1.1.3}$$

parmi tous les couples $(u,\chi) \in H^1(\Omega;\mathbb{R}^N) \times L^\infty(\Omega;\{0,1\})$ tels que $u = w(t_i)$ sur Γ_D et $\chi = 1$ dans $\{\chi_{i-1} = 1\}$, pour chaque $i \in [\![0, I]\!]$ avec $\chi_0 = 0$. Malheureusement, il est désormais communément connu que ce problème de minimisation est mal posé, par manque de convexité de sa densité. En effet, il s'avère énergétiquement favorable que les suites minimisantes forment des microstructures (mélanges fin entre les phases saines et endommagées) convergeant vers des états homogénéisés. La relaxation de l'énergie totale suggère ainsi d'élargir la classe des matériaux admissibles à tout mélange homogénéisé des phases saines en proportion volumique $\theta \in [0,1]$ et endommagées en proportion $1-\theta$. Le modèle finalement proposé par Francfort et Marigo consiste alors à traduire l'irréversibilité de l'endommagement en cherchant les couples $(u_i,\theta_i) \in H^1(\Omega;\mathbb{R}^N) \times L^\infty(\Omega;[0,1])$ qui minimisent

$$\mathcal{E}_i^*(u,\theta) = \int_\Omega \min_{\mathbf{A}^* \in \mathcal{G}_\theta(\mathbf{A}_0,\mathbf{A}_1)} \frac{1}{2}\mathbf{A}^* e(u) : e(u)\, dx + \kappa \int_\Omega \theta\, dx - \int_\Omega f(t_i)\cdot u\, dx - \int_{\Gamma_N} g(t_i)\cdot u\, d\mathcal{H}^{N-1}$$

parmi tout les couples $(u,\theta) \in H^1(\Omega;\mathbb{R}^N) \times L^\infty(\Omega;\{0,1\})$ tels que $u = w(t_i)$ sur Γ_D et $\theta \geq \theta_{i-1}$, pour $i \in [\![0, I]\!]$ avec $\theta_0 = 0$, où $\mathcal{G}_\theta(\mathbf{A}_0,\mathbf{A}_1)$ est l'ensemble défini à la section 1.4.

Bien que ces problèmes de minimisation sous contraintes admettent des solutions, seul le temps initial correspond réellement à la relaxation du problème d'origine (1.1.3). Ce problème est résolu dans [60], où les auteurs définissent et démontrent le caractère bien posé de la formulation variationnelle relaxée de (1.1.3) en temps continu dans le cas quasi-statique. Une évolution quasi-statique d'endommagement brutal est alors donnée par un triplet

$$(u,\Theta,\mathbf{A}) : [0,T] \to H^1(\Omega;\mathbb{R}^N) \times L^\infty(\Omega;[0,1]) \times \mathcal{G}(\mathbf{A}_0,\mathbf{A}_1)$$

tel que

- $\mathbf{A}(t) \in \mathcal{G}_{1-\Theta(t)}(\mathbf{A}_0,\mathbf{A}_1)$ en tout temps $t \in [0,T]$;
- $\mathbf{A}$ et Θ sont décroissants en temps ;
- $(u(t),\mathbf{A}(t),0)$ minimise en tout temps $t \in [0,T]$

  $$\int_\Omega \frac{1}{2}\mathbf{A}^* e(v) : e(v)\, dx + \kappa \int_\Omega \Theta(t)\theta\, dx - \int_\Omega f(t_i)\cdot v\, dx - \int_{\Gamma_N} g(t_i)\cdot v\, d\mathcal{H}^{N-1}$$

  parmi tout les triplets $(v,\mathbf{A}^*,\theta) \in H^1(\Omega;\mathbb{R}^N) \times \mathcal{G}(\mathbf{A}_0,\mathbf{A}(t)) \times L^\infty(\Omega;\{0,1\})$ tels que $v = w(t)$ sur Γ_D et $\mathbf{A}^* \in \mathcal{G}_\theta(\mathbf{A}_0,\mathbf{A}(t))$;
- et l'énergie totale

  $$\mathcal{E} := \int_\Omega \frac{1}{2}\mathbf{A} e(u) : e(u)\, dx + \kappa \int_\Omega (1-\Theta)\, dx - \int_\Omega f\cdot u\, dx - \int_{\Gamma_N} g\cdot u\, d\mathcal{H}^{N-1}$$

  satisfait le bilan d'énergie, en tout temps $t \in [0,T]$:

  $$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega \sigma : e(\dot{w})\, dxds - \int_0^t \int_\Omega \left(\dot{f}\cdot u + f\cdot\dot{w}\right)\, dxds - \int_0^t \int_{\Gamma_N} (\dot{g}\cdot u + g\cdot\dot{w})\, d\mathcal{H}^{N-1}ds.$$

Des formulations variationnelles (relaxées) dans le cadre d'évolutions quasi-statiques et dynamiques ont également été introduites dans [68, 66], pour lesquelles les auteurs démontrent le caractère bien posé et l'existence de solutions. L'assouplissement des contraintes de régularité et la plus grande flexibilité apportés par l'approche variationnelle ont également permis de modéliser des phénomènes où plusieurs types de dissipation entrent en compte, aboutissant notamment à des modèles élastoplastiques mêlés à de l'endommagement ([40, 41]).

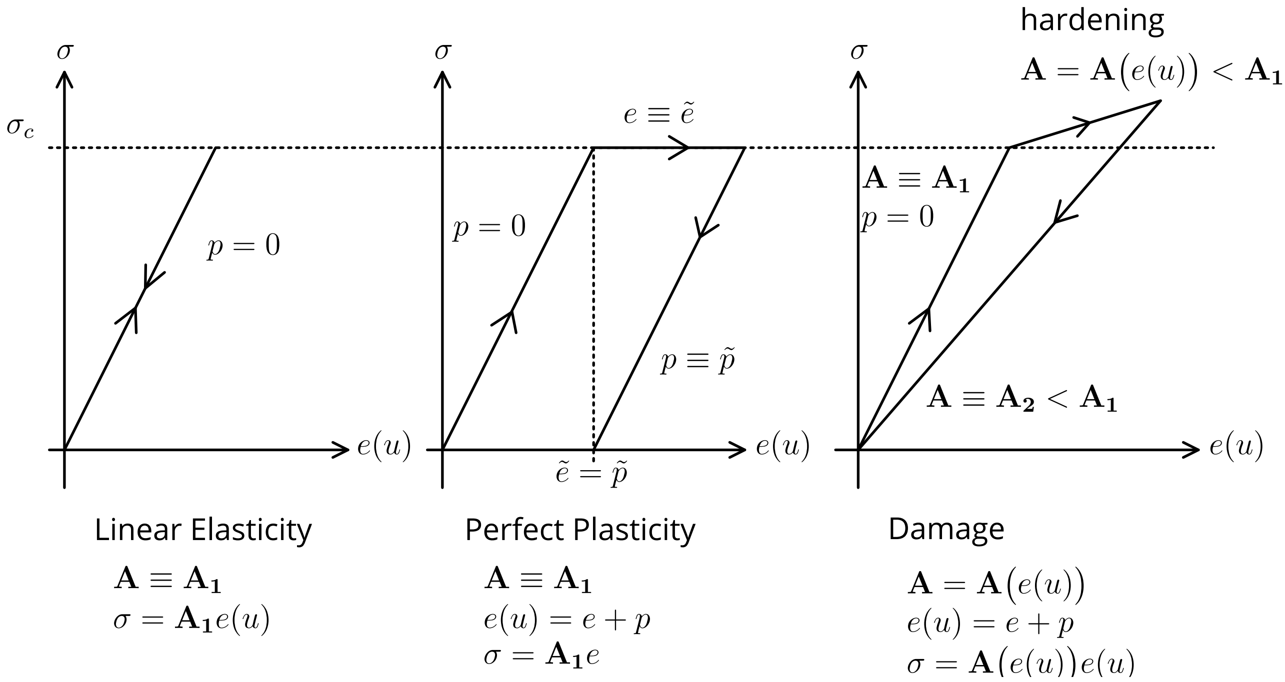

Figure 1.1

1.1.4 . Rupture fragile

La mécanique de la rupture décrit la résistance d'un matériau élastique à la rupture fragile (fracture) ou ductile. Nous décrivons ici la présence de défauts de type fissure dont le cadre théorique, bien qu'introduit dès le début du vingtième siècle dans les travaux fondateurs de Griffith ([72]), continue de poser de nombreuses difficultés aux mathématiciens comme aux mécaniciens (voir [59]). Supposons par simplicité que le problème est planaire ($N = 2$) et que le chargement est suffisamment lent, de sorte que les effets d'inertie peuvent être négligés. Nous nous plaçons donc ici dans le cadre quasi-statique. Le terme générique de fissure désigne les lignes de discontinuité du champ de déplacements. Supposons, en adoptant le postulat de Griffith, que le chemin de fissuration $\hat{\Gamma} \subset \Omega \cup \Gamma_D$ est a priori connu. La fracture est astreinte à demeurer dans $\Omega \cup \Gamma_D$. Par conséquent, sous l'hypothèse que la fissure est suffisamment régulière, celle-ci sera complètement déterminée par sa longueur l, et nous noterons $\Gamma(l)$ la fissure de longueur l dans $\hat{\Gamma}$. Le phénomène de rupture sera donc modélisé par l'intermédiaire de la variable interne l. Du point de vue mécanique, étant donnée une longueur de fissuration $l \geq 0$, le matériau $\Omega \setminus \Gamma(l)$ doit être à l'état d'équilibre avec le chargement à chaque instant $t \in [0,T]$, de sorte que le déplacement $u(t,l)$ minimise l'énergie potentielle

$$\mathcal{E}(t,l,u) := \int_{\Omega\setminus\Gamma(l)} W(e(u))\,dx - \int_{\Omega} f(t)\cdot u\,dx - \int_{\Gamma_N} g(t)\cdot u\,d\mathcal{H}^1$$

parmi tous les déplacements $u \in C^1(\Omega \setminus \Gamma(l);\mathbb{R}^2)$. Par simplicité, nous supposons ici que la densité volumique d'énergie élastique est donnée par

$$W(e(u)) = \frac{1}{2}\mathbf{A}e(u) : e(u)$$

où $\mathbf{A}$ est le module d'élasticité du milieu en dehors de la fissure $\hat{\Gamma}$. L'idée de Griffith pour décrire la loi d'évolution de la fracture consiste alors à postuler que l'énergie dissipée, suite à la création d'une fissure de longueur $l \geq 0$, doit être égale à sa longueur pondérée par la ténacité du matériau $\kappa > 0$ (plus communément notée $G_c > 0$ dans le contexte de la fracture). L'évolution de la fracture est alors gouvernée par le principe selon lequel, si $l(t)$ est la longueur de la fracture au temps $t \in [0,T]$, la création d'une fissure légèrement plus longue dissiperait plus d'énergie surfacique qu'il n'en serait restituée par l'énergie potentielle. Autrement dit, pour toute longueur $0 < \Delta \ll 1$,

$$\mathcal{E}\big(t, l(t)+\Delta, u(t,l(t)+\Delta)\big) + \kappa(l(t)+\Delta) \geq \mathcal{E}\big(t,l(t),u(t,l(t))\big) + \kappa l(t). \tag{1.1.4}$$

Un premier pas pour comprendre l'évolution future de la fracture consiste alors à faire décroître Δ vers 0, de sorte que le *taux de restitution d'énergie potentielle* par unité de surface de zone rompue doit satisfaire :

$$G(t) := -\frac{\partial\mathcal{E}\big(t,l(t),u(t,l(t))\big)}{\partial l} \leq \kappa.$$

Cette condition nécessaire est alors complétée par la condition de croissance de la fissure, traduisant l'hypothèse d'irréversibilité du processus dans les matériaux fragiles :

$$\dot{l} \geq 0,$$

à laquelle on adjoint le postulat selon lequel la fissure ne progresse que lorsque le taux d'énergie restituée atteint la ténacité $\kappa > 0$:

$$\dot{l} = 0 \text{ sauf si } G(l) = \kappa.$$

L'évolution de la fissure est alors dictée par le critère, dit de Griffith :

$$\begin{cases} \dot{l} \geq 0, \\ G(l) \leq \kappa, \\ \big(\kappa - G(l)\big)\dot{l} = 0. \end{cases} \tag{1.1.5}$$

Remark 1.1.1. Bien qu'impossible de faire correspondre ce modèle (global) avec le formalisme présenté précédemment (local), une forte analogie est néanmoins notable, où la force thermodynamique $A = -\partial W/\partial l$ associée à l correspond au taux de restitution de l'énergie potentielle $G(l)$. En effet, en prenant le potentiel de dissipation homogène en contrainte

$$D(l) = \kappa l + I_{[0,+\infty)}(l)$$

et en posant $\sigma = \sigma^r = \mathbf{A}e(u)$ et $\sigma^i = 0$ (ce qui est cohérent avec le fait que les propriétés élastiques du milieu ne s'affaiblissent pas tant que la contrainte demeure inférieure à κ), on vérifie alors que le critère de Griffith correspond bien à

$$G(l) \in \partial D(\dot{l})$$

et à l'inégalité de dissipation

$$\Delta = G(l)\dot{l} = \kappa\dot{l} \geq 0.$$

Encore une fois, notons que le formalisme de (1.1.5) nécessite de pouvoir différentier l'énergie potentielle par rapport à la longueur de la fissure et de pouvoir dériver la longueur de la fissure elle-même en temps. De plus, le modèle de Griffith nécessite de connaître à l'avance la trajectoire de fissuration du matériau, et ne permet donc pas de modéliser l'initiation de fissures. Ces différents inconvénients motivent de préférer une approche variationnelle de la mécanique de la rupture.

Une formulation variationnelle forte a ainsi vu le jour, en remplaçant la loi d'évolution de la fissuration par la conservation de l'énergie totale et en permettant aux fissures de se propager librement tant que leur trajectoire minimise l'énergie totale ([63, 24]). Autrement dit, en notant $\Gamma(t) \subset \Omega \cup \Gamma_D$ la fissure au temps $t \in [0, T]$, la condition de minimalité (locale) (1.1.4) est remplacée par la condition de minimalité (globale)

$$(\Gamma(t), u(t)) \in \underset{(\Gamma,u)}{\operatorname{argmin}} \left\{ \mathcal{E}(t, \Gamma, u) := \int_{\Omega\setminus\Gamma} W(e(u))\, dx - \int_\Omega f(t) \cdot u\, dx - \int_{\Gamma_N} g(t) \cdot u\, d\mathcal{H}^1 + \kappa\mathcal{H}^1(\Gamma) \right\} \tag{1.1.6}$$

parmi tous les couples (Γ, u) avec $\Gamma(t) \subset \Gamma$ de longueur finie et fermé et $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^2)$ tel que $u = w(t)$ sur $\Gamma_D \setminus \Gamma$. En particulier, $u(t) \in H^1(\Omega \setminus \Gamma(t); \mathbb{R}^2)$ est solution au sens faible de

$$\begin{cases} -\mathrm{div}\mathbf{A}e(u(t)) = f(t) & \text{dans } \Omega \setminus \Gamma(t), \\ u(t) = w(t) & \text{sur } \Gamma_D \setminus \Gamma(t), \\ \mathbf{A}e(u(t))\nu = g(t) & \text{sur } \Gamma_N, \\ \mathbf{A}e(u(t))\nu = 0 & \text{sur } \Gamma \cap \overline{\Omega}, \end{cases}$$

puisque $\Gamma(t) \cap \partial\Omega \subset \Gamma_D$ de sorte que $\partial\big(\Omega \setminus \Gamma(t)\big) = (\Gamma_D \setminus \Gamma(t)) \sqcup \Gamma_N \sqcup (\Gamma(t) \cap \overline{\Omega})$. Enfin, la condition d'irréversibilité du processus de fissuration est remplacée par la croissance ensembliste de la fissure :

$$\Gamma(s) \subset \Gamma(t) \quad \text{pour tous } 0 \leq s \leq t \leq T, \tag{1.1.7}$$

tandis que le critère de propagation de Griffith est remplacé par le bilan d'énergie

$$\mathcal{E}(t) := \mathcal{E}\big(t, \Gamma(t), u(t)\big) = \mathcal{E}(0) + \int_0^t \int_{\Gamma_D\setminus\Gamma(t)} \mathbf{A}e(u)\nu \cdot \dot{w}\, d\mathcal{H}^1 ds - \int_0^t \int_\Omega \dot{f} \cdot u\, dx ds - \int_0^t \int_{\Gamma_N} \dot{g} \cdot u\, d\mathcal{H}^1 ds. \tag{1.1.8}$$

L'équivalence entre la formulation variationnelle forte (1.1.6)-(1.1.7)-(1.1.8) et la formulation de Griffith (1.1.5) est détaillée au [24, Chapter 2]. Notons simplement que l'inégalité de dissipation $\Delta = \sigma : e(\dot{u}) - \dot{W} \geq 0$ est automatiquement assurée par le bilan d'énergie, puisque alors

$$\int_0^t \int_\Omega \mathbf{A}e(u) : e(\dot{u})\, dx - \int_\Omega W(e(u(t))\, dx + \int_\Omega W(e(u(0))\, dx = \kappa\mathcal{H}^1(\Gamma(t) \setminus \Gamma(0)) \geq 0.$$

Le caractère bien posé de la formulation variationnelle forte et l'existence de solution sont des questions délicates. Dans le contexte de segmentation d'images introduit par Mumford et Shah, l'homologue scalaire du problème de fracture, l'existence de solution forte s'est avérée équivalente à l'existence de solution faible dans l'espace $SBV(\Omega)$. Ce résultat non trivial fait l'objet de l'article [56]. Comme pour son homologue scalaire, nous décrivons ici une formulation variationnelle faible du problème de fracture, introduite par Francfort et Marigo dans le cas où $f = 0$ et $g = 0$ ([63, 24]). La condition de minimalité (1.1.6) est alors remplacée par

$$(\Gamma(t), u(t)) \in \underset{(\Gamma,u)}{\operatorname{argmin}} \left\{ \mathcal{E}(t, \Gamma, u) := \int_\Omega W(e(u))\, dx + \kappa \mathcal{H}^1(\Gamma) \right\} \tag{1.1.9}$$

parmi tous les couples (Γ, u) avec $\Gamma(t) \subset \Gamma \subset \Omega \cup \Gamma_D$ de longueur finie et fermé et $u \in SBD^2(\Omega; \mathbb{R}^2)$ tel que $u = w(t)$ sur $\mathbb{R}^2 \setminus \overline{\Omega}$ et $J_u \subset \Gamma$. Notons en particulier que des fissures peuvent apparaître sur tout le bord du domaine. L'existence de solution variationnelle faible a été obtenue dans plusieurs contextes, notamment dans le cas de cisaillements anti-planaires où $u : \Omega \to \mathbb{R}$ ([61]), en élasticité non-linéaire ([50]) et avec condition de non-interpénétration de la matière ([52, 53]) ou encore en élasticité linéaire en dimension deux ([65]). Comme pour le problème de Mumford-Shah, prouver l'existence de solutions fortes à partir des solutions faibles requiert l'obtention de régularité plus forte sur le déplacement et son ensemble de saut, ce qui est souvent difficile. Citons quelques avancées récentes dans le cadre statique ([42, 34]) et quasi-statique ([54, 14, 31]).

1.2 . Lois d'échelle et endommagement brutal

Rappelons que dans le contexte de l'endommagement brutal, nous considérons un matériau linéairement élastique composé de deux phases pures : une phase endommagée dont les propriétés élastiques sont affaiblies, et une phase saine. Les propriétés élastiques des régions endommagée et saine sont décrites par leurs tenseurs d'élasticité $\mathbf{A_0}$ et $\mathbf{A_1}$ respectivement, deux tenseurs d'ordre 4 symétriques et coercifs qui satisfont la propriété d'ordre

$$\mathbf{A}_0 < \mathbf{A}_1$$

en tant que formes quadratiques agissant sur $\mathbb{M}^{N\times N}_{\text{sym}}$. Notons $\Omega \subset \mathbb{R}^N$ la configuration au repos du matériau, un ouvert borné. Comme discuté précédemment, il est naturel de choisir la fonction caractéristique de la zone endommagée $\chi \in L^\infty(\Omega; \{0, 1\})$ comme variable interne pour décrire le processus d'endommagement du milieu. Adoptant le modèle variationnel introduit par Francfort-Marigo (voir [64]), l'énergie totale du système associée à un champ de déplacements $u \in H^1(\Omega; \mathbb{R}^N)$ est définie comme la somme de l'énergie élastique emmagasinée dans le corps $\int_\Omega W(e(u), \chi)\, dx$ et de l'énergie de dissipation $\int_\Omega D(\chi)\, dx$:

$$\mathcal{E}(u, \chi) = \frac{1}{2} \int_\Omega (\chi \mathbf{A}_0 + (1 - \chi)\mathbf{A}_1)\, e(u) : e(u)\, dx + \kappa \int_\Omega \chi\, dx, \tag{1.2.1}$$

où $e(u) = \frac{\nabla u + \nabla u^T}{2}$ est le tenseur des déformations linéarisées et $\kappa > 0$ est la ténacité du matériau. Autrement dit, le coût à payer pour avoir endommagé une partie du matériau est égal au volume de la zone endommagée, pondéré par la ténacité du milieu. Il est désormais bien connu que la minimisation de l'énergie totale (1.2.1) en le couple (u, χ) est mal posée, de sorte que l'énergie doit être relaxée. Ce faisant, le caractère brutal de l'endommagement disparaît au profit d'un endommagement progressif, en raison de l'intérêt énergétique que les suites minimisantes ont à former des microstructures. La classe des états admissibles est ainsi étendue à l'ensemble des élasticités homogénéisées, obtenues par des mélanges de plus en plus fins entre les phases saine et endommagée (voir [64, 60, 2, 3]). En minimisant d'abord (1.2.1) par rapport à χ, on vérifie qu'étant donné un déplacement u, la fonction caractéristique optimale est donnée par

$$\chi = \mathbb{1}_{\{(\mathbf{A}_1 - \mathbf{A}_0)e(u) : e(u) \geq 2\kappa\}}.$$

Il est alors équivalent de chercher à minimiser dans $H^1(\Omega; \mathbb{R}^N)$ l'énergie

$$\int_\Omega \underbrace{\min\left(\frac{1}{2}\mathbf{A}_0 e(u) : e(u) + \kappa \,;\, \frac{1}{2}\mathbf{A}_1 e(u) : e(u)\right)}_{=:W(e(u))} dx$$

ou encore son enveloppe semi-continue inférieure

$$\mathcal{E}(u) := \int_\Omega SQW(e(u))\, dx,$$

où

$$SQW(\xi) = \inf\left\{\int_{(0,1)^N} W(x, \xi + \nabla\phi(y))\, dy \,:\, \phi \in H^1_0\left((0,1)^N; \mathbb{R}^N\right)\right\}$$

est l'enveloppe symétrique quasi-convexe de W (voir [3]). L'obtention de formules explicites pour SQW est une question souvent difficile et encore largement ouverte. Notons néanmoins les précisions apportées dans [3, 2], où son expression est donnée par le biais de minimisations dans l'ensemble des matériaux composites admissibles grâce aux bornes de Hashin-Shtrikman.

L'objectif de cette thèse est d'étudier de telles énergies couplées avec différentes lois d'échelle. En d'autres termes, en introduisant des petits paramètres dans la densité d'énergie volumique et dans le domaine de définition de l'énergie totale, nous nous plaçons dans des régimes où la zone endommagée est "petite" et l'endommagement est de plus en plus fort jusqu'à devenir total. La question est alors d'identifier la nature des modèles mécaniques effectifs obtenus par analyse asymptotique (menée par Γ-convergence) lorsque les paramètres tendent vers 0, selon leurs taux de convergence relatifs. Les énergies d'endommagement brutal que nous considérerons seront donc typiquement de la forme

$$\mathcal{E}_\varepsilon(u, \chi) = \frac{1}{2}\int_\Omega \big(\eta_\varepsilon \chi \mathbf{A}_0 + (1-\chi)\mathbf{A}_1\big)e(u) : e(u)\, dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi\, dx \tag{1.2.2}$$

où $\eta_\varepsilon > 0$ et $\varepsilon > 0$ sont deux petits paramètres tels que $\eta_\varepsilon \to 0$ lorsque $\varepsilon \to 0$. On constate alors que les propriétés élastiques de l'état endommagé, $\eta_\varepsilon \mathbf{A}_0$, dégénèrent vers 0 lorsque $\varepsilon \to 0$. Simultanément, pour toute suite $\{(u_\varepsilon, \chi_\varepsilon)\}_{\varepsilon>0}$ uniformément bornée en énergie, le caractère divergeant de la ténacité du matériau $\kappa/\varepsilon \to +\infty$ force la zone endommagée à se concentrer sur des ensembles

Lebesgue-négligeables. L'analyse du comportement asymptotique de ces énergies consiste à comprendre l'effet de la compétition entre les deux phénomènes cités plus haut sur la densité d'énergie limite : d'une part l'intérêt énergétique des suites minimisantes à osciller et développer des microstructures, d'autre part la concentration de l'endommagement. Selon les lois d'échelle considérées, l'interaction entre l'homogénéisation du matériau (due à la nécessité de relaxer l'énergie totale) et la formation de singularités spatiales (due à la concentration de l'endommagement) met en lumière des phénomènes surprenants et non triviaux. Par exemple, dans l'étude statique [15], les auteurs ont mis en évidence trois lois d'échelle pour lesquelles les modèles limites ont des propriétés mécaniques très différentes : un modèle trivial lorsque $\eta_\varepsilon \ll \varepsilon$, un modèle d'élasticité linéaire lorsque $\eta_\varepsilon \gg \varepsilon$ et un modèle de plasticité de Hencky lorsque $\eta_\varepsilon \sim \varepsilon$.

1.3 . Description des résultats

1.3.1 . Approximation discrète de la fonctionnelle de Griffith

Le deuxième chapitre de cette thèse constitue un travail en collaboration avec *Jean-François BABADJIAN* et a donné lieu à une publication [12] dans la revue *SIAM Journal on Mathematical Analysis*. Dans ce travail, nous démontrons une approximation discrète par éléments finis adaptatifs de la fonctionnelle de Griffith isotrope en dimension 2, en termes de Γ-convergence, par une suite de fonctionnelles intégrales de type (1.3.1). La fonctionnelle de Griffith a été initialement introduite dans les travaux de Griffith [72] en mécanique de la rupture fragile, puis revisitée dans un cadre d'évolution variationnelle par Francfort et Marigo dans [63] (voir également [24]). Étant donné $\Omega \subset \mathbb{R}^2$, la configuration de référence d'un matériau élastique se fracturant sous l'effet d'un champ de déplacements discontinu $u : \Omega \to \mathbb{R}^2$, l'énergie de Griffith est définie (dans une formulation faible) comme

$$\mathcal{G}(u) := \int_{\Omega\setminus J_u} |e(u)|^2 \, dx + \kappa \mathcal{H}^1(J_u),$$

où les fissures du matériau élastique correspondent à l'ensemble de saut $J_u \subset \Omega$ du déplacement, et où $e(u)$ est son gradient symétrisé (tous deux définis au sens de la théorie géométrique de la mesure). L'énergie met en compétition un terme d'énergie de volume, représentant l'énergie élastique emmagasinée dans le matériau hors de la fracture, et un terme d'énergie surfacique qui pénalise l'existence de la fracture en payant sa longueur (donnée par sa mesure de Hausdorff 1-dimensionnelle) pondérée par la ténacité $\kappa > 0$.

S'inspirant de l'approche présentée dans [39], nous considérons l'ensemble des éléments finis

$$V_\varepsilon(\Omega) = \left\{ u \in C^0\big(\Omega; \mathbb{R}^2\big) \ : \ \exists \mathbf{T}_\varepsilon \in \mathcal{T}_\varepsilon(\Omega, \theta_0), \ u \text{ est affine sur chaque triangle } T \in \mathbf{T}_\varepsilon \right\}$$

et introduisons les fonctionnelles approximantes

$$\mathcal{E}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon} f\left(\varepsilon \mathbf{A} e(u) : e(u)\right) \, dx$$

où $u \in V_\varepsilon(\Omega)$, $\mathbf{A} \in \mathcal{L}(\mathbb{M}^{2\times 2}_{\rm sym}; \mathbb{M}^{2\times 2}_{\rm sym})$ est le tenseur d'élasticité du matériau, symétrique et coercif, et

$$f : [0, +\infty) \to [0, +\infty)$$

est une fonction continue et décroissante telle que $\lim_{t\to 0^+} f(t)/t = 1$ et $\lim_{t\to\infty} f(t) = \kappa$. Notons que dans le cas particulier où

$$f(t) = t \wedge \kappa,$$

l'énergie s'écrit

$$\mathcal{E}_\varepsilon(u) = \int_\Omega (1-\chi_\varepsilon)\mathbf{A}e(u) : e(u)\, dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi_\varepsilon\, dx$$

où

$$\chi_\varepsilon = \mathbb{1}_{\left\{\mathbf{A}e(u):e(u)\geq \frac{\kappa}{\varepsilon}\right\}} \in L^\infty(\Omega;\{0,1\}),$$

ce qui correspond à l'énergie d'endommagement brutal d'un matériau élastique dont l'élasticité de la phase saine est prise égale à l'identité et celle de la phase endommagée est fixée à 0. En considérant une suite $\{u_\varepsilon\}_\varepsilon$ uniformément bornée en énergie, l'espace d'énergie qui intervient naturellement pour l'obtention de résultats de compacité est l'ensemble des fonctions spéciales généralisées à déformations bornées $GSBD^2(\Omega)$ (introduit dans [48]).

Dans [12], nous démontrons la Γ-convergence (pour la topologie de la convergence en mesure dans $L^0(\Omega;\mathbb{R}^2)$) des énergies d'endommagement brutal $\mathcal{E}_\varepsilon$ vers la fonctionnelle de Griffith :

$$\int_\Omega \mathbf{A}e(u) : e(u)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_u),$$

généralisant les résultats de [39] au cadre vectoriel (2-dimensionnel) de l'élasticité linéaire. La preuve de l'inégalité de la borne supérieure se base sur des résultats de densité récents dans $GSBD(\Omega)$ et sur la construction explicite d'une triangulation admissible optimale, introduite dans [39]. La difficulté principale vient du fait que l'énergie ne contrôle que la partie symétrique du gradient, ce qui rend impossible toute adaptation de la preuve constructive de [39] et nécessite une méthode complètement différente afin d'obtenir l'inégalité de la borne inférieure pour le terme d'énergie surfacique. Nous nous ramenons dans un premier temps au cas d'une interface de saut rectiligne, par un argument de blow-up autour d'un point de saut de u. Notre démonstration consiste alors à quantifier, parmi les triangles de la triangulation globale, le nombre minimal de triangles associés à de grandes variations des déplacements. Pour cela, nous procédons par une méthode de slicing le long des sections (essentiellement) orthogonales à l'interface de saut J_u. L'idée consiste alors à comparer l'amplitude limite du saut de u avec l'estimation de la variation d'une fonction affine le long d'une section (orthogonale à l'interface de saut) d'un tel triangle.

Notons que par définition de l'ensemble des éléments finis, les triangulations admissibles sont adaptatives et ont une structure géométrique flexible (mis à part une contrainte de non-aplatissement traduite par l'angle minimal $\theta_0 > 0$). En effet, minimiser l'énergie à $\varepsilon > 0$ fixé fait appel à une optimisation implicite parmi les ε-triangulations adaptées à un déplacement donné u_ε. Cela permet en particulier d'obtenir une énergie surfacique isotrope dans la Γ-limite, ne dépendant que de la longueur de la fissure et non de son orientation. La ténacité du matériau limite est une constante explicite et déterminée par la géométrie de nos triangles ($\kappa \sin\theta_0$), de sorte que toute ténacité donnée $\mu > 0$ peut être obtenue en remplaçant κ par $\mu/(\sin\theta_0)$ dans la définition de $\mathcal{E}_\varepsilon$.

Finalement, nous étendons ce résultat au cas des énergies avec conditions de Dirichlet au bord et démontrons la convergence des minimiseurs. Plus précisément, afin de donner un sens à la notion de condition au bord à $\varepsilon > 0$ fixé, nous considérons un voisinage ouvert $\Omega \subset\subset \Omega'$ et définissons l'ensemble des éléments finis

$$V_\varepsilon^{\mathrm{Dir}}(\Omega') = \left\{ \begin{array}{r} u \in C^0(\Omega';\mathbb{R}^2) \,:\, \exists \mathbf{T}_\varepsilon \in \mathcal{T}_\varepsilon(\Omega'),\, u \text{ est affine sur chaque } T \in \mathbf{T}_\varepsilon \text{ et} \\ u = w_{\mathbf{T}_\varepsilon} \text{ sur chaque } T \in \mathbf{T}_\varepsilon \text{ intersectant } \Omega' \setminus \Omega. \end{array} \right\}$$

où $w_{\mathbf{T}_\varepsilon}$ est l'interpolation de Lagrange affine par morceaux de w sur $\mathbf{T}_\varepsilon$. Nous démontrons alors que les fonctionnelles

$$\mathcal{G}_\varepsilon : u \in L^0(\Omega';\mathbb{R}^2) \mapsto \begin{cases} \displaystyle\int_\Omega \frac{1}{\varepsilon} f\big(\varepsilon \mathbf{A} e(u) : e(u)\big)\, dx & \text{si } u \in V_\varepsilon^{\mathrm{Dir}}(\Omega') \\ +\infty & \text{sinon} \end{cases}$$

Γ-convergent (pour la topologie de la convergence en mesure dans $L^0(\Omega';\mathbb{R}^2)$) vers l'énergie de Griffith avec condition au bord de Dirichlet

$$\mathcal{G} : u \in L^0(\Omega';\mathbb{R}^2) \mapsto \begin{cases} \displaystyle\int_\Omega \mathbf{A} e(u) : e(u)\, dx + \kappa \sin\theta_0 \big[\mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega \cap \{u \neq w\})\big] \\ \qquad\qquad\qquad \text{si } u \in GSBD^2(\Omega') \text{ et } u = w\, \mathcal{L}^2\text{-p.p. dans } \Omega' \setminus \overline{\Omega} \\ +\infty \quad \text{sinon} \end{cases}$$

lorsque $\varepsilon \searrow 0$. Nous plaçant ensuite dans le cas plus simple où

$$f(t) = \kappa \wedge t,$$

nous prouvons l'existence de minimiseurs, à $\varepsilon > 0$ fixé, pour la fonctionnelle d'énergie $\mathcal{G}_\varepsilon$ avec conditions de Dirichlet et obtenons un résultat de compacité pour les suites de déplacements uniformément bornés en énergie. Pour cela, nous utilisons un théorème de compacité dans $GSBD(\Omega)$ introduit dans [37], qui nécessite le retrait de mouvements rigides par morceaux associés à une partition de Caccioppoli du domaine élargi Ω'. Nous avons alors besoin d'une borne inférieure plus fine, afin de contrôler les éventuels sauts additionnellement créés aux frontières de la partition de Caccioppoli. Finalement, nous démontrons la convergence des minimiseurs (après le retrait des mouvements rigides) vers un minimiseur de la fonctionnelle de Griffith avec condition de Dirichlet.

1.3.2 . Modèles statiques discrets en endommagement brutal

Dans le troisième chapitre de cette thèse, nous complétons l'étude statique [15] dans un cadre discret en espace en dimension 2. En d'autres termes, nous restreignons l'ensemble des couples admissibles aux déplacements u continus affines par morceaux et aux fonctions caractéristiques χ constantes par morceaux, adaptés à une ε-triangulation commune du domaine $\Omega \subset \mathbb{R}^2$. L'énergie totale (1.2.2) est alors une énergie discrète dépendant d'un troisième paramètre $0 < h_\varepsilon \ll 1$, représentant la taille typique des cellules de la triangulation. Selon les différentes lois d'échelle considérées, nous nous attendons à obtenir une plus large classe de modèles effectifs limites. Plus précisément, étant donné un angle $\theta_0 > 0$, une triangulation $\mathbf{T}$ du domaine Ω est admissible, noté

$$\mathbf{T} \in \mathcal{T}_{h_\varepsilon}(\Omega, \theta_0),$$

si elle est formée de triangles dont les longueurs des côtés sont de l'ordre de h_ε et dont tous les angles sont supérieurs ou égaux à θ_0 (ce qui correspond à un critère de non aplatissement des triangles). Nous définissons alors l'ensemble des éléments finis $X_{h_\varepsilon}(\Omega)$ comme l'ensemble des couples

$$(u, \chi) \in C^0(\Omega; \mathbb{R}^2) \times L^\infty(\Omega; \{0, 1\})$$

tels qu'il existe une triangulation admissible commune de Ω adaptée au couple (u, χ) au sens où u est affine et χ constante sur chacun de ses triangles. Nous introduisons alors les fonctionnelles d'énergie $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \to [0, +\infty]$ définies par

$$\mathcal{F}_\varepsilon(u, \chi) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \big(\eta_\varepsilon \chi \mathbf{A}_0 + (1 - \chi)\mathbf{A}_1\big)e(u) : e(u)\, dx + \dfrac{\kappa}{\varepsilon} \displaystyle\int_\Omega \chi\, dx & \text{si } (u, \chi) \in X_{h_\varepsilon}(\Omega), \\ +\infty & \text{sinon} \end{cases} \tag{1.3.1}$$

ainsi que les taux de convergence

$$\alpha = \lim_{\varepsilon \searrow 0} \frac{\eta_\varepsilon}{\varepsilon} \in [0, +\infty] \quad \text{et} \quad \beta = \lim_{\varepsilon \searrow 0} \frac{h_\varepsilon}{\varepsilon} \in [0, +\infty],$$

donnant lieu à cinq lois d'échelle à traiter :

(i) $\alpha = +\infty$ ou $\beta = +\infty$;
(ii) $\alpha = \beta = 0$;
(iii) $\alpha = 0$ et $\beta \in (0, +\infty)$;
(iv) $\alpha \in (0, +\infty)$ et $\beta = 0$;
(v) $\alpha, \beta \in (0, +\infty)$.

L'analyse asymptotique des trois premiers régimes est démontrée aux paragraphes 3.1, 3.2 et 3.3 du deuxième chapitre de cette thèse, tandis que les deux derniers régimes sont encore ouverts et constituent des projets de recherche en cours. Nous discutons des difficultés allant à l'encontre de l'obtention de la borne supérieure dans le quatrième régime au paragraphe 3.4, tandis qu'un premier résultat dans le cadre un-dimensionnel est démontré pour le dernier régime au paragraphe 3.5 du deuxième chapitre.

- Dans le régime (i),

$$\varepsilon \ll \eta_\varepsilon \quad \text{ou} \quad \varepsilon \ll h_\varepsilon.$$

  La concentration de la zone endommagée l'emporte alors sur les phénomènes d'oscillation, de sorte que $\mathcal{F}_\varepsilon$ Γ-converge dans $L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega)$ vers le modèle d'élasticité linéaire

$$\mathcal{F}(u, \chi) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \mathbf{A}_1 e(u) : e(u)\, dx & \text{si } u \in H^1(\Omega; \mathbb{R}^2) \text{ et } \chi = 0, \\ +\infty & \text{sinon.} \end{cases}$$

- Dans le régime (ii),

$$\eta_\varepsilon \ll \varepsilon \quad \text{et} \quad h_\varepsilon \ll \varepsilon.$$

En particulier, l'énergie élastique emmagasinée dans la zone endommagée devient négligeable, de sorte que $\mathcal{F}_\varepsilon$ Γ-converge dans $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$ vers le modèle trivial

$$\mathcal{F}(u,\chi)=\begin{cases}0 & \text{si } \chi=0,\\ +\infty & \text{sinon},\end{cases}$$

sous condition que $\theta_0\leq 45°$.

- Dans le régime (iii),

$$\eta_\varepsilon \ll \varepsilon \quad \text{et} \quad h_\varepsilon\sim\beta\varepsilon.$$

La fonctionnelle $\mathcal{F}_\varepsilon$ Γ-converge alors vers un modèle de rupture fragile, donné par l'énergie

$$\mathcal{F}(u,\chi)=\begin{cases}\displaystyle\frac{1}{2}\int_\Omega \mathbf{A}_1 e(u):e(u)\,dx+\beta\kappa\sin(\theta_0)\mathcal{H}^1(J_u) & \text{si } u\in GSBD^2(\Omega) \text{ et } \chi=0,\\ +\infty & \text{sinon},\end{cases}$$

pour la topologie de la convergence en mesure dans $L^0(\Omega;\mathbb{R}^2)\times L^0(\Omega)$. Nous démontrons ce résultat au paragraphe 3.3 du chapitre 3, en adaptant légèrement les arguments du travail présenté au chapitre 2 (décrit plus haut).

- Les deux derniers régimes (iv) et (v) restent encore ouverts et constituent des projets de recherche en cours. Plus précisément, nous supposons ici que $\mathbf{A}_0$ et $\mathbf{A}_1$ sont des tenseurs isotropes, déterminés par leurs coefficients de Lamé $\lambda_1>\lambda_0>0$ et $\mu_1>\mu_0>0$ en posant, pour tout $\xi\in\mathbb{M}^{2\times 2}_{\rm sym}$:

$$\mathbf{A}_i\xi=\lambda_i\big(\mathrm{tr}\xi\big)I_N+2\mu_i\xi,\quad i\in\{1,2\}.$$

  - Dans le régime (iv),

  $$\eta_\varepsilon\sim\alpha\varepsilon \quad \text{et} \quad h_\varepsilon\ll\varepsilon,$$

  de sorte que la vitesse de convergence de la discrétisation en espace vers le modèle continu est plus rapide que celles des autres phénomènes en jeu. Ainsi, nous nous attendons naturellement à ce que $\mathcal{F}_\varepsilon$ Γ-converge dans $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$ vers le même modèle de plasticité (de Hencky) obtenu dans [15, Theorem 3.1], donné par

  $$\mathcal{F}(u,\chi)=\begin{cases}\displaystyle\int_\Omega \overline{W}_\alpha\big(e(u)\big)\,dx+\int_\Omega \overline{W}_\alpha^\infty\left(\frac{dE^su}{d|E^su|}\right)\,d|E^su| & \text{si } u\in BD(\Omega) \text{ et } \chi=0,\\ +\infty & \text{sinon}.\end{cases}$$

  où $\overline{W}_\alpha$ est l'inf-convolution

  $$\overline{W}_\alpha:\xi\in\mathbb{M}^{2\times 2}_{\rm sym}\mapsto \inf_{\tau\in\mathbb{M}^{2\times 2}_{\rm sym}}\left\{\frac{1}{2}\mathbf{A}_1\tau:\tau+\sqrt{2\alpha\kappa h(\xi-\tau)}\right\},$$

  $\overline{W}_\alpha^\infty$ est sa fonction de récession

  $$\overline{W}_\alpha^\infty:\xi\in\mathbb{M}^{2\times 2}_{\rm sym}\mapsto\lim_{t\to+\infty}\frac{\overline{W}_\alpha\big(t\xi\big)}{t},$$

et
$$h:\xi\in\mathbb{M}^{2\times 2}_{\rm sym}\mapsto \mathbf{A}_0\xi:\xi+4\mu_0\big(\det\xi\big)^+.$$
Un premier pas vers cette conjecture est démontré au Théorème 3.4.1 (paragraphe 3.4), où la contrainte sur le pas du maillage est relaxée en autorisant toute triangulation adaptée à un pas inférieur $0<\overline{h}_\varepsilon<h_\varepsilon$. Ce résultat est néanmoins plus restrictif du point de vue des applications numériques, de par la difficulté à identifier le pas exact du maillage $\overline{h}_\varepsilon$. De plus, la relaxation de la contrainte sur le pas du maillage entraîne une indépendance non naturelle du modèle effectif vis à vis de β. Autrement dit, quel que soit $\beta\in[0,+\infty]$, ce modèle relaxé Γ-converge vers le modèle de plasticité. Par conséquent, ce résultat de convergence ne semble pas assez fin puisqu'il ne décèle pas le modèle d'élasticité linéaire ($\beta=+\infty$). Ces remarques remettent en question notre intuition première, ce qui est confirmé par l'étude de la borne supérieure pour un déplacement affine dans le cas scalaire (voir Exemple 3.4.4).

- Dans le régime (v),
$$\eta_\varepsilon\sim\alpha\varepsilon\quad\text{et}\quad h_\varepsilon\sim\beta\varepsilon.$$
Heuristiquement, aucun des trois phénomènes en jeu n'est négligeable vis-à-vis des deux autres. La convergence de la discrétisation en espace, la concentration des zones endommagées et la dégénérescence des propriétés élastiques de la phase endommagée entrent tous les trois en compétition. Une étude scalaire (démontrée au Théorème 3.5.1) semble indiquer que $\mathcal{F}_\varepsilon$ Γ-converge dans $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$ vers un modèle intermédiaire entre la plasticité de Hencky et la rupture fragile donné par
$$\mathcal{F}(u,\chi)=\begin{cases}\displaystyle\int_\Omega\frac{1}{2}\mathbf{A}_1e(u):e(u)\,dx+\int_{J_u}\phi_{\alpha,\beta}\left(u^+-u^-\right)\,d\mathcal{H}^1 & \text{si } u\in SBD^2(\Omega)\text{ et }\chi=0,\\ +\infty & \text{sinon,}\end{cases}$$
où la densité d'énergie de surface $\phi_{\alpha,\beta}:\mathbb{R}^2\to(0,+\infty)$ est une fonction à croissance linéaire, bornée inférieurement par une constante strictement positive. En dimension un, $\phi_{\alpha,\beta}$ est explicitement donnée par :
$$\phi_{\alpha,\beta}:t\in\mathbb{R}\mapsto\begin{cases}\dfrac{a_0\alpha}{2\beta}t^2+\beta\kappa & \text{si } |t|\le\beta\sqrt{\dfrac{2\kappa}{a_0\alpha}},\\ \sqrt{2\kappa a_0\alpha}\,|t| & \text{sinon.}\end{cases}$$
Ce modèle est qualitativement similaire aux modèles variationnels de fracture avec zone cohésive et de cisaillement plastique anti-plan, introduits par Dal Maso et Iurlano ([51, 74]) et Ambrosio, Lemenant et Royer-Carfagni ([7]) respectivement, où la densité d'énergie surfacique est de la forme $1+|u^+-u^-|$ et l'énergie totale est donnée par
$$\int_\Omega\frac{1}{2}|\nabla u|^2\,dx+\gamma\mathcal{H}^1(J_u)+\sigma_0\int_{J_u}\left|u^+-u^-\right|\,d\mathcal{H}^1$$
pour un déplacement scalaire $u:\Omega\to\mathbb{R}$. Dans ces modèles, le coût des déformations anélastiques est différent selon l'ordre de grandeur de l'amplitude du saut $|u^+-u^-|$. Lorsque le saut est petit, le coût à payer est de l'ordre de
$$\gamma\mathcal{H}^1(J_u),$$

où la ténacité $\gamma > 0$ représente l'énergie par unité de surface dépensée pour fracturer localement le matériau, comme en rupture fragile. Tandis que pour les sauts de grande amplitude, l'énergie de surface dépensée est alors de l'ordre de l'amplitude du saut :

$$\sigma_0 \int_{J_u} |u^+ - u^-| \, d\mathcal{H}^1$$

où la densité surfacique est à croissance linéaire en l'amplitude du saut. Comme en plasticité, $\sigma_0 > 0$ représente le seuil sur les contraintes à partir duquel les déformations élastiques cèdent le pas aux déformations plastiques. Ici, la densité surfacique $\phi_{\alpha,\beta}$ a le même comportement : pour les sauts de petite amplitude, le terme dissipatif prépondérant correspond à l'énergie de fracture (avec $\gamma = \beta\kappa$), alors que celle-ci devient négligeable vis à vis du coût de la déformation plastique (avec $\sigma_0 = \sqrt{2\kappa a_0 \alpha}$) pour les sauts de grande amplitude.

1.3.3 . Modèles d'évolution en endommagement brutal

Les phénomènes mécaniques dissipatifs étant associés à des processus d'évolution, il est naturel de vouloir étendre les travaux de [15], menés dans le cadre statique et continu en espace, au cadre dynamique. Le plus simple étant celui des évolutions quasi-statiques, où l'inertie (l'accélération du déplacement) est négligée. Dans le quatrième chapitre de cette thèse, nous nous intéressons à l'étude quasi-statique du régime

$$\eta_\varepsilon \sim \varepsilon,$$

faisant l'objet de l'article [17] en cours de soumission. Ce papier traite de l'interaction, au cours d'un processus d'évolution, entre les phénomènes de relaxation (*i.e.* de Γ-convergence) et l'irréversibilité de l'endommagement.

Nous nous plaçons en dimension un et considérons un matériau linéairement élastique dont la configuration au repos est un intervalle ouvert

$$\Omega = (0, L)$$

soumis à un déplacement donné aux extrémités $\{0, L\}$,

$$w \in AC([0,T]; H^1(\mathbb{R})).$$

Nous supposons connus les coefficients d'élasticité $a_1 > 0$ et la ténacité $\kappa > 0$ du matériau. L'idée est alors d'adapter la méthode introduite dans [15] au cadre quasi-statique. Pour cela, nous considérons la famille d'évolutions quasi-statiques d'endommagement brutal introduite dans [60] avec la même loi d'échelle, où nous prenons le taux de convergence égal à 1 pour simplifier (*i.e.* $\eta_\varepsilon = \varepsilon$). Plus précisément, à $\varepsilon > 0$ fixé, nous appliquons le [60, Theorem 2] à un matériau élastique de ténacité κ/ε, composé exclusivement de deux phases pures, l'une endommagée et l'autre saine ayant $0 < \varepsilon a_0 < a_1$ pour coefficients d'élasticité respectifs, soumis au déplacement $w(t)$ aux extrémités $\{0, L\}$ en tout temps $t \in [0, T]$. Nous disposons ainsi d'un triplet

$$(u_\varepsilon, \Theta_\varepsilon, a_\varepsilon) : [0,T] \to H^1(\Omega;\mathbb{R}) \times L^\infty(\Omega;[0,1]) \times L^\infty(\Omega;[0,a_1]) \qquad (1.3.2)$$

décrivant, à $\varepsilon > 0$ fixé, l'état du matériau en tout temps $t \in [0,T]$: sa configuration spatiale est donnée par le déplacement $u_\varepsilon(t)$ et ses coefficients d'élasticité sont donnés par

$$a_\varepsilon(t) = \left(\frac{\Theta_\varepsilon(t)}{a_1} + \frac{1-\Theta_\varepsilon(t)}{\varepsilon a_0} \right)^{-1}$$

où $\Theta_\varepsilon(t) \in [0,1]$ est la fraction linéique de matériau sain a_1. La question étudiée dans [17] est alors : en faisant tendre le paramètre ε vers 0 dans les évolutions (1.3.2), obtient-on une évolution quasi-statique de plasticité parfaite

$$(u,e,p) : [0,T] \to BV((0,L)) \times L^2((0,L)) \times \mathcal{M}([0,L])$$

comme définie au paragraphe 1.1.2 de l'Introduction ?

Par analogie, nous nous attendons à ce que le convexe d'élasticité associé à l'évolution limite soit le même que celui obtenu dans l'analyse statique de [15] :

$$K := \big[-\sqrt{2\kappa a_0}, \sqrt{2\kappa a_0}\, \big].$$

Le potentiel de dissipation associé est alors donné par

$$I_K^* = \sqrt{2\kappa a_0}\,|\cdot|\,.$$

Nous démontrons, grâce à l'obtention de bornes uniformes en temps et en ε, l'existence d'une suite extraite (indépendante du temps et toujours notée ε) et d'une évolution limite

$$(u,e,p,\mu) : [0,T] \to BV((0,L)) \times L^2((0,L)) \times \mathcal{M}([0,L]) \times \mathcal{M}^+([0,L])$$

absolument continue en temps, telle que

$$p = \frac{\sigma}{a_0}\mu \quad \text{dans } \mathcal{M}([0,L])$$

et satisfaisant les propriétés suivantes en tout temps $t \in [0,T]$:

i. Décomposition Additive : $Du(t) = e(t)\mathcal{L}^1 \llcorner (0,L) + p(t) \llcorner (0,L)$ dans $\mathcal{M}((0,L))$
ii. Condition de Dirichlet Relaxée : $p(t) \llcorner \{0,L\} = \big(w(t) - u(t)\big)\big(\delta_L - \delta_0\big)$ dans $\mathcal{M}(\{0,L\})$
iii. Loi de Comportement : $\sigma(t) = a_1 e(t)$
iv. Équation d'Équilibre : $\sigma'(t) = 0$ dans $H^{-1}((0,L))$
v. Contrainte sur les forces : $\sigma(t) \in K$.

Contrairement à l'analyse statique, l'interaction entre l'irréversibilité de l'endommagement et la Γ-convergence des énergies n'est pas stable au cours de l'évolution en temps. En effet, en fonction de la donnée au bord w, l'évolution obtenue ci-dessus peut ne pas être compatible avec une évolution de plasticité parfaite. Étonnamment, l'évolution du matériau effectif limite peut être interprétée comme de l'endommagement, en introduisant la variable interne

$$c : t \in [0,T] \mapsto \frac{\mu(t)}{a_0} + \frac{1}{a_1}\mathcal{L}^1 \llcorner (0,L) \in \mathcal{M}^+([0,L]),$$

qui représente la compliance du matériau limite (tenseur de la souplesse élastique) et dont le caractère croissant retranscrit l'irréversibilité de l'endommagement. En effet, la compliance satisfait la Loi de Comportement

$$Du(t) = \sigma(t)c(t) \llcorner (0, L) \text{ dans } \mathcal{M}((0, L))$$

en tout temps $t \in [0, T]$ et vérifie une Loi d'Évolution de type Griffith

$$\dot{c}(t)\left(2\kappa a_0 - \sigma(t)^2\right) = 0 \text{ dans } \mathcal{M}^+([0, L])$$

pour $\mathcal{L}^1$-presque tout $t \in [0, T]$. Ainsi, l'endommagement progresse uniquement lorsque le tenseur des contraintes de Cauchy σ sature la contrainte. Comme mentionné ci-dessus, la réponse du matériau à la condition de Dirichlet w peut ne pas satisfaire les conditions d'une évolution plastique. En effet, l'évolution (u, e, p) satisfait les hypothèses d'une évolution de plasticité parfaite (4.1.3) et (4.1.4) si et seulement si la condition de Dirichlet satisfait, pour tout temps $0 \le s < t \le T$,

$$\left|\left[w(t)\right]_0^L\right| < \left|\left[w(s)\right]_0^L\right| \Rightarrow \left|\left[w(t)\right]_0^L\right| \le \sqrt{2\kappa a_0}\frac{L}{a_1}.$$

Cette étude scalaire est un exemple d'interaction instable au cours d'évolution en temps entre la Γ-convergence des énergies et l'irréversibilité de l'endommagement.

Dans une vision à court terme, j'aimerais généraliser cette étude au cadre multi-dimensionnel. Une étape intermédiaire de ce projet de recherche consisterait à s'intéresser au cadre scalaire anti-plan où $\Omega \subset \mathbb{R}^2$. Il s'agit d'un modèle à la fois mutli-dimensionnel et scalaire, au sens où le déplacement anti-plan

$$u : \Omega \to \mathbb{R}$$

est à valeurs réelles. Les coefficients élastiques du matériau sont donnés par une matrice symétrique $\mathbf{A} \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ satisfaisant les propriétés de croissance et de coercivité

$$0 < a_0 \le \mathbf{A} \le a_1 < +\infty$$

au sens des formes quadratiques agissant sur $\mathbb{R}^2$. Dans ce contexte, les enveloppes quasi-convexes coïncident avec les enveloppes convexes, pour lesquelles il est généralement plus facile d'obtenir des formules explicites, offrant l'espoir que les méthodes menées en dimension un soient adaptables. En effet, l'une des difficultés consiste à calculer l'enveloppe symétrique quasi-convexe des densités d'énergie à $\varepsilon > 0$ fixé, faisant appel à l'utilisation des bornes de Hashin-Shtrikman pour lesquelles l'obtention de formules explicites est difficile dans le cadre vectoriel.

Au plus long terme, j'aimerais travailler sur le cas dynamique, en prenant en compte les effets d'inertie dans les modèles d'évolution d'endommagement brutal. En s'inspirant de la méthode précédente, l'idée serait de coupler les évolutions d'endommagement brutal introduites dans [66] avec la loi d'échelle $\eta_\varepsilon = \varepsilon$ et d'effectuer l'analyse asymptotique de ce régime. Comme dans le cas quasi-statique, la première étape consisterait à étudier le modèle un-dimensionnel pour comprendre les phénomènes en jeu et l'expression de l'irréversibilité dans le modèle limite effectif.

1.4 . Notation and preliminary results

Vectors. If a and $b \in \mathbb{R}^N$, with $N \in \mathbb{N} \setminus \{0\}$, we write $a \cdot b = \sum_{i=1}^N a_i b_i$ for the Euclidean scalar product and $|a| = \sqrt{a \cdot a}$ for the corresponding norm. For $x \in \mathbb{R}^N$ and $\varrho > 0$, we denote by $B_\varrho(x) := \{y \in \mathbb{R}^N : |x-y| < \varrho\}$ the open ball centered at x with radius ϱ. If $x = 0$, we simply write B_ϱ instead of $B_\varrho(0)$. The notation $\mathbb{S}^{N-1}$ stands for the unit sphere ∂B_1.

Matrices. The space of all real $m \times N$ matrices is denoted by $\mathbb{M}^{m \times N}$, and the subspace of symmetric real $N \times N$ matrices by $\mathbb{M}^{N\times N}_{\rm sym}$. It will be endowed with the Froebenius scalar product $A : B = {\rm tr}(A^T B)$ and the corresponding norm $|A| = \sqrt{A : A}$.

Given two vectors a and $b \in \mathbb{R}^N$, the tensor product between a and b is defined as $a \otimes b := ab^T \in \mathbb{M}^{N\times N}$ and the symmetric tensor product by $a \odot b := (a \otimes b + b \otimes a)/2 \in \mathbb{M}^{N\times N}_{\rm sym}$.

Measures. The Lebesgue and k-dimensional Hausdorff measures in $\mathbb{R}^N$ are respectively denoted by $\mathcal{L}^N$ and $\mathcal{H}^k$. If X is a borel subset of $\mathbb{R}^N$ and Y is an Euclidean space, we denote by $\mathcal{M}(X;Y)$ the space of Y-valued bounded Radon measures in X which, according to the Riesz Representation Theorem, can be identified to the dual of $C_0(X;Y)$ (the closure of $C_c(X;Y)$ for the sup-norm in X). The weak-* topology of $\mathcal{M}(X;Y)$ is defined using this duality. The indication of the space Y is omitted when $Y = \mathbb{R}$. For $\mu \in \mathcal{M}(X;Y)$, its total variation is denoted by $|\mu|$ and we denote by $\mu = \mu^a + \mu^s$ the Radon-Nikodým decomposition of μ with respect to Lebesgue, where μ^a is absolutely continuous and μ^s is singular with respect to the Lebesgue measure $\mathcal{L}^N$.

Convex analysis. We recall some definition and standard results from convex analysis (see [88]). Let $f : \mathbb{R}^N \to [0,+\infty]$ be a proper function (*i.e.* not identically $+\infty$). The convex conjugate of f is defined as

$$f^*(x) = \sup_{y \in \mathbb{R}^N} \{x \cdot y - f(y)\}$$

which turns out to be convex and lower semicontinuous. If f is convex and finite, we define its recession function as

$$f^\infty(x) = \lim_{t \nearrow +\infty} \frac{f(tx)}{t} \in [0,+\infty]$$

which is convex and positively 1-homogeneous. If $f, g : \mathbb{R}^N \to [0,+\infty]$ are proper convex functions, then their infimal convolution is defined as

$$f \,\square\, g(x) = \inf_{y \in \mathbb{R}^N} \{f(x-y) + g(y)\}$$

which is convex as well. The indicator function of a set $C \subset \mathbb{R}^N$ is defined as $I_C = 0$ in C and $+\infty$ otherwise. The convex conjugate I_C^* of I_C is called the support function of C.

Functional spaces. We use standard notation for Lebesgue and Sobolev spaces. If U is a bounded open subset of $\mathbb{R}^N$, we denote by $L^0(U;\mathbb{R}^m)$ the set of all $\mathcal{L}^N$-measurable functions from U to $\mathbb{R}^m$.

We recall some properties regarding functions with values in a Banach space and refer to [58, 28, 49] for details and proofs on this matter. If Y is a Banach space and $T > 0$, we denote by $AC([0,T];Y)$ the space of absolutely continuous functions $f : [0,T] \to Y$. If Y is the dual of a separable Banach space X, then every function $f \in AC([0,T];Y)$ is such that the weak-* limit

$$\dot f(t) = \text{w*-}\lim_{s\to t} \frac{f(t)-f(s)}{t-s} \in Y$$

exists for $\mathcal{L}^1$-a.e. $t \in [0,T]$, $\dot f : [0,T] \to Y$ is weakly-* measurable and $t \mapsto \|\dot f(t)\|_Y \in L^1([0,T])$. If $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a function of two variables, time and spatial derivatives will be respectively denoted by $\dot f$ and f'.

We recall that a sequence $\{g_k\}_{k\in\mathbb{N}}$ in $L^0(U;\mathbb{R}^m)$ converges in measure to $g \in L^0(U;\mathbb{R}^m)$ if for all $\varepsilon > 0$,

$$\mathcal{L}^N\left(\{x \in U : |g_k(x) - g(x)| > \varepsilon\}\right) \to 0.$$

Note that, for any fixed constant $M > 0$, we can define the following mapping

$$d_M : (g,h) \in L^0(U;\mathbb{R}^m) \times L^0(U;\mathbb{R}^m) \mapsto \int_U M \wedge |g-h| \; dx \in \mathbb{R}^+ \tag{1.4.1}$$

which turns out to be a distance over $L^0(U;\mathbb{R}^m)$, with the property that g_k converges in measure to g if and only if $d_M(g_k, g) \to 0$. It confers to $L^0(U;\mathbb{R}^m)$ a metric space structure.

Functions of bounded variation and sets of finite perimeter. We refer to [6] for an exhaustive treatment on that subject and just recall few notation. Let $U \subset \mathbb{R}^N$ be a bounded open set. A function $u \in L^1(U;\mathbb{R}^m)$ is a *function of bounded variation* in U, and we write $u \in BV(U;\mathbb{R}^m)$, if its distributional derivative Du belongs to $\mathcal{M}(U;\mathbb{M}^{m\times N})$. We use standard notation for that space, referring to [6] for details. We just recall that a function u belongs to $SBV^2(U;\mathbb{R}^m)$ if $u \in SBV(U;\mathbb{R}^m)$ (the distributional derivative Du has no Cantor part), its approximate gradient ∇u belongs to $L^2(U;\mathbb{M}^{m\times N})$ and its jump set J_u satisfies $\mathcal{H}^{N-1}(J_u) < \infty$.

If U has Lipschitz boundary, every function $u \in BV(U;\mathbb{R}^m)$ has an inner trace on ∂U (still denoted by u and $\mathcal{H}^{N-1}$-integrable on ∂U) and there exists a constant $C > 0$ depending only on U such that

$$\frac{1}{C}\|u\|_{BV(U;\mathbb{R}^m)} \le |Du|(U) + \int_{\partial U} |u| \; d\mathcal{H}^{N-1} \le C\|u\|_{BV(U;\mathbb{R}^m)} \tag{1.4.2}$$

according to [94, Proposition 2.4, Remark 2.5 (ii)].

A Lebesgue measurable set $A \subset \mathbb{R}^N$ is a *set of finite perimeter* in U if its characteristic function $\mathbb{1}_A$ belongs to $BV(U;\mathbb{R}^N)$. The reduced boundary of A is denoted by

$$\partial^* A = \left\{ x \in \operatorname{supp}|D\mathbb{1}_A| \cap U \;:\; \text{the limit } \nu_A(x) = \lim_{\varrho\searrow 0} \frac{D\mathbb{1}_A\big(B_\varrho(x)\big)}{|D\mathbb{1}_A|\big(B_\varrho(x)\big)} \text{ exists and satisfies } |\nu_A(x)| = 1 \right\}$$

and the essential (or measure theoretic) boundary is denoted by

$$\partial_* A = \mathbb{R}^N \setminus \left(A^0 \cup A^1\right)$$

where, for every $t \in [0,1]$, we denote the set of points where A has density t by

$$A^{(t)} = \left\{ x \in \mathbb{R}^N \;:\; \lim_{\varrho\searrow 0} \frac{\mathcal{L}^N\big(A \cap B_\varrho(x)\big)}{\mathcal{L}^N\big(B_\varrho(x)\big)} = t \right\}.$$

We also recall that a partition $\mathcal{P} = \{P_i\}_{i\in\mathbb{N}}$ of an open set U is a *Cacciopoli partition* if each P_i have finite perimeter in U, and $\sum_{i\in\mathbb{N}} |D\mathbb{1}_{P_i}|(U) < \infty$. In that case,

$$\bigcup_{i\in\mathbb{N}} (P_i)^{(1)} \cup \bigcup_{i,j\in\mathbb{N},\, i\neq j} \partial^* P_i \cap \partial^* P_j$$

contains $\mathcal{H}^{N-1}$-almost all of U (see [6, Section 4.4]). In the sequel (as in [37, Theorem 2.5]), we will sometimes use the following notation for Caccioppoli partitions :

$$\mathcal{P}^{(1)} := \bigcup_{i\in\mathbb{N}} P_i^{(1)}, \quad \partial^*\mathcal{P} := \bigcup_{i\in\mathbb{N}} \partial^* P_i.$$

(Generalized) functions of bounded deformation. A function $u \in L^1(U;\mathbb{R}^N)$ is a *function of bounded deformation*, and we write $u \in BD(U)$, if its distributional symmetric gradient $Eu := (Du + Du^T)/2$ belongs to $\mathcal{M}(U;\mathbb{M}^{N\times N}_{\rm sym})$. We refer to [92, 94, 91, 5, 22] for the main properties and notation of that space. The space $SBD^2(U)$ is made of all functions $u \in SBD(U)$ (Eu has no Cantor part) such that the approximate symmetric gradient $e(u)$ (the absolutely continuous part of Eu with respect to $\mathcal{L}^N$) belongs to $L^2(U;\mathbb{M}^{N\times N}_{\rm sym})$ and its jump set J_u satisfies $\mathcal{H}^{N-1}(J_u) < \infty$.

We now recall the definition and the main properties of the space of *generalized functions of bounded deformation* introduced in [48]. We first need to introduce some notation. Let $\xi \in \mathbb{S}^{N-1}$, we denote by $\Pi_\xi := \{y \in \mathbb{R}^N : \ y\cdot\xi = 0\}$ the orthogonal space to ξ and by p_ξ the orthogonal projection onto Π_ξ. For every set $B \subset \mathbb{R}^N$, we define for $\xi \in \mathbb{S}^{N-1}$ and $y \in \mathbb{R}^N$,

$$B^\xi_y := \{t \in \mathbb{R} : \ y + t\xi \in B\}, \quad B^\xi := p_\xi(B)$$

and, for every (vector-valued) function $u : B \to \mathbb{R}^N$ and (scalar-valued) function $f : B \to \mathbb{R}$,

$$u^\xi_y(t) := u(y+t\xi)\cdot\xi, \quad f^\xi_y(t) = f(y+t\xi) \quad \text{for all } y \in \mathbb{R}^N \text{ and all } t \in B^\xi_y.$$

Definition 1.4.1. Let $U \subset \mathbb{R}^N$ be a bounded open set and $u \in L^0(U;\mathbb{R}^N)$. Then, $u \in GBD(U)$ if there exists a nonnegative measure $\lambda \in \mathcal{M}(U)$ such that one of the following equivalent conditions holds true for every $\xi \in \mathbb{S}^{N-1}$:

1. for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$, the partial derivative $D_\xi(\tau(u\cdot\xi)) = D(\tau(u\cdot\xi))\cdot\xi$ belongs to $\mathcal{M}(U)$, and
$$|D_\xi(\tau(u\cdot\xi))|(B) \leq \lambda(B) \quad \text{for every Borel set } B \subset U;$$
2. $u^\xi_y \in BV_{\rm loc}(U^\xi_y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in U^\xi$, and
$$\int_{\Pi_\xi} \Big(|Du^\xi_y|(B^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{u^\xi_y}) \Big)\, d\mathcal{H}^{N-1}(y) \leq \lambda(B) \quad \text{for every Borel set } B \subset U,$$
where $J^1_{u^\xi_y} := \{t \in J_{u^\xi_y} : \ |[u^\xi_y](t)| \geq 1\}$.

The function u belongs to $GSBD(U)$ if $u \in GBD(U)$ and $u^\xi_y \in SBV_{\rm loc}(U^\xi_y)$ for every $\xi \in \mathbb{S}^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in U^\xi$.

Every $u \in GBD(U)$ has an approximate symmetric gradient $e(u) \in L^1(U; \mathbb{M}^{N\times N}_{\rm sym})$ such that for every $\xi \in \mathbb{S}^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in U^\xi$,

$$e(u)(y+t\xi)\xi\cdot\xi = (u^\xi_y)'(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in U^\xi_y.$$

Moreover, the jump set J_u of $u \in GBD(U)$, defined as the set of all $x_0 \in U$ for which there exist $(u^+(x_0), u^-(x_0), \nu_u(x_0)) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}$ with $u^+(x_0) \neq u^-(x_0)$ such that the function

$$y \in B_1 \mapsto u_{x_0,\varrho} := u(x_0 + \varrho y)$$

converges in measure in B_1 as $\varrho \searrow 0$ to

$$y \in B_1 \mapsto \begin{cases} u^+(x_0) & \text{if } y\cdot\nu_u(x_0) > 0, \\ u^-(x_0) & \text{if } y\cdot\nu_u(x_0) \leq 0, \end{cases}$$

is countably $(\mathcal{H}^{N-1}, N-1)$-rectifiable. Finally, the energy space $GSBD^2(U)$ is defined as

$$GSBD^2(U) := \{u \in GSBD(U):\ e(u) \in L^2(U; \mathbb{M}^{N\times N}_{\rm sym}),\ \mathcal{H}^{N-1}(J_u) < \infty\}.$$

Homogenization and H-convergence We refer to [2] for an exhaustive presentation of these notions and only recall minimal results. We denote, for fixed $\alpha, \beta > 0$, the subset of fourth-order symmetric tensors

$$\mathcal{F}_{\alpha,\beta} = \left\{ \mathbf{A} \in \mathbb{R}^{N^4} \ :\ \mathbf{A}_{ijkl} = \mathbf{A}_{klij} = \mathbf{A}_{jikl},\ \alpha\,|\xi|^2 \leq \mathbf{A}\xi : \xi \leq \beta\,|\xi|^2 \ \text{for all } \xi \in \mathbb{M}^{N\times N}_{\rm sym} \right\}.$$

Let Ω be a bounded open set of $\mathbb{R}^N$. We say that $\mathbf{A}_n \in L^\infty(\Omega; \mathcal{F}_{\alpha,\beta})$ H-converges to $\mathbf{A} \in L^\infty(\Omega; \mathcal{F}_{\alpha,\beta})$ if, for every $f \in H^{-1}(\Omega; \mathbb{R}^N)$, the solutions $u_n \in H^1_0(\Omega; \mathbb{R}^N)$ of the equilibrium equations

$$\begin{cases} -\mathrm{div}\big(\mathbf{A}_n e(u_n)\big) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

are such that $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega; \mathbb{R}^N)$ and $\mathbf{A}_n e(u_n) \rightharpoonup \mathbf{A}e(u)$ weakly in $L^2(\Omega; \mathbb{M}^{N\times N}_{\rm sym})$ as $n \nearrow +\infty$, where $u \in H^1_0(\Omega; \mathbb{R}^N)$ is the solution of

$$\begin{cases} -\mathrm{div}\big(\mathbf{A} e(u)\big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Given a volume fraction $\theta \in L^\infty(\Omega; [0,1])$ and $\mathbf{A}, \mathbf{B} \in L^\infty(\Omega; \mathcal{F}_{\alpha,\beta})$ with $\mathbf{A} \leq \mathbf{B}$ as quadratic forms on $\mathbb{M}^{N\times N}_{\rm sym}$, the $\mathcal{G}$-closure set

$$\mathcal{G}_\theta(\mathbf{A}, \mathbf{B})$$

is defined as the set of all possible H-limits of $\chi_n\mathbf{A} + (1-\chi_n)\mathbf{B}$ where $\chi_n \in L^\infty(\Omega; \{0,1\})$ weakly-* converges to θ in $L^\infty(\Omega; [0,1])$. We also denote the set of all possible composite materials by

$$\mathcal{G}(\mathbf{A}, \mathbf{B}) := \{\mathbf{C} \in \mathcal{G}_\theta(\mathbf{A}, \mathbf{B}) \ :\ \theta \in L^\infty(\Omega; \{0,1\})\}.$$

Γ-convergence We refer to [47] for a complete study of this notion and only recall the definition and Fundamental Theorem of Γ-convergence. Let (X, d) be a metric space. We say that a sequence $f_k : X \to \overline{\mathbb{R}}$ Γ-converges in (X, d) to $f : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have

1. (Lower bound) $f(x) \leq \liminf_{k \nearrow \infty} f_k(x_k)$ for any sequence $\{x_k\}_k$ converging to x,
2. (Upper bound) $f(x) \geq \limsup_{k \nearrow \infty} f_k(x_k)$ for some (recovery) sequence $\{x_k\}_k$ converging to x.

In this case, if $x_k^* \in \underset{X}{\operatorname{argmin}} f_k$ converges to some $x^* \in X$, then $x^* \in \underset{X}{\operatorname{argmin}} f$ and $f_k(x_k^*) \to f(x^*)$ as $k \nearrow \infty$.

2 - Discrete approximation of the Griffith functional by adaptive finite elements

This chapter is devoted to show a discrete adaptive finite element approximation result for the isotropic two-dimensional Griffith energy arising in fracture mechanics. The problem is addressed in the geometric measure theoretic framework of generalized special functions of bounded deformation which corresponds to the natural energy space for this functional. It is proved to be approximated in the sense of Γ-convergence by a sequence of discrete integral functionals defined on continuous piecewise affine functions. The main feature of this result is that the mesh is part of the unknown of the problem, and it gives enough flexibility to recover isotropic surface energies. This is joint work with Jean-François Babadjian corresponding to [12] and accepted for publication in the journal SIAM Journal on Mathematical Analysis.

2.1 . Introduction

2.1.1 . The variational approach to fracture

The Griffith functional has been introduced in the context of brittle fracture. It finds its roots in the seminal work of Griffith [72] whose main ideas have been revisited in [63] (see also the monograph [24]) into a variational evolution formulation. The main point is that, in a quasi-static setting and in presence of irreversibility, a constrained global minimization principle together with an energy balance select equilibrium states of an elastic body experiencing brittle fracture. In a nutshell, the Griffith energy is defined by

$$\mathcal{G}(u,K) := \int_{\Omega\setminus K} |e(u)|^2\,dx + \mu\mathcal{H}^{N-1}(K), \tag{2.1.1}$$

where $\Omega \subset \mathbb{R}^N$, a bounded open set, stands for the reference configuration of an elastic material, $K \subset \Omega$ is a codimension-one set representing the crack, $u : \Omega \setminus K \to \mathbb{R}^N$ is the displacement field which might be discontinuous across K, and its symmetric gradient $e(u) := (\nabla u + \nabla u^T)/2$ is the linearized elastic strain. The constant $\mu > 0$ is a material parameter called toughness. This energy puts in competition a bulk energy, representing the elastic energy stored in the body outside the crack, and a surface energy penalizing the presence of the crack K through its $(N-1)$-dimensional Hausdorff measure, henceforth denoted by $\mathcal{H}^{N-1}$.

This problem falls within the framework of so-called free discontinuity problems (according to De Giorgi's terminology), and it presents many formal analogies with its scalar counterpart, the Mumford-Shah functional. Although, thanks to geometric measure theory, the existence theory for the latter is by now quite well understood (see e.g. [6] and references therein), the minimization of the Griffith functional had to face serious additional difficulties. In particular, a satisfactory existence theory has only recently been solved. As for the Mumford-Shah functional, it passes through the introduction of a "weak formulation" where the crack is replaced by the jump set J_u of u. A convenient functional setting to investigate this problem is that of functions of bounded deformation, $BD(\Omega)$, which correspond to (integrable) vector fields $u : \Omega \to \mathbb{R}^N$ whose distributional symmetric gradient Eu is a bounded Radon measure. This space has been introduced in [92] (see also [94, 91]) as a natural space to formulate problems of perfect plasticity. Brittle fracture however requires a finer understanding of this space and especially the introduction of the subspace $SBD(\Omega)$ of special functions of bounded deformation in [5, 22], for which the singular part of Eu with respect to the Lebesgue measure is concentrated on the jump set. Unfortunately, this step forward was still not enough because of lack of control of the values of u (due to the failure of Poincaré-Korn and/or Korn type inequalities in that space). It is only recently that the introduction of the space $GSBD(\Omega)$ of generalized special functions of bounded deformation in [48] (see Section 1.4 for the precise definition) has given a satisfactory mathematical framework to investigate a well founded existence theory for the weak formulation, as well as for the original one. Some further compactness properties of that space have been investigated in [35, 34] which has led to prove the existence of minimizers of the Griffith functional under Dirichlet boundary conditions (formulated in a relaxed sense).

2.1.2 . Approximation of the Griffith energy

The Γ-convergence approximation of free discontinuity problems (e.g. by more tractable ones from a numerical point of view) is of fundamental importance in applications. It has been proven in [27] that it is not possible to approximate free discontinuity functionals by means of local integral functionals. To overcome this difficulty, a first possibility is to introduce an additional variable like, e.g., in phase field approximations where the sharp discontinuity is smoothened into a diffuse discontinuity. It represents one of the most popular methods which have already proven to be successful in other contexts such as the Modica-Mortola approximation of the perimeter functional [82], or the Ambrosio-Tortorelli approximation of the Mumford-Shah functional [8]. In the context of brittle fracture, such approximations, which have a founded mechanical interpretation as a gradient damage model, have only recently been established in full generality in [36] (see also [32, 75]). The main drawback is that, an additional numerical approximation would give rise to a multiscale problem with on the one hand the parameter of approximation, and on the other hand the mesh size (see e.g. [21, 19, 43]). Another possibility is to use nonlocal integral functionals as e.g. in [27, 86, 90].

For what concerns the numerical treatment of free discontinuity problems, the main difficulty is related to the fact that the jump set is part of the unknowns and that standard discontinuous finite element methods do not in general apply in this context. Having this problematic in mind as well as the multiscale issues arising in phase field or nonlocal approximations, one is thus tempted to find single scale discrete approximations of free discontinuity problems. There is a huge literature on this subject and, without being exhaustive, we refer to discrete-to-continuous approximations results [1, 29, 30, 71, 84, 85, 87], nonlocal finite elements approximations [77, 85] or discrete approximations based on stochastic meshes in [20, 89].

Let us focus on the discrete approximation result obtained in [39] for the Mumford-Shah functional in dimension $N = 2$. In that work, the classical Mumford-Shah functional

$$F(u) := \int_\Omega |\nabla u|^2 \, dx + \mu \mathcal{H}^1(J_u)$$

is approximated in the sense of Γ-convergence by a functional of the form

$$F_\varepsilon(u) := \int_\Omega f_\varepsilon(\nabla u)\,dx$$

putting a restriction on the functional space on which F_ε is defined. The functional F_ε is discrete in the sense that u is (a scalar-valued) continuous function and piecewise affine on suitable ε-dependent meshes (see Definition 2.1.1). It consists in an adaptive finite element approximation because there is an implicit mesh optimization whose numerical implementation has been carried out in [23]. The function $f_\varepsilon(\nabla u)$ takes the form $\frac{1}{\varepsilon}f(\varepsilon|\nabla u|^2)$ where f is a nondecreasing function satisfying the standard properties (2.1.4). Typical examples of functions f are, on the one hand the $\arctan$ function (as e.g. in [71] following a conjecture of De Giorgi) and, on the other hand $f(t) = t \wedge \kappa$. The main feature of this result is that, allowing the mesh to move gives enough flexibility to approximate isotropic surface energies. The constant μ appearing in the functional F is explicit and only depends on κ and the geometry of the triangulation.

An analogous analysis has been carried out in [84], where the author constraints the mesh to be made either of equilateral triangles, or of right isosceles ones. In that case, the result is that the functional F_ε Γ-converges to an anisotropic version of the Mumford-Shah functional

$$\int_\Omega |\nabla u|^2\,dx + \int_{J_u} \phi(\nu_u)\,d\mathcal{H}^1,$$

for some function $\phi : \mathbb{S}^1 \to \mathbb{R}$, which can be explicitly computed, depending on the normal ν_u to the jump set J_u. In [87], the same problem is addressed in the two-dimensional vectorial setting. If f_ε is as before, the following approximating energy is considered

$$\int_\Omega f_\varepsilon(e(u))\,dx.$$

As in [84], the ε-dependent mesh is fixed and made either of equilateral triangles, or of right isosceles triangles, and the result is that this functional Γ-converges to an anisotropic version of the Griffith functional

$$\int_\Omega |e(u)|^2\,dx + \int_{J_u} \phi(\nu_u)\,d\mathcal{H}^1,$$

where $\phi : \mathbb{S}^1 \to \mathbb{R}$ is as in [84]. Note that if $f(t) = t \wedge \kappa$, then

$$f_\varepsilon(e(u)) = \begin{cases} \varepsilon|e(u)|^2 & \text{if } \varepsilon|e(u)|^2 \le \kappa, \\ \kappa & \text{if } \varepsilon|e(u)|^2 > \kappa. \end{cases} \tag{2.1.2}$$

In order to recover the isotropic Griffith energy (2.1.1), a similar approximation result is considered in [85] where, now, the meshes are allowed to move as in [39], but the function f_ε now depends on the full gradient ∇u (instead of the symmetric gradient) and behaves like

$$f_\varepsilon(\nabla u) \sim \begin{cases} \varepsilon|e(u)|^2 & \text{if } \varepsilon|\nabla u|^2 \le \kappa, \\ \kappa & \text{if } \varepsilon|\nabla u|^2 > \kappa \end{cases} \tag{2.1.3}$$

(compare with (2.1.2)). In that case, the analysis of [39] can be adapted to show a Γ-convergence result towards the isotropic Griffith energy (2.1.1) with the same geometric multiplicative constant μ (as in [39]) in front of the surface energy.

2.1.3 . Our result

The objective of the present work is to generalize the previous results in the two-dimensional vectorial case to show an analogous statement as in [39], namely an adaptive discrete finite element approximation of the isotropic Griffith functional. To state precisely our main result, Theorem 2.1.3, we need to introduce some notation (we refer to Section 1.4 regarding functional spaces).

Let Ω be a bounded open set of $\mathbb{R}^2$ with Lipschitz boundary. As in [39], we introduce the following class of admissible meshes.

Definition 2.1.1. A triangulation of Ω is a finite family of closed triangles intersecting Ω, whose union contains Ω, and such that, given any two triangles of this family, their intersection, if not empty, is exactly a vertex or an edge common to both triangles. Given some angle θ_0 with $0 < \theta_0 \leq 45° - \arctan(1/2)$, and a function $\varepsilon \mapsto \omega(\varepsilon)$ with $\omega(\varepsilon) \geq 6\varepsilon$ for any $\varepsilon > 0$ and $\lim_{\varepsilon\to 0^+} \omega(\varepsilon) = 0$, we define, for any $\varepsilon > 0$

$$\mathcal{T}_\varepsilon(\Omega) := \mathcal{T}_\varepsilon(\Omega, \omega, \theta_0)$$

as the set of all triangulations of Ω made of triangles whose edges have length between ε and $\omega(\varepsilon)$, and whose angles are all greater than or equal to θ_0. Then we consider the finite element space $V_\varepsilon(\Omega)$ of all continuous functions $u : \Omega \to \mathbb{R}^2$ for which there exists $\mathbf{T} \in \mathcal{T}_\varepsilon(\Omega)$ such that u is affine on each triangle $T \in \mathbf{T}$.

Remark 2.1.2. Imposing $\theta_0 > 0$ and $\omega(\varepsilon) \geq \varepsilon$ corresponds to a non-flatness condition that ensures the existence of a radius $\varrho(\theta_0) > 0$ such that for all triangle $T \in \mathbf{T}$, one can find a point $x \in T$ such that

$$\overline{B_\varrho(x)} \subset T.$$

As for the conditions $\theta_0 \leq 45° - \arctan(1/2)$ and $\omega(\varepsilon) \geq 6\varepsilon$, we will later see that they are crucial to prove the existence of recovery sequences. Indeed, we use the same optimal triangulation introduced in [39, Appendix A], where the authors' explicit construction makes use of triangles with edges of length 6ε and angles equal to $45° - \arctan(1/2)$.

Let us consider a nondecreasing continuous function $f : [0, +\infty) \to [0, +\infty)$ satisfying

$$f(0) = 0, \quad \lim_{t\to 0^+} \frac{f(t)}{t} = 1 \quad \text{and} \quad \lim_{t\to\infty} f(t) = \kappa, \tag{2.1.4}$$

for some constant $\kappa > 0$, and a symmetric fourth order tensor $\mathbf{A} \in \mathscr{L}(\mathbb{M}^{2\times 2}_{\mathrm{sym}}, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that

$$\alpha|\xi|^2 \leq \mathbf{A}\xi : \xi \leq \beta|\xi|^2 \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}, \tag{2.1.5}$$

for some constants α, $\beta > 0$.

Our main result is the following Γ-convergence approximation of the Griffith functional.

Theorem 2.1.3. *The functional $\mathcal{F}_\varepsilon : L^0(\Omega; \mathbb{R}^2) \to [0, +\infty]$ defined by*

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f\big(\varepsilon \mathbf{A} e(u) : e(u)\big)\, dx & \text{if } u \in V_\varepsilon(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{2.1.6}$$

Γ*-converges, with respect to the* $L^0(\Omega;\mathbb{R}^2)$*-topology of convergence in measure, to the Griffith functional* $\mathcal{F} : L^0(\Omega;\mathbb{R}^2) \to [0,+\infty]$ *given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega \mathbf{A}e(u) : e(u)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_u) & \textit{if } u \in GSBD^2(\Omega), \\ +\infty & \textit{otherwise.} \end{cases}$$

Remark 2.1.4. As explained above, a meaningful choice is the function $f(t) = t \wedge \kappa$, for which the energy reduces to

$$\int_\Omega \frac{\kappa}{\varepsilon} \wedge \mathbf{A}e(u) : e(u)\, dx.$$

It corresponds to the brittle damage energy of a linearly elastic material composed of two phases : an undamaged one whose elasticity coefficients are represented by the Hooke tensor $\mathbf{A}$, and a damaged one whose elasticity coefficients are set to 0. The constant κ/ε stands for the toughness of the material whose diverging character as $\varepsilon \to 0$ forces the damaged zones to concentrate on vanishingly small sets (see [15]).

2.1.4 . Strategy of proof

As usual in Γ-convergence, the proof is achieved by combining a compactness result, a lower bound and an upper bound inequality. In order to describe our argument, let us assume for simplicity that $f(t) = t \wedge \kappa$ and $\mathbf{A} = \mathrm{id}$.

Our compactness result, Proposition 2.3.5 rests on the general $GSBD$ compactness result of [37]. Given a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ with uniformly bounded energy, one can apply [37, Theorem 1.1] to the modified function $v_\varepsilon := u_\varepsilon \mathbb{1}_{\{|e(u_\varepsilon)|^2 \leq \kappa/\varepsilon\}}$ which consists in putting the value zero inside each triangle T where the (symmetric) gradient of u_ε is "large". It might thus create a jump on the boundary of T whose perimeter can be estimated by $\mathcal{L}^2(T)/\varepsilon$. It leads to compactness in measure for the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ (up to subtracting a sequence of piecewise rigid motions, leaving the energy unchanged), which thus justifies why it is natural to consider Γ-convergence with respect to this topology.

The upper bound causes no particular difficulty. It consists in using known density results in $GSBD^2(\Omega)$ (see [36, 46]) to reduce to the case where the jump set of u is made of finitely many pairwise disjoint closed line segments, and u is smooth outside. Then, considering a similar optimal triangulation of Ω as in [39] (whose vertices do not cross the jump set) and a piecewise affine Lagrange interpolation of u, it leads to the desired upper bound (see Proposition 2.2.11).

The proof of the lower bound inequality is much more delicate to address and it represents, to our opinion, the main achievement of this work. First of all, the blow-up method allows one to identify separately the bulk part and the singular part. The bulk part can be easily recovered by modifying u_ε into a new function which vanishes in all triangles where $e(u_\varepsilon)$ is "too large" as in the compactness argument (see Proposition 2.2.4). The main difficulty is to get a lower bound for the singular part of the energy.

Before describing our strategy of proof, let us briefly explain why the methods of [39] (and similarly [23]) fail in our situation. The idea of [39] consists in modifying every minimizing sequence $\{u_\varepsilon\}_{\varepsilon>0}$ inside each triangle T of the associated triangulation $\mathbf{T}^\varepsilon \in \mathcal{T}_\varepsilon(\Omega)$ according to its variations along each edge of T. It rests on the introduction of a jump criterion which stipulates that if the variation of u_ε is large enough, it is convenient to create a jump along the edge. More precisely, if x_1, x_2 and x_3 stand for the vertices of the triangle T, it will be energetically favorable to create a jump at the middle point of the segment $[x_i, x_j]$ if

$$\frac{|u_\varepsilon(x_i) - u_\varepsilon(x_j)|}{|x_i - x_j|} > \frac{\sigma}{\sqrt{\varepsilon}},$$

for some constant $\sigma > 0$, while u_ε remains unchanged on $[x_i, x_j]$ otherwise. This criterion has to be defined in such a way that :

(i) the new function, say w_ε, has a jump set in each triangle T which satisfies $\mathcal{H}^1(J_{w_\varepsilon} \cap T) \leq \mathcal{L}^2(T)/(\varepsilon \sin\theta_0)$, where θ_0 is as in Definition 2.1.1, and w_ε does not jump across ∂T;

(ii) the absolutely continuous part of the gradient, ∇w_ε, is controlled in $L^2(T)$ by the energy restricted to T.

This construction ensures that the variation of the new discontinuous and piecewise affine function w_ε is always controlled along at least two edges of each triangle T, and it yields a control of the full gradient ∇w_ε of w_ε inside T. In [39], this is possible thanks of the scalar nature of the problem because the gradient $\nabla w_{\varepsilon|T}$ is a (constant) vector in $\mathbb{R}^2$ (see [39, Remark 3.5]).

In our case, u_ε is not scalar-valued anymore, but vector-valued and the energy only depends on its symmetric gradient $e(u_\varepsilon)$. If one uses the same criterion than in [39], then condition (i) above will be satisfied for the new function w_ε on T. However, one will only be able to estimate the L^2-norm of the (symmetric) gradient of w_ε by that of the full gradient of u_ε which, unfortunately, is not controlled by the energy $\mathcal{F}_\varepsilon(u_\varepsilon)$. Note that in [85], such a control is artificially made possible thanks to the particular form of the energy (see (2.1.3) above). This is however not natural in this linearized elasticity setting where the energy should be expressed in terms of the symmetric gradient of the displacement.

As a consequence, the jump criterion has to be modified. As the energy only depends on the symmetric part of the gradient of u_ε, it would be natural to consider a criterion involving the longitudinal variation of u_ε along the edges of the triangle instead of the full variation. In other words, one could modify the criterion by asking that if

$$\frac{|(u_\varepsilon(x_i) - u_\varepsilon(x_j)) \cdot (x_i - x_j)|}{|x_i - x_j|^2} > \frac{\sigma}{\sqrt{\varepsilon}},$$

then we create a jump at the middle point of $[x_i, x_j]$, while u_ε remains unchanged on $[x_i, x_j]$ otherwise. In that case, it is again not possible to control the symmetric gradient $e(w_\varepsilon)$ of the new function w_ε by that of u_ε. Indeed, in a similar way as in [39], the previous criterion ensures that the longitudinal variation of w_ε along at least two edges of each triangle is controlled by the energy restricted to T. If we call ξ_1 and $\xi_2 \in \mathbb{S}^1$ both (linearly independent) directions associated to these “good” edges, it shows that $e(w_\varepsilon)_{|T} : (\xi_1 \otimes \xi_1)$ and $e(w_\varepsilon)_{|T} : (\xi_2 \otimes \xi_2)$ are controlled by $e(u_\varepsilon)_{|T}$ which is not enough to control the full 2×2 symmetric matrix $e(w_\varepsilon)_{|T}$ which has three degrees of freedom. In addition, some (uncontrolled) discontinuities can also be created at the interface $I := \partial T \cap \partial T'$ between two adjacent triangles T and T' so that condition (i) fails as well.

Overcoming these difficulties seems to be a very serious issue so that we decided to attack this problem from a different angle. First of all, the use of the blow-up method allows one to reduce to the case where $\Omega = B$ is the unit ball, u is a step function of the form

$$u(x) = \begin{cases} a & \text{if } x \cdot \nu < 0, \\ b & \text{if } x \cdot \nu > 0, \end{cases}$$

for some a, $b \in \mathbb{R}^2$ with $a \neq b$ and $\nu \in \mathbb{S}^1$ (with a jump set corresponding to the diameter of B orthogonal to ν), and, see Lemma 2.2.6, such that

$$\int_{\{|e(u_\varepsilon)|^2 \leq \kappa/\varepsilon\}} |e(u_\varepsilon)|^2 \, dx \to 0. \tag{2.1.7}$$

To make our strategy of proof more transparent, we assume that $a \cdot \nu \neq b \cdot \nu$. A standard argument based on Fubini's Theorem shows that the one-dimensional section of u_ε in the direction ν passing through the point y, namely $t \mapsto (u_\varepsilon)^\nu_y(t) := u_\varepsilon(y + t\nu) \cdot \nu$ converges (in measure) to the step function

$$t \mapsto u^\nu_y(t) = a \cdot \nu \mathbb{1}_{\mathbb{R}^-} + b \cdot \nu \mathbb{1}_{\mathbb{R}^+}.$$

Let us denote by $\mathbf{T}^\varepsilon$ the triangulation on which u_ε is (continuous and) piecewise affine. We further denote by $\mathbf{T}^\varepsilon_b$ the familly of all triangles $T \in \mathbf{T}^\varepsilon$ such that $|e(u_\varepsilon)_{|T}|^2 > \kappa/\varepsilon$. Thanks to (2.1.7), we show that almost every line orthogonal to $J_u \cap B$ must cross at least one triangle $T \in \mathbf{T}^\varepsilon_b$ (see Lemma 2.2.7). The reason is that if, for some $y \in J_u \cap B$, the line $y + \mathbb{R}\nu$ intersects no such triangles, then $(u_\varepsilon)^\nu_y$ would be bounded in H^1 (because $|((u_\varepsilon)^\nu_y)'| \leq |e(u_\varepsilon)(y + t\nu)|$) and thus, it would converge weakly in that space to a constant function, contradicting that $a \cdot \nu \neq b \cdot \nu$. This information allows one to get a bad lower bound for the surface energy with $1/2$ multiplicative factor. It suggests to improve the previous argument by showing that "many" lines $y + \mathbb{R}\nu$ passing through $y \in J_u \cap B$ must actually cross at least two triangles in $\mathbf{T}^\varepsilon_b$, which is the object of Lemma 2.2.8. To do that, we show in Lemma 2.2.9 that there are very few points y in $J_u \cap B$ such that the line $y + \mathbb{R}\nu$ crosses exactly one triangle $T \in \mathbf{T}^\varepsilon_b$. Indeed, in that case, up to a small error, the function $(u_\varepsilon)^\nu_y$ would have to pass from the value $a \cdot \nu$ to $b \cdot \nu$ inside T. Due to the particular shape of a triangle and of the fact that u_ε is affine inside T, this could only happen for at most two values of y. Moreover, if y is far away from these two values, the variation of $(u_\varepsilon)^\nu_y$ across the triangle T is not sufficient, and it becomes necessary to cross an additional triangle T' in $\mathbf{T}^\varepsilon_b$. With this improvement, we can now construct two disjoint families of triangles with the property that both families project onto $J_u \cap B$ into two sets of almost full $\mathcal{H}^1$ measure (see Lemma 2.2.10). It enables one to compensate the bad multiplicative factor $1/2$ in the previous argument, and obtain the expected lower bound with the correct constant corresponding to $\kappa \sin\theta_0$ (see Proposition 2.2.5). In [8], the right factor in the lower estimate of the jump part comes from the two transitions of the phase field approximating J_u, one from each side of the jump set. Similarily here, the same role is played by the characteristic function $\chi_\varepsilon = \mathbb{1}_{\{|e(u_\varepsilon)|^2 > \kappa/\varepsilon\}}$. Indeed, having in mind the optimal triangulation computed in the upper bound (see [39, Appendix A]) and knowing that almost every line orthogonal to $J_u \cap B$ crosses at least two distinct triangles of $\mathbf{T}^\varepsilon_b$, we expect these triangles to form a neighbourhood of J_u as in Figure 2.1. Therefore, $\mathcal{L}^1\left(\left\{(\chi_\varepsilon)^\nu_y = 1\right\}\right) = \varepsilon \sin\theta_0$ which leads to the right lower bound, independently of the amplitude of the jump $(b - a) \cdot \nu$.

To conclude this introduction, let us mention that the originality of this work is twofold. First of all, we are able to provide a deterministic discrete finite element approximation result of the Griffith

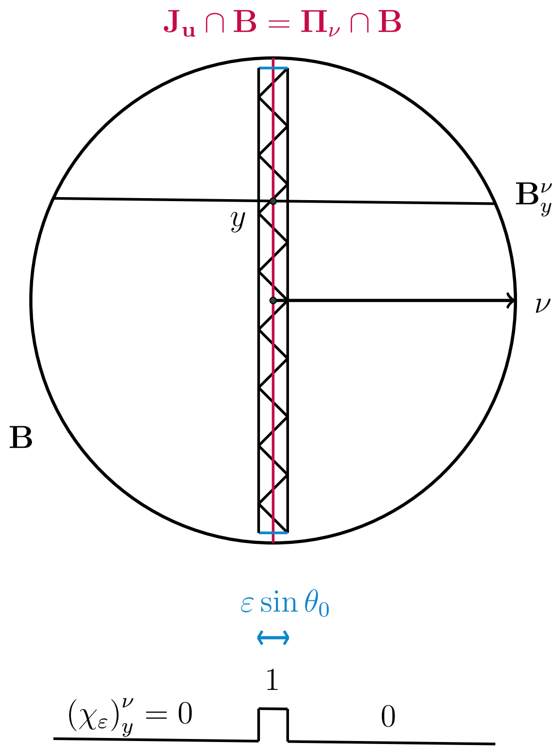

Figure 2.1

functional with isotropic surface energies. In particular, our approach does not require any unnatural dependence of the approximating energy with respect to the skew symmetric part of the gradient (in the context of linear elasticity) nor the use of stochastic meshes. Second, our method relies on an unusual application of the slicing method, which is rather employed in Γ-convergence analysis to reduce the dimension of the problem to a one-dimension study. Here, we instead use this method as a tool to enumerate in a non trivial way the number of triangles needed to derive the correct multiplicity in the surface energy.

2.1.5 . Organisation of the paper

In Section 1.4, we collect useful notation and preliminary results that will be useful in the subsequent sections. Section 3 is devoted to show our main result, Theorem 2.1.3. It is divided into three parts : a first one consisting in a compactness result, Proposition 2.2.1, a second one corresponding to the lower bound inequality, Proposition 2.2.3, and a last one for the upper bound inequality, Proposition 2.2.11 through the construction of a recovery sequence. Eventually, in Section 2.3, we extend the previous Γ-convergence analysis allowing for Dirichlet boundary conditions formulated in a suitable way at the discrete and continuum levels (see Theorem 2.3.1). We then deduce the fundamental property of Γ-convergence, Corollary 2.3.2, in our specific setting, i.e., the convergence of minimizers as well as the minimum value.

2.2 . Proof of the main result

Let us introduce the Γ-lower and upper limits (with respect to the topology of convergence in measure) $\mathcal{F}'$ and $\mathcal{F}'' : L^0(\Omega;\mathbb{R}^2) \to [0,+\infty]$ defined by

$$\mathcal{F}'(u) := \inf\left\{\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u \text{ in measure in } \Omega\right\},$$

and

$$\mathcal{F}''(u) := \inf\left\{\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u \text{ in measure in } \Omega\right\},$$

for all $u \in L^0(\Omega;\mathbb{R}^2)$.

2.2.1 . Domain of the Γ-limit

We begin our analysis by identifying the domain of finiteness of the Γ-limit.

Proposition 2.2.1. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \to 0$, $u \in L^0(\Omega;\mathbb{R}^2)$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^0(\Omega;\mathbb{R}^2)$ be such that $M := \sup_k \mathcal{F}_{\varepsilon_k}(u_k) < \infty$ and $u_k \to u$ in measure in Ω. Then, $u \in GSBD^2(\Omega)$.*

Proof. According to the properties (2.1.4) satisfied by f, for all $\delta > 0$, there exists a constant $0 < K < \kappa$ such that

$$f(t) \geq K \wedge [(1-\delta)t] \quad \text{for all } t \geq 0. \tag{2.2.1}$$

Indeed, since $f(t)/t \to 1$ as $t \to 0^+$, there exists $t^* > 0$ such that $f(t)/t \geq 1-\delta$ for all $t \in [0,t^*]$ and $K := (1-\delta)t^* < \kappa$. Hence, for all $t \in [0,t^*]$, we have $f(t) \geq (1-\delta)t$, while for all $t > t^*$, as f is nondecreasing, $f(t) \geq f(t^*) \geq K$.

By definition of $\mathcal{F}_{\varepsilon_k}$, there exists a triangulation $\mathbf{T}^k \in \mathcal{T}_{\varepsilon_k}(\Omega)$ such that $u_k \in V_{\varepsilon_k}(\Omega)$ is affine on each triangle $T \in \mathbf{T}^k$. We introduce the characteristic functions

$$\chi_k := \mathbb{1}_{\left\{(1-\delta)\mathbf{A}e(u_k):e(u_k) \geq \frac{K}{\varepsilon_k}\right\}} \in L^\infty(\Omega;\{0,1\})$$

which are constant on each triangle $T \in \mathbf{T}^k$, so that

$$D_k := \{\chi_k = 1\} \cap \Omega = \bigcup_{i=1}^{N_k} (T_i^k \cap \Omega)$$

for some triangles $T_i^k \in \mathbf{T}^k$. Remark that this choice of χ_k implies that

$$M \geq \mathcal{F}_{\varepsilon_k}(u_k) \geq (1-\delta)\int_\Omega (1-\chi_k)\mathbf{A}e(u_k) : e(u_k)\,dx + \frac{K}{\varepsilon_k}\int_\Omega \chi_k\,dx,$$

forcing χ_k to converge to 0 in $L^1(\Omega)$ since $0 \leq \int_\Omega \chi_k\,dx \leq K^{-1}M\varepsilon_k \to 0$.

Let $v_k := (1-\chi_k)u_k$ so that, by [6, Theorem 3.84], $v_k \in SBV^2(\Omega;\mathbb{R}^2)$ with $\nabla v_k = (1-\chi_k)\nabla u_k$ and

$$J_{v_k} \subset \Omega \cap \partial D_k \subset \bigcup_{i=1}^{N_k} \partial T_i^k.$$

Note that

$$v_k \to u \text{ in measure in } \Omega \text{ and } A := \{x \in \Omega : |u_k(x)| \to \infty\} \text{ is } \mathcal{L}^2\text{-negligible.} \tag{2.2.2}$$

Indeed, since $u_k \to u$ in measure in Ω and $\{u_k \neq v_k\} \subset D_k$ with $\mathcal{L}^2(D_k) \to 0$, for all $\eta > 0$, we get that $\mathcal{L}^2(\{|v_k - u| > \eta\}) \leq \mathcal{L}^2(\{|u_k - u| > \eta\}) + \mathcal{L}^2(D_k) \to 0$. Additionally, up to a subsequence (not relabeled), $u_k(x) \to u(x) \in \mathbb{R}^2$ for $\mathcal{L}^2$-a.e. $x \in \Omega$.

On the one hand, using the energy bound $\mathcal{F}_{\varepsilon_k}(u_k) \leq M$ and the ellipticity property (2.1.5) of $\mathbf{A}$, we infer that

$$\int_\Omega |e(v_k)|^2 \, dx \leq \frac{M}{(1-\delta)\alpha}. \tag{2.2.3}$$

On the other hand, by definition of an admissible triangulation, the edges of each triangle T_i^k have length greater than or equal to ε_k and their angles are all greater than or equal to θ_0, so that the heights of such triangles must be greater than or equal to $\varepsilon_k \sin\theta_0$. Therefore, for all $1 \leq i \leq N_k$,

$$\mathcal{L}^2(T_i^k) \geq \frac{1}{2}(\varepsilon_k \sin\theta_0)\frac{\mathcal{H}^1(\partial T_i^k)}{3}$$

which implies that for all open subset $U \subset\subset \Omega$:

$$\mathcal{H}^1(J_{v_k} \cap U) \leq \frac{6}{\sin\theta_0} \sum_{i \in \{1,\dots,N_k\},\, T_i^k \cap U \neq \emptyset} \frac{\mathcal{L}^2(T_i^k)}{\varepsilon_k}.$$

Let $k_U \geq 1$ (depending on U) be such that for all $k \geq k_U$, any triangle $T \in \mathbf{T}^k$ intersecting U is contained in Ω, then it follows that for all $k \geq k_U$,

$$\mathcal{H}^1(J_{v_k} \cap U) \leq \frac{6}{\varepsilon_k \sin\theta_0} \int_\Omega \chi_k \, dx \leq \frac{6M}{K \sin\theta_0}, \tag{2.2.4}$$

where we used once more the energy bound $\mathcal{F}_{\varepsilon_k}(u_k) \leq M$.

Gathering (2.2.3) and (2.2.4), we can apply the $GSBD^2$-compactness Theorem ([35, Theorem 1.1]). Together with (2.2.2), it ensures the existence of a subsequence (depending on the open subset U, which we do not relabel) such that $u_{|U} \in GSBD^2(U)$,

$$e(v_k)_{|U} \rightharpoonup e(u_{|U}) \text{ weakly in } L^2(U; \mathbb{M}^{2\times 2}_{\text{sym}}) \text{ and } \mathcal{H}^1(J_u \cap U) \leq \liminf_{k\to\infty} \mathcal{H}^1(J_{v_k} \cap U).$$

We then consider an exhaustion of Ω by a sequence of open subsets $\{U_m\}_{m\in\mathbb{N}}$ satisfying $U_m \subset\subset U_{m+1} \subset\subset \Omega$ for all $m \in \mathbb{N}$ and $\bigcup_m U_m = \Omega$. Using a diagonal extraction argument, we can find a subsequence (still denoted by $\{v_k\}_{k\in\mathbb{N}}$) such that for all $m \in \mathbb{N}$, $u_{|U_m} \in GSBD^2(U_m)$ and

$$e(v_k)_{|U_m} \rightharpoonup e(u_{|U_m}) \text{ weakly in } L^2(U_m; \mathbb{M}^{2\times 2}_{\text{sym}}) \text{ and } \mathcal{H}^1(J_u \cap U_m) \leq \liminf_{k\to\infty} \mathcal{H}^1(J_{v_k} \cap U_m). \tag{2.2.5}$$

Let us now check that u belongs to $GSBD^2(\Omega)$. Indeed, let $\xi \in \mathbb{S}^1$ and $\tau \in C^1(\mathbb{R})$ be such that $|\tau| \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$. For all test function $\phi \in C_c^\infty(\Omega)$, there exists $m \in \mathbb{N}$ such that $\operatorname{supp}\phi \subset U_m$ so that, owing to the dominated convergence Theorem,

$$\langle D_\xi(\tau(u\cdot\xi)), \phi\rangle = -\int_{U_m} \tau(u\cdot\xi) D_\xi\phi \, dx = -\lim_{k\to\infty}\int_{U_m} \tau(v_k\cdot\xi) D_\xi\phi \, dx = \lim_{k\to\infty}\langle D_\xi(\tau(v_k\cdot\xi)), \phi\rangle.$$

Since $v_k \cdot \xi \in SBV^2(\Omega)$, using the chain rule formula in BV ([6, Theorem 3.96]), we get that $\tau(v_k \cdot \xi) \in SBV^2(\Omega)$ with

$$D_\xi\left(\tau(v_k \cdot \xi)\right) = \tau'(v_k \cdot \xi)e(v_k) : (\xi \otimes \xi)\,\mathcal{L}^2 \llcorner \Omega + \left(\tau(v_k^+ \cdot \xi) - \tau(v_k^- \cdot \xi)\right)\nu_{v_k} \cdot \xi\,\mathcal{H}^1 \llcorner J_{v_k}.$$

Taking the variation, we infer that

$$|D_\xi\left(\tau(v_k \cdot \xi)\right)| \leq |e(v_k)|\mathcal{L}^2 \llcorner \Omega + \mathcal{H}^1 \llcorner J_{v_k} =: \lambda_k.$$

As a consequence of (2.2.3) together with (2.2.4), the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(U_m)$ for all $m \in \mathbb{N}$, with

$$\sup_{k \geq k_{U_m}} \lambda_k(U_m) \leq \frac{M}{(1-\delta)\alpha} + \frac{6M}{K \sin\theta_0} =: M_\delta < +\infty,$$

so that, up to a further diagonal extraction, $\lambda_k \llcorner U_m \rightharpoonup \lambda^{(m)}$ weakly* in $\mathcal{M}(\Omega)$ for some nonnegative measure $\lambda^{(m)} \in \mathcal{M}(\Omega)$ satisfying, for all $m \in \mathbb{N}$,

$$\lambda^{(m)}(\Omega) \leq \liminf_{k\to\infty} \lambda_k(U_m) \leq M_\delta.$$

Therefore, we can introduce the following nonnegative measure $\lambda \in \mathcal{M}(\Omega)$ defined by

$$\lambda(B) := \sup_{m\in\mathbb{N}} \lambda^{(m)}(B) = \lim_{m\to\infty} \lambda^{(m)}(B) \text{ for all Borel subset } B \subset \Omega.$$

We thus obtain that

$$|\langle D_\xi\left(\tau(u\cdot\xi)\right), \phi\rangle| \leq \lim_{k\to\infty} \langle \lambda_k \llcorner U_m, |\phi|\rangle = \langle \lambda^{(m)}, |\phi|\rangle \leq \langle \lambda, |\phi|\rangle,$$

implying both that $D_\xi\left(\tau(u\cdot\xi)\right) \in \mathcal{M}(\Omega)$ according to Riesz Representation Theorem and that

$$|D_\xi(\tau(u\cdot\xi))| \leq \lambda \quad \text{in } \mathcal{M}(\Omega),$$

which shows that $u \in GBD(\Omega)$. Using next that $u \in GSBD(U_m)$ for all $m \in \mathbb{N}$ and [48, Definition 4.2], we deduce that $u \in GSBD(\Omega)$. Eventually, by locality of the definition of the approximate symmetric gradient $e(u)$ (see [48, Formula (9.1)]), as a consequence of (2.2.3) together with (2.2.5), we infer that $e(v_k) \rightharpoonup e(u)$ weakly in $L^2(\Omega; \mathbb{M}^{2\times 2}_{\rm sym})$ with $e(u) \in L^2(\Omega; \mathbb{M}^{2\times 2}_{\rm sym})$. Passing to the limit as $m \to \infty$ in the last property of (2.2.5) and using (2.2.4) shows that $\mathcal{H}^1(J_u) < \infty$. All of this establishes that $u \in GSBD^2(\Omega)$ and completes the proof of the Proposition. □

Remark 2.2.2. We will later improve the previous result (see Proposition 2.3.5) by getting rid-off the a priori knowledge that u_k converges in measure in Ω. The price to pay will be to subtract a sequence of piecewise rigid body motions. Proposition 2.3.5 will a posteriori justify why the topology of convergence in measure is the natural one to address the Γ-convergence analysis.

2.2.2 . The lower bound

The proof of the lower bound inequality relies on the blow up method which consists in identifying separately the Lebesgue and jump parts of the energy.

Proposition 2.2.3. *For all* $u \in L^0(\Omega;\mathbb{R}^2)$*,*

$$\mathcal{F}(u) \le \mathcal{F}'(u).$$

Proof. Without loss of generality, we can assume that $\mathcal{F}'(u) < \infty$. For any $\zeta > 0$, there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon \to u$ in measure in Ω and

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \le \mathcal{F}'(u) + \zeta < \infty.$$

Let us extract a subsequence $\{u_k\}_{k\in\mathbb{N}} := \{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ from $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_k \to u$ $\mathcal{L}^2$-a.e. in Ω and

$$\lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k) = \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty.$$

This implies that, for k large enough, $u_k \in V_{\varepsilon_k}(\Omega)$ and $\sup_k \mathcal{F}_{\varepsilon_k}(u_k) < \infty$. By definition of the finite element space $V_{\varepsilon_k}(\Omega)$, there exists a triangulation $\mathbf{T}^k \in \mathcal{T}_{\varepsilon_k}(\Omega)$ such that u_k is affine on each $T \in \mathbf{T}^k$.

We first note that, according to Proposition 2.2.1, $u \in GSBD^2(\Omega)$. Let us show the lower bound inequality $\mathcal{F}'(u) \ge \mathcal{F}(u)$. To this aim, we introduce the following sequence of Radon measures on Ω

$$\lambda_k := \frac{1}{\varepsilon_k} f\big(\varepsilon_k \mathbf{A}e(u_k) : e(u_k)\big)\mathcal{L}^2 \llcorner \Omega.$$

Since the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{M}(\Omega)$, up to a subsequence (not relabeled), we have $\lambda_k \overset{*}{\rightharpoonup} \lambda$ weakly* in $\mathcal{M}(\Omega)$ for some nonnegative measure $\lambda \in \mathcal{M}(\Omega)$. Thanks to the lower semicontinuity of weak* convergence in $\mathcal{M}(\Omega)$ along open sets, we have that

$$\mathcal{F}'(u) + \zeta \ge \lim_{k\to\infty} \lambda_k(\Omega) \ge \lambda(\Omega). \tag{2.2.6}$$

Using that the measures $\mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner J_u$ are mutually singular, it is enough to show that

$$\frac{d\lambda}{d\mathcal{L}^2} \ge \mathbf{A}e(u) : e(u) \quad \mathcal{L}^2\text{-a.e. in } \Omega, \tag{2.2.7}$$

and

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u} \ge \kappa \sin\theta_0 \quad \mathcal{H}^1\text{-a.e. on } J_u. \tag{2.2.8}$$

Indeed, once (2.2.7) and (2.2.8) are satisfied, it follows from the Radon-Nikodým decomposition and the Besicovitch differentiation Theorems that

$$\lambda = \frac{d\lambda}{d\mathcal{L}^2}\mathcal{L}^2 \llcorner \Omega + \frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}\mathcal{H}^1 \llcorner J_u + \lambda^s,$$

for some nonnegative measure λ^s which is singular with respect to both $\mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner J_u$. Thus, after integration over Ω and recalling (2.2.6), we get that

$$\mathcal{F}'(u) + \zeta \ge \int_\Omega \mathbf{A}e(u) : e(u)\,dx + \kappa \sin\theta_0 \mathcal{H}^1(J_u) = \mathcal{F}(u).$$

Taking the limit as $\zeta \to 0$, we obtain the desired lower bound inequality. □

The rest of this section is devoted to the establishment of (2.2.7) and (2.2.8). We start by identifying the lower bound for the bulk energy.

Proposition 2.2.4 (**Lower bound for the Lebesgue part).** *For $\mathcal{L}^2$-a.e. $x_0 \in \Omega$,*

$$\frac{d\lambda}{d\mathcal{L}^2}(x_0) \geq \mathbf{A}e(u)(x_0) : e(u)(x_0).$$

Proof. Let $x_0 \in \Omega$ be such that

$$\frac{d\lambda}{d\mathcal{L}^2}(x_0) = \lim_{\varrho \searrow 0} \frac{\lambda\big(B_\varrho(x_0)\big)}{\pi \varrho^2}$$

exists and is finite, and

$$\lim_{\varrho \searrow 0} \frac{1}{\varrho^2} \int_{B_\varrho(x_0)} |e(u)(y) - e(u)(x_0)|^2 \, dy = 0.$$

According to Besicovitch and Lebesgue differentiation Theorems, $\mathcal{L}^2$-almost every point x_0 in Ω satisfies these properties. We next consider a sequence of radii $\{\varrho_j\}_{j \in \mathbb{N}}$ such that $\varrho_j \searrow 0$ and $\lambda(\partial B_{\varrho_j}(x_0)) = 0$ for all $j \in \mathbb{N}$.

As in the proof of Proposition 2.2.1, according to the properties (2.1.4) satisfied by f, for all $\delta > 0$, there exists a constant $0 < K < \kappa$ such that $f(t) \geq K \wedge [(1-\delta)t]$ for all $t \geq 0$. Moreover, using the characteristic functions

$$\chi_k := \mathbb{1}_{\left\{(1-\delta)\mathbf{A}e(u_k):e(u_k) \geq \frac{K}{\varepsilon_k}\right\}} \in L^\infty(\Omega; \{0,1\})$$

we have for every Borel set $B \subset \Omega$,

$$\lambda_k(B) \geq (1-\delta) \int_B (1-\chi_k)\mathbf{A}e(u_k) : e(u_k)\, dx + \frac{K}{\varepsilon_k} \int_B \chi_k \, dx.$$

Note that because u_k is affine on each triangle $T \in \mathbf{T}^k$, χ_k is constant on each triangle $T \in \mathbf{T}^k$. Following the proof of Proposition 2.2.1, the sequence $v_k := (1-\chi_k)u_k \in SBV^2(\Omega; \mathbb{R}^2)$ satisfies $v_k \to u$ in measure in Ω and $e(v_k) \rightharpoonup e(u)$ weakly in $L^2(\Omega; \mathbb{M}^{2\times 2}_{\rm sym})$. Then, for all $j \in \mathbb{N}$,

$$\begin{aligned} \lambda(B_{\varrho_j}(x_0)) = \lim_{k\to\infty} \lambda_k(B_{\varrho_j}(x_0)) &\geq (1-\delta) \liminf_{k\to\infty} \int_{B_{\varrho_j}(x_0)} \mathbf{A}e(v_k) : e(v_k)\, dx \\ &\geq (1-\delta) \int_{B_{\varrho_j}(x_0)} \mathbf{A}e(u) : e(u)\, dx. \end{aligned}$$

Dividing the previous inequality by $\pi\varrho_j^2$ and passing to the limit as $j \to \infty$ implies by the choice of the point x_0 that

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^2}(x_0) = \lim_{j\to\infty} \frac{\lambda(B_{\varrho_j}(x_0))}{\pi\varrho_j^2} &\geq (1-\delta) \lim_{j\to\infty} \frac{1}{\pi\varrho_j^2} \int_{B_{\varrho_j}(x_0)} \mathbf{A}e(u) : e(u)\, dx \\ &= (1-\delta)\mathbf{A}e(u)(x_0) : e(u)(x_0). \end{aligned}$$

Taking the limit as $\delta \to 0^+$ completes the proof of the lower bound for the Lebesgue part. □

We next pass to the lower bound inequality for the jump part of the energy which represents the most difficult and original part of our result.

Proposition 2.2.5 (**Lower bound for the jump part).** *For $\mathcal{H}^1$-a.e. $x_0 \in J_u$,*

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \kappa \sin \theta_0.$$

The proof of Proposition 2.2.5 is quite long and involved. It necessitates the introduction of some tools in order to carry out the blow-up analysis coupled with the slicing method.

Let $x_0 \in J_u$ be such that

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \lim_{\varrho \searrow 0} \frac{\lambda\big(B_\varrho(x_0)\big)}{\mathcal{H}^1\big(J_u \cap B_\varrho(x_0)\big)}$$

exists and is finite, and

$$\lim_{\varrho \searrow 0} \frac{\mathcal{H}^1(J_u \cap B_\varrho(x_0))}{2\varrho} = 1.$$

According to the Besicovitch differentiation Theorem and the countably $(\mathcal{H}^1, 1)$-rectifiability of J_u (see [6, Theorem 2.83]), it follows that $\mathcal{H}^1$-almost every point x_0 in J_u fulfills these conditions. The point $x_0 \in J_u$ being fixed throughout the rest of the proof of Proposition 2.2.5, we sometimes intentionally omit to write the dependence with respect to x_0.

By definition of the jump set J_u, there exist $\nu := \nu_u(x_0) \in \mathbb{S}^1$ and $u^\pm(x_0) \in \mathbb{R}^2$ with $u^+(x_0) \neq u^-(x_0)$ such that the function

$$u_{x_0,\varrho} := u(x_0 + \varrho\,\cdot)$$

converges in measure in $B := B_1(0)$ to the jump function

$$\overline{u} : y \in B \mapsto \begin{cases} u^+(x_0) & \text{if } y \cdot \nu > 0, \\ u^-(x_0) & \text{if } y \cdot \nu < 0, \end{cases}$$

as $\varrho \searrow 0$ (see [48, Definition 2.3]). Note that, the jump set $J_{\overline{u}}$ in B coincides with the diameter $B^\nu = p_\nu(B)$ orthogonal to ν. Moreover, since $\mathcal{H}^1\left(\{\xi \in \mathbb{S}^1 : [u](x_0) \cdot \xi = 0\}\right) = 0$, for any $\eta > 0$, there exists $\xi \in \mathbb{S}^1$ such that

$$|\nu - \xi| \leq \eta, \quad \nu \cdot \xi \geq \frac{1}{2}, \quad \left|\nu \cdot \xi^\perp\right| \leq \eta \quad \text{and} \quad [u](x_0) \cdot \xi \neq 0, \tag{2.2.9}$$

where $[u](x_0) := u^+(x_0) - u^-(x_0)$. If $[u](x_0) \cdot \nu \neq 0$, we can simply take $\xi = \nu$. We then set

$$M_\eta := |u^+(x_0) \cdot \xi| + |u^-(x_0) \cdot \xi| > 0. \tag{2.2.10}$$

From now on, when working with the convergence in measure, we will use the distance d_{M_η} defined in (1.4.1) associated to this precise value of M_η. As before, we consider a sequence of radii $\{\varrho_j\}_{j \in \mathbb{N}}$ such that $\varrho_j \searrow 0$ and $\lambda(\partial B_{\varrho_j}(x_0)) = 0 = \mathcal{H}^1(J_u \cap \partial B_{\varrho_j}(x_0))$ for all $j \in \mathbb{N}$.

By our choice of x_0, we have

$$\begin{cases} \lim_{j\to\infty}\lim_{k\to\infty} u_k(x_0+\varrho_j\,\cdot) = \lim_{j\to\infty} u_{x_0,\varrho_j} = \overline{u} \quad \text{in measure in } B,\\ \lim_{j\to\infty}\lim_{k\to\infty} \dfrac{\lambda_k(B_{\varrho_j}(x_0))}{2\varrho_j} = \lim_{j\to\infty} \dfrac{\lambda(B_{\varrho_j}(x_0))}{2\varrho_j} = \dfrac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0),\\ \lim_{j\to\infty}\lim_{k\to\infty} \dfrac{\varepsilon_k}{\varrho_j} = \lim_{j\to\infty}\lim_{k\to\infty} \dfrac{\omega(\varepsilon_k)}{\varrho_j} = 0. \end{cases}$$

The metrizability of the convergence in measure in B shows the existence of an increasing sequence $\{k_j\}_{j\in\mathbb{N}}$ (depending on η) such that $k_j \nearrow \infty$ as $j \to \infty$ and

$$\begin{cases} v_j := u_{k_j}(x_0+\varrho_j\,\cdot) \to \overline{u} \quad \text{in measure in } B, & \text{(2.2.11a)}\\ \dfrac{\lambda_{k_j}(B_{\varrho_j}(x_0))}{2\varrho_j} \to \dfrac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0), & \text{(2.2.11b)}\\ \dfrac{\varepsilon_{k_j}}{\varrho_j} \to 0, \quad \dfrac{\omega(\varepsilon_{k_j})}{\varrho_j} \to 0. & \text{(2.2.11c)} \end{cases}$$

In particular, using a change of variables, we get that

$$\begin{aligned} 2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) &= \lim_{j\to\infty} \frac{1}{\varrho_j\varepsilon_{k_j}} \int_{B_{\varrho_j}(x_0)} f\left(\varepsilon_{k_j}\mathbf{A}e(u_{k_j}) : e(u_{k_j})\right)\,dx\\ &= \lim_{j\to\infty} \frac{\varrho_j}{\varepsilon_{k_j}} \int_B f\left(\frac{\varepsilon_{k_j}}{\varrho_j^2}\mathbf{A}e(v_j) : e(v_j)\right)\,dy\\ &\geq \limsup_{j\to\infty} \frac{\varrho_j}{\varepsilon_{k_j}} \int_B f\left(\frac{\varepsilon_{k_j}}{\varrho_j^2}\alpha|e(v_j)\xi\cdot\xi|^2\right)\,dy, \end{aligned}$$

where, in the last inequality, we used the ellipticity property (2.1.5) of $\mathbf{A}$, the nondecreasing character of f and that $\xi \in \mathbb{S}^1$.

According to the properties (2.1.4) satisfied by f, for all $\delta \in (0,1)$, there exists a constant $A > 0$ such that

$$f(t) \geq (At) \wedge [(1-\delta)\kappa] \quad \text{for all } t \geq 0.$$

Indeed, since $f(t) \to \kappa$ as $t \to \infty$, there exists $t^* \geq 0$ such that for all $t \geq t^*$, $f(t) \geq (1-\delta)\kappa$. The function $f(t)/t$ being continuous over $[0,t^*]$ (extended by the value 1 at $t=0$), it reaches its minimum value $A > 0$ over this segment so that $f(t) \geq At$ for all $t \in [0,t^*]$.

Let us introduce the characteristic functions

$$\chi_j := \mathbb{1}_{\left\{\frac{A\alpha\varepsilon_{k_j}}{\varrho_j^2}|e(v_j)\xi\cdot\xi|^2 \geq (1-\delta)\kappa\right\}} \in L^\infty(B;\{0,1\}),$$

so that

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \limsup_{j\to\infty}\left\{\frac{\alpha A}{\varrho_j}\int_B (1-\chi_j)|e(v_j)\xi\cdot\xi|^2\,dy + \frac{(1-\delta)\kappa\varrho_j}{\varepsilon_{k_j}}\int_B \chi_j\,dy\right\}. \qquad \text{(2.2.12)}$$

We then introduce the translated and rescaled triangulations

$$\mathbf{T}^{x_0,j} := \frac{1}{\varrho_j}\left(\mathbf{T}^{k_j} - x_0\right), \quad \mathbf{T}_b^{x_0,j} := \left\{T \in \mathbf{T}^{x_0,j} : \frac{\alpha A}{\varrho_j}|e(v_j)_{|T}\xi \cdot \xi|^2 \geq \frac{(1-\delta)\kappa\varrho_j}{\varepsilon_{k_j}}\right\}. \tag{2.2.13}$$

Note that v_j is affine on each $T \in \mathbf{T}^{x_0,j}$. Let us point out that

$$\chi_{j|T} := \begin{cases} 1 & \text{if } T \in \mathbf{T}_b^{x_0,j}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.14}$$

Since $((v_j)_z^\xi)'(t) = e(v_j)(z + t\xi)\xi \cdot \xi$, then for $\mathcal{H}^1$-a.e. $z \in B^\xi$,

$$(\chi_j)_z^\xi(t) = \begin{cases} 1 & \text{if } \frac{\alpha A}{\varrho_j}|((v_j)_z^\xi)'(t)|^2 \geq \frac{(1-\delta)\kappa\varrho_j}{\varepsilon_{k_j}}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in B_z^\xi. \tag{2.2.15}$$

The triangles belonging to the collection $\mathbf{T}_b^{x_0,j}$ correspond to the sets where the longitudinal slope of v_j in the direction ξ is "very large". They, roughly speaking, represent the places where it will be energetically convenient to introduce a jump because of the sharp transition.

The following result, which will play a major role in the proof of Proposition 2.2.5, shows that for many points $y \in J_{\overline{u}} \cap B$, the one-dimensional energy on B_y^ξ is arbitrarily small uniformly with respect to y.

Lemma 2.2.6. *For all $\eta > 0$, there exist a subset $Z \subset J_{\overline{u}} \cap B$ with $\mathcal{H}^1(Z) \leq \eta$ and a subsequence (not relabeled, depending on x_0) such that the following property holds : for all $\gamma > 0$, there exists $j_0 = j_0(\gamma) \in \mathbb{N}$ such that for all $y \in J_{\overline{u}} \cap B \setminus Z$ and all $j \geq j_0$,*

$$\begin{cases} \displaystyle\int_{B_y^\xi} (\chi_j)_y^\xi \, dt \leq \gamma, & (2.2.16a) \\ \displaystyle\int_{B_y^\xi} \big(1 - (\chi_j)_y^\xi\big)|((v_j)_y^\xi)'|^2 \, dt \leq \gamma^2, & (2.2.16b) \\ \displaystyle\int_{B_y^\xi} M_\eta \wedge |(v_j - \overline{u})_y^\xi| \, dt \leq \gamma. & (2.2.16c) \end{cases}$$

Proof. According to Fubini's Theorem, the convergence in measure (2.2.11a) and (2.2.12), we infer that

$$\begin{aligned} \int_{B^\xi} \left(\int_{B_z^\xi} M_\eta \wedge |(v_j - \overline{u})_z^\xi| \, dt + \int_{B_z^\xi} (1 - (\chi_j)_z^\xi)|((v_j)_z^\xi)'|^2 \, dt + \int_{B_z^\xi} (\chi_j)_z^\xi \, dt \right) d\mathcal{H}^1(z) \\ \leq \int_B M_\eta \wedge |v_j - \overline{u}| \, dx + \int_B (1 - \chi_j)|e(v_j)\xi \cdot \xi|^2 \, dx + \int_B \chi_j \, dx \to 0. \end{aligned}$$

As a consequence, up to a subsequence (not relabeled), there exists an $\mathcal{H}^1$-negligible set $N \subset B^\xi$ such that

$$\int_{B_z^\xi} M_\eta \wedge |(v_j - \overline{u})_z^\xi| \, dt + \int_{B_z^\xi} (1 - (\chi_j)_z^\xi)|((v_j)_z^\xi)'|^2 \, dt + \int_{B_z^\xi} (\chi_j)_z^\xi \, dt \to 0 \quad \text{for all } z \in B^\xi \setminus N.$$

In order to pass from arbitrary points $z \in B^\xi$ to arbitrary points $y \in J_{\overline{u}} \cap B = B^\nu$, let us consider the following mapping (see Figure 2.2)

$$\Phi : z \in \mathbb{R}^2 \longmapsto z - \frac{\nu \cdot z}{\nu \cdot \xi}\xi \in \Pi_\nu \tag{2.2.17}$$

which corresponds to the linear projection onto Π_ν in the direction ξ. Thanks to (2.2.9), we can check that the Lipschitz constant of Φ is bounded by $\sqrt{1+4\eta^2}$. Moreover, since for all $z \in B^\xi$ we have $B^\xi_z + \frac{\nu \cdot z}{\nu\cdot\xi} = B^\xi_{\Phi(z)} = \left\{s \in \mathbb{R} \,:\, z + \left(s - \frac{\nu\cdot z}{\nu\cdot\xi}\right)\xi \in B\right\}$, we deduce that

$$\begin{aligned}\int_{B^\xi_z} M_\eta \wedge |(v_j - \overline{u})^\xi_z|\,dt + \int_{B^\xi_z} (1-(\chi_j)^\xi_z)|((v_j)^\xi_z)'|^2\,dt + \int_{B^\xi_z} (\chi_j)^\xi_z\,dt \\ = \int_{B^\xi_{\Phi(z)}} M_\eta \wedge |(v_j - \overline{u})^\xi_{\Phi(z)}|\,ds + \int_{B^\xi_{\Phi(z)}} (1-(\chi_j)^\xi_{\Phi(z)})|((v_j)^\xi_{\Phi(z)})'|^2\,ds + \int_{B^\xi_{\Phi(z)}} (\chi_j)^\xi_{\Phi(z)}\,ds\end{aligned}$$

thanks to the change of variables $s = t + \frac{\nu\cdot z}{\nu\cdot\xi}$. Since $B^\nu \subset \Phi(B^\xi)$, setting $N' := \Phi(N) \subset \Pi_\nu$, we get that $\mathcal{H}^1(N') = 0$ and

$$\int_{B^\xi_y} M_\eta \wedge |(v_j - \overline{u})^\xi_y|\,ds + \int_{B^\xi_y} (1-(\chi_j)^\xi_y)|((v_j)^\xi_y)'|^2\,ds + \int_{B^\xi_y} (\chi_j)^\xi_y\,ds \to 0 \quad \text{for all } y \in B^\nu \setminus N'.$$

Applying Egoroff's theorem, for all $\eta > 0$, there exists a subset $Z \subset B^\nu$ such that $\mathcal{H}^1(Z) \le \eta$ and the above convergence is uniform with respect to $y \in B^\nu \setminus Z$. □

Let us consider the subsequence introduced in Lemma 2.2.6. For all $y \in \left(B_{1-\frac{\eta}{4}}\right)^\nu = J_{\overline{u}} \cap B_{1-\frac{\eta}{4}}$, we define the end points of the section passing through y in the direction ξ (see the Figure 2.2) :

$$\begin{cases} a(y) := \min\left\{t \in [-2,2] \,:\, y + t\xi \in \overline{B_{1-\frac{\eta}{4}}}\right\} \in [-2,0], \\ b(y) := \max\left\{t \in [-2,2] \,:\, y + t\xi \in \overline{B_{1-\frac{\eta}{4}}}\right\} \in [0,2], \end{cases} \tag{2.2.18}$$

so that $\left(B_{1-\frac{\eta}{4}}\right)^\xi_y = (a(y), b(y))$. Note that, for all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \subset \left(B_{1-\frac{\eta}{4}}\right)^\nu$,

$$0 < L_\eta := \sqrt{\left(1-\frac{\eta}{2}\right)^2 |\xi\cdot\nu^\perp|^2 + \frac{\eta(8-3\eta)}{16}} - \left(1-\frac{\eta}{2}\right)\left|\xi\cdot\nu^\perp\right| \le |a(y)|,\ |b(y)| \le 2. \tag{2.2.19}$$

We introduce the family

$$\mathbf{T}^{x_0,j}_{b,int} := \left\{T \in \mathbf{T}^{x_0,j}_b \,:\, T \cap \overline{B_{1-\frac{\eta}{4}}} \neq \emptyset\right\}$$

of triangles which intersect $\overline{B_{1-\frac{\eta}{4}}}$ and where v_j varies enough in the direction ξ. Note that for $j \in \mathbb{N}$ large enough (depending on η), each $T \in \mathbf{T}^{x_0,j}_{b,int}$ is contained in B, since the lengths of all triangles's edges are controlled by $\omega(\varepsilon_{k_j})/\varrho_j \to 0$. The collection $\mathbf{T}^{x_0,j}_{b,int}$ is introduced for technical reasons to deal with triangles which intersect the boundary of the ball B.

In the following result, we show that, for some subset of $Z' \subset J_{\overline{u}} \cap B_{1-\frac{\eta}{2}}$ of arbitrarily small $\mathcal{H}^1$ measure, and along a subsequence (only depending on η), all the sections in the direction ξ passing through $J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'$ must cross at least one triangle $T \in \mathbf{T}^{x_0,j}_{b,int}$ contained in B, and on which the

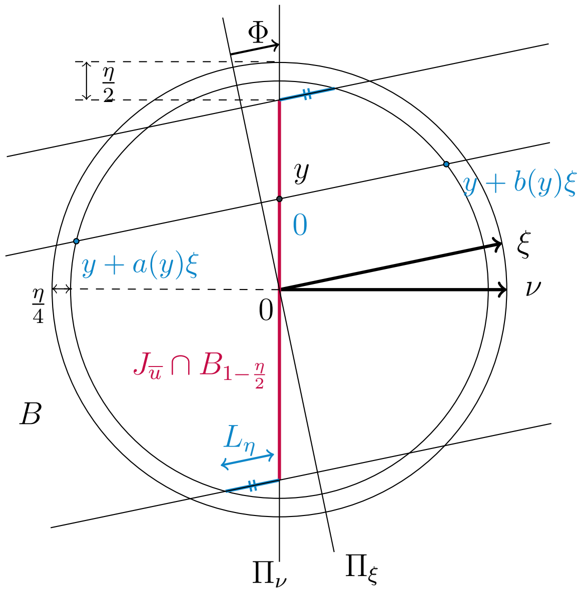

Figure 2.2

longitudinal slope of v_j in the direction ξ is "large". The formal idea of the proof consists in observing that, if for some $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}}$ the one-dimensional section B^ξ_y intersects no triangle in the collection $\mathbf{T}^{x_0,j}_{b,int}$ for infinitely many j's, then the function $(v_j)^\xi_y$ would be bounded in $H^1(B^\xi_y)$. Lemma 2.2.6 would then entail that $(v_j)^\xi_y$ converges (weakly in $H^1(B^\xi_y)$ and also $\mathcal{L}^1$-a.e. in B^ξ_y) to a constant function. This property contradicts the fact that $(v_j)^\xi_y \to \overline{u}^\xi_y$ $\mathcal{L}^1$-a.e. in B^ξ_y, where $\overline{u}^\xi_y$ is a step function taking two different values $u^\pm(x_0) \cdot \xi$.

Lemma 2.2.7. *For all $\eta > 0$, there exist a subset $Z' \subset J_{\overline{u}} \cap B$ containing Z with $\mathcal{H}^1(Z') \leq \eta$, and a subsequence (not relabeled) such that the following property holds : for all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'$ and all $j \in \mathbb{N}$, there exists a triangle $T = T(y,j) \in \mathbf{T}^{x_0,j}_{b,int}$ such that $(\mathring{T} \cap B)^\xi_y \neq \emptyset$.*

Proof. Let Z be the exceptional set given by Lemma 2.2.6. We first show the weaker result that there exists an increasing mapping $\phi : \mathbb{N} \to \mathbb{N}$ with the following property : for all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z$ and all $j \in \mathbb{N}$, there exists a triangle $T = T(y, \phi(j)) \in \mathbf{T}^{x_0,\phi(j)}_{b,int}$ such that $(T \cap B)^\xi_y \neq \emptyset$.

Suppose by contradiction that such is not the case, and define

$$\gamma_1^* := L_\eta M_\eta > 0, \quad \gamma_2^* := \frac{L_\eta |[u](x_0) \cdot \xi|}{1 + 2L_\eta} > 0 \quad \text{and} \quad \gamma^* = \gamma^*(\eta) := \frac{\gamma_1^* \wedge \gamma_2^*}{4} > 0,$$

where we recall that the constants M_η and L_η are defined in (2.2.10) and (2.2.19), respectively. Thanks to Lemma 2.2.6, there exists $j^* = j^*(\gamma^*) \in \mathbb{N}$ such that for all $y \in J_{\overline{u}} \cap B \setminus Z$ and all $j \geq j^*$,

$$\int_{B^\xi_y} (1 - (\chi_j)^\xi_y)|((v_j)^\xi_y)'|^2 \, dt \leq \gamma^{*2} \text{ and } \int_{B^\xi_y} M_\eta \wedge |(v_j - \overline{u})^\xi_y| \, dt \leq \gamma^*.$$

We then consider the extraction $\phi : j \in \mathbb{N} \longmapsto j + j^* \in \mathbb{N}$ which only depends on η. By assumption, there exists $y = y(\phi) \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z$ and $j = j(\phi) \in \mathbb{N}$ such that $(T \cap B)^\xi_y = \emptyset$ for all $T \in \mathbf{T}^{x_0, j+j^*}_{b,int}$. Remembering (2.2.14), we deduce that $(\chi_{j+j^*})^\xi_y \equiv 0$ on $(a(y), b(y))$. Moreover, since $\phi(j) = j + j^* \geq j^*$, we have

$$\int_{a(y)}^{b(y)} |((v_{j+j^*})^\xi_y)'|^2 \, dt \leq \gamma^{*2} \text{ and } \int_{a(y)}^{b(y)} M_\eta \wedge |(v_{j+j^*} - \overline{u})^\xi_y| \, dt \leq \gamma^*.$$

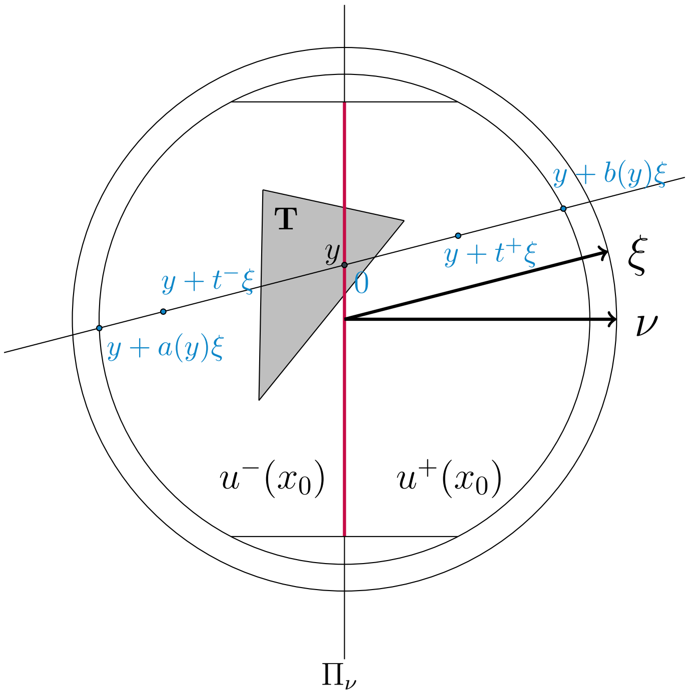

Figure 2.3

By continuity of $(v_{j+j^*})^\xi_y$ on the compact $[a(y), b(y)]$, $(v_{j+j^*})^\xi_y$ being in $H^1(B^\xi_y)$, there exist two points $t^\pm \in [a(y), b(y)] \cap \mathbb{R}^\pm$ such that

$$\min_{[a(y),b(y)] \cap \mathbb{R}^\pm} \left(M_\eta \wedge |\,(v_{j+j^*})^\xi_y - u^\pm(x_0) \cdot \xi| \right) = M_\eta \wedge |\,(v_{j+j^*})^\xi_y\,(t^\pm) - u^\pm(x_0) \cdot \xi|.$$

Hence,

$$\begin{aligned}
\frac{\gamma^*}{L_\eta} &\geq \frac{1}{L_\eta} \int_{a(y)}^0 M_\eta \wedge |\,(v_{j+j^*})^\xi_y - u^-(x_0) \cdot \xi| \, dt + \frac{1}{L_\eta} \int_0^{b(y)} M_\eta \wedge |\,(v_{j+j^*})^\xi_y - u^+(x_0) \cdot \xi| \, dt \\
&\geq M_\eta \wedge |\,(v_{j+j^*})^\xi_y\,(t^-) - u^-(x_0) \cdot \xi| \, + \, M_\eta \wedge |\,(v_{j+j^*})^\xi_y\,(t^+) - u^+(x_0) \cdot \xi| \\
&\geq M_\eta \wedge \left(|\,(v_{j+j^*})^\xi_y\,(t^-) - u^-(x_0) \cdot \xi| \, + \, |\,(v_{j+j^*})^\xi_y\,(t^+) - u^+(x_0) \cdot \xi| \right) \\
&\geq M_\eta \wedge \left(|[u](x_0) \cdot \xi| - \left| \int_{t^-}^{t^+} \left((v_{j+j^*})^\xi_y \right)'(t) \, dt \right| \right) \\
&\geq M_\eta \wedge \left(|[u](x_0) \cdot \xi| - 2\gamma^* \right),
\end{aligned}$$

which is impossible thanks to of our choice of γ^*.

We are now in position to complete the proof of Lemma 2.2.7. For all $j \in \mathbb{N}$, let

$$\begin{aligned} Z_j := \Big\{ & y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} : \text{ there exists } T \in \mathbf{T}^{x_0,j}_{b,int} \text{ such that} \\ & (T \cap B)^\xi_y \text{ is contained in an edge or a vertex of } T \Big\}, \end{aligned}$$

and

$$Z' := Z \cup \bigcup_{j \in \mathbb{N}} Z_j.$$

We notice that $\bigcup_j Z_j$ is $\mathcal{H}^1$-negligible (each Z_j being finite), hence $\mathcal{H}^1(Z') \le \eta$. Moreover, for all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'$ and all $j \in \mathbb{N}$, there exists a triangle $T \in \mathbf{T}^{x_0,\phi(j)}_{b,int}$ such that $(T \cap B)^\xi_y$ is non-empty, and it is neither reduced to a vertex of T nor contained in an edge of it. It thus implies that $(\mathring{T} \cap B)^\xi_y \neq \emptyset$. □

Let us consider the further subsequence introduced in Lemma 2.2.7. As a consequence, for all $j \in \mathbb{N}$, the family of triangles

$$\mathscr{F}_j := \left\{ T \in \mathbf{T}^{x_0,j}_{b,int} : \text{ there exists } y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \text{ such that } (\mathring{T} \cap B)^\xi_y \neq \emptyset \right\} \tag{2.2.20}$$

is nonempty. Thanks to Lemma 2.2.7, it is possible to obtain a bad lower bound. Indeed, from that result, we infer that $J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z' \subset \bigcup_{T \in \mathscr{F}_j} \Phi(p_\xi(T))$ with Φ the projection onto Π_ν in the direction ξ defined in (2.2.17). Using next that $\mathcal{L}^2(T) \ge \mathcal{H}^1(p_\xi(T))(\varepsilon_{k_j}/\varrho_j)\sin\theta_0/2$ and that the Lipschitz constant of Φ is bounded by $\sqrt{1+4\eta^2}$, we deduce from (2.2.12) and our choice of x_0 that

$$\begin{aligned} 2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) &\ge \liminf_{j\to\infty} \frac{(1-\delta)\kappa\varrho_j}{\varepsilon_{k_j}} \int_B \chi_j \, dy \ge \liminf_{j\to\infty} \sum_{T \in \mathscr{F}_j} \frac{(1-\delta)\kappa\varrho_j \mathcal{L}^2(T)}{\varepsilon_{k_j}} \\ &\ge \frac{(1-\delta)\kappa\sin\theta_0}{2\sqrt{1+4\eta^2}} \liminf_{j\to\infty} \mathcal{H}^1\left(\bigcup_{T\in\mathscr{F}_j} \Phi(p_\xi(T)) \right) \\ &\ge \frac{(1-\delta)\kappa\sin\theta_0}{2\sqrt{1+4\eta^2}} \mathcal{H}^1\left(J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z' \right) \ge \frac{(1-\delta)\kappa\sin\theta_0}{\sqrt{1+4\eta^2}}(1-\eta). \end{aligned}$$

Letting $\eta \to 0$ and $\delta \to 0$ leads to

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \ge \frac{\kappa \sin\theta_0}{2}$$

which corresponds to a too low lower bound because of the factor $1/2$ in the right-hand side of the previous inequality. In order to improve the previous argument, we need to establish that many lines B^ξ_y parallel to ξ and passing through the jump set at some point $y \in J_{\overline{u}} \cap B$ must actually intersect at least two triangles of the collection $\mathbf{T}^{x_0,j}_{b,int}$, where the longitudinal variation of v_j in the direction ξ is "large". This idea is precisely formulated in the following result which is an improvement of Lemma 2.2.7.

Lemma 2.2.8. *For all $\eta > 0$, there exist $Z'' \subset J_{\overline{u}} \cap B$ containing Z' with $\mathcal{H}^1(Z'') \le 3\eta$, and a (not relabeled) subsequence such that for all $j \in \mathbb{N}$ and for all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$,*

$$\# \left\{ T \in \mathbf{T}^{x_0,j}_{b,int} : (\mathring{T} \cap B)^\xi_y \neq \emptyset \right\} \ge 2.$$

The proof of Lemma 2.2.8 consists in constructing both Z'' and the subsequence inductively by means of the following technical result, Lemma 2.2.9. It stipulates that the set of all points $y \in J_{\overline{u}} \cap B$ such that B_y^ξ intersects exactly one triangle T in the collection $\mathbf{T}_{b,int}^{x_0,j}$, has arbitrarily small $\mathcal{H}^1$ measure. To establish this property, we first show that if such situation arises, then the function $(v_j)_y^\xi$ is uniformly close (with respect to y) to the step function $\overline{u}_y^\xi$ taking the values $u^\pm(x_0) \cdot \xi$. Thus, up to a small error which is uniform in y, the function $(v_j)_y^\xi$ must pass from the value $u^-(x_0) \cdot \xi$ to $u^+(x_0) \cdot \xi$ in an affine way inside the only triangle $T \in \mathbf{T}_{b,int}^{x_0,j}$ which is crossed by B_y^ξ. However, due to the shape of a triangle, this can happen for at most two different values of y, say z_1 and z_2. Then, if $y \in J_{\overline{u}} \cap B$ is far away from these two values z_1 and z_2, the variation of $(v_j)_y^\xi$ is not sufficient to connect the values $u^\pm(x_0) \cdot \xi$ in an affine way. It thus becomes necessary for B_y^ξ to intersect an additional triangle $T' \in \mathbf{T}_{b,int}^{x_0,j}$, where the variation of $(v_j)_y^\xi$ is substantial, in order to recover the full jump.

Lemma 2.2.9. *For all $\eta > 0$, there exist constants $C_* = C_*(\eta) > 0$, $\gamma_* = \gamma_*(\eta) > 0$ and a subset $Z_* = Z_*(\eta) \subset J_{\overline{u}} \cap B$ containing Z' and satisfying $\mathcal{H}^1(Z_*) \leq 2\eta$ such that the following property holds : for all $0 < \gamma < \gamma_*$, there exists $j(\gamma) \in \mathbb{N}$ such that for all $j \geq j(\gamma)$, the set*

$$Y_j := \left\{ y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z' : \text{ there exists a unique } T \in \mathbf{T}_{b,int}^{x_0,j} \text{ such that } (\mathring{T} \cap B)_y^\xi \neq \emptyset \right\} \tag{2.2.21}$$

satisfies

$$\mathcal{H}^1(Y_j \setminus Z_*) \leq C_* \gamma.$$

Proof of Lemma 2.2.9. The proof is divided into three steps.

Step 1. In this first step, we show that for j large enough and for many points $y \in Y_j$, the set $(B \cap T)_y^\xi$ (where T is the only triangle in $\mathbf{T}_{b,int}^{x_0,j}$ which crosses B_y^ξ) is close to $(J_{\overline{u}})_y^\xi$, uniformly with respect to y.

For all $j \in \mathbb{N}$ and all $y \in Y_j$, let $T_j(y) \in \mathbf{T}_{b,int}^{x_0,j}$ be the unique triangle such that $(\mathring{T}_j(y) \cap B)_y^\xi \neq \emptyset$. We define the end points of the section in the direction ξ passing through y inside $T_j(y)$ (see the Figure 2.4) by

$$\begin{cases} a_j(y) := \min \{ t \in [-2, 2] : \ y + t\xi \in T_j(y) \} , \\ b_j(y) := \max \{ t \in [-2, 2] : \ y + t\xi \in T_j(y) \} , \end{cases} \tag{2.2.22}$$

so that $(T_j(y))_y^\xi = [a_j(y), b_j(y)]$. Note that $T_j(y) \subset B$ (since $T_j(y) \cap \overline{B_{1-\frac{\eta}{4}}} \neq \emptyset$), hence $-2 \leq a_j(y) \leq b(y)$ and $2 \geq b_j(y) \geq a(y)$. Let us show that

$$f_j(y) := (|a_j(y)| + |b_j(y)|) \, \mathbb{1}_{Y_j}(y) \to 0 \quad \text{for all } y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'. \tag{2.2.23}$$

Let $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'$ and set $\ell := \limsup_j f_j(y) \in [0, 4]$. Assume by contradiction that $\ell > 0$ and extract a subsequence depending on y (not relabeled) such that $f_j(y) \to \ell$. Then, there exists $j_0 \in \mathbb{N}$ such that $y \in Y_j$ for all $j \geq j_0$. Moreover, according to Lemma 2.2.6 and setting $I_j(y) := (a(y), b(y)) \setminus (a_j(y), b_j(y)) \subset B_y^\xi$, we have

$$\begin{cases} |b_j(y) - a_j(y)| \leq \displaystyle\int_{B_y^\xi} (\chi_j)_y^\xi \, dt \to 0, \\ \displaystyle\int_{I_j(y)} |((v_j)_y^\xi)'|^2 \, dt \leq \int_{B_y^\xi} \left(1 - (\chi_j)_y^\xi\right) |((v_j)_y^\xi)'|^2 \, dt \to 0, \\ \displaystyle\int_{a(y)}^{b(y)} M_\eta \wedge |(v_j - \overline{u})_y^\xi| \, dt \leq \int_{B_y^\xi} M_\eta \wedge |(v_j - \overline{u})_y^\xi| \, dt \to 0. \end{cases} \tag{2.2.24}$$

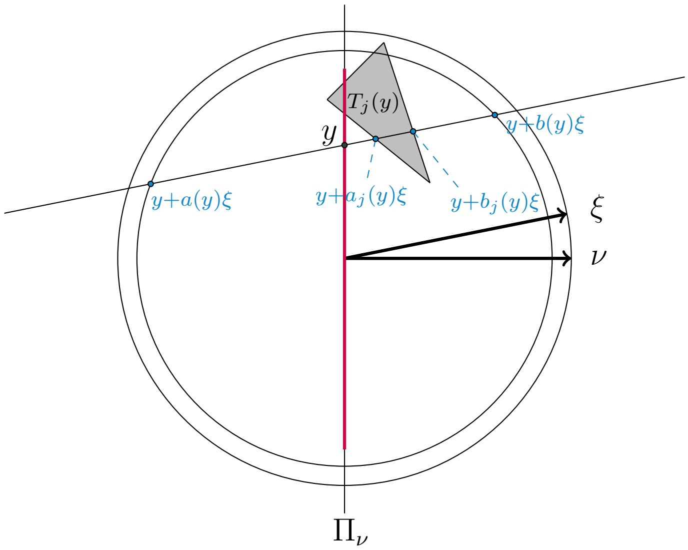

Figure 2.4

Up to another subsequence (still not relabeled), the first condition in (2.2.24) ensures that $a_j(y) \to m$ and $b_j(y) \to m$ for some $m \in [a(y), b(y)]$. Thus, for all $\tau > 0$, there exists $j_1 = j_1(\tau) \geq j_0$ such that for all $j \geq j_1$,

$$I_\tau := (a(y), m - \tau) \cup (m + \tau, b(y)) \subset I_j(y),$$

with the convention that $(x, y) = \emptyset$ if $y < x$. We set

$$I_\tau^- := (a(y), m - \tau)\,, \quad I_\tau^+ := (m + \tau, b(y))\,,$$

so that $(v_j)^\xi_{y|I^\pm_\tau} \in H^1(I^\pm_\tau)$ and the truncated function $w_j := \left(M_\eta \wedge (v_j)^\xi_y\right) \vee (-M_\eta) \in H^1(I^\pm_\tau)$ satisfies $w_j' = ((v_j)^\xi_y)' \mathbb{1}_{\{|(v_j)^\xi_y| \leq M_\eta\}}$. According to the second condition in (2.2.24), the sequence $\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H^1(I^\pm_\tau)$ and $w_j' \to 0$ in $L^2(I^\pm_\tau)$. As a consequence, up to a subsequence, there exist constants $c^\pm \in \mathbb{R}$ such that $w_j \to c^\pm$ in $H^1(I^\pm_\tau)$ and $\mathcal{L}^1$-a.e. in $I^\pm_\tau$. Yet, as $(v_j)^\xi_y$ converges in measure to $\overline{u}^\xi_y$ in $I^\pm_\tau$, up to another subsequence (still not relabeled), we have that $(v_j)^\xi_y$ pointwise converges to $\overline{u}^\xi_y$ $\mathcal{L}^1$-a.e. in $I^\pm_\tau$. Hence $c^\pm = (M_\eta \wedge u^\pm(x_0)\cdot\xi) \vee (-M_\eta) = u^\pm(x_0)\cdot\xi$ by our choice (2.2.10) of M_η. Thus, for all $\tau > 0$,

$$u^-(x_0)\cdot\xi\,\mathbb{1}_{(a(y),m-\tau)} + u^+(x_0)\cdot\xi\,\mathbb{1}_{(m+\tau,b(y))} = \overline{u}^\xi_{y|I_\tau} \quad \mathcal{L}^1\text{-a.e. in } I_\tau.$$

Taking the limit as $\tau \to 0^+$, we obtain that

$$u^-(x_0)\cdot\xi\,\mathbb{1}_{(a(y),m)} + u^+(x_0)\cdot\xi\,\mathbb{1}_{(m,b(y))} = \overline{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } (a(y), b(y)),$$

leading to $m = 0$ since $[u](x_0)\cdot\xi \neq 0$ by our choice (2.2.9) of ξ. As a consequence $f_j(y) = (|a_j(y)| + |b_j(y)|)\mathbb{1}_{Y_j}(y) \to 0$ which is against $\ell > 0$.

Using (2.2.23), Lemma 2.2.6 and owing to Egoroff's Theorem, we can find a set $Z_* \subset J_{\overline{u}} \cap B$ containing Z' with $\mathcal{H}^1(Z_*) \leq 2\eta$ such that for all $\gamma > 0$, there exists $j_0(\gamma) \in \mathbb{N}$ satisfying

$$\begin{cases} \displaystyle\int_{B_y^\xi} (1 - (\chi_j)_y^\xi)|((v_j)_y^\xi)'|^2 \, dt \leq \gamma^2, \\ \displaystyle\int_{B_y^\xi} M_\eta \wedge |(v_j - \overline{u})_y^\xi| \, dt \leq \gamma, \\ (|a_j(y)| + |b_j(y)|) \, \mathbb{1}_{Y_j}(y) \leq \gamma \end{cases} \qquad \text{for all } y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z_* \text{ and all } j \geq j_0(\gamma). \tag{2.2.25}$$

Step 2. In this step, we show that for many points $y \in Y_j$, the variation of $(v_j)_y^\xi$ inside the only triangle T in $\mathbf{T}_{b,int}^{x_0,j}$ which is crossed by B_y^ξ, is uniformly close with respect to y to the jump of $\overline{u}_y^\xi$. More precisely, let

$$C_\eta := 8\left(1 + \frac{1}{L_\eta}\right) > 0, \quad \gamma_* = \gamma_*(\eta) := \frac{1}{2}\min\left(1, \frac{M_\eta}{C_\eta}, L_\eta, \frac{|[u](x_0) \cdot \xi|}{4C_\eta}\right) > 0. \tag{2.2.26}$$

Let us show that for all $0 < \gamma < \gamma_*$, there exists $j_1(\gamma) \in \mathbb{N}$ such that

$$\left|(v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y)) - [u](x_0) \cdot \xi\right| \leq C_\eta \gamma \quad \text{for all } j \geq j_1(\gamma) \text{ and all } y \in Y_j \setminus Z_*. \tag{2.2.27}$$

Fix $0 < \gamma < \gamma_*$ and, by (2.2.25), let $j_0(\gamma) \in \mathbb{N}$ be such that

$$\begin{cases} \displaystyle\int_{(a(y),b(y)) \setminus (a_j(y),b_j(y))} |((v_j)_y^\xi)'|^2 \, dt \leq \gamma^2, \\ \displaystyle\int_{B_y^\xi} M_\eta \wedge |(v_j - \overline{u})_y^\xi| \, dt \leq \gamma, \\ (|a_j(y)| + |b_j(y)|) \leq \gamma \end{cases} \qquad \text{for all } j \geq j_0(\gamma) \text{ and all } y \in Y_j \setminus Z_*.$$

In particular, recalling (2.2.18), (2.2.19) and by the choice (2.2.26) of γ_*, we get that $2 \geq |b(y)| \geq L_\eta > \gamma_* > \gamma \geq |b_j(y)|$ and $2 \geq |a(y)| \geq L_\eta > \gamma_* > \gamma \geq |a_j(y)|$, hence

$$a(y) < a_j(y) \leq b_j(y) < b(y).$$

Writing

$$\begin{aligned} M_\eta \wedge \left|(v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y)) - [u](x_0) \cdot \xi\right| \quad &\leq \quad M_\eta \wedge \left|(v_j)_y^\xi\,(0 \vee b_j(y)) - u^+(x_0) \cdot \xi\right| \\ &\quad + M_\eta \wedge \left|(v_j)_y^\xi(0 \vee b_j(y)) - (v_j)_y^\xi(b_j(y))\right| \\ &\quad + M_\eta \wedge \left|(v_j)_y^\xi\,(0 \wedge a_j(y)) - u^-(x_0) \cdot \xi\right| \\ &\quad + M_\eta \wedge \left|(v_j)_y^\xi(a_j(y)) - (v_j)_y^\xi(0 \wedge a_j(y))\right| \\ &=: \quad J_1 + J_2 + J_3 + J_4, \end{aligned}$$

it remains to control each of the last four terms.

Let us first estimate the terms J_2 and J_4. If $b_j(y) \geq 0$, $J_2 = 0$. Otherwise, by the Cauchy-Schwarz inequality,

$$J_2 = M_\eta \wedge \left| \int_{b_j(y)}^0 ((v_j)_y^\xi)' \, dt \right| \leq \sqrt{|b_j(y)|} \left(\int_{b_j(y)}^0 |\big((v_j)_y^\xi\big)'|^2 \, dt \right)^{\frac{1}{2}} \leq \gamma^{3/2} \leq \gamma.$$

Similarly, we have that $J_4 \leq \gamma$.

Let us now estimate the term J_1. We consider the function

$$z_j := M_\eta \wedge |(v_j)_y^\xi - u^+(x_0) \cdot \xi| \in H^1(B_y^\xi)$$

with

$$z_j' = ((v_j)_y^\xi)' \mathbb{1}_{\{0 \leq (v_j)_y^\xi - u^+(x_0)\cdot\xi \leq M_\eta\}} - ((v_j)_y^\xi)' \mathbb{1}_{\{0 \leq u^+(x_0)\cdot\xi - (v_j)_y^\xi \leq M_\eta\}},$$

and the nonempty open interval $I^+ := (0 \vee b_j(y), b(y))$. By the Sobolev embedding and (2.2.26), we have that for all $t \in I^+$,

$$\begin{aligned}
|z_j(t)| &\leq \sqrt{b(y) - 0 \vee b_j(y)}\, \|z_j'\|_{L^2(I^+)} + \frac{1}{b(y) - 0 \vee b_j(y)} \|z_j\|_{L^1(I^+)} \\
&\leq \sqrt{2} \|((v_j)_y^\xi)'\|_{L^2(I^+)} + \frac{2}{L_\eta} \|M_\eta \wedge |(v_j)_y^\xi - u^+(x_0) \cdot \xi|\|_{L^1(I^+)} \\
&\leq \left(\sqrt{2} + \frac{2}{L_\eta} \right) \gamma.
\end{aligned}$$

By continuity of $(v_j)_y^\xi$ in B_y^ξ, the above inequality remains true up to the end point $0 \vee b_j(y)$ of I^+, so that $J_1 \leq (\sqrt{2} + \frac{2}{L_\eta})\gamma$. A similar argument shows that $J_3 \leq (\sqrt{2} + \frac{2}{L_\eta})\gamma$, and thus $J_1 + J_2 + J_3 + J_4 \leq 8(1 + \frac{1}{L_\eta})\gamma = C_\eta \gamma$, which shows that

$$M_\eta \wedge \left| (v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y)) - [u](x_0) \cdot \xi \right| \leq C_\eta \gamma.$$

Eventually, as $C_\eta \gamma < M_\eta$ for all $0 < \gamma < \gamma^*$ by (2.2.26), we conclude the validity of (2.2.27).

Step 3. We now show that it is possible to include $Y_j \setminus Z_*$ inside a finite union of arbitrarily small segments contained in $J_{\overline{u}} \cap B$ (see Figure 2.7).

Let $0 < \gamma < \gamma_*$ and $j_1(\gamma) \in \mathbb{N}$ be given by (2.2.27). For all $j \geq j_1(\gamma)$, we define

$$\widehat{\mathbf{T}}_j := \{T \in \mathbf{T}_{b,int}^{x_0,j} : \text{ there exists } y \in Y_j \setminus Z_* \text{ such that } (\mathring{T} \cap B)_y^\xi \neq \emptyset\},$$

and, for all $T \in \widehat{\mathbf{T}}_j$, we introduce both following quantities :

$$\begin{cases} L^{\mathrm{ref}}(T) := \frac{|[u](x_0)\cdot\xi| - 4C_\eta\gamma}{|e(v_j)_{|T} : (\xi \otimes \xi)|} \text{ the reference length of } T, \\ L^{\max}(T) := \max\limits_{z \in p_\xi(T)} \mathcal{L}^1(T_z^\xi) \text{ the maximal section's length of } T \text{ along the direction } \xi. \end{cases} \quad (2.2.28)$$

Note that because $T \in \mathbf{T}_b^{x_0,j}$, see (2.2.13), then $|e(v_j)_{|T}\xi \cdot \xi|^2 \geq (1-\delta)\kappa\varrho_j^2/(\alpha A \varepsilon_{k_j}) > 0$, so that $L^{\mathrm{ref}}(T)$ is well defined, and positive by (2.2.26) since $\gamma < \gamma_*$. The quantity $L^{\mathrm{ref}}(T)$ stands for the required length of the section T_y^ξ in order for the (affine) function $(v_j)_y^\xi$ to pass exactly from the values $u^-(x_0)\cdot\xi$ to $u^+(x_0)\cdot\xi$ across T, up to the error $4C_\eta\gamma$. Note that $\mathcal{L}^1(T_y^\xi) = L^{\mathrm{ref}}(T)$ for at most two values of y, say z_1 and z_2, only depending on j and T. If $y \in Y_j \setminus Z_*$ is such that $(\mathring{T}\cap B)_y^\xi \neq \emptyset$, we know from Step 2 that the variation of $(v_j)_y^\xi$ across T is close to $[u](x_0)\cdot\xi$, up to a small error of order $O(\gamma)$ which is uniform with respect to y. Therefore, we will show that if y is far away from z_1 and z_2, then the variation of $(v_j)_y^\xi$ across T is not sufficient to recover the full jump $[u](x_0)\cdot\xi$.

Let x_1, x_2 and $x_3 \in T$ be the three vertices of T and $X_i := p_\xi(x_i) \in B^\xi$. We easily see that there exists $i_0 \in \{1,2,3\}$ such that $X_{i_0} = \operatorname*{arg\,max}_{z\in p_\xi(T)} \mathcal{L}^1(T_z^\xi)$. Up to a permutation of $\{x_1,x_2,x_3\}$, there is no loss of generality to assume that $i_0 = 3$ and $X_3\cdot\xi^\perp \le X_1\cdot\xi^\perp$, with $\xi^\perp \in \mathbb{S}^1$ being one of the two orthogonal vectors to ξ.

Let $h_T \ge \frac{\varepsilon_{k_j}\sin(\theta_0)}{\varrho_j} > 0$ be the smallest height of T. We claim that, for all z, $z' \in p_\xi(T)$ be such that $\mathring{T}_z^\xi \neq \emptyset$, $\mathring{T}_{z'}^\xi \neq \emptyset$ and either $z\cdot\xi^\perp, z'\cdot\xi^\perp \ge X_3\cdot\xi^\perp$ or $z\cdot\xi^\perp, z'\cdot\xi^\perp \le X_3\cdot\xi^\perp$, then,

$$|z-z'| \le \frac{2\mathcal{L}^2(T)}{h_T}\,\frac{|\mathcal{L}^1(T_z^\xi)-\mathcal{L}^1(T_{z'}^\xi)|}{\max(\mathcal{L}^1(T_z^\xi),\mathcal{L}^1(T_{z'}^\xi))}. \tag{2.2.29}$$

Indeed, consider for instance the case where

$$X_1\cdot\xi^\perp \ge z\cdot\xi^\perp > z'\cdot\xi^\perp \ge X_3\cdot\xi^\perp$$

(see Figure 2.5). Let

$$L := \mathcal{L}^1(T_z^\xi), \quad L' := \mathcal{L}^1(T_{z'}^\xi), \quad d := |X_1 - z|, \quad d' := |X_1 - z'|.$$

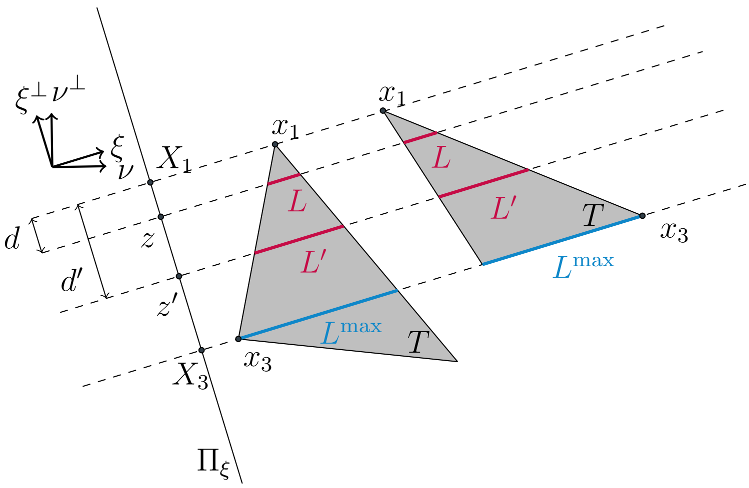

Figure 2.5

Then, $L' > L > 0$, $d' > d > 0$ and using Thalès' Theorem, we have that

$$\frac{d}{d'} = \frac{d' - |z-z'|}{d'} = \frac{L}{L'}.$$

Since $d' \leq |X_1 - X_3| = \mathcal{H}^1(p_\xi([x_1, x_3])) \leq |x_1 - x_3| \leq \frac{2\mathcal{L}^2(T)}{h_T}$, we obtain that

$$|z - z'| = d' \, \frac{L' - L}{L'} \leq \frac{2\mathcal{L}^2(T)}{h_T} \, \frac{|L - L'|}{L'},$$

so that (2.2.29) holds in that case. The proof of the other case $X_2 \cdot \xi^\perp \leq z \cdot \xi^\perp, z' \cdot \xi^\perp \leq X_3 \cdot \xi^\perp$ is similar and we omit it.

For all $j \geq j_1(\gamma)$ and all $T \in \widehat{\mathbf{T}}_j$, we have $L^{\max}(T) > L^{\text{ref}}(T)$. Indeed, if such would not be the case, denoting by $y \in Y_j \setminus Z_*$ a point such that $(\mathring{T} \cap B)^\xi_y \neq \emptyset$, then $\mathcal{L}^1(T^\xi_{p_\xi(y)}) = \mathcal{L}^1(T^\xi_y) = b_j(y) - a_j(y) \leq L^{\max}(T) \leq L^{\text{ref}}(T)$, entailing that

$$|(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y))| = \left|e(v_j)_{|T} : (\xi \otimes \xi)\right| (b_j(y) - a_j(y)) \leq |[u](x_0) \cdot \xi| - 4C_\eta \gamma,$$

by definition (2.2.28) of $L^{\text{ref}}(T)$. Therefore, we would obtain that

$$4C_\eta\gamma \leq |[u](x_0) \cdot \xi| - \left|(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y))\right| \leq \left|(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y)) - [u](x_0) \cdot \xi\right|,$$

which is against (2.2.27). Applying the Intermediate Value Theorem to the strictly monotone and continuous functions $y \in [X_1, X_3] \mapsto \mathcal{L}^1(T^\xi_y) \in [0, L^{\max}(T)]$ and $y \in [X_2, X_3] \mapsto \mathcal{L}^1(T^\xi_y) \in [0, L^{\max}(T)]$, there are at least one and at most two points $z^1_{\text{ref}}, z^2_{\text{ref}} \in p_\xi(T)$ (according to whether T has an edge along the direction ξ or not, see Figure 2.6), only depending on j and T, such that

$$\mathcal{L}^1(T^\xi_{z^1_{\text{ref}}}) = \mathcal{L}^1(T^\xi_{z^2_{\text{ref}}}) = L^{\text{ref}}(T).$$

(We set $z^1_{\text{ref}} = z^2_{\text{ref}}$ in the case where T has an edge along the direction ξ). Without loss of generality, we can assume that $z^1_{\text{ref}} \cdot \xi^\perp \geq z^2_{\text{ref}} \cdot \xi^\perp$.

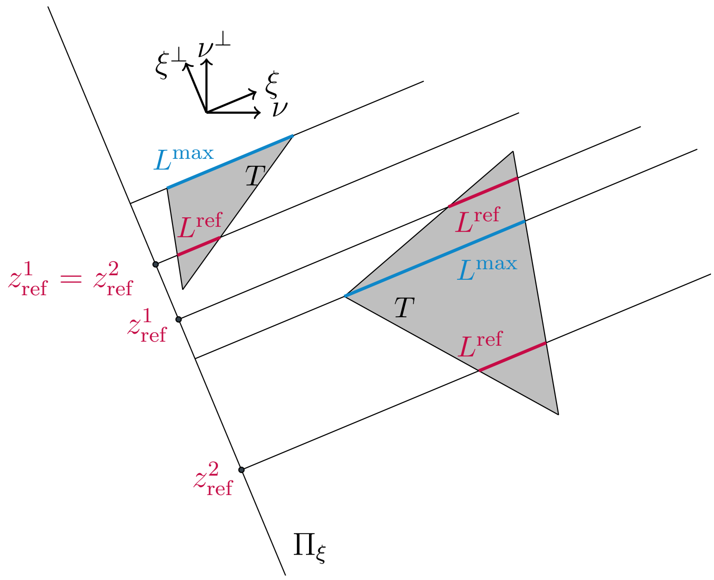

Figure 2.6 – Two possible configurations of T.

Let us introduce the following segments (orthogonal to ξ) associated to T (see Figure 2.7),

$$\mathfrak{T}_i(T) := \left\{ z \in \Pi_\xi : \ \left|z - z^i_{\rm ref}\right| \le C'_\eta \, \frac{\varrho_j \mathcal{L}^2(T)}{\varepsilon_{k_j}} \, \gamma \right\} \qquad \text{for } i \in \{1,2\}, \tag{2.2.30}$$

where

$$C'_\eta := \frac{20 C_\eta}{\sin\theta_0 \, |[u](x_0)\cdot\xi|}$$

is a constant only depending on η.

For every $j \ge j_1(\gamma)$ and every $y \in Y_j \setminus Z_*$, let $T \in \mathbf{T}^{x_0,j}_{b,int}$ be such that $(\mathring{T} \cap B)^\xi_y \neq \emptyset$. Note that $T \in \widehat{\mathbf{T}}_j$. If $p_\xi(y)\cdot\xi^\perp \ge X_3\cdot\xi^\perp$ and $z^1_{\rm ref}\cdot\xi^\perp \ge X_3\cdot\xi^\perp$ (the other cases being treated similarly), applying (2.2.29) above, with $z = p_\xi(y)$ and $z' = z^1_{\rm ref}$, we get that

$$\begin{aligned}
\left|p_\xi(y) - z^1_{\rm ref}\right| &\le \frac{2\mathcal{L}^2(T)}{h_T} \, \frac{\left|(b_j(y)-a_j(y)) - L^{\rm ref}(T)\right|}{\max\left(b_j(y)-a_j(y), L^{\rm ref}(T)\right)} \\
&\le \frac{2\mathcal{L}^2(T)}{h_T} \, \frac{\left| |(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y))| - |[u](x_0)\cdot\xi| + 4C_\eta\gamma \right|}{\left|e(v_j)_{|T} : (\xi\otimes\xi)\right| \, L^{\rm ref}(T)} \\
&\le \frac{2\mathcal{L}^2(T)}{h_T} \, \frac{5C_\eta\gamma}{|[u](x_0)\cdot\xi| - 4C_\eta\gamma} \\
&\le C'_\eta \, \frac{\varrho_j\mathcal{L}^2(T)}{\varepsilon_{k_j}} \, \gamma,
\end{aligned}$$

where we also used (2.2.27) and (2.2.26).

We have just shown that for all $j \ge j_1(\gamma)$ and all $y \in Y_j \setminus Z_*$, there exists $T \in \widehat{\mathbf{T}}_j$ such that $p_\xi(y) \in \mathfrak{T}_1(T) \cup \mathfrak{T}_2(T)$. Since $y \in \Pi_\nu$, then $y = \Phi(p_\xi(y)) \in \Phi\left(\mathfrak{T}_1(T) \cup \mathfrak{T}_2(T)\right)$, with Φ introduced in (2.2.17). Recalling that the Lipschitz constant of Φ is less than $\sqrt{1+4\eta^2} \le 2$ for η small enough, we deduce that

$$\mathcal{H}^1(\Phi\left(\mathfrak{T}_1(T)\cup\mathfrak{T}_2(T)\right)) \le 2\mathcal{H}^1(\mathfrak{T}_1(T)\cup\mathfrak{T}_2(T)) \le 8C'_\eta \, \frac{\varrho_j\mathcal{L}^2(T)}{\varepsilon_{k_j}} \, \gamma.$$

Together with the fact that each triangle in $\widehat{\mathbf{T}}_j \subset \mathbf{T}^{x_0,j}_{b,int}$ is contained in B, we obtain that for all $j \ge j_1(\gamma)$,

$$\begin{aligned}
\mathcal{H}^1(Y_j \setminus Z_*) &\le \sum_{T\in\widehat{\mathbf{T}}_j} \mathcal{H}^1(\Phi\left(\mathfrak{T}_1(T)\cup\mathfrak{T}_2(T)\right)) \\
&\le 8C'_\eta\,\gamma\,\frac{\varrho_j}{\varepsilon_{k_j}} \sum_{T\in\widehat{\mathbf{T}}_j} \mathcal{L}^2(T) \le \frac{8C'_\eta\gamma}{\kappa(1-\delta)} \, \frac{(1-\delta)\kappa\varrho_j}{\varepsilon_{k_j}} \int_B \chi_j\,dx \le \frac{8C'_\eta\gamma}{\kappa(1-\delta)} \, \frac{\lambda_{k_j}(B_{\varrho_j}(x_0))}{\varrho_j}.
\end{aligned}$$

Possibly taking a larger $j_1(\gamma) \in \mathbb{N}$, we finally get that for all $j \ge j_1(\gamma)$,

$$\mathcal{H}^1(Y_j \setminus Z_*) \le \frac{8C'_\eta}{\kappa(1-\delta)} \left(2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) + 1 \right) \gamma =: C_*\gamma,$$

for some constant $C_* > 0$ only depending on η. $\square$

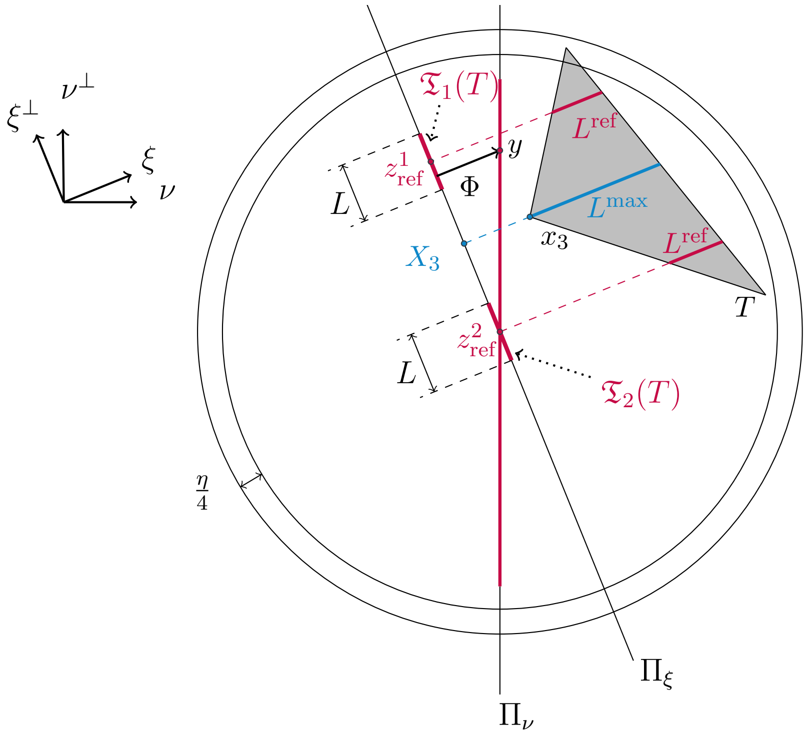

Figure 2.7 – The length of $\mathfrak{T}_i(T)$ is given by $L = \frac{2C'_\eta \varrho_j \mathcal{L}^2(T)}{\varepsilon_{k_j}}\gamma$.

We are now in position to prove Lemma 2.2.8.

Proof of Lemma 2.2.8. Let $j_0 = j_0(\eta) \in \mathbb{N}$ such that $\frac{\eta}{2^{j_0}C_*} < \gamma_*$, where C_* and γ_* are given by Lemma 2.2.9. For all $j \geq j_0$, as $0 < \gamma_j := \frac{\eta}{2^j C_*} < \gamma_*$, Lemma 2.2.9 ensures the existence of an integer $i(\gamma_j) \geq j_0$ such that for all $i \geq i(\gamma_j)$,

$$\mathcal{H}^1(Y_i \setminus Z_*) \leq C_* \gamma_j = \frac{\eta}{2^j}.$$

Thereby, we define the following extraction

$$\begin{cases} \phi(j_0) := i(\gamma_{j_0}) \geq j_0, \\ \phi(j+1) := \max\left(\phi(j)+1, i(\gamma_{j+1})\right) \quad \text{for all } j \geq j_0. \end{cases} \tag{2.2.31}$$

Since $\phi(j) \geq i(\gamma_j)$, then $\mathcal{H}^1(Y_{\phi(j)} \setminus Z_*) \leq \frac{\eta}{2^j}$. Hence, we set

$$Z'' := Z_* \cup \bigcup_{j=j_0}^{+\infty} Y_{\phi(j)}, \tag{2.2.32}$$

which satisfies

$$\mathcal{H}^1(Z'') \leq 2\eta + \sum_{j \geq j_0} \frac{\eta}{2^j} \leq 3\eta.$$

Moreover, for all $j \geq j_0$ and all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$, Lemma 2.2.7 ensures that

$$\# \left\{ T \in \mathbf{T}^{x_0,\phi(j)}_{b,int} : \; (\mathring{T} \cap B)^\xi_y \neq \emptyset \right\} \geq 1,$$

and since $y \notin Y_{\phi(j)}$ for all $j \geq j_0$, it actually follows that

$$\# \left\{ T \in \mathbf{T}^{x_0,\phi(j)}_{b,int} : \; (\mathring{T} \cap B)^\xi_y \neq \emptyset \right\} \geq 2,$$

concluding the proof of Lemma 2.2.8. □

Let us consider the further subsequence introduced in Lemma 2.2.8. In order to derive a lower bound for the surface energy without the factor $1/2$, we now construct two disjoint subfamilies $\mathscr{F}^1_j$ and $\mathscr{F}^2_j$ from $\mathscr{F}_j$ (see (2.2.20)) with the property that both sets

$$\bigcup_{T \in \mathscr{F}^1_j} T, \qquad \bigcup_{T \in \mathscr{F}^2_j} T$$

project onto $B^\nu = J_{\overline{u}} \cap B$, thanks to the mapping $\Phi \circ p_\xi$, into two sets of almost full $\mathcal{H}^1$ measure in $J_{\overline{u}} \cap B$. This is the object of the following technical result.

Lemma 2.2.10. *Let $K \subset J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$ be a compact set. For all $j \in \mathbb{N}$, there exist two disjoint subfamilies $\mathscr{F}^1_j$ and $\mathscr{F}^2_j$ of $\mathscr{F}_j$ such that*

$$K \subset \Phi \left(\bigcup_{T \in \mathscr{F}^1_j} p_\xi(\mathring{T}) \right) \cap \Phi \left(\bigcup_{T \in \mathscr{F}^2_j} p_\xi(\mathring{T}) \right).$$

Proof. For the sake of clarity, we omit to write the explicit dependance on j for the different objects considered herafter (triangles, intervals, and so forth).

For all $y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$, we consider a pair of distinct triangles of $\mathscr{F}_j$ satisfying

$$\begin{aligned} \left\{ T^1(y), T^2(y) \right\} \quad \in \quad & \arg\min \Big\{ \mathcal{H}^1 \left(\Phi(p_\xi(\mathring{T}^1) \cap p_\xi(\mathring{T}^2)) \right) : \\ & T^1, \, T^2 \in \mathbf{T}^{x_0,j}_{b,int}, \, \mathring{T}^1 \cap \mathring{T}^2 = \emptyset, \, (\mathring{T}^1 \cap B)^\xi_y \neq \emptyset, \, (\mathring{T}^2 \cap B)^\xi_y \neq \emptyset \Big\}. \end{aligned} \tag{2.2.33}$$

Note that Lemma 2.2.8 ensures that the set

$$\left\{ \{T^1, T^2\} \subset \mathbf{T}^{x_0,j}_{b,int} : \; (\mathring{T}^i \cap B)^\xi_y \neq \emptyset \text{ for all } i \in \{1,2\} \right\}$$

is nonempty and finite, hence the minimum in (2.2.33) is achieved and we have at our disposal such a pair of distinct triangles $\left\{ T^1(y), T^2(y) \right\}$. Then, we introduce the following open segment in $B^\nu = J_{\overline{u}} \cap B$

$$I(y) := \Phi \left(p_\xi(\mathring{T}^1(y)) \cap p_\xi(\mathring{T}^2(y)) \right) \subset B^\nu = J_{\overline{u}} \cap B. \tag{2.2.34}$$

Since $y \in I(y)$, it follows that

$$K \subset J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'' \subset \bigcup_{y \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''} I(y).$$

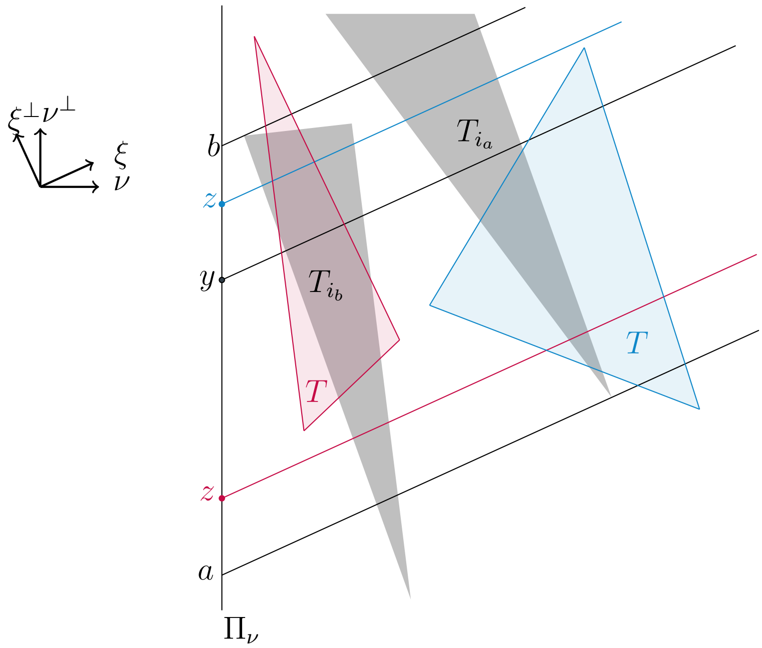

Figure 2.8

Furthermore, $I(y)$ is optimal in the sense that any other triangle $T \in \mathbf{T}^{x_0,j}_{b,int}$ satisfying $(\mathring{T} \cap B)^\xi_y \neq \emptyset$ is such that

$$I(y) \subset \Phi(p_\xi(\mathring{T})). \tag{2.2.35}$$

Indeed, setting $J := \Phi(p_\xi(\mathring{T}))$, there exist points a, b, $\tilde{a}$ and $\tilde{b}$ in $B^\nu = J_{\overline{u}} \cap B$ such that $I(y) = (a, b)$, $J = (\tilde{a}, \tilde{b})$ with $a \cdot \nu^\perp < b \cdot \nu^\perp$ and $\tilde{a} \cdot \nu^\perp < \tilde{b} \cdot \nu^\perp$. By construction, there exist i_a (resp. i_b) $\in \{1, 2\}$ such that a (resp. b) is the image by $\Phi \circ p_\xi$ of a vertex of $T^{i_a}(y)$ (resp. $T^{i_b}(y)$), see Figure 2.8. Assume by contradiction that there exists a point $z \in I(y) \setminus J$. If $y \cdot \nu^\perp < z \cdot \nu^\perp$, then $J \subset (\tilde{a}, z)$ since J is a segment containing y. In particular, $\Phi(p_\xi(\mathring{T}^{i_a}(y))) \cap J \subset (a, z)$. Together with (2.2.33) and recalling that $J := \Phi(p_\xi(\mathring{T}))$, it ensures that

$$\mathcal{H}^1((a,z)) \geq \mathcal{H}^1\left(\Phi(p_\xi(\mathring{T}^{i_a}(y))) \cap \Phi(p_\xi(\mathring{T}))\right) \geq \mathcal{H}^1(I(y)) = \mathcal{H}^1((a,z)) + \mathcal{H}^1((z,b)) > \mathcal{H}^1((a,z)),$$

which is impossible. A similar argument shows that the other situation $y \cdot \nu^\perp > z \cdot \nu^\perp$ is also impossible. This shows the validity of (2.2.35).

By compactness of K, there exist an integer $N = N(j, K) \geq 1$ and points $y_1, \ldots, y_N \in J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$ such that

$$K \subset \bigcup_{i=1}^N I(y_i). \tag{2.2.36}$$

Up to relabeling the points y_i, we can assume that $y_1 \cdot \nu^\perp < \cdots < y_N \cdot \nu^\perp$ (See Figure 2.9). Let us now construct two disjoint subfamilies $\mathscr{F}^1_j$ and $\mathscr{F}^2_j$ of $\mathscr{F}_j$ by induction in N iterations.

Iteration 1. Set $\mathscr{F}^1(1) := \{T^1(y_1)\}$ and $\mathscr{F}^2(1) := \{T^2(y_1)\}$. Clearly $\mathscr{F}^1(1) \cap \mathscr{F}^2(1) = \emptyset$ and, for all $k \in \{1, 2\}$, there is $T \in \mathscr{F}^k(1)$ such that $(B \cap \mathring{T})^\xi_{y_1} \neq \emptyset$ and $I(y_1) \subset \Phi(p_\xi(\mathring{T}))$.

Iteration 2. We distinguish two cases :

i) If $\{T^1(y_2), T^2(y_2)\} \cap \big(\mathscr{F}^1(1) \cup \mathscr{F}^2(1)\big) = \emptyset$, then we set $\mathscr{F}^1(2) := \mathscr{F}^1(1) \cup \{T^1(y_2)\}$ and $\mathscr{F}^2(2) := \mathscr{F}^2(1) \cup \{T^2(y_2)\}$. We have that $\mathscr{F}^1(2) \cap \mathscr{F}^2(2) = \emptyset$ and, for all $i, k \in \{1, 2\}$, there exists $T \in \mathscr{F}^k(2)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$.

ii) Otherwise, there exist $i, k \in \{1, 2\}$ such that $T^i(y_2) \in \mathscr{F}^k(1)$, i.e. $T^i(y_2) = T^k(y_1)$, and $T^{3-i}(y_2) \notin \mathscr{F}^k(1)$. In that case, we set $\mathscr{F}^k(2) := \mathscr{F}^k(1)$ and $\mathscr{F}^{3-k}(2) := \mathscr{F}^{3-k}(1) \cup \{T^{3-i}(y_2)\}$. Note that, it might be the case that $T^{3-i}(y_2) \in \mathscr{F}^{3-k}(1)$. We have that $\mathscr{F}^1(2) \cap \mathscr{F}^2(2) = \emptyset$ and, for all $i, k \in \{1, 2\}$, there exists $T \in \mathscr{F}^k(2)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$.

Iteration $n+1$ for some $n \in \{1, \ldots, N-1\}$. Assume that we have constructed two disjoint subfamilies $\mathscr{F}^1(n)$ and $\mathscr{F}^2(n)$ of $\mathscr{F}_j$ with the following properties : for all $k \in \{1, 2\}$ and all $i \in \{1, \ldots, n\}$, there exists $T \in \mathscr{F}^k(n)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$. Let us now construct $\mathscr{F}^1(n+1)$ and $\mathscr{F}^2(n+1)$:

i) If $\{T^1(y_{n+1}), T^2(y_{n+1})\} \cap \big(\mathscr{F}^1(n) \cup \mathscr{F}^2(n)\big) = \emptyset$, then we set $\mathscr{F}^1(n+1) := \mathscr{F}^1(n) \cup \{T^1(y_{n+1})\}$ and $\mathscr{F}^2(n+1) := \mathscr{F}^2(n) \cup \{T^2(y_{n+1})\}$. In that case, we have that $\mathscr{F}^1(n+1) \cap \mathscr{F}^2(n+1) = \emptyset$ and that for all $k \in \{1, 2\}$ and all $i \in \{1, \ldots, n+1\}$, there exists $T \in \mathscr{F}^k(n+1)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$. For $i \in \{1, \ldots, n\}$, it follows from the previous iteration n and because $\mathscr{F}^k(n) \subset \mathscr{F}^k(n+1)$, while for $i = n+1$, it is a consequence of the fact that $T^k(y_{n+1}) \in \mathscr{F}^k(n+1)$.

ii) Otherwise, there exist $p, q \in \{1, 2\}$ such that $T^p(y_{n+1}) \in \mathscr{F}^q(n)$. Let us further distinguish two subcases :

   (a) If $T^{3-p}(y_{n+1}) \notin \mathscr{F}^q(n)$, then we set $\mathscr{F}^q(n+1) := \mathscr{F}^q(n)$ and $\mathscr{F}^{3-q}(n+1) := \mathscr{F}^{3-q}(n) \cup \{T^{3-p}(y_{n+1})\}$. Then $\mathscr{F}^1(n+1) \cap \mathscr{F}^2(n+1) = \emptyset$ and, for all $k \in \{1, 2\}$ and all $i \in \{1, \ldots, n+1\}$, there exists $T \in \mathscr{F}^k(n+1)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$. Indeed, for $i \in \{1, \ldots, n\}$ this is a consequence of the previous iteration n and of the fact that $\mathscr{F}^k(n) \subset \mathscr{F}^k(n+1)$, while, for $i = n+1$, it results from $T^p(y_{n+1}) \in \mathscr{F}^q(n+1)$ and $T^{3-p}(y_{n+1}) \in \mathscr{F}^{3-q}(n+1)$.

   (b) If both $T^1 := T^1(y_{n+1}) \in \mathscr{F}^q(n)$ and $T^2 := T^2(y_{n+1}) \in \mathscr{F}^q(n)$, we introduce the indexes

$$i_1 := \arg\min \Big\{ i \in \{1, \ldots, n+1\} : \ (B \cap \mathring{T^1})^{\xi}_{y_i} \neq \emptyset \Big\}$$

   and

$$i_2 := \arg\min \Big\{ i \in \{1, \ldots, n+1\} : \ (B \cap \mathring{T^2})^{\xi}_{y_i} \neq \emptyset \Big\} .$$

   Up to interchanging T_1 and T_2, there is no loss of generality to assume that $i_1 \geq i_2$ (see Figure 2.9). We set $\mathscr{F}^q(n+1) := \mathscr{F}^q(n) \setminus \{T^1\}$ and $\mathscr{F}^{3-q}(n+1) := \mathscr{F}^{3-q}(n) \cup \{T^1\}$. Once more, we have $\mathscr{F}^1(n+1) \cap \mathscr{F}^2(n+1) = \emptyset$ and, for all $k \in \{1, 2\}$ and all $i \in \{1, \ldots, n+1\}$, there exists $T \in \mathscr{F}^k(n+1)$ such that $(B \cap \mathring{T})^{\xi}_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$. This is immediate for $i = n+1$ because $T^2 = T^2(y_{n+1}) \in \mathscr{F}^q(n+1)$ and $T^1 = T^1(y_{n+1}) \in \mathscr{F}^{3-q}(n+1)$. For $i \in \{1, \ldots, n\}$, there are two possibilities :

   - for $k = 3 - q$, it follows from $\mathscr{F}^{3-q}(n) \subset \mathscr{F}^{3-q}(n+1)$.

- for $k = q$ and $i \in \{1, \dots, i_1 - 1\}$, by the previous iteration n there exists $T \in \mathscr{F}^q(n)$ such that $(B \cap \mathring{T})^\xi_{y_i} \neq \emptyset$ and $I(y_i) \subset \Phi(p_\xi(\mathring{T}))$. As $(B \cap \mathring{T}^1)^\xi_{y_i} = \emptyset$, by definition of i_1, this implies that $T \neq T^1$, so that actually $T \in \mathscr{F}^q(n+1)$ satisfies the above requirements. Assuming next that $i \in \{i_1, \dots, n\}$, we deduce that $n + 1 > i \geq i_2$. By definition of i_2, we have $(B \cap \mathring{T}^2)^\xi_{y_{i_2}} \neq \emptyset$ while $(B \cap \mathring{T}^2)^\xi_{y_{n+1}} \neq \emptyset$ so that the convexity of $\mathring{T}^2$ together with the ordering of the points y_i lead to $(B \cap \mathring{T}^2)^\xi_{y_i} \neq \emptyset$. Hence, owing to (2.2.35), we infer that $I(y_i) \subset \Phi(p_\xi(\mathring{T}^2))$ and $T^2 \in \mathscr{F}^q(n+1)$ satisfies the above requirements.

We proceed this construction up to the N^{th} iteration, and finally define

$$\mathscr{F}^k_j := \mathscr{F}^k(N) \text{ for } k \in \{1, 2\}, \tag{2.2.37}$$

which define two disjoint subfamilies of $\mathscr{F}_j$ satisfying in particular, thanks to (2.2.36),

$$\begin{aligned} K \;&\subset\; \bigcup_{i=1}^N I(y_i) \subset \left(\bigcup_{T \in \mathscr{F}^1_j} \Phi(p_\xi(\mathring{T})) \right) \cap \left(\bigcup_{T \in \mathscr{F}^2_j} \Phi(p_\xi(\mathring{T})) \right) \\ &= \Phi\left(\bigcup_{T \in \mathscr{F}^1_j} p_\xi(\mathring{T}) \right) \cap \Phi\left(\bigcup_{T \in \mathscr{F}^2_j} p_\xi(\mathring{T}) \right). \end{aligned}$$

The proof of Lemma 2.2.10 is now complete. □

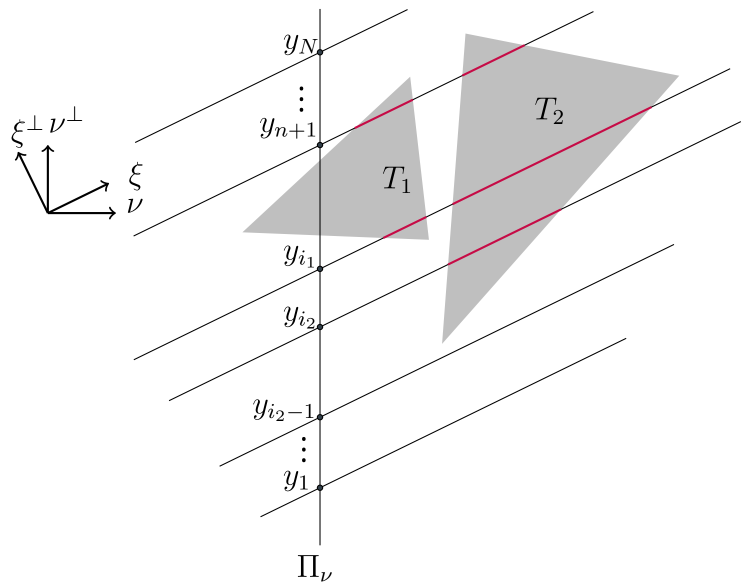

Figure 2.9

We are now in position to complete the proof of Proposition 2.2.3.

Proof of Proposition 2.2.3. Since all triangles T in $\mathscr{F}_j$ are contained in B, we get from (2.2.11b) and (2.2.12),

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \lim_{j\to\infty} \frac{\lambda_{k_j}(B_{\varrho_j}(x_0))}{2\varrho_j} \geq \liminf_{j\to\infty} \frac{(1-\delta)\kappa\varrho_j}{2\varepsilon_{k_j}} \int_B \chi_j \, dx \geq \liminf_{j\to\infty} \frac{(1-\delta)\kappa\varrho_j}{2\varepsilon_{k_j}} \sum_{T\in\mathscr{F}_j} \mathcal{L}^2(T).$$

Using next the inequality $\mathcal{L}^2(T) \geq \mathcal{H}^1(p_\xi(T))(\varepsilon_{k_j}/\varrho_j)\sin\theta_0/2$, we deduce that

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \frac{(1-\delta)\kappa\sin\theta_0}{2} \liminf_{j\to\infty} \sum_{T\in\mathscr{F}_j} \mathcal{H}^1(p_\xi(T)).$$

Let $K \subset J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$ be a compact set and $\mathscr{F}_j^1$ and $\mathscr{F}_j^2$ be two disjoint subfamilies of $\mathscr{F}_j$ given by Lemma 2.2.10. Thus

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \frac{(1-\delta)\kappa\sin\theta_0}{2} \liminf_{j\to\infty} \left\{ \sum_{T\in\mathscr{F}_j^1} \mathcal{H}^1(p_\xi(T)) + \sum_{T\in\mathscr{F}_j^2} \mathcal{H}^1(p_\xi(T)) \right\},$$

and remembering that Φ has a Lipschitz constant bounded by $\sqrt{1+4\eta^2}$,

$$\begin{aligned} 2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) &\geq \frac{(1-\delta)\kappa\sin\theta_0}{2\sqrt{1+4\eta^2}} \liminf_{j\to\infty} \left\{ \sum_{T\in\mathscr{F}_j^1} \mathcal{H}^1(\Phi(p_\xi(T))) + \sum_{T\in\mathscr{F}_j^2} \mathcal{H}^1(\Phi(p_\xi(T))) \right\} \\ &\geq \frac{(1-\delta)\kappa\sin\theta_0}{2\sqrt{1+4\eta^2}} \liminf_{j\to\infty} \left\{ \mathcal{H}^1\left(\bigcup_{T\in\mathscr{F}_j^1} \Phi\left(p_\xi(T)\right) \right) + \mathcal{H}^1\left(\bigcup_{T\in\mathscr{F}_j^2} \Phi\left(p_\xi(T)\right) \right) \right\} \\ &\geq \frac{(1-\delta)\kappa\sin\theta_0}{\sqrt{1+4\eta^2}} \mathcal{H}^1(K). \end{aligned}$$

By inner regularity of the Radon measure $\mathcal{H}^1 \llcorner (J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'')$, passing to the supremum with respect to all compact sets $K \subset J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z''$, we get that

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \frac{(1-\delta)\kappa\sin\theta_0}{\sqrt{1+4\eta^2}} \mathcal{H}^1(J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'').$$

Remembering that $J_{\overline{u}} \cap B = B^\nu$, we have

$$2 = \mathcal{H}^1(J_{\overline{u}} \cap B) = \mathcal{H}^1(J_{\overline{u}} \cap B_{1-\frac{\eta}{2}}) + \eta \leq \mathcal{H}^1(J_{\overline{u}} \cap B_{1-\frac{\eta}{2}} \setminus Z'') + 4\eta$$

because $\mathcal{H}^1(Z'') \leq 3\eta$. Hence

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \frac{(1-\delta)\kappa\sin\theta_0}{\sqrt{1+4\eta^2}}(2-4\eta).$$

Finally passing to the limit as $\eta \to 0$ and $\delta \to 0$, we deduce that

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \kappa\sin\theta_0,$$

which corresponds to the desired lower bound with the correct multiplicative constant. □

2.2.3 . The upper bound

The proof of the Γ-$\limsup$ inequality relies on suitable approximation results in $GSBD$ (see [32, 75, 36, 46]) which allow us to reduce to the case where the jump set of u is a finite union of pairwise disjoint closed line segments, and u is smooth outside the jump set. Then, an explicit mesh construction introduced in [39], adapted to this simple geometrical situation, provides the desired upper bound.

Proposition 2.2.11. *For all $u \in L^0(\Omega;\mathbb{R}^2)$,*

$$\mathcal{F}''(u) \le \mathcal{F}(u).$$

Proof. We can assume that $\mathcal{F}(u) < \infty$, and thus that $u \in GSBD^2(\Omega)$. Using the density result for $GSBD$ functions (see [36, Theorem 1.1]) as well as the lower semicontinuity of $\mathcal{F}''$ with respect to the convergence in measure (see [47, Proposition 6.8]), we can further assume without loss of generality that $u \in SBV^2(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$.

Writing $u = (u_1, u_2)$, we can apply [39, Lemma 4.2] to both components u_1 and $u_2 \in SBV^2(\Omega) \cap L^\infty(\Omega)$ of u. For $\Omega' := (a,b) \times (c,d) \subset \mathbb{R}^2$ with $\Omega \subset\subset \Omega'$, we can find an extension $v \in SBV^2(\Omega';\mathbb{R}^2) \cap L^\infty(\Omega';\mathbb{R}^2)$ such that

$$v_{|\Omega} = u, \quad \|v\|_{L^\infty(\Omega';\mathbb{R}^2)} \le \sqrt{2}\|u\|_{L^\infty(\Omega;\mathbb{R}^2)} \quad \text{and} \quad \mathcal{H}^1(\partial\Omega \cap J_v) = 0. \tag{2.2.38}$$

Next owing to the density result in SBV (see [46, Theorem 3.1]), there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$ in $SBV^2(\Omega';\mathbb{R}^2) \cap L^\infty(\Omega';\mathbb{R}^2)$ as well as N_k disjoint closed segments $L_1^k, \dots, L_{N_k}^k \subset \overline{\Omega'}$ with the following properties :

$$\overline{J_{v_k}} = \bigcup_{i=1}^{N_k} L_i^k, \quad \mathcal{H}^1(\overline{J_{v_k}} \setminus J_{v_k}) = 0, \quad v_k \in W^{2,\infty}(\Omega' \setminus \overline{J_{v_k}};\mathbb{R}^2)$$

and

$$\begin{cases} v_k \to v \quad \text{strongly in } L^1(\Omega';\mathbb{R}^2), \\ \nabla v_k \to \nabla v \quad \text{strongly in } L^2(\Omega';\mathbb{M}^{2\times 2}), \\ \limsup_k \mathcal{H}^1(\overline{A} \cap J_{v_k}) \le \mathcal{H}^1(\overline{A} \cap J_v) \text{ for all open subset } A \subset\subset \Omega'. \end{cases} \tag{2.2.39}$$

Using (2.2.39) and the lower semicontinuity of $\mathcal{F}''$ in $L^0(\Omega;\mathbb{R}^2)$ with respect to the convergence in measure, we obtain that

$$\mathcal{F}''(u) \le \liminf_{k\to\infty} \mathcal{F}''(v_{k|\Omega}).$$

The proof is complete once we know that $\liminf_k \mathcal{F}''(v_{k|\Omega}) \le \mathcal{F}(u)$. This follows from Lemma 2.2.12 below, applied to each function v_k. Indeed, using that result, we get that

$$\liminf_{k\to\infty} \mathcal{F}''(v_{k|\Omega}) \le \liminf_{k\to\infty} \left\{ \int_\Omega \mathbf{A}e(v_k) : e(v_k)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_{v_k} \cap \overline{\Omega}) \right\}.$$

Recalling the convergences (2.2.39), we conclude that

$$\mathcal{F}''(u) \le \int_\Omega \mathbf{A}e(v) : e(v)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v \cap \overline{\Omega}) = \int_\Omega \mathbf{A}e(u) : e(u)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_u) = \mathcal{F}(u),$$

where we used $\mathcal{H}^1(J_v \cap \partial\Omega) = 0$ and that $v = u$ in Ω. □

We are back to establishing the following result.

Lemma 2.2.12. *Let $v \in SBV^2(\Omega';\mathbb{R}^2) \cap L^\infty(\Omega';\mathbb{R}^2)$ be such that*

$$\overline{J_v} = \bigcup_{i=1}^N L_i, \quad \mathcal{H}^1(\overline{J_v} \setminus J_v) = 0, \quad v \in W^{2,\infty}(\Omega' \setminus \overline{J_v};\mathbb{R}^2),$$

for some pairwise disjoint closed segments $L_1, \ldots, L_N \subset \overline{\Omega'}$. Then,

$$\mathcal{F}''(v_{|\Omega}) \leq \int_\Omega \mathbf{A}e(v) : e(v)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v \cap \overline{\Omega}).$$

Proof. Since $\Omega \subset\subset \Omega'$, then $d := \operatorname{dist}(\Omega, \mathbb{R}^2 \setminus \Omega') > 0$. For all $\delta \in (0,d)$, let us consider the open sets

$$\Omega_\delta := \{x \in \Omega' :\ \operatorname{dist}(x, \mathbb{R}^2 \setminus \Omega') > \delta\}$$

which satisfy $\Omega \subset\subset \Omega_\delta \subset\subset \Omega'$. We introduce a cut-off function $\phi_\delta \in C_c^\infty(\mathbb{R}^2;[0,1])$ which is supported in Ω' and such that $\phi_\delta = 1$ in Ω_δ, $\phi_\delta = 0$ in $\mathbb{R}^2 \setminus \Omega_{\frac{\delta}{2}}$. We next introduce the function $\overline{v} := \phi_\delta v \in SBV^2(\mathbb{R}^2;\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^2)$. We remark that

$$\overline{v} \in W^{2,\infty}\left(\mathbb{R}^2 \setminus \bigcup_{i=1}^N \overline{L_i};\mathbb{R}^2\right), \quad J_{\overline{v}} \subset J_v, \quad \text{and} \quad J_v \setminus J_{\overline{v}} \subset J_v \setminus \Omega_\delta. \tag{2.2.40}$$

Since $J_v \subset \overline{J_v}$ and $\mathcal{H}^1(\overline{J_v}) < \infty$, the disjoint closed segments $L_i \subset \overline{\Omega'}$ satisfy

$$\mathcal{H}^1(J_v) = \mathcal{H}^1(\overline{J_v}) = \sum_{i=1}^N \mathcal{H}^1(L_i).$$

Then according to [39, Appendix A], since θ_0 is smaller than or equal to $\Theta_0 := 45^\circ - \arctan(1/2)$, for all $\varepsilon > 0$ there exists an admissible triangulation $\mathbf{T}_\varepsilon \in \mathcal{T}_\varepsilon(\mathbb{R}^2, \omega, \theta_0)$ such that, setting $\mathbf{T}'_\varepsilon := \{T \in \mathbf{T}_\varepsilon : T \cap \bigcup_i L_i \neq \emptyset\}$,

- The vertices of $\mathbf{T}_\varepsilon$ are never situated on any L_i :

$$\text{for all } i \in \{1, \ldots, N\}, \quad L_i \cap \operatorname{Vertices}(\mathbf{T}_\varepsilon) = \emptyset,$$

- Using [39, Formula (4.9)],

$$\sum_{T \in \mathbf{T}'_\varepsilon} \frac{\mathcal{L}^2(T)}{\varepsilon} \to \sin\theta_0 \mathcal{H}^1(J_v). \tag{2.2.41}$$

We define the set $D_\varepsilon := \bigcup_{T \in \mathbf{T}'_\varepsilon} T$ and $\chi_\varepsilon := \mathbb{1}_{D_\varepsilon} \in L^\infty(\mathbb{R}^2;\{0,1\})$, while $\overline{v}_\varepsilon$ is the Lagrange interpolation of the values of $\overline{v}$ at the vertices of the triangulation $\mathbf{T}_\varepsilon$. Note that, if x_1, x_2 and x_3 are the vertices of $T \in \mathbf{T}_\varepsilon$, the values $\overline{v}(x_i)$ are well defined since, by construction of the triangulation $\mathbf{T}_\varepsilon$, the points x_1, x_2 and x_3 do not belong to $\bigcup_i L_i$. In particular, $\overline{v}_\varepsilon \in V_\varepsilon(\Omega', \omega, \theta_0)$ and

$$\begin{cases} \chi_\varepsilon \to 0 & \text{strongly in } L^1(\Omega'), \\ \overline{v}_\varepsilon \to \overline{v} & \text{strongly in } L^2(\Omega';\mathbb{R}^2), \\ e(\overline{v}_\varepsilon)\mathbb{1}_{\Omega' \setminus D_\varepsilon} \to e(\overline{v}) & \text{strongly in } L^2(\Omega';\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \end{cases} \tag{2.2.42}$$

Indeed, the first convergence is a consequence of (2.2.41) since

$$\|\chi_\varepsilon\|_{L^1(\Omega')} \leq \mathcal{L}^2(D_\varepsilon) = \sum_{T\in\mathbf{T}'_\varepsilon} \mathcal{L}^2(T) \to 0.$$

Next, noticing that every $T \in \mathbf{T}_\varepsilon \setminus \mathbf{T}'_\varepsilon$ is contained in $\mathbb{R}^2 \setminus \bigcup_i L_i$ and $\overline{v} \in W^{2,\infty}(\mathbb{R}^2 \setminus \bigcup_i L_i; \mathbb{R}^2)$, we infer that for all $\varepsilon > 0$ and $T \in \mathbf{T}_\varepsilon \setminus \mathbf{T}'_\varepsilon$,

$$\|\overline{v}_\varepsilon - \overline{v}\|_{H^1(T;\mathbb{R}^2)} \leq C\varepsilon \|D^2\overline{v}\|_{L^2(T)}, \tag{2.2.43}$$

for some constant $C = C(\theta_0) > 0$ depending only on θ_0 (see e.g. [44, Theorem 3.1.5]). On the one hand, since $\|\overline{v}_\varepsilon\|_{L^\infty(T;\mathbb{R}^2)} \leq \|\overline{v}\|_{L^\infty(T;\mathbb{R}^2)}$ for all $T \in \mathbf{T}_\varepsilon$, we get that

$$\|\overline{v}_\varepsilon - \overline{v}\|^2_{L^2(\Omega';\mathbb{R}^2)} \leq 2\|\overline{v}\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \sum_{T\in\mathbf{T}'_\varepsilon} \mathcal{L}^2(T) + \sum_{T\in\mathbf{T}_\varepsilon\setminus\mathbf{T}'_\varepsilon,\, T\cap\Omega'\neq\emptyset} \int_T |\overline{v}_\varepsilon - \overline{v}|^2\, dx.$$

Then, using (2.2.43) yields

$$\begin{aligned}\|\overline{v}_\varepsilon - \overline{v}\|^2_{L^2(\Omega';\mathbb{R}^2)} &\leq 2\|\overline{v}\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\mathcal{L}^2(D_\varepsilon) + C^2\varepsilon^2 \sum_{T\in\mathbf{T}_\varepsilon\setminus\mathbf{T}'_\varepsilon,\, T\cap\Omega'\neq\emptyset} \|D^2\overline{v}\|^2_{L^2(T)} \\ &\leq 2\|\overline{v}\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\mathcal{L}^2(D_\varepsilon) + C^2\varepsilon^2\|D^2\overline{v}\|^2_{L^2(\mathbb{R}^2\setminus\bigcup_i L_i)} \to 0,\end{aligned}$$

leading to the second convergence in (2.2.42). Note in particular that $\overline{v}_\varepsilon$ converges in measure to $\overline{v}$ in Ω'. On the other hand, writing

$$\|e(\overline{v}_\varepsilon)\mathbb{1}_{\Omega'\setminus D_\varepsilon} - e(\overline{v})\|^2_{L^2(\Omega';\mathbb{M}^{2\times2}_{\rm sym})} = \int_{\Omega'\cap D_\varepsilon} |e(\overline{v})|^2\, dx + \int_{\Omega'\setminus D_\varepsilon} |e(\overline{v}_\varepsilon) - e(\overline{v})|^2\, dx,$$

and using that $\mathcal{L}^2(D_\varepsilon) \to 0$, that $\nabla\overline{v} \in L^2(\mathbb{R}^2; \mathbb{M}^{2\times2})$ (because $\overline{v} \in SBV^2(\mathbb{R}^2;\mathbb{R}^2)$) as well as (2.2.43), we get that

$$\|e(\overline{v}_\varepsilon)\mathbb{1}_{\Omega'\setminus D_\varepsilon} - e(\overline{v})\|^2_{L^2(\Omega';\mathbb{M}^{2\times2}_{\rm sym})} \leq \int_{\Omega'\cap D_\varepsilon} |e(\overline{v})|^2\, dx + C^2\varepsilon^2\|D^2\overline{v}\|^2_{L^2(\mathbb{R}^2\setminus\bigcup_i L_i)} \to 0,$$

which implies the third convergence in (2.2.42).

We next show that

$$\mathcal{F}''(v_{|\Omega}) \leq \int_\Omega \mathbf{A}e(v) : e(v)\, dx + \kappa\sin\theta_0 \left(\mathcal{H}^1(J_v \cap \overline{\Omega}) + \mathcal{H}^1(J_v \setminus \Omega_\delta)\right). \tag{2.2.44}$$

Indeed, as $f \leq \kappa$ thanks to the growth properties (2.1.4), we get

$$\int_{\Omega'} \frac{1}{\varepsilon} f\left(\varepsilon\mathbf{A}e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)\right)\, dx \leq \sum_{T\in\mathbf{T}_\varepsilon\setminus\mathbf{T}'_\varepsilon,\, T\cap\Omega'\neq\emptyset} \mathcal{L}^2(T\cap\Omega')\frac{1}{\varepsilon} f\left(\varepsilon\mathbf{A}e(\overline{v}_\varepsilon)_{|T} : e(\overline{v}_\varepsilon)_{|T}\right) + \frac{\kappa}{\varepsilon}\sum_{T\in\mathbf{T}'_\varepsilon} \mathcal{L}^2(T).$$

On the one hand, (2.2.41) implies that

$$\frac{\kappa}{\varepsilon}\sum_{T\in\mathbf{T}'_\varepsilon} \mathcal{L}^2(T) \to \kappa\sin\theta_0\mathcal{H}^1(J_v). \tag{2.2.45}$$

On the other hand, since every triangle $T \in \mathbf{T}_\varepsilon \setminus \mathbf{T}'_\varepsilon$ is contained in $\mathbb{R}^2 \setminus \bigcup_{i=1}^N L_i$, then

$$\left| \nabla \overline{v}_{\varepsilon|T} \frac{x_i - x_j}{|x_i - x_j|} \right| = \frac{|\overline{v}(x_i) - \overline{v}(x_j)|}{|x_i - x_j|} \leq \|\nabla \overline{v}\|_{L^\infty(\mathbb{R}^2 \setminus \bigcup_i L_i; \mathbb{M}^{2\times 2})},$$

where x_1, x_2 and x_3 are the vertices of T. Hence, applying [39, Remark 3.5], it results that

$$\|e(\overline{v}_\varepsilon)\|_{L^\infty(\mathbb{R}^2 \setminus D_\varepsilon; \mathbb{M}^{2\times 2}_{\rm sym})} \leq \frac{\sqrt{5}}{\sin\theta_0} \|\nabla \overline{v}\|_{L^\infty(\mathbb{R}^2 \setminus \bigcup_i L_i; \mathbb{M}^{2\times 2})} =: K < \infty. \tag{2.2.46}$$

Therefore, setting

$$\delta_\varepsilon := \sup_{0 < t < \varepsilon \beta K^2} \frac{f(t)}{t},$$

we deduce, using $f(0) = 0$ and the property (2.1.5) of $\mathbf{A}$, that

$$\frac{1}{\varepsilon} f\left(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)(1 - \chi_\varepsilon)\right) \leq \delta_\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)(1 - \chi_\varepsilon) \quad \text{in } \Omega'$$

From the properties (2.1.4) of f, we infer that $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$. Hence, using the third convergence in (2.2.42), it ensures that

$$\begin{aligned}
\sum_{T \in \mathbf{T}_\varepsilon \setminus \mathbf{T}'_\varepsilon,\, T \cap \Omega' \neq \emptyset} \mathcal{L}^2(T \cap \Omega') \frac{1}{\varepsilon} f\left(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon)_{|T} : e(\overline{v}_\varepsilon)_{|T}\right) &= \int_{\Omega'} \frac{1}{\varepsilon} f\left(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)(1 - \chi_\varepsilon)\right) dx \\
&\leq \delta_\varepsilon \int_{\Omega'} \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon) \mathbb{1}_{\Omega' \setminus D_\varepsilon} \, dx \\
&\to \int_{\Omega'} \mathbf{A} e(\overline{v}) : e(\overline{v}) \, dx.
\end{aligned} \tag{2.2.47}$$

Gathering (2.2.45) and (2.2.47), we obtain that

$$\limsup_{\varepsilon \to 0^+} \int_{\Omega'} \frac{1}{\varepsilon} f\left(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)\right) dx \leq \int_{\Omega'} \mathbf{A} e(\overline{v}) : e(\overline{v}) \, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v). \tag{2.2.48}$$

Besides, after decomposing the above integral over $\Omega' \setminus \overline{\Omega}$ and $\overline{\Omega}$, we can apply the lower bound estimate of Propostion 2.2.3 to the open bounded set with Lipschitz boundary $\Omega' \setminus \overline{\Omega}$ (for which $\mathbf{T}_\varepsilon$ is also an admissible triangulation, $\overline{v}_{\varepsilon|\Omega' \setminus \overline{\Omega}} \in V_\varepsilon(\Omega' \setminus \overline{\Omega})$ and $\overline{v}_\varepsilon$ converges in measure to $\overline{v}$ in $\Omega' \setminus \overline{\Omega}$), which leads to

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \int_{\Omega'} \frac{1}{\varepsilon} f(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)) \, dx \\
&\quad \geq \limsup_{\varepsilon \to 0} \int_{\Omega} \frac{1}{\varepsilon} f(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)) \, dx + \liminf_{\varepsilon \to 0} \int_{\Omega' \setminus \overline{\Omega}} \frac{1}{\varepsilon} f(\varepsilon \mathbf{A} e(\overline{v}_\varepsilon) : e(\overline{v}_\varepsilon)) \, dx \\
&\qquad \geq \mathcal{F}''(\overline{v}_{|\Omega}) + \int_{\Omega' \setminus \overline{\Omega}} \mathbf{A} e(\overline{v}) : e(\overline{v}) \, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_{\overline{v}} \cap \Omega' \setminus \overline{\Omega}).
\end{aligned}$$

Gathering (2.2.48) and (2.2.40), as by construction $\overline{v}_{|\Omega} = v_{|\Omega}$, we deduce that

$$\begin{aligned}
\mathcal{F}''(v_{|\Omega}) &\leq \int_{\Omega} \mathbf{A} e(v) : e(v) \, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v \setminus J_{\overline{v}}) + \kappa \sin\theta_0 \mathcal{H}^1(J_v \cap \overline{\Omega}) \\
&\leq \int_{\Omega} \mathbf{A} e(v) : e(v) \, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v \setminus \Omega_\delta) + \kappa \sin\theta_0 \mathcal{H}^1(J_v \cap \overline{\Omega}),
\end{aligned}$$

which settles (2.2.44). Passing to the limit as $\delta \searrow 0^+$ thanks to the monotone convergence Theorem, we obtain that $\mathcal{H}^1(J_v \setminus \Omega_\delta) \to \mathcal{H}^1(J_v \setminus \Omega') = 0$, hence

$$\mathcal{F}''(v_{|\Omega}) \leq \int_\Omega \mathbf{A}e(v) : e(v)\,dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v \cap \overline{\Omega}),$$

which completes the proof of Lemma 2.2.12. □

2.3 . Convergence of minimizers

In order to investigate the approximation of minimizers for the Griffith energy, it is natural to impose boundary conditions to avoid trivial minimizers such as rigid displacements. This section is devoted to an approximation of the Griffith functional under a Dirichlet boundary condition by means of brittle damage energies.

2.3.1 . Griffith energy with Dirichlet boundary condition

In order to formulate a Dirichlet boundary condition, we need to consider a larger bounded Lipschitz open set Ω' such that $\overline{\Omega} \subset \Omega'$. Let $w \in W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)$ be a prescribed boundary displacement. Given an admissible triangulation $\mathbf{T}_\varepsilon \in \mathcal{T}_\varepsilon(\Omega')$ of Ω', we define $w_{\mathbf{T}_\varepsilon}$ as the piecewise affine Lagrange interpolation of w on $\mathbf{T}_\varepsilon$. Note that by standard finite element estimates (see [44, Theorem 3.1.5]),

$$w_{\mathbf{T}_\varepsilon} \in V_\varepsilon(\Omega'), \quad w_{\mathbf{T}_\varepsilon} \to w \text{ strongly in } H^1(\Omega';\mathbb{R}^2) \quad \text{and} \quad \sup_{\varepsilon>0} \mathcal{F}_\varepsilon(w_{\mathbf{T}_\varepsilon}) < +\infty. \tag{2.3.1}$$

We define $V_\varepsilon^{\mathrm{Dir}}(\Omega')$ to be the set of all continuous functions $u : \Omega' \to \mathbb{R}^2$ for which there exists a triangulation $\mathbf{T}_\varepsilon \in \mathcal{T}_\varepsilon(\Omega')$ so that u is affine on each triangle $T \in \mathbf{T}_\varepsilon$ and $u = w_{\mathbf{T}_\varepsilon}$ on each triangle $T \in \mathbf{T}_\varepsilon$ such that $T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$. We consider the following discrete functionals

$$\mathcal{G}_\varepsilon : u \in L^0(\Omega';\mathbb{R}^2) \mapsto \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f\big(\varepsilon \mathbf{A}e(u) : e(u)\big)\,dx & \text{if } u \in V_\varepsilon^{\mathrm{Dir}}(\Omega'), \\ +\infty & \text{otherwise.} \end{cases}$$

The Griffith energy with Dirichlet boundary condition w is defined, for $u \in L^0(\Omega';\mathbb{R}^2)$, by

$$\mathcal{G}(u) := \begin{cases} \displaystyle\int_\Omega \mathbf{A}e(u) : e(u)\,dx & \\ \quad +\kappa \sin\theta_0 \left[\mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega \cap \{u \neq w\})\right] & \text{if } \begin{cases} u \in GSBD^2(\Omega'), \\ u = w \ \mathcal{L}^2\text{-a.e. in } \Omega' \setminus \overline{\Omega}, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that the additional boundary term accounts for possible jumps at the boundary, where the boundary condition fails to be satisfied. In the previous expression and in the sequel, we still denote by u the trace of $u_{|\Omega} \in GSBD^2(\Omega)$ on $\partial\Omega$ (see [48, Theorem 5.5]).

We will first prove the following result, generalizing Theorem 2.1.3 to the case of Dirichlet boundary conditions.

Theorem 2.3.1 (**Γ-convergence under Dirichlet boundary conditions**)**.** *The family $\{\mathcal{G}_\varepsilon\}_{\varepsilon>0}$ Γ-converges, with respect to the $L^0(\Omega';\mathbb{R}^2)$-topology, to the Griffith functional $\mathcal{G}$.*

Next, we will show a compactness result for sequences of dispacements u_ε with uniformly bounded energy, with respect to the $L^0(\Omega';\mathbb{R}^2)$-topology of convergence in measure, under the simplifying assumption that $f:[0,+\infty)\to[0,+\infty)$ reduces to

$$f(t)=\kappa\wedge t$$

for $t\in\mathbb{R}^+$. Considering eventually a sequence of minimizers of $\mathcal{G}_\varepsilon$ (see Lemma 2.3.13), we will show that, up to a subsequence and up to subtracting a sequence of piecewise rigid body motions, it converges in measure in Ω' to a minimizer of $\mathcal{G}$ and the minimal value of $\mathcal{G}_\varepsilon$ converges to the minimal value of $\mathcal{G}$. In other words, we obtain the fundamental theorem of Γ-convergence in our specific context.

Corollary 2.3.2 (**Convergence of minimizers).** *Assume further that Ω and Ω' are connected. For each $\varepsilon>0$ small, let $u_\varepsilon\in V_\varepsilon^{\rm Dir}(\Omega')$ be a minimizer of $\mathcal{G}_\varepsilon$. Then, there exist a subsequence (not relabeled), a sequence of piecewise rigid body motions $\{r_\varepsilon\}_{\varepsilon>0}$ and a function $u\in GSBD^2(\Omega')$ with $u=w$ $\mathcal{L}^2$-a.e. in $\Omega'\setminus\overline{\Omega}$, such that $u_\varepsilon-r_\varepsilon\to u$ in measure in Ω, $\mathcal{G}_\varepsilon(u_\varepsilon)\to\mathcal{G}(u)$ and u is a minimizer of $\mathcal{G}$.*

Remark 2.3.3. Let us clarify that the improved lower bound (2.3.6) in Proposition 2.3.5 is crucial only for the compactness and convergence of minimizing sequences, to ensure that after the removal of piecewise rigid body motions, minimizers of the approximating functionals converge to a minimizer of the Griffith functional and their energies converge as well. Instead, the Γ-convergence result under Dirichlet boundary conditions directly follows from Theorem 2.1.3 and Proposition 2.2.1. Indeed, the proof of the lower bound in Theorem 2.3.1 is the consequence of the lower bound in Theorem 2.1.3 applied in Ω' together with the identification of the volume terms in $\Omega'\setminus\overline{\Omega}$.

2.3.2 . Γ-limit under Dirichlet boundary conditions

Let us introduce the Γ-lower and upper limits $\mathcal{G}'$ and $\mathcal{G}''$ defined, for all $u\in L^0(\Omega';\mathbb{R}^2)$, by

$$\mathcal{G}'(u):=\inf\left\{\liminf_{\varepsilon\to0}\mathcal{G}_\varepsilon(u_\varepsilon):\ u_\varepsilon\to u \text{ in measure in } \Omega'\right\},$$

and

$$\mathcal{G}''(u):=\inf\left\{\limsup_{\varepsilon\to0}\mathcal{G}_\varepsilon(u_\varepsilon):\ u_\varepsilon\to u \text{ in measure in } \Omega'\right\}.$$

Proof of Theorem 2.3.1. **Lower bound.** Let $u\in L^0(\Omega';\mathbb{R}^2)$. Without loss of generality, we can assume that $\mathcal{G}'(u)<+\infty$. For any $\zeta>0$, there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon\to u$ in measure in Ω' and

$$\liminf_{\varepsilon\to0}\mathcal{G}_\varepsilon(u_\varepsilon)\le\mathcal{G}'(u)+\zeta<+\infty.$$

Let us extract a subsequence $\{u_k\}_{k\in\mathbb{N}}:=\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ such that $u_k\to u$ $\mathcal{L}^2$-a.e. in Ω' and

$$\lim_{k\to\infty}\mathcal{G}_{\varepsilon_k}(u_k)=\liminf_{\varepsilon\to0}\mathcal{G}_\varepsilon(u_\varepsilon)<+\infty.$$

This implies that, for k large enough, $u_k\in V_{\varepsilon_k}^{\rm Dir}(\Omega')$ and $\sup_k\mathcal{G}_{\varepsilon_k}(u_k)<+\infty$. Especially, we get that $u_k\in V_{\varepsilon_k}(\Omega')$. Hence, according to Proposition 2.2.1 and Theorem 2.1.3 applied in Ω', we infer that

$$u\in GSBD^2(\Omega')$$

and

$$\liminf_{k\to\infty}\int_{\Omega'}\frac{1}{\varepsilon_k}f\big(\varepsilon_k\mathbf{A}e(u_k):e(u_k)\big)\,dx\geq\int_{\Omega'}\mathbf{A}e(u):e(u)\,dx+\kappa\sin\theta_0\mathcal{H}^1(J_u).$$

Setting $w_k := w_{\mathbf{T}_{\varepsilon_k}}$ and using (2.3.1) together with the convergence in measure of $u_k = w_k$ to u in $\Omega'\setminus\overline{\Omega}$, we get that $u = w$ $\mathcal{L}^2$-a.e. in $\Omega'\setminus\overline{\Omega}$. Setting now $\delta_k := \sup\left\{f(t)/t\,:\,0<t<\varepsilon_k\beta K^2\right\}$ where

$$K=\frac{\sqrt{5}}{\sin\theta_0}\|\nabla w\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}<\infty,$$

one can check that δ_k converges to 1 as $k\to\infty$ according to (2.1.4) and

$$\int_{\Omega'}\frac{1}{\varepsilon_k}f\big(\varepsilon_k\mathbf{A}e(u_k):e(u_k)\big)\,dx\leq\mathcal{G}_{\varepsilon_k}(u_k)+\int_{\Omega'\setminus\overline{\Omega}}\delta_k\,\mathbf{A}e(w_k):e(w_k)\,dx.$$

Hence, the Dominated Convergence Theorem ensures that

$$\zeta+\mathcal{G}'(u)+\int_{\Omega'\setminus\overline{\Omega}}\mathbf{A}e(w):e(w)\,dx\geq\int_{\Omega'}\mathbf{A}e(u):e(u)\,dx+\kappa\sin\theta_0\mathcal{H}^1(J_u).$$

Recalling that $J_u\cap\partial\Omega=\{u\neq w\}\cap\partial\Omega$ and $J_u\cap(\Omega'\setminus\overline{\Omega})=J_w\cap(\Omega'\setminus\overline{\Omega})=\emptyset$, it entails that $\zeta+\mathcal{G}'(u)\geq\mathcal{G}(u)$, and the conclusion follows letting $\zeta\searrow 0$.

Upper bound. Let $u\in L^0(\Omega';\mathbb{R}^2)$. We can assume that $\mathcal{G}(u)<+\infty$ so that $u\in GSBD^2(\Omega')$ and $u=w$ $\mathcal{L}^2$-a.e. in $\Omega'\setminus\overline{\Omega}$. According to the density results for $GSBD$ functions (see [36, Theorem 1.1] and [36, Formula (5.11)]), there exists a sequence of functions $u_n\in SBV^2(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$ such that

$$\begin{cases}u_n\to u \text{ in measure in } \Omega,\\ u_n=w \text{ in an open bounded neighborhood of } \partial\Omega,\\ \limsup_n\mathcal{G}(u_n)\leq\mathcal{G}(u).\end{cases}\tag{2.3.2}$$

Extending (continuously) u_n by w on $\Omega'\setminus\overline{\Omega}$, we get that $u_n\to u$ in measure in Ω'. The proof is thus complete once we know that $\mathcal{G}''(u_n)\leq\mathcal{G}(u_n)$, for all $n\in\mathbb{N}$, as it would imply $\mathcal{G}''(u)\leq\mathcal{G}(u)$, using the lower semicontinuity of $\mathcal{G}''$ in $L^0(\Omega';\mathbb{R}^2)$ with respect to the convergence in measure together with the last point of (2.3.2).

Therefore, we can assume without loss of generality that $u\in SBV^2(\Omega';\mathbb{R}^2)\cap L^\infty(\Omega';\mathbb{R}^2)$ and $u=w$ on $V\cup(\Omega'\setminus\overline{\Omega})$ with V an open bounded neighborhood of $\partial\Omega$. Next according to Proposition 5.0.1 (see Appendix) there exist a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $SBV^2(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$ as well as N_k disjoint closed segments $L_1^k,\dots,L_{N_k}^k\subset\Omega$ satisfying :

$$\overline{J_{u_k}}=\bigcup_{i=1}^{N_k}L_i^k,\quad\mathcal{H}^1(\overline{J_{u_k}}\setminus J_{u_k})=0,\quad u_k\in W^{2,\infty}(\Omega\setminus\overline{J_{u_k}};\mathbb{R}^2),$$

$$\begin{cases}u_k=w \text{ in an open neighborhood of } \partial\Omega,\\ u_k\to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2),\\ \nabla u_k\to\nabla u \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{2\times 2}),\\ \limsup_{k\to\infty}\mathcal{H}^1(J_{u_k})\leq\mathcal{H}^1(J_u).\end{cases}\tag{2.3.3}$$

Thus, extending continuously u_k to Ω' by setting $u_k = w$ on $\Omega' \setminus \Omega$ and using again the lower semicontinuity of $\mathcal{G}''$ in $L^0(\Omega';\mathbb{R}^2)$ with respect to the convergence in measure together with the convergences in (2.3.3), we are back to establishing that $\mathcal{G}''(u_k) \leq \mathcal{G}(u_k)$ as the conclusion follows letting $k \to +\infty$. Arguing almost word for word as in the proof of Lemma 2.2.12, we show the following Lemma 2.3.4, which leads to the desired inequality. □

Lemma 2.3.4. *Let $v \in SBV^2(\Omega';\mathbb{R}^2) \cap L^\infty(\Omega';\mathbb{R}^2)$ and W be a bounded open neighborhood of $\partial\Omega$ be such that $v = w$ on $W \cup (\Omega' \setminus \Omega)$ and*

$$\Omega \setminus W \supset \overline{J_v} = \bigcup_{i=1}^N L_i, \quad \mathcal{H}^1(\overline{J_v} \setminus J_v) = 0, \quad v \in W^{2,\infty}(\Omega' \setminus \overline{J_v};\mathbb{R}^2),$$

for some pairwise disjoint closed segments $L_1, \ldots, L_N \subset \Omega \setminus W$. Then,

$$\mathcal{G}''(v) \leq \int_\Omega \mathbf{A}e(v) : e(v)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_v) = \mathcal{G}(v).$$

We do not detail the proof of Lemma 2.3.4. We only stress that, following Lemma 2.2.12, for $\varepsilon > 0$ small enough, if $\mathbf{T}_\varepsilon$ is the admissible triangulation given by [39, Appendix A] and v_ε is the Lagrange interpolation of the values of v at the vertices of $\mathbf{T}_\varepsilon$, each triangle $T \in \mathbf{T}_\varepsilon$ such that $T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$ is contained in $W \cup \mathbb{R}^2 \setminus \Omega$, so that $v_\varepsilon = w_{\mathbf{T}_\varepsilon}$ on T. In particular, it ensures that $v_\varepsilon \in V_\varepsilon^{\mathrm{Dir}}(\Omega')$.

2.3.3 . Compactness for sequences with uniformly bounded energy and convergence of minimizers

In this paragraph, the density f reduces to $f(t) = \kappa \wedge t$ for $t \in \mathbb{R}$, so that the energy $\mathcal{G}_\varepsilon$ corresponds to

$$\mathcal{G}_\varepsilon(u) = \int_\Omega \frac{\kappa}{\varepsilon} \wedge \mathbf{A}e(u) : e(u)\, dx \quad \text{for } u \in V_\varepsilon^{\mathrm{Dir}}(\Omega').$$

The reason of this simplifying assumption on f comes from the difficulty to obtain compactness for sequences with uniformly bounded energies and from the difficulty to prove the existence of minimizers, as it will be detailed below. This is however a meaningful case since it corresponds to a brittle damage type energy from the mechanical point of view.‘

The following result shows a compactness and lower bound estimate for any sequence with uniformly bounded energy.

Proposition 2.3.5. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \to 0$ and let $\{u_k\}_{k\in\mathbb{N}} \subset L^0(\Omega';\mathbb{R}^2)$ be such that $M := \sup_k \mathcal{G}_{\varepsilon_k}(u_k) < \infty$. Then there exist a subsequence (not relabeled), a Caccioppoli partition $\mathcal{P} = \{P_j\}_{j\in\mathbb{N}}$ of Ω', a sequence of piecewise rigid motions $\{r_k\}_{k\in\mathbb{N}}$ with*

$$r_k := \sum_{j\in\mathbb{N}} r_k^j \mathbb{1}_{P_j},$$

and a function $u \in GSBD^2(\Omega')$ such that $u = w$ $\mathcal{L}^2$-a.e. in $\Omega' \setminus \overline{\Omega}$,

$$|r_k^i(x) - r_k^j(x)| \to +\infty \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega', \quad \text{for all } i \neq j, \tag{2.3.4}$$

$$u_k - r_k \to u \quad \textit{in measure in } \Omega', \tag{2.3.5}$$

and

$$\liminf_{k\to\infty} \int_\Omega \frac{\kappa}{\varepsilon_k} \wedge \mathbf{A}e(u_k) : e(u_k)\, dx \geq \int_\Omega \mathbf{A}e(u) : e(u)\, dx + \kappa \sin\theta_0 \mathcal{H}^1(J_u \cup \partial^* \mathcal{P}). \tag{2.3.6}$$

Remark 2.3.6. The lower bound inequality (2.3.6) strongly relies on the simplifying assumption

$$f(t) = \kappa \wedge t \quad \text{for } t \in \mathbb{R}.$$

Indeed, when working with a more general density $f : [0, +\infty) \to [0, +\infty)$ satisfying (2.1.4), the main issue arises when one needs to fix some $\delta > 0$ to use (2.2.1), in order to exhibit an extraction, a Caccioppoli partition, rigid motions and a limit displacement which satisfy (2.3.4) and (2.3.5). As all of them depend on $\delta > 0$, it becomes difficult to derive the lower bound, even for the Lebesgue part (2.3.9) below, since one simultaneously needs δ to be fixed (so that $\mathcal{P}$, $\{r_k\}_{k\in\mathbb{N}}$ and u are well defined) and to converge to 0 (in order to recover (2.3.9) as in the proof of Proposition 2.2.4).

Proof. By definition of $V^{\mathrm{Dir}}_{\varepsilon_k}(\Omega')$, there exists an admissible triangulation $\mathbf{T}_k \in \mathcal{T}_{\varepsilon_k}(\Omega')$ such that u_k is affine on each triangle $T \in \mathbf{T}_k$ and $u_k = w_{\mathbf{T}_k}$ on each triangle $T \in \mathbf{T}_k$ intersecting $\Omega' \setminus \overline{\Omega}$. We introduce the characteristic functions

$$\chi_k := \mathbb{1}_{\{\mathbf{A}e(u_k):e(u_k) \geq \frac{\kappa}{\varepsilon_k}\}} \in L^\infty(\Omega'; \{0,1\})$$

which are constant on each triangle $T \in \mathbf{T}_k$. Since $u_k = w_{\mathbf{T}_k}$ on each triangle $T \in \mathbf{T}_k$ intersecting $\Omega' \setminus \overline{\Omega}$ and $w \in W^{2,\infty}(\mathbb{R}^2; \mathbb{R}^2)$, we deduce that, for k large enough, $\chi_k = 0$ in $\Omega' \setminus \overline{\Omega}$. Thus

$$D_k := \{\chi_k = 1\} = \bigcup_{i=1}^{N_k} T_i^k \subset \overline{\Omega}$$

for some triangles $T_i^k \in \mathbf{T}_k$, and $\mathcal{L}^2(D_k) = \int_\Omega \chi_k\, dx \to 0$.

Let $v_k := (1-\chi_k)u_k \in SBV^2(\Omega'; \mathbb{R}^2)$ with $\nabla v_k = (1-\chi_k)\nabla u_k$ and $J_{v_k} \subset \bigcup_{i=1}^{N_k} \partial T_i^k \subset \overline{\Omega}$. Arguing as in the proof of Proposition 2.2.1, we infer that

$$\sup_{k\in\mathbb{N}} \left\{ \int_{\Omega'} |e(v_k)|^2\, dx + \mathcal{H}^1(J_{v_k}) \right\} < \infty.$$

In view of the $GSBD^2$-compactness Theorem ([37, Theorem 1.1]), there exist a subsequence (not relabeled), a Caccioppoli partition $\mathcal{P} = \{P_j\}_{j\in\mathbb{N}}$ of Ω', a sequence of piecewise rigid motions $\{\tilde{r}_k\}_{k\in\mathbb{N}}$ with

$$\tilde{r}_k := \sum_{j\in\mathbb{N}} \tilde{r}_k^j\, \mathbb{1}_{P_j},$$

and a function $\tilde{u} \in GSBD^2(\Omega')$ such that

$$\begin{cases} |\tilde{r}_k^i(x) - \tilde{r}_k^j(x)| \to +\infty & \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega', \text{ for all } i \neq j, \\ v_k - \tilde{r}_k \to \tilde{u} & \text{in measure in } \Omega', \\ e(v_k) \rightharpoonup e(\tilde{u}) & \text{weakly in } L^2(\Omega'; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \end{cases}$$

Since $\mathcal{L}^2(D_k) \to 0$, we deduce that $u_k - \tilde{r}_k \to \tilde{u}$ in measure in Ω'.

For all $j \in \mathbb{N}$ such that $\mathcal{L}^2(P_j \cap \Omega' \setminus \overline{\Omega}) > 0$, the convergence in measure of $u_k - \tilde{r}_k^j$ to $\tilde{u}$ together with the convergence in measure of u_k to w in $P_j \cap \Omega' \setminus \overline{\Omega} =: V_j$ ensure that $\tilde{r}_k^j \to w - \tilde{u}$ in measure in V_j. Since the space of rigid body motions is a closed finite dimensional subspace of $L^0(\Omega'; \mathbb{R}^2)$, we can find a rigid body motion r^j such that $r^j_{|V_j} = w - \tilde{u}$ $\mathcal{L}^2$-a.e. in V_j. Therefore, with

$$r := \sum_{j \in \mathbb{N},\, \mathcal{L}^2(P_j \cap \Omega' \setminus \overline{\Omega}) > 0} r^j \, \mathbb{1}_{P_j},$$

the piecewise rigid body motion $r_k := \tilde{r}_k - r$ and the function $u = \tilde{u} + r \in GSBD^2(\Omega')$ are such that

$$\begin{cases} u_k - r_k \to u \text{ in measure in } \Omega', \\ u = w \quad \mathcal{L}^2\text{-a.e. in } \Omega' \setminus \overline{\Omega}, \\ e(v_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega'; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \end{cases} \tag{2.3.7}$$

We are now back to prove (2.3.6). As in the proof of Proposition 2.2.3, we define the following Radon measures on Ω'

$$\lambda_k := \frac{\kappa}{\varepsilon_k} \wedge \mathbf{A} e(u_k) : e(u_k) \, \mathcal{L}^2 \llcorner \Omega' \in \mathcal{M}(\Omega').$$

Using (2.3.1) and the energy bound assumption on u_k, we obtain that the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $\mathcal{M}(\Omega')$. Thus, up to a subsequence (not relabeled), we have $\lambda_k \overset{*}{\rightharpoonup} \lambda$ weakly* in $\mathcal{M}(\Omega')$ for some nonnegative measure $\lambda \in \mathcal{M}(\Omega')$. Thanks to the lower semicontinuity of weak* convergence in $\mathcal{M}(\Omega')$ along open sets, we have that

$$\liminf_{k \to \infty} \int_{\Omega'} \frac{\kappa}{\varepsilon_k} \wedge \mathbf{A} e(u_k) : e(u_k) \, dx = \liminf_{k \to \infty} \lambda_k(\Omega') \geq \lambda(\Omega'). \tag{2.3.8}$$

Recalling that $\mathcal{P}^{(1)} \cup \partial^* \mathcal{P}$ contains $\mathcal{H}^1$-almost all of Ω', and using that the measures $\mathcal{L}^2 \llcorner \Omega'$, $\mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u)$ and $\mathcal{H}^1 \llcorner \partial^* \mathcal{P}$ are mutually singular, it is enough to show that

$$\frac{d\lambda}{d\mathcal{L}^2 \llcorner \Omega'} \geq \mathbf{A} e(u) : e(u) \quad \mathcal{L}^2\text{-a.e. in } \Omega', \tag{2.3.9}$$

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u)} \geq \kappa \sin \theta_0 \quad \mathcal{H}^1\text{-a.e. in } \mathcal{P}^{(1)} \cap J_u, \tag{2.3.10}$$

and

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^* \mathcal{P}} \geq \kappa \sin \theta_0 \quad \mathcal{H}^1\text{-a.e. in } \partial^* \mathcal{P}. \tag{2.3.11}$$

Indeed, once (2.3.9), (2.3.10) and (2.3.11) are satisfied, it follows from the Radon-Nikodým decomposition and the Besicovitch differentiation Theorems that

$$\lambda = \frac{d\lambda}{d\mathcal{L}^2 \llcorner \Omega'} \mathcal{L}^2 \llcorner \Omega' + \frac{d\lambda}{d\mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u)} \mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u) + \frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^* \mathcal{P}} \mathcal{H}^1 \llcorner \partial^* \mathcal{P} + \lambda^s,$$

for some nonnegative measure λ^s which is singular with respect to $\mathcal{L}^2 \llcorner \Omega'$, $\mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u)$ and $\mathcal{H}^1 \llcorner \partial^* \mathcal{P}$. Thus, after integration over Ω' and recalling (2.3.8), we would get that

$$\liminf_{k \to \infty} \int_{\Omega'} \frac{\kappa}{\varepsilon_k} \wedge \mathbf{A} e(u_k) : e(u_k) \, dx \geq \int_{\Omega'} \mathbf{A} e(u) : e(u) \, dx + \kappa \sin \theta_0 \mathcal{H}^1\big((J_u \cap \mathcal{P}^{(1)}) \cup \partial^* \mathcal{P}\big).$$

On the one hand, the convergence in $H^1(\Omega';\mathbb{R}^2)$ of $w_{\mathbf{T}_k}$ to w (see (2.3.1)) ensures that

$$\begin{aligned}\liminf_{k\to\infty}\int_{\Omega'}\frac{\kappa}{\varepsilon_k}\wedge\mathbf{A}e(u_k):e(u_k)\,dx &\le \limsup_{k\to\infty}\int_{\Omega'\setminus\Omega}\mathbf{A}e(w_{\mathbf{T}_k}):e(w_{\mathbf{T}_k})\,dx+\liminf_{k\to\infty}\mathcal{G}_{\varepsilon_k}(u_k)\\ &\le \int_{\Omega'\setminus\Omega}\mathbf{A}e(w):e(w)\,dx+\liminf_{k\to\infty}\mathcal{G}_{\varepsilon_k}(u_k). \qquad (2.3.12)\end{aligned}$$

On the other hand, using that $u = w$ in $\Omega'\setminus\overline{\Omega}$ and that $\mathcal{P}^{(1)}\cup\partial^*\mathcal{P}$ covers $\mathcal{H}^1$ almost every Ω', we obtain that

$$\begin{aligned}&\int_{\Omega'}\mathbf{A}e(u):e(u)\,dx+\kappa\sin\theta_0\mathcal{H}^1\big((J_u\cap\mathcal{P}^{(1)})\cup\partial^*\mathcal{P}\big)\\ &\qquad=\int_{\Omega'\setminus\Omega}\mathbf{A}e(w):e(w)\,dx+\int_{\Omega}\mathbf{A}e(u):e(u)\,dx+\kappa\sin\theta_0\mathcal{H}^1(J_u\cup\partial^*\mathcal{P}). \qquad (2.3.13)\end{aligned}$$

Gathering (2.3.12) and (2.3.13) leads to (2.3.6), which completes the proof of Proposition 2.3.5. □

Using the last convergence in (2.3.7), we easily get inequality (2.3.9) arguing in an identical manner than in the proof of Proposition 2.2.4. We do not reproduce the argument. The rest of this section is devoted to the establishment of (2.3.10) and (2.3.11). We start with the lower bound inequality for the jump part of the energy in the measure theoretic interior of $\mathcal{P}$.

Proposition 2.3.7 (**Lower bound for the jump part in $\mathcal{P}^{(1)}$).** *For $\mathcal{H}^1$-a.e. $x_0\in\mathcal{P}^{(1)}\cap J_u$,*

$$\frac{d\lambda}{d\mathcal{H}^1\llcorner(\mathcal{P}^{(1)}\cap J_u)}(x_0)\ge\kappa\sin\theta_0.$$

Proof. The proof is very similar to that of Proposition 2.2.5. We just sketch it, underlying the main differences.

Let $x_0\in\mathcal{P}^{(1)}\cap J_u$ be such that

$$\frac{d\lambda}{d\mathcal{H}^1\llcorner(\mathcal{P}^{(1)}\cap J_u)}(x_0)=\lim_{\varrho\searrow 0}\frac{\lambda\big(B_\varrho(x_0)\big)}{\mathcal{H}^1\big(\mathcal{P}^{(1)}\cap J_u\cap B_\varrho(x_0)\big)}$$

exists and is finite, and

$$\lim_{\varrho\searrow 0}\frac{\mathcal{H}^1(\mathcal{P}^{(1)}\cap J_u\cap B_\varrho(x_0))}{2\varrho}=1.$$

According to the Besicovitch differentiation Theorem and the countably $(\mathcal{H}^1,1)$-rectifiability of $\mathcal{P}^{(1)}\cap J_u$, it follows that $\mathcal{H}^1$-almost every point x_0 in $\mathcal{P}^{(1)}\cap J_u$ fulfills these conditions.

By definition of the jump set J_u, there exist $\nu:=\nu_u(x_0)\in\mathbb{S}^1$ and $u^\pm(x_0)\in\mathbb{R}^2$ with $u^+(x_0)\neq u^-(x_0)$ such that the function

$$u_{x_0,\varrho}:=u(x_0+\varrho\,\cdot)$$

converges in measure in $B:=B_1(0)$ to the jump function

$$\overline{u}:y\in B\mapsto\begin{cases}u^+(x_0) & \text{if } y\cdot\nu>0,\\ u^-(x_0) & \text{if } y\cdot\nu<0,\end{cases}$$

as $\varrho \searrow 0$. As before, we consider a sequence of radii $\{\varrho_j\}_{j\in\mathbb{N}}$ such that $\varrho_j \searrow 0$ and $\lambda(\partial B_{\varrho_j}(x_0)) = 0 = \mathcal{H}^1(\mathcal{P}^{(1)} \cap J_u \cap \partial B_{\varrho_j}(x_0))$ for all $j \in \mathbb{N}$. Arguing as in Proposition 2.2.5, there exists an increasing sequence $\{k_j\}_{j\in\mathbb{N}}$ such that $k_j \nearrow \infty$ as $j \to \infty$ and

$$\begin{cases} \big(u_{k_j} - r_{k_j}\big)(x_0 + \varrho_j \,\cdot) \to \overline{u} & \text{in measure in } B,\\ \dfrac{\lambda_{k_j}(B_{\varrho_j}(x_0))}{2\varrho_j} \to \dfrac{d\lambda}{d\mathcal{H}^1 \llcorner (\mathcal{P}^{(1)} \cap J_u)}(x_0), \\ \varepsilon_{k_j}/\varrho_j \to 0, \quad \omega(\varepsilon_{k_j})/\varrho_j \to 0. \end{cases}$$

By definition of $\mathcal{P}^{(1)}$, there exists $i_0 \in \mathbb{N}$ such that $x_0 \in {P_{i_0}}^{(1)}$. We thus infer that the function $v_j := \big(u_{k_j} - r^{i_0}_{k_j}\big)(x_0 + \varrho_j \,\cdot) \in H^1(B;\mathbb{R}^2)$ converges in measure to $\overline{u}$ in B. Indeed, for all $\eta > 0$,

$$\begin{aligned} &\mathcal{L}^2\big(B \cap \{|v_j - \overline{u}| > \eta\}\big) \\ &\qquad \le \mathcal{L}^2\big(B \cap \{|\big(u_{k_j} - r_{k_j}\big)(x_0 + \varrho_j \,\cdot) - \overline{u}| > \eta\}\big) + \mathcal{L}^2\left(B \setminus \left(\frac{P_{i_0} - x_0}{\varrho_j}\right)\right) \to 0, \end{aligned}$$

where we used that x_0 is a point of density 1 for P_{i_0}. We are now back to an analogous situation than (2.2.11), since v_j is continuous on B and piecewise affine on each triangle $T \in \big(\mathbf{T}_{k_j} - x_0\big)/\varrho_j$. Therefore, from here the conclusion of Proposition 2.3.7 results from the proof of Proposition 2.2.5. $\square$

We next pass to the lower bound inequality for the energy on the reduced boundary of $\mathcal{P}$, which presents some non trivial adaptations of the proof of Proposition 2.2.5.

Proposition 2.3.8 (Lower bound on the reduced boundary $\partial^*\mathcal{P}$). *For $\mathcal{H}^1$-a.e. $x_0 \in \partial^*\mathcal{P}$,*

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0) \ge \kappa \sin\theta_0.$$

The rest of this subsection is devoted to prove Proposition 2.3.8, with essentially the same structure than the proof of Proposition 2.2.5.

Blow-up. Let $x_0 \in \partial^*\mathcal{P}$ be such that

$$x_0 \in \partial^* P_{i_0} \cap \partial^* P_{j_0} \quad \text{for some } i_0 \neq j_0,$$

$$\nu := \nu_{P_{i_0}}(x_0) = -\nu_{P_{j_0}}(x_0) \quad \text{where } \nu_{P_k}(x_0) := \lim_{\varrho\searrow 0} \frac{D\mathbb{1}_{P_k}\big(B_\varrho(x_0)\big)}{|D\mathbb{1}_{P_k}|\big(B_\varrho(x_0)\big)} \quad \text{for } k \in \{i_0, j_0\},$$

$$\frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0) = \lim_{\varrho\searrow 0} \frac{\lambda\big(B_\varrho(x_0)\big)}{\mathcal{H}^1\big(\partial^*\mathcal{P} \cap B_\varrho(x_0)\big)} \quad \text{exists and is finite,}$$

$$\lim_{\varrho\searrow 0} \frac{\mathcal{H}^1(\partial^*\mathcal{P} \cap B_\varrho(x_0))}{2\varrho} = 1,$$

and there exist traces $u^\pm(x_0) \in \mathbb{R}^2$ such that the function

$$u_{x_0,\varrho} := u(x_0 + \varrho\,\cdot)$$

converges in measure in $B := B_1(0)$ to

$$y \in B \mapsto \overline{u}(y) := \begin{cases} u^+(x_0) & \text{if } y \cdot \nu > 0, \\ u^-(x_0) & \text{if } y \cdot \nu < 0, \end{cases} \qquad \text{as } \varrho \searrow 0.$$

The previous properties turn out to be satisfied for $\mathcal{H}^1$-a.e. $x_0 \in \partial^*\mathcal{P}$. This is a consequence of the countably $(\mathcal{H}^1, 1)$-rectifiability of that set, the Besicovitch differentiation Theorem, the fact that $\mathcal{P}^{(1)} \cup \bigcup_{i \neq j}(\partial^* P_i \cap \partial^* P_j)$ covers $\mathcal{H}^1$ almost all of Ω' ([6, Theorem 4.17]), and the existence of traces on $(\mathcal{H}^1, 1)$-rectifiable sets (see [48, Theorem 5.2] in the case of 1-dimensional C^1 submanifolds which may be extended to countably $(\mathcal{H}^1, 1)$-rectifiable sets arguing as in [10, Proposition 4.1]).

To simplify notation, let us denote by $P^+ := P_{i_0}$ and $P^- := P_{j_0}$. According to De Giorgi's Theorem (see [6, Theorem 3.59]) we infer that

$$\mathbb{1}_{\frac{P^\pm - x_0}{\varrho}} \to \mathbb{1}_{H^\pm} \text{ strongly in } L^1(B) \text{ as } \varrho \searrow 0, \tag{2.3.14}$$

where $H^\pm \subset \mathbb{R}^2$ denote the halfspaces orthogonal to ν and containing $\pm\nu$. With these notation, we have that

$$\overline{u} = u^+(x_0)\mathbb{1}_{H^+ \cap B} + u^-(x_0)\mathbb{1}_{H^- \cap B}.$$

Note also that contrary to Proposition 2.2.5 where jump points were considered, it might be the case that $u^+(x_0) = u^-(x_0)$, i.e. that $\overline{u}$ is constant.

Extraction of diagonal subsequences. As before, we consider a sequence of radii $\{\varrho_j\}_{j \in \mathbb{N}}$ such that $\varrho_j \searrow 0$ and $\lambda(\partial B_{\varrho_j}(x_0)) = 0 = \mathcal{H}^1\big(\partial^*\mathcal{P} \cap \partial B_{\varrho_j}(x_0)\big)$ for all $j \in \mathbb{N}$. By our choice of x_0, (2.3.4) and (2.3.5), with $r_k^\pm := r_{k|P^\pm}$, we have :

$$\begin{cases} \lim\limits_{j\to\infty}\lim\limits_{k\to\infty} \big(u_k - r_k\big)(x_0 + \varrho_j \cdot) = \lim\limits_{j\to\infty} u_{x_0,\varrho_j} = \overline{u} & \text{in measure in } B, \\ \lim\limits_{j\to\infty}\lim\limits_{k\to\infty} \arctan|r_k^+ - r_k^-|(x_0 + \varrho_j \cdot) = \dfrac{\pi}{2} & \text{in measure in } B, \\ \lim\limits_{j\to\infty}\lim\limits_{k\to\infty} \dfrac{\lambda_k(B_{\varrho_j}(x_0))}{2\varrho_j} = \lim\limits_{j\to\infty} \dfrac{\lambda(B_{\varrho_j}(x_0))}{2\varrho_j} = \dfrac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0), \\ \lim\limits_{j\to\infty}\lim\limits_{k\to\infty} \dfrac{\varepsilon_k}{\varrho_j} = \lim\limits_{j\to\infty}\lim\limits_{k\to\infty} \dfrac{\omega(\varepsilon_k)}{\varrho_j} = 0. \end{cases}$$

We can thus find an increasing sequence $\{k_j\}_{j \in \mathbb{N}}$ such that $k_j \nearrow \infty$ as $j \to \infty$ and

$$\begin{cases} \big(u_{k_j} - r_{k_j}\big)(x_0 + \varrho_j \cdot) \to \overline{u} \quad \text{in measure in } B, & \text{(2.3.15a)} \\ \arctan|r_{k_j}^+ - r_{k_j}^-|(x_0 + \varrho_j \cdot) \to \dfrac{\pi}{2} \quad \text{in measure in } B, & \text{(2.3.15b)} \\ \dfrac{\lambda_{k_j}(B_{\varrho_j}(x_0))}{2\varrho_j} \to \dfrac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0), & \text{(2.3.15c)} \\ \dfrac{\varepsilon_{k_j}}{\varrho_j} \to 0, \quad \dfrac{\omega(\varepsilon_{k_j})}{\varrho_j} \to 0. & \text{(2.3.15d)} \end{cases}$$

Let $v_j := u_{k_j}(x_0 + \varrho_j \cdot)$, $r_j^\pm := r_{k_j}^\pm(x_0 + \varrho_j \cdot)$ and $r_j := r_j^+ \mathbb{1}_{H^+ \cap B} + r_j^- \mathbb{1}_{H^- \cap B}$. By (2.3.14) and (2.3.15), we have for all $\eta > 0$,

$$\begin{aligned}\mathcal{L}^2\big(\{|v_j - r_j^\pm - u^\pm(x_0)| > \eta\} \cap B \cap H^\pm\big) \\ \le \mathcal{L}^2\left(B \cap H^\pm \setminus (P^\pm - x_0)/\varrho_j\right) + \mathcal{L}^2\left(\{|v_j - r_{k_j}(x_0 + \varrho_j \cdot) - \overline{u}| > \eta\} \cap B\right) \to 0.\end{aligned}$$

Thus, up to a subsequence

$$\begin{cases} v_j - r_j^\pm \to u^\pm(x_0) & \text{in measure in } B \cap H^\pm, \\ |r_j^+ - r_j^-| \to +\infty & \mathcal{L}^2\text{-a.e. in } B. \end{cases} \tag{2.3.16}$$

Selection of a slicing direction. According to [37, Lemma 2.8], there exist an $\mathcal{H}^1$-negligible set $N \subset B^\nu$ and a countable dense subset D of $\mathbb{S}^1$ such that for all $\xi \in D$ and all $y \in B^\nu \setminus N$,

$$|(r_j^+ - r_j^-)(y) \cdot \xi| \to +\infty$$

as $j \to +\infty$. Note that $\nabla r_j^\pm \xi \cdot \xi = 0$, so that the quantity $t \mapsto (r_j^+ - r_j^-)_y^\xi(t) = (r_j^+ - r_j^-)(y) \cdot \xi$ is independent of $t \in B_y^\xi$. Thus, for all $y \in B^\nu \setminus N$, we have

$$\arctan|(r_j^+ - r_j^-)_y^\xi| \to \frac{\pi}{2} \quad \text{uniformly in } t \in B_y^\xi. \tag{2.3.17}$$

For any $\eta > 0$, let $\xi \in \mathbb{S}^1 \cap D$ be such that

$$|\nu - \xi| \le \eta, \quad \nu \cdot \xi \ge \frac{1}{2}, \quad \left|\nu \cdot \xi^\perp\right| \le \eta. \tag{2.3.18}$$

As in the proof of Proposition 2.2.5, using a change of variables and the ellipticity property (2.1.5) of $\mathbf{A}$ and (2.3.15d), we get

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0) \ge \limsup_{j\to\infty} \varrho_j \int_B \frac{\kappa}{\varepsilon_{k_j}} \wedge \frac{\alpha}{\varrho_j^2}|e(v_j)\xi \cdot \xi|^2\, dy$$

so that, introducing the following characteristic functions,

$$\chi_j := \mathbb{1}_{\left\{\frac{\alpha}{\varrho_j^2}|e(v_j)\xi\cdot\xi|^2 \ge \frac{\kappa}{\varepsilon_{k_j}}\right\}} \in L^\infty(B; \{0,1\}),$$

we obtain that

$$2\frac{d\lambda}{d\mathcal{H}^1 \llcorner \partial^*\mathcal{P}}(x_0) \ge \limsup_{j\to\infty} \left\{\frac{\alpha}{\varrho_j}\int_B (1 - \chi_j)|e(v_j)\xi \cdot \xi|^2\, dy + \frac{\kappa\varrho_j}{\varepsilon_{k_j}}\int_B \chi_j\, dy\right\}. \tag{2.3.19}$$

We define the translated and rescaled triangulations :

$$\mathbf{T}^{x_0,j} := \frac{1}{\varrho_j}\left(\mathbf{T}^{k_j} - x_0\right), \quad \mathbf{T}_b^{x_0,j} := \left\{T \in \mathbf{T}^{x_0,j} : \frac{\alpha}{\varrho_j}|e(v_j)_{|T}\xi \cdot \xi|^2 \ge \frac{\kappa\varrho_j}{\varepsilon_{k_j}}\right\},$$

and the family of triangles which intersect $\overline{B_{1-\frac{\eta}{4}}}$:

$$\mathbf{T}^{x_0,j}_{b,int} := \left\{ T \in \mathbf{T}^{x_0,j}_b : \ T \cap \overline{B_{1-\frac{\eta}{4}}} \neq \emptyset \right\}.$$

Note that $v_j - r_j^{\pm}$ is affine and χ_j is constant on each $T \in \mathbf{T}^{x_0,j}$, and that (2.3.15d) ensures that for $j \in \mathbb{N}$ large enough (depending on η), each $T \in \mathbf{T}^{x_0,j}_{b,int}$ is contained in B. As in (2.2.18), for all $y \in (B_{1-\frac{\eta}{4}})^\nu$, we denote by $a(y)$ and $b(y)$ the end points of the section passing through y in the direction ξ (see the Figure 2.2) in such a way that $(B_{1-\frac{\eta}{4}})^\xi_y = (a(y), b(y))$. We also recall that L_η is defined as in (2.2.19) and satisfies $0 < L_\eta \leq |a(y)|, \, |b(y)| \leq 2$.

Using (2.3.16), (2.3.17), (2.3.19), Fubini's and Egoroff's Theorem, we can adapt the proof of Lemma 2.2.6 to show the following result.

Lemma 2.3.9. *For all $\eta > 0$, there exist a subset $Z \subset B^\nu$ containing N with $\mathcal{H}^1(Z) \leq \eta$, and a subsequence (not relabeled) such that the following property holds : for all $\gamma > 0$, there exists $j_0 = j_0(\gamma) \in \mathbb{N}$ such that for all $y \in B^\nu \setminus Z$ and all $j \geq j_0$,*

$$\int_{B^\xi_y} (\chi_j)^\xi_y \, dt \leq \gamma, \quad \int_{B^\xi_y} \big(1 - (\chi_j)^\xi_y\big) |((v_j)^\xi_y)'|^2 \, dt \leq \gamma^2,$$

and

$$\int_{B^\xi_y \cap \mathbb{R}^\pm} 1 \wedge |(v_j - r_j^\pm)^\xi_y - u^\pm(x_0) \cdot \xi| \, dt \leq \frac{\gamma}{2}, \quad |(r_j^+ - r_j^-)^\xi_y| \geq |[u](x_0) \cdot \xi| + 1.$$

As in the proof of Proposition 2.2.5, we next show that, for some subset $Z' \subset (B_{1-\frac{\eta}{2}})^\nu$ of arbitrarily small $\mathcal{H}^1$ measure, and along a subsequence (only depending on η), all the sections in the direction ξ passing through $(B_{1-\frac{\eta}{2}})^\nu \setminus Z'$ must cross at least one triangle $T \in \mathbf{T}^{x_0,j}_{b,int}$ contained in B.

Lemma 2.3.10. *For all $\eta > 0$, there exist a subset $Z' \subset B^\nu$ containing Z with $\mathcal{H}^1(Z') \leq \eta$, and a subsequence (not relabeled) such that the following property holds : for all $y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z'$ and all $j \in \mathbb{N}$, there exists a triangle $T = T(y,j) \in \mathbf{T}^{x_0,j}_{b,int}$ such that $(\mathring{T} \cap B)^\xi_y \neq \emptyset$.*

Proof. Let Z be the exceptional set given by Lemma 2.3.9. We first show that there exists an increasing mapping $\phi : \mathbb{N} \to \mathbb{N}$ such that : for all $y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z$ and all $j \in \mathbb{N}$, there exists a triangle $T = T(y, \phi(j)) \in \mathbf{T}^{x_0,\phi(j)}_{b,int}$ such that $(T \cap B)^\xi_y \neq \emptyset$. Suppose by contradiction that such is not the case, and define

$$\gamma_1^* := L_\eta > 0, \quad \gamma_2^* := \frac{L_\eta}{1 + 2L_\eta} > 0 \quad \text{and} \quad \gamma^* = \gamma^*(\eta) := \frac{\gamma_1^* \wedge \gamma_2^*}{4} > 0.$$

Thanks to Lemma 2.3.9, there exists $j^* = j^*(\gamma^*) \in \mathbb{N}$ such that for all $y \in B^\nu \setminus Z$ and all $j \geq j^*$, $\left|(r_j^+ - r_j^-)^\xi_y\right| \geq |[u](x_0) \cdot \xi| + 1$ and

$$\int_{B^\xi_y} (1 - (\chi_j)^\xi_y) |((v_j)^\xi_y)'|^2 \, dt \leq \gamma^{*2}, \quad \int_{B^\xi_y \cap \mathbb{R}^\pm} 1 \wedge |(v_j - r_j^\pm)^\xi_y - u^\pm(x_0) \cdot \xi| \, dt \leq \frac{\gamma^*}{2}.$$

As in Lemma 2.2.7, we consider the extraction $\phi : j \in \mathbb{N} \longmapsto j + j^* \in \mathbb{N}$. By assumption, there exist $y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z$ and $j \in \mathbb{N}$ such that $(T \cap B)^\xi_y = \emptyset$ for all $T \in \mathbf{T}^{x_0, j+j^*}_{b,int}$, implying that $(\chi_{j+j^*})^\xi_y \equiv 0$ on $(a(y), b(y))$, $\left|(r^+_{j+j^*} - r^-_{j+j^*})^\xi_y\right| \geq |[u](x_0) \cdot \xi| + 1$ and

$$\int_{a(y)}^{b(y)} |((v_{j+j^*})^\xi_y)'|^2 \, dt \leq \gamma^{*2}, \quad \int_{[a(y),b(y)]\cap\mathbb{R}^\pm} 1 \wedge |(v_{j+j^*} - r^\pm_{j+j^*})^\xi_y - u^\pm(x_0) \cdot \xi| \, dt \leq \frac{\gamma^*}{2},$$

since $\phi(j) = j + j^* \geq j^*$. By continuity of $(v_{j+j^*} - r^\pm_{j+j^*})^\xi_y$ on the compact sets $[a(y), b(y)] \cap \mathbb{R}^\pm$, there exist two points

$$t^\pm \in \operatorname*{arg\,min}_{[a(y),b(y)]\cap\mathbb{R}^\pm} \left(1 \wedge |\left(v_{j+j^*} - r^\pm_{j+j^*}\right)^\xi_y - u^\pm(x_0) \cdot \xi|\right).$$

Hence, recalling (2.2.19)

$$\begin{aligned}
\frac{\gamma^*}{L_\eta} &\geq 1 \wedge |\left(v_{j+j^*} - r^-_{j+j^*}\right)^\xi_y (t^-) - u^-(x_0) \cdot \xi| + 1 \wedge |\left(v_{j+j^*} - r^+_{j+j^*}\right)^\xi_y (t^+) - u^+(x_0) \cdot \xi| \\
&\geq 1 \wedge \left(\left|[u](x_0) \cdot \xi + (r^+_{j+j^*} - r^-_{j+j^*})^\xi_y\right| - \left|\int_{t^-}^{t^+} \left((v_{j+j^*})^\xi_y\right)'(t) \, dt\right|\right) \geq 1 - 2\gamma^*,
\end{aligned}$$

which is impossible thanks to of our choice of γ^*. We conclude the proof of Lemma 2.3.10 in the same way as for Lemma 2.2.7. ☐

As a consequence of Lemma 2.3.10, introducing the family of triangles

$$\mathscr{F}_j := \left\{T \in \mathbf{T}^{x_0,j}_{b,int} : \text{ there exists } y \in (B_{1-\frac{\eta}{2}})^\nu \text{ such that } (\mathring{T} \cap B)^\xi_y \neq \emptyset\right\}$$

for all $j \in \mathbb{N}$, it is possible to obtain a too low lower bound, roughly speaking because Lemma 2.3.10 does not exhibit enough triangles in $\mathbf{T}^{x_0,j}_{b,int}$, as explained after (2.2.20). Therefore, we need to establish that many lines B^ξ_y parallel to ξ and passing through B^ν must actually intersect at least two triangles of the collection $\mathbf{T}^{x_0,j}_{b,int}$. To this aim, we show that the set of points $y \in B^\nu$ such that B^ξ_y intersects exactly one triangle T in the collection $\mathbf{T}^{x_0,j}_{b,int}$, has arbitrarily small $\mathcal{H}^1$ measure.

Lemma 2.3.11. *For all $\eta > 0$, there exist constants $C_* = C_*(\eta) > 0$, $\gamma_* = \gamma_*(\eta) > 0$ and a subset $Z_* = Z_*(\eta) \subset B^\nu$ containing Z' and satisfying $\mathcal{H}^1(Z_*) \leq 3\eta$ such that the following property holds : for all $0 < \gamma < \gamma_*$, there exists $j(\gamma) \in \mathbb{N}$ such that for all $j \geq j(\gamma)$, the set*

$$Y_j := \left\{y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z' : \text{ there exists a unique } T \in \mathbf{T}^{x_0,j}_{b,int} \text{ such that } (\mathring{T} \cap B)^\xi_y \neq \emptyset\right\}$$

satisfies

$$\mathcal{H}^1(Y_j \setminus Z_*) \leq C_*\gamma.$$

Proof of Lemma 2.3.11. We follow the same three steps structuring the proof of Lemma 2.2.9.

Step 1. We start by showing that for j large enough and for many points $y \in Y_j$, the only triangle T in $\mathbf{T}^{x_0,j}_{b,int}$ crossing B^ξ_y is getting closer to the diameter B^ν.

For all $j \in \mathbb{N}$ and all $y \in Y_j$, let $T_j(y) \in \mathbf{T}^{x_0,j}_{b,int}$ be the unique triangle such that $(\mathring{T}_j(y) \cap B)^\xi_y \neq \emptyset$. We keep the notation (2.2.22) for the end points $a_j(y)$ and $b_j(y)$ of the section in the direction ξ passing through y inside $T_j(y)$ (see the Figure 2.4). Let us show that

$$f_j(y) := (|a_j(y)| + |b_j(y)|)\, \mathbb{1}_{Y_j}(y) \to 0 \quad \text{for all } y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z'.$$

Let $y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z'$. Assume by contradiction that $\ell := \limsup_j f_j(y) > 0$ and extract a subsequence (depending on y, not relabeled) such that $f_j(y) \to \ell$. Then, there exists $j_0 \in \mathbb{N}$ such that $y \in Y_j$ for all $j \geq j_0$. Moreover, according to Lemma 2.3.9 and setting $I_j(y) := (a(y), b(y)) \setminus (a_j(y), b_j(y)) \subset B^\xi_y$, we infer that

$$|b_j(y) - a_j(y)| + \int_{I_j(y)} |((v_j)^\xi_y)'|^2\, dt + \int_{[a(y),b(y)] \cap \mathbb{R}^\pm} 1 \wedge |(v_j - r_j^\pm)^\xi_y - u^\pm(x_0) \cdot \xi|\, dt \to 0. \tag{2.3.20}$$

Up to another subsequence (still not relabeled), it ensures that $a_j(y), b_j(y) \to m$ for some $m \in [a(y), b(y)]$. Thus, for all $\tau > 0$, $I_\tau := (a(y), m - \tau) \cup (m + \tau, b(y)) \subset I_j(y)$ for $j \in \mathbb{N}$ sufficiently large. In particular, we deduce that $m - \tau \leq 0$. Indeed, assuming that $m - \tau > 0$, by continuity of $(v_j - r_j^+)^\xi_y$ on $\big(0, m-\tau\big)$ and of $(v_j - r_j^-)^\xi_y$ on $\big(a(y), 0\big)$, there exist

$$t_j^+ \in \operatorname*{arg\,min}_{(0,m-\tau)} 1 \wedge |(v_j - r_j^+)^\xi_y - u^+(x_0) \cdot \xi| \quad \text{and} \quad t_j^- \in \operatorname*{arg\,min}_{(a(y),0)} 1 \wedge |(v_j - r_j^-)^\xi_y - u^-(x_0) \cdot \xi|,$$

which satisfy

$$\begin{aligned} &1 \wedge |(v_j - r_j^+)^\xi_y(t_j^+) - u^+(x_0) \cdot \xi| + 1 \wedge |(v_j - r_j^-)^\xi_y(t_j^-) - u^-(x_0) \cdot \xi| \\ &\qquad \geq 1 \wedge \left(\left| [u](x_0) \cdot \xi + (r_j^+ - r_j^-)^\xi_y \right| - 2 \sqrt{\int_{I_j(y)} |((v_j)^\xi_y)'|^2\, dt} \right) \to 1 \end{aligned}$$

according to (2.3.20) and (2.3.17). However,

$$\begin{aligned} &1 \wedge |(v_j - r_j^+)^\xi_y(t_j^+) - u^+(x_0) \cdot \xi| + 1 \wedge |(v_j - r_j^-)^\xi_y(t_j^-) - u^-(x_0) \cdot \xi| \\ &\quad \leq \frac{1}{L_\eta} \int_{a(y)}^0 1 \wedge |(v_j - r_j^-)^\xi_y - u^-(x_0) \cdot \xi|\, dt + \frac{1}{m - \tau} \int_0^{m-\tau} 1 \wedge |(v_j - r_j^+)^\xi_y - u^+(x_0) \cdot \xi|\, dt \to 0 \end{aligned}$$

according again to (2.3.20), which leads to a contradiction. We similarly show that $m + \tau \geq 0$, leading to $|m| \leq \tau$. Taking the limit as $\tau \to 0^+$, we obtain that $m = 0$ which is against $\ell > 0$.

Therefore, owing to Lemma 2.3.9 and Egoroff's Theorem, we can find a set $Z^1_* \subset B^\nu$ containing Z' with $\mathcal{H}^1(Z^1_*) \leq 2\eta$ such that for all $\gamma > 0$, there exists $j_0(\gamma) \in \mathbb{N}$ satisfying

$$\begin{cases} \displaystyle\int_{B^\xi_y} (1 - (\chi_j)^\xi_y) |((v_j)^\xi_y)'|^2\, dt \leq \gamma^2, \\ \displaystyle\int_{B^\xi_y \cap \mathbb{R}^\pm} 1 \wedge |(v_j - r_j^\pm)^\xi_y - u^\pm(x_0) \cdot \xi|\, dt \leq \frac{\gamma}{2}, \\ (|a_j(y)| + |b_j(y)|)\, \mathbb{1}_{Y_j}(y) \leq \gamma \end{cases} \tag{2.3.21}$$

for all $y \in (B_{1-\frac{\eta}{2}})^\nu \setminus Z^1_*$ and all $j \geq j_0(\gamma)$.

Step 2. Arguing in the same manner as for (2.2.27), one can show that for many points $y \in Y_j$, the variation of $(v_j - r_j)^\xi_y$ inside the only triangle T in $\mathbf{T}^{x_0,j}_{b,int}$ which is crossed by B^ξ_y, is close to that of $\overline{u}^\xi_y$. Precisely, setting the constants

$$C_\eta := 8\left(1 + \frac{1}{L_\eta}\right) > 0, \quad \gamma_* = \gamma_*(\eta) := \frac{1}{2}\min\left(1, \frac{1}{C_\eta}, L_\eta\right) > 0,$$

we get that for all $0 < \gamma < \gamma_*$, there exists $j_1(\gamma) \in \mathbb{N}$ such that for all $j \geq j_1(\gamma)$ and all $y \in Y_j \setminus Z^1_*$,

$$\left|(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y)) - (r^+_j - r^-_j)^\xi_y - [u](x_0) \cdot \xi\right| \leq C_\eta \gamma. \tag{2.3.22}$$

Step 3. We now show that, after enlarging slightly the set Z^1_* into a set $Z_* \subset B^\nu$ with $\mathcal{H}^1(Z_*) \leq 3\eta$, it is possible to include $Y_j \setminus Z_*$ inside a finite union of arbitrarily small segments contained in B^ν (see Figure 2.7).

By definition of rigid body motions, there exist skew symmetric matrices $M_j \in \mathbb{M}^{2\times 2}_{\mathrm{skew}}$ and vectors $m_j \in \mathbb{R}^2$ such that

$$r^+_j - r^-_j = M_j \, \mathrm{id}_{\mathbb{R}^2} + m_j$$

for all $j \in \mathbb{N}$. Recalling (2.3.17), we get that for all $y \in B^\nu \setminus Z^1_* \subset B^\nu \setminus N$,

$$\left|(r^+_j - r^-_j)^\xi_y\right| = \left|\left(M^T_j \xi\right) \cdot y + m_j \cdot \xi\right| \to +\infty$$

as $j \to +\infty$. In particular, setting $\alpha_j := M^T_j \xi \in \mathbb{R}^2$ and $\beta_j := m_j \cdot \xi \in \mathbb{R}$, we get that $\mu_j := |\alpha_j| + |\beta_j| \to +\infty$ as $j \to +\infty$. Hence, up to a subsequence (depending only on ξ, not relabeled), there exist $\alpha \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$ such that

$$\frac{\alpha_j}{\mu_j} \to \alpha, \quad \frac{\beta_j}{\mu_j} \to \beta, \quad \text{and} \quad |\alpha| + |\beta| = 1.$$

In particular,

$$\frac{1}{\mu_j}\left(\alpha_j \cdot \mathrm{id}_{\mathbb{R}^2} + \beta_{\mathrm{j}}\right) \to \alpha \cdot \mathrm{id}_{\mathbb{R}^2} + \beta \text{ uniformly on } B. \tag{2.3.23}$$

Notice that the affine line $\Delta := \{z \in \mathbb{R}^2 : \alpha \cdot z + \beta = 0\}$ cannot coincide with Π_ν. Indeed, if such would be the case, it would entail that $\beta = 0$ and $\alpha = \pm\nu \in \mathbb{S}^1$. Yet, $M_j \in \mathbb{M}^{2\times 2}_{\mathrm{skew}}$ being skew symmetric, we would obtain that

$$0 = \frac{M_j \xi \cdot \xi}{\mu_j} = \frac{\alpha_j}{\mu_j} \cdot \xi \to \alpha \cdot \xi = \pm\nu \cdot \xi$$

which is against our choice (2.3.18) of $\xi \in \mathbb{S}^1 \cap D$. As a consequence Δ intersects B^ν in at most one point z_*.

If $\Delta \cap \Pi_\nu = \{z_*\}$, we define $Z^2_* := B^\nu \cap B_{\frac{\eta}{2}}(z_*)$ while if $\Delta \cap \Pi_\nu = \emptyset$, we define $Z^2_* = \emptyset$. The continuity of $\alpha \cdot \mathrm{id}_{\mathbb{R}^2} + \beta$ on the compact set $B^\nu \setminus Z^2_*$ entails that

$$0 < m_{\alpha,\beta}(\eta) := \min\{|\alpha \cdot y + \beta| : y \in B^\nu \setminus Z^2_*\}.$$

Set $Z_* := Z^1_* \cup Z^2_* \subset B^\nu$, which satisfies $\mathcal{H}^1(Z_*) \leq 3\eta$, and for all $j \in \mathbb{N}$, we define

$$\widehat{\mathbf{T}}_j := \{T \in \mathbf{T}^{x_0,j}_{b,int} : \text{ there exists } y \in Y_j \setminus Z_* \text{ such that } (\mathring{T} \cap B)^\xi_y \neq \emptyset\}.$$

Thus, for all $j \in \mathbb{N}$ and for each triangle $T \in \widehat{\mathbf{T}}_j$, there exists a point $y_T \in \Phi \circ p_\xi(\mathring{T}) \setminus Z_*^2 \subset B^\nu \setminus Z_*^2$ which satisfies

$$|\alpha \cdot y_T + \beta| \geq m_{\alpha,\beta}(\eta) > 0,$$

with Φ introduced in (2.2.17).

Remembering that $\omega(\varepsilon_{k_j})/\varrho_j \to 0$ and that the Lipschitz constant of Φ is less than $\sqrt{1+4\eta^2} \leq 2$ for η small enough, together with the uniform convergence (2.3.23) and (2.3.22), it follows that for all $\gamma > 0$, there exists $j_2(\gamma) \geq j_1(\gamma)$ such that for all $j \geq j_2(\gamma)$,

$$\begin{cases} \left| \dfrac{1}{\mu_j}(r_j^+ - r_j^-)_y^\xi - \big(\alpha \cdot y + \beta\big) \right| \leq \dfrac{m_{\alpha,\beta}}{8}\gamma \quad \text{for all } y \in B^\nu \setminus Z_*, & \text{(2.3.24a)} \\ \mathcal{H}^1(\Phi \circ p_\xi(T)) \leq 2\omega(\varepsilon_{k_j})/\varrho_j \leq \dfrac{m_{\alpha,\beta}}{8}\gamma \quad \text{for all } T \in \mathbf{T}^{x_0,j}. & \text{(2.3.24b)} \end{cases}$$

Therefore, for all $j \geq j_2(\gamma)$ and all $T \in \widehat{\mathbf{T}}_j$, we introduce the following quantities :

$$\begin{cases} L^{\mathrm{ref}}(T) := \dfrac{\big|[u](x_0) \cdot \xi + \mu_j\big(\alpha \cdot y_T + \beta\big)\big| - \big(C_\eta + \frac{m_{\alpha,\beta}}{2}\mu_j\big)\gamma}{\big|e(v_j)_{|T} : (\xi \otimes \xi)\big|} \text{ the reference length of } T, \\ L^{\max}(T) := \max\limits_{z \in p_\xi(T)} \mathcal{L}^1(T_z^\xi) \text{ the maximal section's length of } T \text{ along the direction } \xi. \end{cases} \tag{2.3.25}$$

Note that $L^{\mathrm{ref}}(T)$ is well defined (since $|e(v_j)_{|T}\xi \cdot \xi|^2 \geq \kappa\varrho_j^2/(\alpha\varepsilon_{k_j}) > 0$ as $T \in \mathbf{T}_b^{x_0,j}$) and positive for j large enough since $\mu_j \to +\infty$ and $|\alpha \cdot y_T + \beta| > m_{\alpha,\beta}\,\gamma/2 > 0$. Moreover, we have $L^{\max}(T) > L^{\mathrm{ref}}(T)$. Indeed, if such would not be the case, denoting by $y \in Y_j \setminus Z_*$ a point such that $(\mathring{T} \cap B)_y^\xi \neq \emptyset$, then $\mathcal{L}^1(T_{p_\xi(y)}^\xi) = \mathcal{L}^1(T_y^\xi) = b_j(y) - a_j(y) \leq L^{\max}(T) \leq L^{\mathrm{ref}}(T)$, entailing that

$$\begin{aligned} |(v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y))| &= \big|e(v_j)_{|T} : (\xi \otimes \xi)\big|\,(b_j(y) - a_j(y)) \\ &\leq \big|[u](x_0) \cdot \xi + \mu_j\big(\alpha \cdot y_T + \beta\big)\big| - \big(C_\eta + \frac{m_{\alpha,\beta}}{2}\mu_j\big)\gamma, \end{aligned}$$

by definition (2.3.25) of $L^{\mathrm{ref}}(T)$. Therefore, we would obtain that

$$\begin{aligned} C_\eta\gamma + \frac{m_{\alpha,\beta}}{2}\mu_j\gamma &\leq \big|[u](x_0) \cdot \xi + \mu_j\big(\alpha \cdot y_T + \beta\big)\big| - \Big|(v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y))\Big| \\ &\leq \Big|(v_j)_y^\xi(b_j(y)) - (v_j)_y^\xi(a_j(y)) - [u](x_0) \cdot \xi - (r_j^+ - r_j^-)_y^\xi\Big| \\ &\quad + \Big|(r_j^+ - r_j^-)_y^\xi - \mu_j\big(\alpha \cdot y + \beta\big)\Big| + |\mu_j\alpha \cdot (y - y_T)| \\ &\leq C_\eta\gamma + \frac{m_{\alpha,\beta}}{8}\mu_j\gamma + \frac{m_{\alpha,\beta}}{8}\mu_j\gamma, \end{aligned}$$

where we used (2.3.22), (2.3.24a) and (2.3.24b) (since $y, y_T \in \Phi \circ p_\xi(T)$), leading to a contradiction.

Therefore, arguing as in the proof of Lemma 2.2.9, there are exactly one or two points $z_{\mathrm{ref}}^1, z_{\mathrm{ref}}^2 \in p_\xi(T)$, only depending on j and T, such that $\mathcal{L}^1(T_{z_{\mathrm{ref}}^1}^\xi) = \mathcal{L}^1(T_{z_{\mathrm{ref}}^2}^\xi) = L^{\mathrm{ref}}(T)$. Then, as in (2.2.30), we introduce the following segments (orthogonal to ξ) associated to T (see Figure 2.7),

$$\mathfrak{T}_i(T) := \left\{ z \in \Pi_\xi : \ \big|z - z_{\mathrm{ref}}^i\big| \leq C'_\eta\, \frac{\varrho_j \mathcal{L}^2(T)}{\varepsilon_{k_j}}\,\gamma \right\} \quad \text{for } i \in \{1, 2\},$$

where the constant C'_η, only depending on η, now changes into

$$C'_\eta := \frac{8}{\sin\theta_0}\left(\frac{2C_\eta}{m_{\alpha,\beta}}+1\right).$$

For every $j \geq j_2(\gamma)$ and every $y \in Y_j \setminus Z_*$, let $T \in \mathbf{T}^{x_0,j}_{b,int}$ be such that $(\mathring{T}\cap B)^\xi_y \neq \emptyset$. In particular, note that $T \in \widehat{\mathbf{T}}_j$. Arguing in the same way as in the proof of Lemma 2.2.9, we get that there exists $i \in \{1,2\}$ such that

$$\begin{aligned}
&|p_\xi(y) - z^i_{\rm ref}| \\
&\quad \leq \frac{2\mathcal{L}^2(T)}{h_T}\,\frac{\big|(b_j(y)-a_j(y)) - L^{\rm ref}(T)\big|}{L^{\rm ref}(T)} \\
&\quad = \frac{2\mathcal{L}^2(T)}{h_T}\,\frac{\Big||(v_j)^\xi_y(b_j(y)) - (v_j)^\xi_y(a_j(y))| - |[u](x_0)\cdot\xi + \mu_j(\alpha\cdot y_T+\beta)| + \big(C_\eta + \frac{m_{\alpha,\beta}}{2}\mu_j\big)\gamma\Big|}{\big|[u](x_0)\cdot\xi + \mu_j\big(\alpha\cdot y_T+\beta\big)\big| - \big(C_\eta + \frac{m_{\alpha,\beta}}{2}\mu_j\big)\gamma} \\
&\quad \leq \frac{2\mathcal{L}^2(T)}{h_T}\,\frac{C_\eta\gamma + \Big|(r^+_j - r^-_j)^\xi_y - \mu_j(\alpha\cdot y_T+\beta)\Big| + \big(C_\eta + \frac{m_{\alpha,\beta}}{2}\mu_j\big)\gamma}{\mu_j\, m_{\alpha,\beta}/4} \\
&\quad \leq \frac{2\mathcal{L}^2(T)}{h_T}\,\frac{\big(2C_\eta + m_{\alpha,\beta}\mu_j\big)\gamma}{\mu_j\, m_{\alpha,\beta}/4} \leq C'_\eta\,\frac{\varrho_j\mathcal{L}^2(T)}{\varepsilon_{k_j}}\,\gamma,
\end{aligned}$$

where we used (2.3.22), (2.3.24a), (2.3.24b) and the fact that

$$\big|[u](x_0)\cdot\xi + \mu_j\big(\alpha\cdot y_T+\beta\big)\big| - \left(C_\eta + \frac{\mu_j\, m_{\alpha,\beta}}{2}\right)\gamma \geq \frac{\mu_j\, m_{\alpha,\beta}}{4}$$

up to enlarging $j_2(\gamma)\in\mathbb{N}$. As in the proof of Lemma 2.2.9, we deduce that for all $j \geq j_2(\gamma)$,

$$\mathcal{H}^1(Y_j\setminus Z_*) \leq \sum_{T\in\widehat{\mathbf{T}}_j} \mathcal{H}^1(\Phi\,(\mathfrak{T}_1(T)\cup\mathfrak{T}_2(T))) \leq 8C'_\eta\,\gamma\,\frac{\varrho_j}{\varepsilon_{k_j}}\sum_{T\in\widehat{\mathbf{T}}_j}\mathcal{L}^2(T) \leq \frac{8C'_\eta\gamma}{\kappa}\,\frac{\kappa\varrho_j}{\varepsilon_{k_j}}\int_B \chi_j\,dx.$$

Recalling (2.3.19) and possibly taking a larger $j_2(\gamma)\in\mathbb{N}$, we finally get that for all $j \geq j_2(\gamma)$,

$$\mathcal{H}^1(Y_j\setminus Z_*) \leq \frac{8C'_\eta}{\kappa}\left(2\frac{d\lambda}{d\mathcal{H}^1\llcorner\partial^*\mathcal{P}}(x_0)+1\right)\gamma =: C_*\gamma,$$

for some constant $C_*>0$ only depending on η, which settles Lemma 2.3.11. □

Arguing exactly as in the proof of Lemma 2.2.8, having Lemma 2.3.11 at hand, we deduce the following result.

Lemma 2.3.12. *For all $\eta>0$, there exist $Z''\subset B^\nu$ containing Z' with $\mathcal{H}^1(Z'')\leq 4\eta$, and a (not relabeled) subsequence such that for all $j\in\mathbb{N}$ and for all $y\in(B_{1-\frac{\eta}{2}})^\nu\setminus Z''$,*

$$\#\left\{T\in\mathbf{T}^{x_0,j}_{b,int}:\ (\mathring{T}\cap B)^\xi_y\neq\emptyset\right\}\geq 2.$$

Finally, owing to Lemma 2.3.12, the proof of Proposition 2.3.8 is identical to that of Proposition 2.2.5.

In the following result, we prove the existence of minimizers of the discrete brittle damage energy $\mathcal{G}_\varepsilon$ on $V_\varepsilon^{\rm Dir}(\Omega')$.

Lemma 2.3.13. *Assume that Ω and Ω' are connected. For $\varepsilon > 0$ sufficiently small, there exists a minimizer $u_\varepsilon \in V_\varepsilon^{\rm Dir}(\Omega')$ of $\mathcal{G}_\varepsilon$.*

Proof. Let $\varepsilon_0 := \kappa/(\beta\|\nabla w\|^2_{L^\infty(\mathbb{R}^2;\mathbb{M}^{2\times 2})})$ and fix $\varepsilon < \varepsilon_0$. Since $\mathcal{G}_\varepsilon(w_{\mathbf{T}_\varepsilon}) < +\infty$, we can consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset V_\varepsilon^{\rm Dir}(\Omega')$ satisfying

$$\lim_{n\to\infty} \mathcal{G}_\varepsilon(u_n) = \inf_{L^0(\Omega;\mathbb{R}^2)} \mathcal{G}_\varepsilon \in [0,+\infty). \tag{2.3.26}$$

By definition of the finite element space $V_\varepsilon^{\rm Dir}(\Omega')$, there exists a triangulation $\mathbf{T}^n \in \mathcal{T}_\varepsilon(\Omega')$ such that u_n is affine on each $T \in \mathbf{T}^n$ and $u_n = w_{\mathbf{T}^n}$ on every triangle $T \in \mathbf{T}^n$ such that $T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$.

Let Ω'' be a bounded open set such that $\Omega' \subset\subset \Omega''$ and $\bigcup_{T\in\mathbf{T}_n} T \subset \Omega''$ for all $n \in \mathbb{N}$. Since, for all $T \in \mathbf{T}^n$, $\mathcal{L}^2(T) \geq \varepsilon^2 \sin\theta_0/2$, it is easily seen that

$$\#\mathbf{T}^n \leq \frac{2\mathcal{L}^2(\Omega'')}{\varepsilon^2 \sin\theta_0}.$$

As a consequence, the sequence of integers $\{\#\mathbf{T}^n\}_{n\in\mathbb{N}}$ admits a subsequence converging as $n \to +\infty$ to an integer $N \in \mathbb{N}$. We can thus assume, without loss of generality, that

$$\#\mathbf{T}^n = N \quad \text{for all } n \in \mathbb{N}.$$

We write $\mathbf{T}^n = \{T_1^n, \dots, T_N^n\}$ for all $n \in \mathbb{N}$. Up to a subsequence, we can check that for all $i \in \{1,\dots,N\}$, the closed triangle T_i^n converges to a closed limit triangle T_i in the sense of Hausdorff, with the property that the limit triangulation $\mathbf{T} := \{T_1,\dots,T_N\} \in \mathcal{T}_\varepsilon(\Omega')$ remains an admissible triangulation of Ω'.

Introducing the characteristic functions $\chi_n := \mathbb{1}_{\{\varepsilon\mathbf{A}e(u_n):e(u_n)\geq\kappa\}} \in L^\infty(\Omega';\{0,1\})$, we can write the energy as

$$\mathcal{G}_\varepsilon(u_n) = \int_\Omega (1-\chi_n)\mathbf{A}e(u_n) : e(u_n)\,dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi_n\,dx. \tag{2.3.27}$$

First, by definition of ε_0 and since $\varepsilon < \varepsilon_0$, we have that $\chi_n = 0$ in $\Omega' \setminus \overline{\Omega}$ for all $n \in \mathbb{N}$, since

$$\mathbf{A}e(u_n) : e(u_n) = \mathbf{A}e(w_{\mathbf{T}_n}) : e(w_{\mathbf{T}_n}) \leq \beta|e(w_{\mathbf{T}_n})|^2 \leq \beta|\nabla w_{\mathbf{T}_n}|^2 \leq \beta\|\nabla w\|^2_{L^\infty(\mathbb{R}^2;\mathbb{M}^{2\times 2})} < \frac{\kappa}{\varepsilon}$$

on that set. Being constant equal to 1 or 0 on each triangle of $\mathbf{T}^n$, χ_n can be identified with a vector $V_n \in \{0,1\}^N$. Hence, up to a subsequence, there exists $V \in \{0,1\}^N$ such that $V_n \to V$ in $\mathbb{R}^N$. In particular, there exists $n_0 \in \mathbb{N}$ such that $V_n = V$ for all $n \geq n_0$. Up to reordering the triangles, we can thus find a integer $0 \leq M < N$ such that

$$\{\chi_n = 1\} = \bigcup_{i=1}^M T_i^n, \quad \{\chi_n = 0\} = \bigcup_{i=M+1}^N T_i^n \quad \text{for all } n \geq n_0.$$

By the Hausdorff convergence property, we infer that

$$\chi_n \to \chi := \mathbb{1}_{\bigcup_{i=1}^M T_i} \quad \text{strongly in } L^1(\Omega'). \tag{2.3.28}$$

We next show some compactness on the sequence of displacements $\{u_n\}_{n\in\mathbb{N}}$, carefully overcoming the lack of control on $\{\chi_n e(u_n)\}_n$ in $L^2(\Omega';\mathbb{M}^{2\times 2}_{\rm sym})$. Remembering that $(1-\chi_n)|e(u_n)|^2 \le \kappa/(\alpha\varepsilon)$ for all $n \ge n_0$ and that the sequence $\{(1-\chi_n)e(u_n)\}_{n\in\mathbb{N}}$ lives in the finite dimensional space $(\mathbb{M}^{2\times 2})^N$, up to a new subsequence (not relabeled), there exists a function $\xi \in L^\infty(\Omega';\mathbb{M}^{2\times 2}_{\rm sym})$ which is constant on each triangle $T \in \mathbf{T}$ such that

$$(1-\chi_n)e(u_n) \to \xi \text{ strongly in } L^2(\Omega';\mathbb{M}^{2\times 2}_{\rm sym}), \tag{2.3.29}$$

and $\xi = 0$ on $\bigcup_{i=1}^M T_i$. Let us define the set

$$\omega_0 := \bigcup_{i=M+1}^N T_i.$$

Note that $\Omega' \setminus \overline{\Omega} \subset \omega_0$. Indeed, if $x \in \Omega' \setminus \overline{\Omega}$, then for all $n \ge n_0$, there exists $M+1 \le i_n \le N$ such that $x \in T^n_{i_n}$. At the expense of extracting a further subsequence, there is no loss of generality to assume that $i_n = i$ is independent of n. By the Hausdorff convergence of T_i^n to T_i, we infer that $x \in T_i \subset \omega_0$. By connectedness of $\Omega' \setminus \overline{\Omega} \subset \omega_0$, we can consider ω the connected component of ω_0 containing $\Omega' \setminus \overline{\Omega}$. Let $M \le K < N$ be such that $\omega = \bigcup_{i=K+1}^N T_i$, up to reordering the triangles again.

Observe that for all $T \in \mathbf{T}$ such that $T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$, then $T \in \{T_{K+1}, \dots, T_N\}$. Thus, for all $T \in \{T_1, \dots, T_K\}$, $T \cap (\Omega' \setminus \overline{\Omega}) = \emptyset$ so that $T \subset \overline{\Omega}$. Therefore, for all open set $W \subset\subset \Omega'$ with $\bigcup_{i=1}^K T_i \subset W$, having that

$$\bigcup_{i=1}^K T_i^n \to \bigcup_{i=1}^K T_i \quad \text{in the sense of Hausdorff,}$$

there exists $n_1 \ge n_0$ such that $\bigcup_{i=1}^K T_i^n \subset W$ for all $n \ge n_1$. Since

$$\Omega' \setminus \overline{W} \subset \bigcup_{i=K+1}^N T_i^n \subset \bigcup_{i=M+1}^N T_i^n = \{\chi_n = 0\},$$

owing to (2.3.26), (2.3.27) and that $u_n = w_{\mathbf{T}^n}$ in $\Omega' \setminus \overline{\Omega}$, we infer that

$$\int_{\Omega' \setminus \overline{W}} |e(u_n)|^2 \, dx \le C_*,$$

for some constant $C_* > 0$ independent of n and W. Using that $u_n - w_{\mathbf{T}^n} \in H^1(\Omega' \setminus \overline{W};\mathbb{R}^2)$ is equal to 0 on the open set $(\Omega' \setminus \overline{W}) \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$, the Poincaré–Korn inequality ensures that (up to a subsequence) there exists $u \in H^1(\Omega' \setminus \overline{W};\mathbb{R}^2)$ such that

$$u_n \rightharpoonup u \text{ weakly in } H^1(\Omega' \setminus \overline{W};\mathbb{R}^2),$$

$u = w_{\mathbf{T}}$ on $(\Omega' \setminus \overline{W}) \cap (\Omega' \setminus \overline{\Omega})$ since $w_{\mathbf{T}^n} \to w_{\mathbf{T}}$ strongly in $H^1(\Omega';\mathbb{R}^2)$ and, thanks to (2.3.29), $e(u) = \xi$ in $\Omega' \setminus \overline{W}$. In addition, by weak lower semicontinuity of the norm, we get that

$$\int_{\Omega' \setminus \overline{W}} |e(u)|^2 \, dx \leq \liminf_{n \to \infty} \int_{\Omega' \setminus \overline{W}} |e(u_n)|^2 \, dx \leq C_*.$$

Considering a decreasing sequence of open sets $\{W_j\}_{j \in \mathbb{N}}$ such that $\bigcup_{i=1}^K T_i \subset W_j \subset\subset \Omega'$ for each $j \in \mathbb{N}$, and $\bigcap_j W_j = \bigcup_{i=1}^K T_i$, we deduce through a diagonalisation argument that there exists $u \in H^1(\mathring{\omega} \cap \Omega';\mathbb{R}^2)$ such that $u = w_{\mathbf{T}}$ on $\mathring{\omega} \cap (\Omega' \setminus \overline{\Omega}) = \Omega' \setminus \overline{\Omega}$ and $e(u) = \xi$ in $\mathring{\omega} \cap \Omega'$. In particular, since ξ is constant in each triangle of ω, we infer that u is affine in the interior of each triangle of ω. Being in $H^1(\mathring{\omega} \cap \Omega';\mathbb{R}^2)$, we get that u is continuous at the interfaces of each triangle in ω. Moreover, since $u = w_{\mathbf{T}}$ on $\Omega' \setminus \overline{\Omega}$, we deduce that $u_{|T} = w_{\mathbf{T}}$ on each triangle $T \in \mathbf{T}$ such that $T \cap (\Omega' \setminus \overline{\Omega}) \neq \emptyset$. Note that u is defined on such triangles T, as they are included in ω.

In order to extend u outside ω, we introduce the family of triangles which are at a distance of at least one triangle from ω, i.e.

$$\mathbf{T}^{\text{far}} := \{T \in \mathbf{T} : \ T \cap \omega = \emptyset\} \subset \{T_1, \ldots, T_K\},$$

so that every remaining triangle $T \notin \mathbf{T}^{\text{far}}$ and such that $T \not\subset \omega$, has its three vertices in $\mathrm{Vertices}(\omega) \cup \mathrm{Vertices}\left(\mathbf{T}^{\text{far}}\right)$. Note that $\{T_{M+1}, \ldots T_K\} \subset \mathbf{T}^{\text{far}}$ since, by construction of the connected component ω of ω_0, each triangle $T \in \mathbf{T}$ included in $\omega_0 \setminus \mathring{\omega}$ is at a distance of at least one triangle from ω. We extend the function u to all triangles by setting $u \equiv 0$ on every triangle $T \in \mathbf{T}^{\text{far}}$, and by interpolating on each remaining triangle which happens to have its three vertices' values imposed. It defines a function $u \in V_\varepsilon^{\text{Dir}}(\Omega')$ which satisfies $e(u) = \xi$ on ω and $e(u) \equiv 0$ on each triangle $T \in \{T_{M+1}, \ldots T_K\} \subset \mathbf{T}^{\text{far}}$.

On the one hand, $\xi = 0$ in $\{\chi = 1\}$, hence $\xi = (1-\chi)\xi$. On the other hand, $e(u) = \xi$ in ω and $e(u) = 0$ in $\omega_0 \setminus \omega$, so that $(1-\chi)\mathbf{A}\xi : \xi \geq (1-\chi)\mathbf{A}e(u) : e(u)$ by positivity of $\mathbf{A}$. Thus, by (2.3.28) together with (2.3.29),

$$\begin{aligned}\inf_{L^0(\Omega;\mathbb{R}^2)} \mathcal{G}_\varepsilon &= \lim_{n \to \infty} \left\{ \int_\Omega (1-\chi_n)\mathbf{A}e(u_n) : e(u_n) \, dx + \frac{\kappa}{\varepsilon} \int_\Omega \chi_n \, dx \right\} \\ &= \int_\Omega (1-\chi)\mathbf{A}\xi : \xi \, dx + \frac{\kappa}{\varepsilon} \int_\Omega \chi \, dx \geq \int_\Omega (1-\chi)\mathbf{A}e(u) : e(u) \, dx + \frac{\kappa}{\varepsilon} \int_\Omega \chi \, dx = \mathcal{G}_\varepsilon(u),\end{aligned}$$

which settles that u is a minimizer of $\mathcal{G}_\varepsilon$. □

Remark 2.3.14. The above proof strongly relies on the choice of the density $f(t) = \kappa \wedge t$, mainly because of the identification (2.3.27), which would unfortunately result into a too low lower bound on the energy for a general f. Indeed for a generic f satisfying (2.1.4) one only gets, for all $\delta > 0$, the existence of a constant $0 < K^\delta < \kappa$ such that

$$\mathcal{G}_\varepsilon(u_n) \geq (1-\delta) \int_\Omega (1-\chi_n^\delta)\mathbf{A}e(u_n) : e(u_n) \, dx + \frac{K^\delta}{\varepsilon} \int_\Omega \chi_n^\delta \, dx$$

where the characteristic function $\chi_n^\delta := \mathbb{1}_{\left\{\varepsilon(1-\delta)\mathbf{A}e(u_n):e(u_n) \geq K^\delta\right\}} \in L^\infty(\Omega';\{0,1\})$ depends on δ. Even in the case where the above proof could be adapted to show the existence of a displacement $u^\delta \in$

$V_\varepsilon^{\mathrm{Dir}}(\Omega')$ and a characteristic function χ^δ such that (up to a subsequence, not relabeled)

$$\lim_{n\to\infty} \left\{(1-\delta)\int_\Omega (1-\chi_n^\delta)\mathbf{A}e(u_n) : e(u_n)\,dx + \frac{K^\delta}{\varepsilon}\int_\Omega \chi_n^\delta\,dx\right\}$$
$$\geq (1-\delta)\int_\Omega (1-\chi^\delta)\mathbf{A}e(u^\delta) : e(u^\delta)\,dx + \frac{K^\delta}{\varepsilon}\int_\Omega \chi^\delta\,dx,$$

the above lower bound might be too low since $f(t) > \sup_{\delta>0}\left\{K^\delta \wedge (1-\delta)t\right\}$ a priori.

We are now in position to prove the fundamental property of Γ-convergence.

Proof of Corollary 2.3.2. On the one hand, for all $\varepsilon > 0$, we remark that $\mathcal{G}_\varepsilon(u_\varepsilon) \leq \mathcal{G}_\varepsilon(w_{\mathbf{T}_\varepsilon})$ is uniformly bounded due to (2.3.1). Therefore, Proposition 2.3.5 implies that, up to a subsequence, there exist a sequence of piecewise rigid motions $\{r_\varepsilon\}_{\varepsilon>0}$ and a function $u \in GSBD^2(\Omega')$ with $u = w$ $\mathcal{L}^2$-a.e. in $\Omega' \setminus \overline{\Omega}$, such that $u_\varepsilon - r_\varepsilon \to u$ in measure in Ω' and $\liminf_\varepsilon \mathcal{G}_\varepsilon(u_\varepsilon) \geq \mathcal{G}(u)$.

On the other hand, the Γ-convergence of $\mathcal{G}_\varepsilon$ to $\mathcal{G}$ ensures that, for all $v \in GSBD^2(\Omega')$ with $v = w$ $\mathcal{L}^2$-a.e. in $\Omega' \setminus \overline{\Omega}$, there exists a recovery sequence $v_\varepsilon \in L^0(\Omega;\mathbb{R}^2)$ such that $v_\varepsilon \to v$ in measure in Ω' and $\mathcal{G}_\varepsilon(v_\varepsilon) \to \mathcal{G}(v)$. Hence

$$\mathcal{G}(v) = \lim_{\varepsilon\to 0}\mathcal{G}_\varepsilon(v_\varepsilon) \geq \limsup_{\varepsilon\to 0}\mathcal{G}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0}\mathcal{G}_\varepsilon(u_\varepsilon) \geq \mathcal{G}(u),$$

implying both that $u \in \arg\min \mathcal{G}$ and $\mathcal{G}_\varepsilon(u_\varepsilon) \to \mathcal{G}(u)$. □

3 - Discrete models in static brittle damage

This chapter is concerned with partial results on the asymptotic analysis of discrete brittle damage energies in different regimes. These are work in progress.

Let Ω be a bounded open set of $\mathbb{R}^2$ with Lipschitz boundary. As in [39], we introduce the following class of admissible meshes.

Definition 3.0.1. A triangulation of Ω is a finite family of closed triangles intersecting Ω, whose union contains Ω, and such that, given any two triangles of this family, their intersection, if not empty, is exactly a vertex or an edge common to both triangles. Given some angle $\theta_0 > 0$ and a function $h \mapsto \omega(h)$ with $\omega(h) \geq 6h$ for any $h > 0$ and $\lim_{h\to 0^+} \omega(h) = 0$, we define

$$\mathcal{T}_h(\Omega) := \mathcal{T}_h(\Omega, \omega, \theta_0)$$

as the set of all triangulations of Ω made of triangles whose edges have length between h and $\omega(h)$, and whose angles are all greater than or equal to θ_0. Then we consider the finite element space $X_h(\Omega)$ of all couples $(u, \chi) \in C^0(\Omega; \mathbb{R}^2) \times L^\infty(\Omega; \{0, 1\})$ for which there exists $\mathbf{T} \in \mathcal{T}_h(\Omega)$ such that u is affine and χ is constant on each triangle $T \in \mathbf{T}$.

Let $\varepsilon > 0$, $\eta_\varepsilon > 0$ and $h_\varepsilon > 0$. As mentionned before, we introduce the brittle damage functionals $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \to [0, +\infty]$ defined by (1.3.1) :

$$\mathcal{F}_\varepsilon(u, \chi) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega (\eta_\varepsilon \chi \mathbf{A}_0 + (1-\chi)\mathbf{A}_1) e(u) : e(u)\, dx + \dfrac{\kappa}{\varepsilon} \int_\Omega \chi\, dx & \text{if } (u, \chi) \in X_{h_\varepsilon}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\kappa > 0$ and $\mathbf{A}_0$, $\mathbf{A}_1$ are symmetric fourth order tensors satisfying

$$a_i \,\mathrm{Id} \leq \mathbf{A}_i \leq a_i' \,\mathrm{Id} \quad \text{for } i \in \{0, 1\}$$

as quadratic forms over $\mathbb{M}^{N\times N}_{\text{sym}}$, for some constants $a_0, a_1, a_0', a_1' > 0$. We denote by

$$\alpha = \lim_{\varepsilon\to 0} \frac{\eta_\varepsilon}{\varepsilon} \in [0, +\infty], \quad \beta = \lim_{\varepsilon\to 0} \frac{h_\varepsilon}{\varepsilon} \in [0, +\infty]$$

and let $\mathcal{F}'_{\alpha,\beta}$ and $\mathcal{F}''_{\alpha,\beta} : L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \to [0, +\infty]$ be the Γ-lower and Γ-upper limits respectively, *i.e.*, for all $(u, \chi) \in L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega)$,

$$\mathcal{F}'_{\alpha,\beta}(u, \chi) := \inf \left\{ \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) : (u_\varepsilon, \chi_\varepsilon) \to (u, \chi) \text{ in } L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \right\},$$

and

$$\mathcal{F}''_{\alpha,\beta}(u, \chi) := \inf \left\{ \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) : (u_\varepsilon, \chi_\varepsilon) \to (u, \chi) \text{ in } L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \right\}.$$

The choice of the $L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega)$-topology might sometimes be improved (see the regime of brittle fracture 3.3), in the sense that the natural topology should be the one for which sequences of displacements with uniformly bounded energies prove to be compact. Here, for simplicity, we confine ourselves to the strong $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$-topolgy wich in particular provides compactness, whatever is the regime under consideration, for sequences with uniformly bounded energies (with or without boundary condition). We begin with the following result, which in particular gives the domain of the Γ-limit in $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega;\mathbb{R})$ of the functionals (1.3.1) according to the values of α and β.

Proposition 3.0.2. *Let $(u,\chi)\in L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$ be such that $\mathcal{F}'_{\alpha,\beta}(u,\chi)<+\infty$. Then $\chi=0$ a.e. in Ω and if $\alpha=\infty$ or $\beta=\infty$, then $u\in H^1(\Omega;\mathbb{R}^2)$.*

Proof. Let $\delta>0$ and $(u_\varepsilon,\chi_\varepsilon)_{\varepsilon>0}$ be a sequence such that $(u_\varepsilon,\chi_\varepsilon)\to(u,\chi)$ in $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$ as $\varepsilon\searrow 0$ and

$$\liminf_{\varepsilon\searrow 0}\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon)\le\mathcal{F}'_{\alpha,\beta}(u,\chi)+\delta<+\infty.$$

Up to a subsequence (not relabeled), we can assume that the limit

$$\lim_{\varepsilon\searrow 0}\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon)\le\mathcal{F}'_{\alpha,\beta}(u,\chi)+\delta<+\infty$$

exists,

$$M:=\sup_{\varepsilon>0}\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon)<+\infty$$

and $(u_\varepsilon,\chi_\varepsilon)\in X_{h_\varepsilon}(\Omega)$ for all $\varepsilon>0$. Thus, there exists $\mathbf{T}_\varepsilon\in\mathcal{T}_{h_\varepsilon}(\Omega)$ such that for all $T\in\mathbf{T}_\varepsilon$, u_ε is affine and χ_ε is constant on T. Using the energy bound, we infer that

$$\int_\Omega\chi_\varepsilon\,dx\le\frac{M}{\kappa}\varepsilon\to 0\quad\text{as } \varepsilon\searrow 0$$

hence

$$\chi=0\quad\mathcal{L}^2\text{-a.e. in } \Omega.$$

Next, if $\alpha=\infty$, the result follows from the lower bound inequality of [15, Theorem 5.1]. Let us assume that $\beta=\infty$ and show that $u\in H^1(\Omega;\mathbb{R}^2)$. We consider the displacements

$$v_\varepsilon=(1-\chi_\varepsilon)u_\varepsilon\in SBV^2(\Omega;\mathbb{R}^2)$$

and the sets

$$D_\varepsilon=\bigcup_{T\in\mathbf{T}_\varepsilon,\,\chi_{\varepsilon|T}\equiv 1}T$$

which satisfy

$$\nabla v_\varepsilon=(1-\chi_\varepsilon)\nabla u_\varepsilon,\quad J_{v_\varepsilon}\subset\Omega\cap\partial D_\varepsilon\quad\text{and}\quad v_\varepsilon\to u\text{ in } L^1(\Omega;\mathbb{R}^2).$$

Moreover, by coercivity of $\mathbf{A}_1$, we get that

$$\sup_{\varepsilon>0}\int_\Omega|e(v_\varepsilon)|^2\,dx\le\frac{2M}{a_1}<+\infty.\tag{3.0.1}$$

We then consider an exhaustion of Ω by a sequence of smooth open subsets $\{U_m\}_{m\in\mathbb{N}}$ satisfying $U_m\subset\subset U_{m+1}\subset\subset\Omega$ for all $m\in\mathbb{N}$ and $\bigcup_m U_m=\Omega$. In particular, for all $m\in\mathbb{N}$, there exists $\varepsilon(m)>0$ such that for all $0<\varepsilon<\varepsilon(m)$,

$$U_m\cap\partial D_\varepsilon\subset\bigcup_{T\in\mathbf{T}_\varepsilon,\,T\cap U_m\neq\emptyset}\partial T\subset\Omega.$$

Since for all $T \in \mathbf{T}_\varepsilon$

$$\mathcal{H}^1(\partial T) \leq \frac{6\mathcal{L}^2(T)}{h_\varepsilon \sin\theta_0},$$

we obtain that for all $m \in \mathbb{N}$ and all $0 < \varepsilon < \varepsilon(m)$

$$\mathcal{H}^1\big(J_{v_\varepsilon} \cap U_m\big) \leq \frac{6}{h_\varepsilon \sin\theta_0} \int_\Omega \chi_\varepsilon \, dx \leq \frac{6M}{\sin\theta_0} \frac{\varepsilon}{h_\varepsilon}.$$

As $\beta = \infty$, we deduce that for all $m \in \mathbb{N}$

$$\limsup_{\varepsilon \searrow 0} \mathcal{H}^1\big(J_{v_\varepsilon} \cap U_m\big) = 0.$$

In particular, [48, Theorem 11.3] entails that $u_{|U_m} \in GSBD^2(U_m)$ and, up to a subsequence (not relabeled, depending on m),

$$e(v_\varepsilon)_{|U_m} \rightharpoonup e(u_{|U_m}) \quad \text{weakly in } L^2(U_m; \mathbb{M}^{2\times 2}_{\rm sym})$$

when $\varepsilon \searrow 0$ and

$$\mathcal{H}^1\big(J_{u_{|U_m}}\big) = 0.$$

Therefore, one can actually check that $v := u_{|U_m} \in H^1(U_m; \mathbb{R}^2)$ for all $m \in \mathbb{N}$. Indeed, due to the Generalized area formula [6, Theorem 2.91], we have for all $\xi \in \mathbb{S}^1$:

$$0 = \mathcal{H}^1\left(J_v^\xi\right) = \int_{J_v^\xi} |\nu_v \cdot \xi| \, d\mathcal{H}^1 = \int_{\Pi_\xi} \#\left(\left(J_v^\xi\right)_y^\xi\right) \, d\mathcal{H}^1(y)$$

where

$$J_v^\xi := \{x \in J_v \;:\; \xi \cdot \big(v^+(x) - v^-(x)\big) \neq 0\}.$$

In particular, [48, Theorem 8.1] entails that

$$J^1_{v_y^\xi} := \left\{x \in J_{v_y^\xi} \;:\; \left|(v_y^\xi)^+ - (v_y^\xi)^-\right| \geq 1\right\} \subset J_{v_y^\xi} = \emptyset$$

are empty for $\mathcal{H}^1$-a.e. $y \in \Pi_\xi$. Moreover, since $u \in L^1(\Omega)$, we get that $v_y^\xi \in L^1((U_m)_y^\xi)$ for $\mathcal{H}^1$-a.e. $y \in \Pi_\xi$ as well. Therefore, according to [48, Definition 4.2], we infer that for $\mathcal{H}^1$-a.e. $y \in \Pi_\xi$,

$$v_y^\xi \in SBV_{\rm loc}((U_m)_y^\xi) \quad \text{and} \quad J_{v_y^\xi} = \emptyset = J^1_{v_y^\xi}.$$

In particular, [48, Definition 4.1 (b)] entails that

$$\int_{\Pi_\xi} |Dv_y^\xi|\big((U_m)_y^\xi\big) \, d\mathcal{H}^1(y) < +\infty$$

so that

$$v_y^\xi \in BV((U_m)_y^\xi)$$

for $\mathcal{H}^1$-a.e. $y \in \Pi_\xi$. Especially, since $D\big(v_y^\xi\big) \in \mathcal{M}\big((U_m)_y^\xi\big)$ and $|D^s\big(v_y^\xi\big)|(I) = 0$ for all open set $I \subset\subset (U_m)_y^\xi$, we infer that Dv_y^ξ has no Cantor part and no jump part in $(U_m)_y^\xi$, so that

$$v_y^\xi \in W^{1,1}\big((U_m)_y^\xi\big).$$

Hence, [5, Proposition 3.2 and Theorem 4.5] entail that

$$v \in LD(U_m) \cap GSBD^2(U_m),$$

so that

$$v = u_{|U_m} \in H^1(U_m; \mathbb{R}^2)$$

thanks to Korn-Poincaré's inequality (see [33, Theorem 1.1]). Finally, we deduce from the weak convergence of the symmetric gradients in $L^2(U_m; \mathbb{M}^{2\times 2}_{\rm sym})$ and the uniform bound (3.0.1) that

$$\sup_{m\in\mathbb{N}} \|e(u)\|_{L^2(U_m;\mathbb{M}^{2\times 2}_{\rm sym})} < +\infty.$$

Hence, we deduce that the distributional derivative Eu is in $L^2(\Omega; \mathbb{M}^{2\times 2}_{\rm sym})$. As $u \in L^1(\Omega; \mathbb{R}^2)$, applying Korn-Poincaré's inequality once more in Ω leads to

$$u \in H^1(\Omega; \mathbb{R}^2).$$

Also note that, using a diagonal extraction argument, we can find a subsequence $\varepsilon_k \searrow 0$ as $k \nearrow \infty$ such that $v_k := v_{\varepsilon_k}$ satisfies

$$v_k \to u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^2) \quad \text{and} \quad e(v_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{2\times 2}_{\rm sym})$$

when $k \nearrow \infty$. □

3.1 . Linear elasticity

Theorem 3.1.1. *If $\alpha = \infty$ or $\beta = \infty$, then the functional $\mathcal{F}_\varepsilon$ Γ-converges for the strong $L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega)$-topology to the functional $\Psi_{\alpha,\beta} : L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega) \to [0, +\infty]$ defined by*

$$\Psi_{\alpha,\beta}(u,\chi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbf{A}_1 e(u) : e(u)\, dx & \text{if } \begin{cases} \chi = 0 \text{ a.e. in } \Omega, \\ u \in H^1(\Omega; \mathbb{R}^2), \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. According to Proposition 3.0.2, it is enough to identify the Γ-limit for $u \in H^1(\Omega; \mathbb{R}^2)$ and $\chi = 0$.

Lower bound. If $\alpha = \infty$, Theorem [15, Theorem 5.1] ensures that for all $\beta \in [0, \infty]$,

$$\mathcal{F}'_{\infty,\beta}(u, 0) \geq \frac{1}{2}\int_\Omega \mathbf{A}_1 e(u) : e(u)\, dx = \Psi_{\infty,\beta}(u, 0).$$

Now if $\beta = \infty$ and $\alpha \in [0, \infty]$ is arbitrary, arguing as in the proof of Proposition 3.0.2, there exists a sequence $(u_k, \chi_k)_{k\in\mathbb{N}} \subset X_{h_{\varepsilon_k}}(\Omega; \mathbb{R}^2)$ such that

$$(u_k, \chi_k) \to (u, 0) \quad \text{in } L^1(\Omega; \mathbb{R}^2) \times L^1(\Omega),$$

$$\mathcal{F}'_{\alpha,\infty}(u, 0) = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, \chi_k)$$

and the sequence $(v_k)_{k\in\mathbb{N}} = ((1-\chi_k)u_k)_{k\in\mathbb{N}} \subset SBV^2(\Omega;\mathbb{R}^2)$ satisfies

$$v_k \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2) \quad \text{and} \quad e(v_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{2\times 2}_{\rm sym}) \quad \text{as } k \nearrow \infty.$$

Therefore,

$$\mathcal{F}'_{\alpha,\infty}(u,0) \geq \liminf_{k\to\infty} \frac{1}{2}\int_\Omega \mathbf{A}_1 e(v_k) : e(v_k)\,dx \geq \frac{1}{2}\int_\Omega \mathbf{A}_1 e(u) : e(u)\,dx = \Psi_{\alpha,\infty}(u,0).$$

Upper bound. Since Ω has a Lipschitz boundary, we can extend u and assume that $u \in H^1(\mathbb{R}^2;\mathbb{R}^2)$. By lower semi continuity of $\mathcal{F}''_{\alpha,\beta}$ and density of $C^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$ in $H^1(\mathbb{R}^2;\mathbb{R}^2)$, we can assume without loss of generality that $u \in C^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$. For all $\varepsilon > 0$, let us fix any triangulation $\mathbf{T}_\varepsilon \in \mathcal{T}_{h_\varepsilon}(\Omega)$ and let $u_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$ be the Lagrange interpolation of the values of u at the vertices of $\mathbf{T}_\varepsilon$. According to [44, Theorem 3.1.5], there exists a constant $C(\theta_0) > 0$ such that for all $\varepsilon > 0$ and $T \in \mathbf{T}_\varepsilon$

$$\|u_\varepsilon - u\|_{H^1(T;\mathbb{R}^2)} \leq C\omega(h_\varepsilon)\|D^2 u\|_{L^2(T)}.$$

Therefore, $(u_\varepsilon, 0) \in X_{h_\varepsilon}(\Omega)$ and

$$u_\varepsilon \to u \quad \text{strongly in } H^1(\Omega;\mathbb{R}^2),$$

so that

$$\mathcal{F}_\varepsilon(u_\varepsilon,0) = \frac{1}{2}\int_\Omega \mathbf{A}_1 e(u_\varepsilon) : e(u_\varepsilon)\,dx \to \frac{1}{2}\int_\Omega \mathbf{A}_1 e(u) : e(u)\,dx,$$

which shows that $\mathcal{F}''_{\alpha,\beta}(u,0) \leq \Psi_{\alpha,\beta}(u,0)$. □

3.2 . Trivial regime

Theorem 3.2.1. *Assume that $\theta_0 \leq 45°$. If $\alpha = \beta = 0$, then the functional $\mathcal{F}_\varepsilon$ Γ-converges for the strong $L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega)$-topology to the functional $\Psi_{0,0} : L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \to [0,+\infty]$ defined by*

$$\Psi_{0,0}(u,\chi) = \begin{cases} 0 & \text{if } \chi = 0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. According to Proposition 3.0.2, it is enough to identify the Γ-limit for $u \in L^1(\Omega;\mathbb{R}^2)$ and $\chi = 0$. The proof of the lower bound is then straightforward, as

$$\mathcal{F}'_{0,0}(u,0) \geq 0.$$

Let us prove the upper bound inequality. By a rescaling and translation argument, we can assume without loss of generality that Ω is the unit cube $Q := (0,1)^2$. Next, arguing as in the proof of [15, Theorem 4.1], we can assume that u is a piecewise constant function of the form

$$u = \sum_{i\in[\![0,N-1]\!]^2} u_i \mathbb{1}_{Q_i}$$

for some constant vectors $u_i \in \mathbb{R}^2$ and where

$$Q_i := \frac{1}{N}(i + Q)$$

for all $i \in [\![0, N-1]\!]^2$. For all $\varepsilon > 0$, we define the parameter $\delta_\varepsilon := \sqrt{\varepsilon\,(\eta_\varepsilon + h_\varepsilon)} > 0$ which satisfies

$$\eta_\varepsilon \ll \delta_\varepsilon \ll \varepsilon \quad \text{and} \quad h_\varepsilon \ll \delta_\varepsilon \ll \varepsilon. \tag{3.2.1}$$

We will first construct a background triangulation $\mathbf{T}_{\delta_\varepsilon} \in \mathcal{T}_{\delta_\varepsilon}(Q)$ and a recovery sequence $(u_\varepsilon, \chi_\varepsilon) \in L^1(Q;\mathbb{R}^2) \times L^1(Q)$ adapted to this triangulation in the sense of Definition (3.0.1). Next, since the mesh-size $\delta_\varepsilon \gg h_\varepsilon$, we will further subdivide the triangulation $\mathbf{T}_{\delta_\varepsilon}$ in order to obtain an admissible triangulation $\mathbf{T}_\varepsilon^{\rm rec} \in \mathcal{T}_{h_\varepsilon}(Q)$ (see Figure 3.2).

Assume that $\varepsilon > 0$ is small enough so that $\delta_\varepsilon < 1/(2N)$ and define

$$N_\varepsilon := \left\lfloor \frac{1}{\delta_\varepsilon N} \right\rfloor \geq 2 \quad \text{and} \quad l_\varepsilon := \frac{1}{N_\varepsilon N},$$

so that

$$\delta_\varepsilon \leq l_\varepsilon \leq 2\delta_\varepsilon.$$

We denote by $Q_M := \frac{1}{M} Q$ the cube of side length $\frac{1}{M} > 0$. We then subdivide the cube Q_N into N_ε^2 subcubes of side length l_ε, so that

$$Q_N = \bigcup_{j \in [\![0, N_\varepsilon - 1]\!]^2} \left(\frac{j}{N_\varepsilon N} + Q_{N_\varepsilon N} \right) \quad \text{and} \quad Q = \bigcup_{\substack{i \in [\![0, N-1]\!]^2 \\ j \in [\![0, N_\varepsilon - 1]\!]^2}} \left(\frac{i}{N} + \frac{j}{N_\varepsilon N} + Q_{N_\varepsilon N} \right)$$

up to a $\mathcal{L}^2$-negligible set. These $(N_\varepsilon N)^2$ rectangles are then divided into two isoceles right triangles with edges of length l_ε and $\sqrt{2} l_\varepsilon$. In particular,

$$\delta_\varepsilon \leq l_\varepsilon \leq \sqrt{2} l_\varepsilon \leq 2\sqrt{2}\delta_\varepsilon \leq \omega(\delta_\varepsilon)$$

so that it defines a triangulation $\mathbf{T}_{\delta_\varepsilon} \in \mathcal{T}_{\delta_\varepsilon}(Q)$ as illustrated in Figure 3.1. We consider the cut-off function $\phi_N : \mathbb{R}^2 \to [0,1]$ defined as the Lagrange interpolation of the values 0 at the vertices on the boundary ∂Q_N and 1 at the vertices inside Q_N, which we extend by 0 outside $\overline{Q}_N$. Then, ϕ_N is continuous and piecewise affine, affine on each triangle $T \in \mathbf{T}_{\delta_\varepsilon}$, such that

$$\phi_N \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus Q_N \quad \text{and} \quad \phi_N \equiv 1 \quad \text{in } Q_N^\varepsilon := \left[l_\varepsilon, \frac{1}{N} - l_\varepsilon \right]^2$$

as illustrated in Figure 3.1. Defining the characteristic function

$$\chi_\varepsilon = \mathbb{1}_{D_\varepsilon}$$

of the set

$$D_\varepsilon = \bigcup_{i \in [\![0, N-1]\!]^2} \left(\frac{i}{N} + Q_N \setminus Q_N^\varepsilon \right)$$

and the displacement

$$u_\varepsilon = \sum_{i\in[\![0,N-1]\!]^2} u_i \phi_N\left(\cdot - \frac{i}{N}\right),$$

one can check that $u_\varepsilon \in C^0(Q;\mathbb{R}^2)$, u_ε is affine and χ_ε is constant on each triangle $T \in \mathbf{T}_{\delta_\varepsilon}$ and

$$\chi_\varepsilon \to 0 \quad \text{in } L^1(Q), \tag{3.2.2}$$

$$u_\varepsilon \to u \quad \text{in } L^1(Q;\mathbb{R}^2), \tag{3.2.3}$$

$$\frac{1}{2}\int_Q (\eta_\varepsilon\chi_\varepsilon \mathbf{A}_0 + (1-\chi_\varepsilon)\mathbf{A}_1)\, e(u_\varepsilon) : e(u_\varepsilon)\, dx + \frac{\kappa}{\varepsilon}\int_Q \chi_\varepsilon\, dx \to 0 \quad \text{when } \varepsilon \searrow 0. \tag{3.2.4}$$

Indeed,

$$\|\chi_\varepsilon\|_{L^1(Q)} = \mathcal{L}^2(D_\varepsilon) = N^2\mathcal{L}^2\left(Q_N \setminus Q_N^\varepsilon\right) \le 4Nl_\varepsilon \le 8N\delta_\varepsilon \to 0$$

and

$$\|u - u_\varepsilon\|_{L^1(Q;\mathbb{R}^2)} = \sum_{i\in[\![0,N-1]\!]^2} |u_i| \int_{Q_i} \left|1 - \phi_N\left(x - \frac{i}{N}\right)\right| dx \le \|u\|_{L^\infty(Q;\mathbb{R}^2)}\, \mathcal{L}^2(D_\varepsilon) \to 0$$

when $\varepsilon \searrow 0$, which proves (3.2.2) and (3.2.3). On the other hand, by definition of ϕ_N and $\mathbf{T}_{\delta_\varepsilon}$, there exists a constant $C(\theta_0) > 0$ such that

$$\|\nabla\phi_N\|_{L^\infty} \le \frac{C}{\delta_\varepsilon}.$$

Since

$$e(u_\varepsilon) = \sum_{i\in[\![0,N-1]\!]^2} u_i \odot \nabla\phi_N\left(\cdot - \frac{i}{N}\right)$$

is supported in $D_\varepsilon = \{\chi_\varepsilon = 1\}$, we infer that

$$\begin{aligned}\frac{1}{2}\int_Q (\eta_\varepsilon\chi_\varepsilon \mathbf{A}_0 + (1-\chi_\varepsilon)\mathbf{A}_1)\, e(u_\varepsilon) : e(u_\varepsilon)\, dx + \frac{\kappa}{\varepsilon}\int_Q \chi_\varepsilon\, dx &\le \frac{a_0'}{2}\|u\|^2_{L^\infty(Q;\mathbb{R}^2)}\frac{\eta_\varepsilon C^2}{\delta_\varepsilon^2}\mathcal{L}^2(D_\varepsilon) + \frac{\kappa}{\varepsilon}\mathcal{L}^2(D_\varepsilon)\\ &\le C\big(\theta_0, a_0', u, \kappa\big)\left(\frac{\eta_\varepsilon}{\delta_\varepsilon} + \frac{\delta_\varepsilon}{\varepsilon}\right)\end{aligned}$$

which proves (3.2.4) due to (3.2.1).

It remains to construct an admissible triangulation $\mathbf{T}_\varepsilon^{\text{rec}} \in \mathcal{T}_{h_\varepsilon}(Q)$ still adapted to $(u_\varepsilon, \chi_\varepsilon)$ in the sense of Definition 3.0.1. Note that because $(u_\varepsilon, \chi_\varepsilon)$ are adapted to the background triangulation $\mathbf{T}_{\delta_\varepsilon}$, as long as we modify $\mathbf{T}_{\delta_\varepsilon}$ by subdividing some of its triangles, $(u_\varepsilon, \chi_\varepsilon)$ will remain adapted to the modified triangulation. Assume that $\varepsilon > 0$ is sufficiently small so that

$$\frac{h_\varepsilon}{\delta_\varepsilon} < \frac{1}{\sqrt{2}}$$

and define

$$n_\varepsilon := \max\left\{n \in \mathbb{N} \,:\, \frac{h_\varepsilon}{\delta_\varepsilon} \le \sqrt{2}^{-n}\right\}.$$

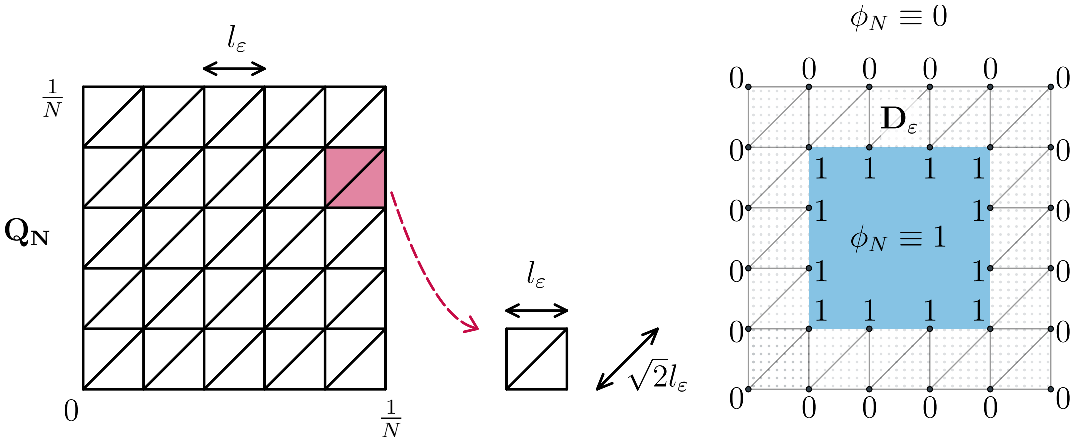

Figure 3.1

In particular, one can check that $n_\varepsilon \geq 1$ and

$$h_\varepsilon \leq \sqrt{2}^{-n_\varepsilon}\delta_\varepsilon < \sqrt{2}h_\varepsilon.$$

We will now construct $\mathbf{T}_\varepsilon^{\text{rec}} \in \mathcal{T}_{h_\varepsilon}(Q)$ recursively in n_ε steps, by subdividing each triangle $T \in \mathbf{T}_{\delta_\varepsilon}$ into 2^{n_ε} isocele right subtriangles as follows : we subdivide T along the height corresponding to its right angle, resulting into two adjacent closed subtriangles, as in Figure 3.2. One can check that these subtriangles are both isocele and right triangles, homotetic to T by a factor $\sqrt{2}^{-1}$ up to translations and rotations, *i.e.* : there sides lengths are given by $\sqrt{2}^{-1}l_\varepsilon$ and l_ε for their shortest and hypothenus edges respectively. We repeat this procedure for each new subtriangle, in total n_ε times. Thus, we obtain 2^{n_ε} isocele right substriangles of T, homotetic to T by a factor $\sqrt{2}^{-n_\varepsilon}$. In particular, all their edges have lenghth $\sqrt{2}^{-n_\varepsilon}l_\varepsilon$ or $\sqrt{2}^{-n_\varepsilon+1}l_\varepsilon$, which satisfy

$$h_\varepsilon \leq \sqrt{2}^{-n_\varepsilon}l_\varepsilon \leq \sqrt{2}^{-n_\varepsilon+1}l_\varepsilon \leq 4h_\varepsilon \leq \omega(h_\varepsilon).$$

Therefore, we obtain an admissible triangulation $\mathbf{T}_\varepsilon^{\text{rec}} \in \mathcal{T}_{h_\varepsilon}(Q)$ such that u_ε is affine and χ_ε is constant on each of its triangle, and $u_\varepsilon \in C^0(Q;\mathbb{R}^2)$. We infer that $(u_\varepsilon, \chi_\varepsilon) \in X_{h_\varepsilon}(Q)$ and

$$\mathcal{F}''_{0,0}(u,0) \leq \limsup_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) = 0$$

thanks to (3.2.4), which completes the proof of the upper bound inequality. □

3.3 . Brittle fracture

Theorem 3.3.1. *If $\alpha = 0$ and $\beta \in (0,\infty)$, then the functional $\mathcal{F}_\varepsilon$ Γ-converges for the strong $L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega)$-topology to the functional $\Psi_{0,\beta} : L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \to [0,+\infty]$ defined by*

$$\Psi_{0,\beta}(u,\chi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbf{A}_1 e(u) : e(u)\,dx + \beta\kappa\sin\theta_0\mathcal{H}^1(J_u) & \textit{if } \begin{cases}\chi = 0 \textit{ a.e. in } \Omega,\\ u \in GSBD^2(\Omega) \cap L^1(\Omega;\mathbb{R}^2),\end{cases} \\ +\infty & \textit{otherwise.}\end{cases}$$

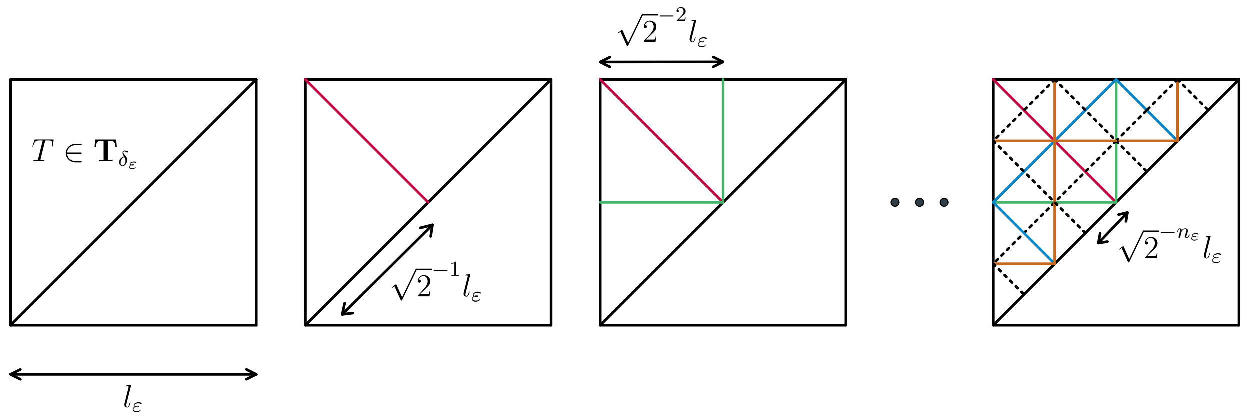

Figure 3.2

Note that the same result holds in $L^0(\Omega;\mathbb{R}^2)\times L^0(\Omega)$ for the topology of convergence in measure, up to enlarging the domain of the Γ-limit to

$$\left\{(u,\chi)\in L^0(\Omega;\mathbb{R}^2)\times L^0(\Omega)\,:\,u\in GSBD^2(\Omega),\,\chi=0\right\}.$$

The proof is almost exactly the same as below.

Proof. According to Proposition 3.0.2, it is enough to identify the Γ-limit for $\chi=0$.

Lower bound. First note that, according to [12, Theorem 1.3] (which is the core result proved in Chapter 2), the functional $\mathcal{G}_\varepsilon : L^0(\Omega;\mathbb{R}^2)\to[0,+\infty]$ defined by

$$\mathcal{G}_\varepsilon(u)=\begin{cases}\displaystyle\int_\Omega \min\left(\frac{1}{2}\mathbf{A}_1 e(u):e(u)\,;\,\frac{\beta\kappa}{h_\varepsilon}\right)dx & \text{if } u\in V_{h_\varepsilon}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

Γ-converges, for the topology of convergence in measure in $L^0(\Omega;\mathbb{R}^2)$, to $\mathcal{G}: L^0(\Omega;\mathbb{R}^2)\to[0,+\infty]$ defined by

$$\mathcal{G}(u)=\begin{cases}\displaystyle\frac{1}{2}\int_\Omega \mathbf{A}_1 e(u):e(u)\,dx+\beta\kappa\sin\theta_0\mathcal{H}^1(J_u) & \text{if } u\in GSBD^2(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

In particular, the lower bound inequality presents no difficulty. Indeed, for all $u\in L^1(\Omega;\mathbb{R}^2)$ and all sequences $(u_\varepsilon,\chi_\varepsilon)_{\varepsilon>0}\in X_{h_\varepsilon}(\Omega)$ converging to $(u,0)$ in $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$, one can check that $u_\varepsilon\in V_{h_\varepsilon}(\Omega)$ converges in measure to u. Hence, $\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon)\geq\mathcal{G}_\varepsilon(u_\varepsilon)$ and passing to the infimum among such sequences leads to

$$\mathcal{F}'_{0,\beta}(u,0)\geq\mathcal{G}(u)=\Psi_{0,\beta}(u,0).$$

Upper bound. We can assume that $\Psi_{0,\beta}(u,0)<+\infty$ and thus that $u\in GSBD^2(\Omega)\cap L^1(\Omega;\mathbb{R}^2)$. Using the density result for $GSBD$ functions (see [36, Theorem 1.1]) as well as the lower semicontinuity of $\mathcal{F}''_{0,\beta}$ with respect to the convergence in $L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega)$, we can further assume without loss of generality that $u \in SBV^2(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$. Looking at the constructive proof for the recovery sequence in [12, Proposition 3.11], there exists $u_\varepsilon \in V_{h_\varepsilon}(\Omega)$ and a triangulation $\mathbf{T}_{h_\varepsilon} \in \mathcal{T}_{h_\varepsilon}(\Omega)$ such that u_ε is affine on each of its triangle and

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^2)}, \tag{3.3.1}$$

$$u_\varepsilon \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2) \quad \text{as } \varepsilon \searrow 0,$$

$$\mathcal{G}_\varepsilon(u_\varepsilon) \to \mathcal{G}(u) \quad \text{as } \varepsilon \searrow 0. \tag{3.3.2}$$

In particular, since u_ε is affine on each triangle $T \in \mathbf{T}_{h_\varepsilon}$, the fact that the length of every edge of T is larger than h_ε together with (3.3.1) imply that there exists a constant $C(\theta_0) > 0$ such that

$$\|e(u_\varepsilon)\|_{L^\infty(T;\mathbb{R}^2)} \le \frac{C\|u\|_{L^\infty(\Omega;\mathbb{R}^2)}}{h_\varepsilon}. \tag{3.3.3}$$

Introducing the characteristic functions

$$\chi_\varepsilon = \mathbb{1}_{\{\mathbf{A}_1 e(u_\varepsilon) : e(u_\varepsilon) \ge 2\kappa\beta/h_\varepsilon\}},$$

one can check that $(u_\varepsilon,\chi_\varepsilon) \in X_{h_\varepsilon}(\Omega)$ and

$$\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon) = \mathcal{G}_\varepsilon(u_\varepsilon) + \frac{1}{2}\int_\Omega \eta_\varepsilon \chi_\varepsilon \mathbf{A}_0 e(u_\varepsilon) : e(u_\varepsilon)\,dx + \left(\frac{h_\varepsilon}{\varepsilon} - \beta\right)\frac{\kappa}{h_\varepsilon}\int_\Omega \chi_\varepsilon\,dx.$$

On the one hand, (3.3.2) entails that

$$M := \sup_{\varepsilon>0} \frac{\kappa}{h_\varepsilon}\int_\Omega \chi_\varepsilon\,dx < +\infty$$

is bounded, so that

$$(u_\varepsilon,\chi_\varepsilon) \to (u,0) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2)\times L^1(\Omega) \text{ when } \varepsilon \searrow 0.$$

On the other hand, using the growth condition of $\mathbf{A}_0$ together with (3.3.3), we infer that there exists a constant $C'(a_0';\kappa;\theta_0;u) > 0$ such that

$$\mathcal{F}_\varepsilon(u_\varepsilon,\chi_\varepsilon) \le \mathcal{G}_\varepsilon(u_\varepsilon) + C'M\left(\frac{\eta_\varepsilon}{h_\varepsilon} + \frac{h_\varepsilon}{\varepsilon} - \beta\right) \to \mathcal{G}(u) = \Psi_{0,\beta}(u,0) \quad \text{when } \varepsilon \searrow 0,$$

where we also used the facts that $\lim_{\varepsilon\searrow 0}(\eta_\varepsilon/\varepsilon) = \alpha = 0$ and $\lim_{\varepsilon\searrow 0}(h_\varepsilon/\varepsilon) = \beta \in (0,+\infty)$. □

3.4 . Plasticity

We begin the study of the regime of plasticity with the following naive result, where the constraint on the mesh-size is relaxed as we allow any scale smaller than h_ε.

Theorem 3.4.1. *We define, for all $\varepsilon > 0$, the functionals $\widehat{\mathcal{F}_\varepsilon} : L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \to [0,+\infty]$ given by*

$$\widehat{\mathcal{F}_\varepsilon}(u,\chi) = \begin{cases} \mathcal{F}_\varepsilon(u,\chi) & \text{if } (u,\chi) \in X_h(\Omega) \text{ for some } 0 < h \leq h_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

If $\alpha \in (0,+\infty)$, then the functional $\widehat{\mathcal{F}_\varepsilon}$ Γ-converges for the strong $L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega)$-topology to the functional

$$\widehat{\Psi}_\alpha(u,\chi) = \begin{cases} \displaystyle\int_\Omega \overline{W}_\alpha(e(u))\,dx + \int_\Omega \overline{W}_\alpha^\infty\left(\frac{dE^s u}{d\,|E^s u|}\right) d\,|E^s u| & \text{if } u \in BD(\Omega) \text{ and } \chi = 0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 3.4.2. Note that the above statement is independent of the converging rate $\beta \in [0,+\infty]$. This yields to a painful lack of accuracy, since the elasticity regime ($\beta = +\infty$) and the fracture-plasticity coupled regime ($\beta \in (0,+\infty)$ for the dimension one) are included and yet indistinguishable in the above convergence. In other words, this convergence result does not detect the regimes of elasticity and coupled fracture-plasticity.

Proof. Following a minor adaptation of Proposition 3.0.2, it is enough to assume that $\chi = 0$. We denote by $\widehat{\mathcal{F}}'_{\alpha,0}$ and $\widehat{\mathcal{F}}''_{\alpha,0} : L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \to [0,+\infty]$ the Γ-lower and Γ-upper limits respectively.

Lower bound. Let $\delta > 0$ and $u \in L^1(\Omega;\mathbb{R}^2)$ such that $\widehat{F}'_{\alpha,0}(u,0) < +\infty$. Consider a sequence $(u_\varepsilon,\chi_\varepsilon) \in L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega)$ such that

$$(u_\varepsilon,\chi_\varepsilon) \to (u,0) \quad \text{in } L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \text{ as } \varepsilon \searrow 0$$

and, up to a subsequence (not relabeled), the following limit exists and satisfies

$$\lim_{\varepsilon\searrow 0} \widehat{\mathcal{F}_\varepsilon}(u_\varepsilon,\chi_\varepsilon) \leq \widehat{\mathcal{F}}'_{\alpha,0}(u,0) + \delta < +\infty.$$

In particular, there exists $0 < \overline{h}_\varepsilon \leq h_\varepsilon$ such that $(u_\varepsilon,\chi_\varepsilon) \in X_{\overline{h}_\varepsilon}(\Omega) \subset H^1(\Omega;\mathbb{R}^2) \times L^\infty(\Omega;\{0,1\})$ for all $\varepsilon > 0$. Hence, the lower bound inequality of [15, Theorem 3.1] entails that $u \in BD(\Omega)$ and

$$\delta + \widehat{\mathcal{F}}'_{\alpha,0}(u,0) \geq \widehat{\Psi}_\alpha(u,0)$$

which completes the proof of the lower bound by letting $\delta \searrow 0$.

Upper bound. The result being trivial when $\widehat{\Psi}_\alpha(u,0) = +\infty$, we can assume that $u \in BD(\Omega)$. The Γ-convergence of [15, Theorem 3.1] then ensures the existence of a recovery sequence

$$(u_\varepsilon,\chi_\varepsilon) \in H^1(\Omega;\mathbb{R}^2) \times L^\infty(\Omega;\{0,1\})$$

such that

$$(u_\varepsilon,\chi_\varepsilon) \to (u,0) \quad \text{in } L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega)$$

and

$$E_\varepsilon(u_\varepsilon,\chi_\varepsilon) := \frac{1}{2}\int_\Omega (\eta_\varepsilon\chi_\varepsilon\mathbf{A}_0 + (1-\chi_\varepsilon)\mathbf{A}_1)\,e(u_\varepsilon) : e(u_\varepsilon)\,dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi_\varepsilon\,dx \to \widehat{\Psi}_\alpha(u,0) \quad \text{as } \varepsilon \searrow 0.$$

On the other hand, according to standard finite element approximation, one can find a sequence $u_\varepsilon^h \in V_h(\Omega)$ such that $u_\varepsilon^h \to u_\varepsilon$ in $H^1(\Omega;\mathbb{R}^2)$ as $h \searrow 0$, for all $\varepsilon > 0$. Therefore, one can find a sequence $\widehat{h}_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$, with

$$0 < \widehat{h}_\varepsilon \leq h_\varepsilon,$$

and displacements $v_\varepsilon := u_\varepsilon^{\widehat{h}_\varepsilon} \in V_{\widehat{h}_\varepsilon}(\Omega)$ such that

$$\|v_\varepsilon - u_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)} \leq \varepsilon \quad \text{and} \quad v_\varepsilon \to u \text{ in } L^1(\Omega;\mathbb{R}^2) \text{ as } \varepsilon \searrow 0.$$

Then, we consider the characteristic functions

$$\widehat{\chi}_\varepsilon = \mathbb{1}_{\{(\mathbf{A}_1 - \eta_\varepsilon \mathbf{A}_0)\, e(v_\varepsilon) : e(v_\varepsilon) \geq 2\kappa/\varepsilon\}} \in L^\infty(\Omega;\{0,1\}),$$

which satisfy

$$(v_\varepsilon, \widehat{\chi}_\varepsilon) \in X_{\widehat{h}_\varepsilon}(\Omega)$$

and

$$\mathcal{E}_\varepsilon(v_\varepsilon, \chi_\varepsilon) \geq \overline{\mathcal{F}}_\varepsilon(v_\varepsilon, \widehat{\chi}_\varepsilon).$$

One can check that

$$\mathcal{E}_\varepsilon(v_\varepsilon, \chi_\varepsilon) = \frac{1}{2}\int_\Omega (\eta_\varepsilon \chi_\varepsilon \mathbf{A}_0 + (1-\chi_\varepsilon)\mathbf{A}_1)\, e(v_\varepsilon) : e(v_\varepsilon)\, dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi_\varepsilon\, dx \to \Psi_{\alpha,0}(u,0) \quad \text{as } \varepsilon \searrow 0$$

since $e(v_\varepsilon) - e(u_\varepsilon) \to 0$ in $L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ as $\varepsilon \searrow 0$. In particular, we infer that

$$\frac{\kappa}{\varepsilon}\int_\Omega \widehat{\chi}_\varepsilon\, dx \leq \mathcal{E}_\varepsilon(v_\varepsilon, \chi_\varepsilon)$$

is bounded, hence

$$(v_\varepsilon, \widehat{\chi}_\varepsilon) \to (u, 0) \quad \text{in } L^1(\Omega;\mathbb{R}^2) \times L^1(\Omega) \quad \text{when } \varepsilon \searrow 0.$$

Therefore,

$$\widehat{\mathcal{F}}''_{\alpha,0}(u,0) \leq \limsup_{\varepsilon \searrow 0} \widehat{\mathcal{F}}_\varepsilon(v_\varepsilon, \overline{\chi}_\varepsilon) \leq \widehat{\Psi}_\alpha(u,0)$$

which completes the proof of the upper bound. □

Remark 3.4.3. Note that imposing $\widehat{h}_\varepsilon \leq h_\varepsilon$ in the construction of the recovery sequence is not a difficulty and does not rely on the convergence rate $\beta \in [0,+\infty]$. The issue is rather to ensure the reverse inequality, which one would intuit to be true in the specific regime $\beta = 0$. Indeed, in this regime, $\varepsilon > 0$ is heuristically interpreted as constant with respect to h_ε, so that the convergence of the discrete model to the continuous setting in space occurs faster than the interplay between concentration and elastic degeneracy of the weak material. Looking at the computation of the Hashin-Shtrikman bounds (see [2, Proposition 2.3.20]), we get that recovery sequences happen to be mixtures, obtained by successive laminations between the phases $\eta_\varepsilon \mathbf{A}_0$ and $\mathbf{A}_1$. Therefore, one could hope to solve the discretization scale issue thanks to the directions of oscillations of the recovery sequence, the idea being to adapt the scale of the spatial discretization to the scale of the lamination directions. Yet, as illustrated in the following example, ensuring that $\widehat{h}_\varepsilon \geq h_\varepsilon$ seems not guaranted by this procedure, even in the simplified one-dimensional setting.

Example 3.4.4. Let us assume that $\Omega = (0, L)$ is a bounded open interval of $\mathbb{R}$ and $u \in H^1(\Omega)$ is affine, so that

$$u' =: \xi \in \mathbb{R} \quad \text{is a constant.}$$

Let $a_0, a_1 > 0$ be the elasticity tensors of the weak and strong material respectively. As proved in [15, Proposition 3.3], the convex envelope $\mathcal{C}W_\varepsilon : \mathbb{R} \to [0, +\infty)$ of

$$W_\varepsilon : \tau \in \mathbb{R} \mapsto \min\left(\frac{\kappa}{\varepsilon} + \frac{1}{2}\eta_\varepsilon a_0 \tau^2 \, ; \, \frac{1}{2} a_1 \tau^2\right)$$

pointwise converges to $\overline{W}_\alpha = \left(\frac{1}{2} a_1 |\cdot|^2\right) \square \sqrt{2\kappa a_0 \alpha}\, |\cdot|$ when $\varepsilon \searrow 0$. Thus, $\mathcal{C}W_\varepsilon(\xi) \to \overline{W}_\alpha(\xi)$ as $\varepsilon \searrow 0$. As we are working in the scalar setting, [2, Lemma 1.3.32, Formula (1.109)] stipulates that the $\mathcal{G}$-closure set of all possible mixtures between the phases $\eta_\varepsilon a_0$ and a_1 with proportions $\theta \in [0, 1]$ and $1 - \theta$ respectively, $\mathcal{G}_\theta(\eta_\varepsilon a_0, a_1)$, is reduced to the singleton

$$a_\theta^\varepsilon := \left(\frac{\theta}{\eta_\varepsilon a_0} + \frac{1 - \theta}{a_1}\right)^{-1}.$$

In particular, according to [3, Lemma 3.1] we deduce that

$$\mathcal{C}W_\varepsilon(\xi) = \min_{\theta \in [0,1]} \left\{\frac{\kappa\theta}{\varepsilon} + \frac{1}{2} a_\theta^\varepsilon \xi^2\right\}.$$

The above minimization being over a strictly convex function, the minimum is indeed achieved at a unique minimizer $\theta_\varepsilon \in [0, 1]$. Denoting $a_{\theta_\varepsilon} := a_{\theta_\varepsilon}^\varepsilon$, we infer that

$$\mathcal{C}W_\varepsilon(\xi) = \frac{\kappa\theta_\varepsilon}{\varepsilon} + \frac{1}{2} a_{\theta_\varepsilon} \xi^2.$$

The idea is now to exploit the fact that a_{θ_ε} is also a laminate. Indeed, for fixed $\varepsilon > 0$, let us consider the characteristic function

$$\chi_\varepsilon := \begin{cases} 1 & \text{in } (0, \theta_\varepsilon), \\ 0 & \text{in } (\theta_\varepsilon, 1), \end{cases} \quad \text{in } (0, 1)$$

which we extend to $\mathbb{R}$ by 1-periodicity. Then, for all $h > 0$, we consider the rescaled characteristic function

$$\chi_\varepsilon^h := \chi_\varepsilon\left(\frac{\cdot}{h}\right) \in L^\infty(\mathbb{R}; \{0, 1\}).$$

Applying Riemann-Lebesgue's Theorem (see [18, Example 2.7]), we have that $\chi_\varepsilon^h \overset{*}{\rightharpoonup} \theta_\varepsilon$ weakly-* in $L^\infty(\mathbb{R})$ as $h \searrow 0$. Therefore, [2, Lemma 1.4.10] entails that

$$a_{\chi_\varepsilon^h} := \left(\chi_\varepsilon^h \eta_\varepsilon a_0 + (1 - \chi_\varepsilon^h) a_1\right)_{|\Omega}$$

H-converges in Ω to a_{θ_ε} when $h \searrow 0$. In particular, the solutions $u_\varepsilon^h \in u + H_0^1(\Omega)$ of

$$\begin{cases} -\left(a_\varepsilon^h (u_\varepsilon^h)'\right)' = 0 & \text{in } H^{-1}(\Omega), \\ u_\varepsilon^h = u & \text{on } \partial\Omega, \end{cases}$$

are such that $\{u_\varepsilon^h - u\}_h$ is bounded in $H_0^1(\Omega)$. Hence, there exists a subsequence $h_k \searrow 0$ as $k \nearrow \infty$ and $v \in H_0^1(\Omega)$ such that $u_\varepsilon^{h_k} \rightharpoonup u+v$ weakly in $H^1(\Omega)$. By the property of independence with respect to the boundary conditions for the H-convergence (see [2, Proposition 1.4.6]), we infer that

$$a_\varepsilon^{h_k}(u_\varepsilon^{h_k})' \rightharpoonup a_{\theta_\varepsilon}(u+v)' \quad \text{weakly in } L^2(\Omega) \text{ when } k \nearrow \infty.$$

Yet, since $(a_{\theta_\varepsilon}(u+v)')' = 0 = a_{\theta_\varepsilon} v''$ in $H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$, we infer by Lax-Milgram that $v = 0$. Since the limit is independent of the subsequence, we deduce that the whole sequence is such that

$$\begin{cases} u_\varepsilon^h \rightharpoonup u & \text{weakly in } H^1(\Omega), \\ u_\varepsilon^h \to u & \text{strongly in } L^2(\Omega), \\ a_\varepsilon^h(u_\varepsilon^h)' \rightharpoonup a_{\theta_\varepsilon}\xi & \text{weakly in } L^2(\Omega), \end{cases} \qquad \text{when } h \searrow 0$$

where we also used Rellich's Theorem for the second convergence. Thus, one can check that

$$\frac{1}{2}\int_\Omega a_\varepsilon^h \left|(u_\varepsilon^h)'\right|^2 dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi_\varepsilon^h \, dx \underset{h\searrow 0}{\longrightarrow} \frac{1}{2}\int_\Omega a_{\theta_\varepsilon}\xi^2 \, dx + \frac{\kappa}{\varepsilon}\int_\Omega \theta_\varepsilon \, dx = \int_\Omega \mathcal{C}W_\varepsilon(\xi)\, dx$$
$$\underset{\varepsilon\searrow 0}{\longrightarrow} \int_\Omega \overline{W}_\alpha(\xi)\, dx$$

and therefore

$$\theta_\varepsilon \to 0 \quad \text{strongly in } L^1(\Omega) \text{ when } \varepsilon \searrow 0.$$

Moreover, χ_ε^h and u_ε^h are respectively piecewise constant and piecewise affine on each subinterval $m + (0, \theta_\varepsilon h)$ and $m + (\theta_\varepsilon h, h)$ for $m \in [\![0, \lfloor L/h \rfloor + 1]\!]$, as illustrated in Figure 3.3.

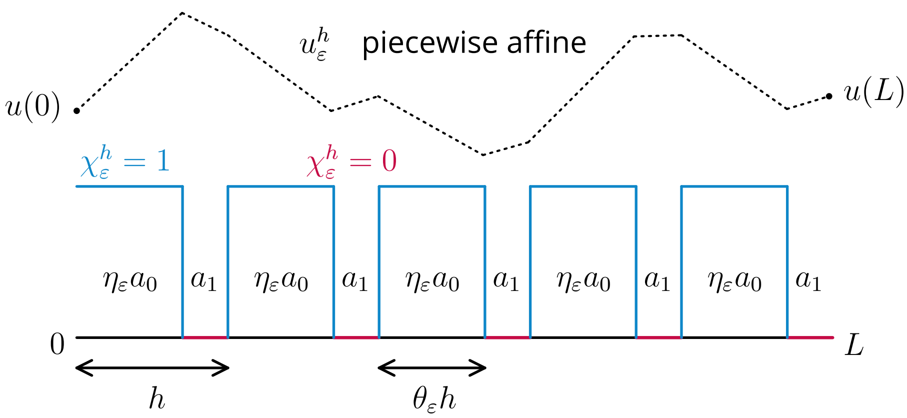

Figure 3.3

The hope is then to find an adequate diagonal extraction $\overline{h}_\varepsilon \searrow 0$ as $\varepsilon \searrow$ such that

$$h_\varepsilon \leq \min\left(\theta_\varepsilon \overline{h}_\varepsilon \,;\, (1-\theta_\varepsilon)\overline{h}_\varepsilon\right) \leq \overline{h}_\varepsilon \leq \omega(h_\varepsilon)$$

so that $(\overline{u}_\varepsilon, \overline{\chi}_\varepsilon) := \left(u_\varepsilon^{\overline{h}_\varepsilon}, \chi_\varepsilon^{\overline{h}_\varepsilon}\right) \in X_{h_\varepsilon(\Omega)}$. Yet, one only gets that for all $\varepsilon > 0$, there exists $\overline{h}_\varepsilon > 0$ such that

$$(\overline{u}_\varepsilon, \overline{\chi}_\varepsilon) := \left(u_\varepsilon^{\overline{h}_\varepsilon}, \chi_\varepsilon^{\overline{h}_\varepsilon}\right) \in X_{\overline{h}_\varepsilon(\Omega)},$$
$$\|\overline{u}_\varepsilon - u\|_{L^1(\Omega)} + \|\overline{\chi}_\varepsilon\|_{L^1(\Omega)} + \left|\mathcal{F}_\varepsilon(\overline{u}_\varepsilon, \overline{\chi}_\varepsilon) - \int_\Omega \overline{W}_\alpha(\xi)\, dx\right| \leq \varepsilon,$$

without guarantee that $\overline{h}_\varepsilon \geq h_\varepsilon$.

3.5 . In between plasticity and brittle fracture

In the remaining case, $\alpha \in (0, +\infty)$ and $\beta \in (0, +\infty)$, the one-dimensional case leads to a limit model, intermediate between plasticity and brittle fracture, which puts in competition the elastic energy stored in the body and a surface energy whose density is of the form

$$1 + \left|u^+ - u^-\right|,$$

so that it controles the number of jump points and has linear growth at infinity (see Figure 3.4). This is the content of Theorem 3.5.1 below. This first result is a motivation for the study of the vectorial case.

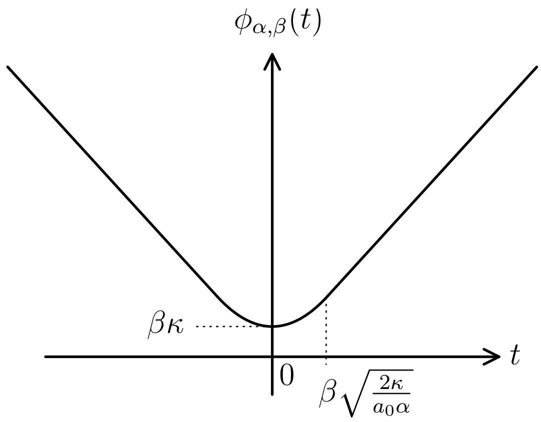

Figure 3.4

On the one hand, a naive expectation would be to derive the lower bound with a slicing method, in order to reduce the problem to a one-dimension study. Part of the difficulty here comes from the lack of control from below for the length of the triangles' sections. On the other hand, the construction of a recovery sequence for the upper bound seems much harder and is not clear yet in the vectorial case.

Let $\Omega = (0, L) \subset \mathbb{R}$ be a bounded open interval and $0 < a_0 < a_1 < +\infty$ be the elasticity coefficients of the weak and strong materials respectively. We define the following surface density

$$\phi_{\alpha,\beta} : t \in \mathbb{R} \mapsto \begin{cases} \dfrac{a_0\alpha}{2\beta} t^2 + \beta\kappa & \text{if } |t| \leq \beta\sqrt{\dfrac{2\kappa}{a_0\alpha}}, \\ \sqrt{2\kappa a_0 \alpha}\, |t| & \text{otherwise.} \end{cases} \tag{3.5.1}$$

Then, we consider the finite element set, still denoted by $X_{h_\varepsilon}(\Omega)$ by analogy with the two-dimensional setting, made of all pairs

$$(u,\chi) \in C^0(\Omega) \times L^\infty(\Omega;\{0,1\})$$

for which there exists a subdivision of $(0,L)$

$$0 = x_0 < \dots < x_n = L \quad \text{for some } n \geq 1,$$

such that for all $i \in [\![0, n-1]\!]$:

$$h_\varepsilon \leq x_{i+1} - x_i \leq \omega(h_\varepsilon), \quad u \text{ is affine on} (x_i, x_{i+1}) \quad \text{and} \quad \chi \text{ is constant on } (x_i, x_{i+1}).$$

Theorem 3.5.1. *If $\alpha \in (0,+\infty)$ and $\beta \in (0,+\infty)$, then the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ defined by*

$$\mathcal{F}_\varepsilon(u,\chi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L (\eta_\varepsilon a_0 \chi + a_1(1-\chi)) \left|u'\right|^2 \, dx + \dfrac{\kappa}{\varepsilon}\displaystyle\int_0^L \chi \, dx & \text{if } (u,\chi) \in X_{h_\varepsilon}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ-converges for the $L^1(\Omega) \times L^1(\Omega)$ topology to the energy $\Psi_{\alpha,\beta} : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ given by

$$\Psi_{\alpha,\beta}(u,\chi) = \begin{cases} \dfrac{a_1}{2}\displaystyle\int_0^L \left|u'\right|^2 \, dx + \displaystyle\int_{J_u} \phi_{\alpha,\beta}(u^+ - u^-) \, d\mathcal{H}^0 & \text{if } \chi = 0 \text{ and } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. According to Proposition 3.0.2, we can assume that $\chi = 0$. We denote by $\mathcal{F}'_{\alpha,\beta} : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ and $\mathcal{F}''_{\alpha,\beta} : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ the Γ-lower and Γ-upper limits respectively.

Lower bound. Let $\delta > 0$ and $u \in L^1(\Omega)$ be such that $\mathcal{F}'_{\alpha,\beta}(u,0) < +\infty$. Then, one can find a sequence $(u_\varepsilon, \chi_\varepsilon) \in X_{h_\varepsilon}(\Omega)$ such that

$$(u_\varepsilon, \chi_\varepsilon) \to (u, 0)$$

in $L^1(\Omega) \times L^1(\Omega)$ as $\varepsilon \searrow 0$ and, up to a subsequence (not relabeled), the following limit exists and satisfies

$$\lim_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) \leq \mathcal{F}'_{\alpha,\beta}(u,0) + \delta =: M < +\infty.$$

Hence, for all $\varepsilon > 0$, there exists a subdivision of $\Omega = (0,L)$

$$0 = x_0^\varepsilon < \dots < x_{n_\varepsilon}^\varepsilon = L$$

for some $n_\varepsilon \in \mathbb{N} \setminus \{0\}$, such that u_ε is affine and χ_ε is constant on each subinterval $(x_i^\varepsilon, x_{i+1}^\varepsilon)$ and

$$h_\varepsilon \leq x_{i+1}^\varepsilon - x_i^\varepsilon \leq \omega(h_\varepsilon) \quad \text{for all } i \in [\![0, n_\varepsilon - 1]\!].$$

Noticing that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) \geq \int_0^L \min\left(\frac{1}{2}\eta_\varepsilon a_0 |u_\varepsilon'|^2 + \frac{\kappa}{\varepsilon}; \frac{1}{2}a_1|u_\varepsilon'|^2\right) dx = \mathcal{F}_\varepsilon(u_\varepsilon, \tilde{\chi}_\varepsilon)$$

for the characteristic function

$$\tilde{\chi}_\varepsilon := \mathbb{1}_{\left\{\frac{1}{2}a_1|u'_\varepsilon|^2 \geq \frac{1}{2}\eta_\varepsilon a_0|u'_\varepsilon|^2 + \frac{\kappa}{\varepsilon}\right\}}$$

which is still constant on each subinterval $(x_i^\varepsilon, x_{i+1}^\varepsilon)$, we can assume without loss of generality that $\chi_\varepsilon = \tilde{\chi}_\varepsilon$. In particular, there exist $m_\varepsilon \in \mathbb{N}$ and $0 \leq b_j^\varepsilon < c_j^\varepsilon \leq L$ for $j \in [\![1, m_\varepsilon]\!]$ such that

$$\bigcup_{j=1}^{m_\varepsilon} \{b_j^\varepsilon, c_j^\varepsilon\} \subset \bigcup_{i=0}^{n_\varepsilon} \{x_i^\varepsilon\}$$

and

$$D_\varepsilon := \{\chi_\varepsilon = 1\} = \bigsqcup_{j=1}^{m_\varepsilon} \left(b_j^\varepsilon, c_j^\varepsilon\right) \subset \left\{u'_\varepsilon \neq 0\right\}.$$

Hence

$$M \geq \frac{\kappa}{\varepsilon} \int_0^L \chi_\varepsilon \, dx = \frac{\kappa}{\varepsilon} \mathcal{L}^1\big(D_\varepsilon\big) \geq \frac{\kappa h_\varepsilon}{\varepsilon} m_\varepsilon.$$

Since $h_\varepsilon/\varepsilon \to \beta \in (0, +\infty)$ when $\varepsilon \searrow 0$, we infer that $m_\varepsilon \leq M/(\beta\kappa)$ is bounded. Up to a further subsequence, still not relabeled, we infer that there exists $m \in \mathbb{N}$ such that

$$m_\varepsilon = m \quad \text{for all } \varepsilon > 0.$$

The idea then consists in modifying the minimizing sequence $\{u_\varepsilon\}_{\varepsilon>0}$ on each subinterval where the variation of u_ε is large enough. Indeed, we will see that it is energetically favorable to create a jump at the end point c_j^ε of each segment $[b_j^\varepsilon, c_j^\varepsilon]$ in D_ε. First notice that if $m = 0$, then $\chi_\varepsilon = 0$ for all $\varepsilon > 0$ hence

$$M \geq \frac{a_1}{2} \int_0^L \left|u'_\varepsilon\right|^2 \, dx,$$

and

$$\frac{1}{L} \int_0^L u_\varepsilon \, dx \to \frac{1}{L} \int_0^L u \, dx \quad \text{as } \varepsilon \searrow 0.$$

Therefore, Poincaré-Wirtinger's inequality entails that $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(\Omega)$. In particular, up to a further subsequence, we infer that $u \in H^1(\Omega)$ and

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega)$$

so that

$$\delta + \mathcal{F}'_{\alpha,\beta}(u, 0) \geq \frac{a_1}{2} \int_0^L \left|u'\right|^2 \, dx = \Psi_{\alpha,\beta}(u, 0).$$

We conclude the proof of the lower bound in this simpler case, by letting δ tend to 0. Let us now assume that

$$m \geq 1.$$

First, one can check that

$$\tilde{E}_\varepsilon := \frac{1}{2} \int_0^L \eta_\varepsilon a_0 \chi_\varepsilon \left|u'_\varepsilon\right|^2 \, dx + \frac{\kappa}{\varepsilon} \int_0^L \chi_\varepsilon \, dx = \sum_{j=1}^m f_j^\varepsilon \left(\frac{c_j^\varepsilon - b_j^\varepsilon}{\varepsilon}\right)$$

where

$$f_j^\varepsilon : l \in (0,+\infty) \mapsto \kappa l + \frac{a_0 \eta_\varepsilon}{2\varepsilon} \left| u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right|^2 \frac{1}{l}$$

is strictly convex. Defining

$$\phi_\varepsilon : t \in \mathbb{R} \mapsto \begin{cases} \dfrac{h_\varepsilon}{\varepsilon}\kappa + \dfrac{\eta_\varepsilon}{h_\varepsilon}\dfrac{a_0}{2}t^2 & \text{if } |t| \le \dfrac{h_\varepsilon}{\sqrt{\varepsilon\eta_\varepsilon}}\sqrt{\dfrac{2\kappa}{a_0}}, \\ \sqrt{2\kappa a_0}\sqrt{\dfrac{\eta_\varepsilon}{\varepsilon}}\,|t| & \text{otherwise,} \end{cases}$$

one can check that

$$\tilde{E}_\varepsilon \ge \sum_{j=1}^m \phi_\varepsilon\left(u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right). \tag{3.5.2}$$

Indeed, the only critical point of f_j^ε on $(0,+\infty)$ being

$$l_{j,\varepsilon}^* := \sqrt{\frac{a_0\eta_\varepsilon}{2\varepsilon\kappa}} \left| u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right| > 0,$$

f_j^ε is increasing on $[l_{j,\varepsilon}^*, +\infty)$. Then,

- either $h_\varepsilon/\varepsilon \ge l_{j,\varepsilon}^*$, so that $\left|u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right| \le \sqrt{\frac{2\kappa}{a_0}}\frac{h_\varepsilon}{\sqrt{\varepsilon\eta_\varepsilon}}$ and

$$f_j^\varepsilon\left(\frac{c_j^\varepsilon - b_j^\varepsilon}{\varepsilon}\right) \ge f_j^\varepsilon\left(\frac{h_\varepsilon}{\varepsilon}\right) = \frac{h_\varepsilon}{\varepsilon}\kappa + \frac{\eta_\varepsilon}{h_\varepsilon}\frac{a_0}{2}\left|u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right|^2 = \phi_\varepsilon\left(u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right)$$

  since $\left(c_j^\varepsilon - b_j^\varepsilon\right) \ge h_\varepsilon$.
- Otherwise, $\left|u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right| > \sqrt{\frac{2\kappa}{a_0}}\frac{h_\varepsilon}{\sqrt{\varepsilon\eta_\varepsilon}}$ and

$$f_j^\varepsilon\left(\frac{c_j^\varepsilon - b_j^\varepsilon}{\varepsilon}\right) \ge f_j^\varepsilon\left(l_{j,\varepsilon}^*\right) = \sqrt{2\kappa a_0\frac{\eta_\varepsilon}{\varepsilon}}\left|u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right| = \phi_\varepsilon\left(u_\varepsilon\left(c_j^\varepsilon\right) - u_\varepsilon\left(b_j^\varepsilon\right)\right).$$

We next consider the modified functions $w_\varepsilon \in SBV^2(\Omega)$ given by

$$w_\varepsilon := \begin{cases} u_\varepsilon & \text{in } \Omega \setminus D_\varepsilon, \\ u_\varepsilon(b_j^\varepsilon) & \text{in } (b_j^\varepsilon, c_j^\varepsilon) \text{ for all } j \in [\![1,m]\!]. \end{cases}$$

Especially, we get that

$$w_\varepsilon' = (1-\chi_\varepsilon)u_\varepsilon', \quad J_{w_\varepsilon} = \bigcup_{j=1}^m \{c_j^\varepsilon\},$$

$$\left[w_\varepsilon(c_j^\varepsilon)\right] := w_\varepsilon^+(c_j^\varepsilon) - w_\varepsilon^-(c_j^\varepsilon) = u_\varepsilon(c_j^\varepsilon) - u_\varepsilon(b_j^\varepsilon) \ne 0$$

and

$$(w_\varepsilon, \chi_\varepsilon) \to (u,0) \quad \text{in } L^1(\Omega)\times L^1(\Omega) \quad \text{as } \varepsilon \searrow 0.$$

Indeed, we infer by (3.5.2) together with Young's inequality that

$$M \geq \int_{J_{w_\varepsilon}} \phi_\varepsilon\left(\left[w_\varepsilon\right]\right)\, d\mathcal{H}^0 \geq \sqrt{2\kappa a_0}\sqrt{\frac{\eta_\varepsilon}{\varepsilon}} \int_{J_{w_\varepsilon}} \left|\left[w_\varepsilon\right]\right|\, d\mathcal{H}^0$$

and

$$\int_{D_\varepsilon} |w_\varepsilon|\, dx \leq \int_{D_\varepsilon} |u_\varepsilon|\, dx + \int_{D_\varepsilon} |u_\varepsilon - w_\varepsilon|\, dx \leq \int_{D_\varepsilon} |u_\varepsilon|\, dx + \omega(h_\varepsilon) \int_{J_{w_\varepsilon}} \left|\left[w_\varepsilon\right]\right|\, d\mathcal{H}^0.$$

Remembering that $\eta_\varepsilon/\varepsilon \to \alpha \in (0, +\infty)$, we thus infer that $\{w_\varepsilon\}_{\varepsilon>0}$ is bounded in $BV(\Omega)$. Hence, Ambrosio's compactness Theorem [6, Theorem 4.7] entails the existence of a subsequence (not relabeled) and a function $w \in SBV^2(\Omega)$ such that

$$w_\varepsilon \rightharpoonup w \quad \text{weakly-* in } BV(\Omega) \quad \text{as } \varepsilon \searrow 0.$$

On the other hand, since $u_\varepsilon \to u$ in measure in Ω and $\{u_\varepsilon \neq w_\varepsilon\} \subset D_\varepsilon$ with $\mathcal{L}^1(D_\varepsilon) \to 0$, we infer that for all $\eta > 0$,

$$\mathcal{L}^1\big(\{|w_\varepsilon - u| > \eta\}\big) \leq \mathcal{L}^1\big(\{|u_\varepsilon - u| > \eta\}\big) + \mathcal{L}^1\big(D_\varepsilon\big) \to 0 \quad \text{when } \varepsilon \searrow 0.$$

Hence, $w_\varepsilon \to u$ in measure in Ω as $\varepsilon \searrow 0$, so that

$$w = u \in SBV^2(\Omega).$$

Moreover, since the limit is independent of the subsequence, we infer that the whole sequence

$$w_\varepsilon \rightharpoonup u \quad \text{weakly-* in } BV(\Omega) \quad \text{as } \varepsilon \searrow 0.$$

Finally, we get that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) \geq \int_0^L \frac{a_1}{2}|w_\varepsilon'|^2\, dx + \int_{J_{w_\varepsilon}} \phi_\varepsilon\left(\left[w_\varepsilon\right]\right)\, d\mathcal{H}^0 = \Psi_{\alpha,\beta}(w_\varepsilon, 0) + o_{\varepsilon\searrow 0}(1)$$

where we used the fact that ϕ_ε uniformly converges to $\phi_{\alpha,\beta}$ in $\mathbb{R}$ and $\mathcal{H}^0(J_{w_\varepsilon}) = m$. Hence, by lower semi-continuity of $\Psi_{\alpha,\beta}$ for the weak-* convergence in $BV(\Omega)$ (see [6, Theorem 5.4]), we infer that

$$\delta + \mathcal{F}'_{\alpha,\beta}(u, 0) \geq \liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) \geq \Psi_{\alpha,\beta}(u, 0)$$

and we conclude the proof of the lower bound inequality by letting δ tend to 0.

Upper bound. Let $u \in SBV^2(\Omega)$. We denote its jump set by

$$J_u = \bigcup_{j=1}^m \{x_j\}$$

with $0 < x_1 < ... < x_m < L$. Using standard approximation results (see [46, Theorem 3.1] for instance) and the lower semi-continuity of $\mathcal{F}''_{\alpha,\beta}$ in $L^1(\Omega) \times L^1(\Omega)$ and of $\Psi_{\alpha,\beta}$ for the weak-* convergence in $BV(\Omega)$, we can assume that

$$u \in C^\infty(\Omega \setminus J_u) \cap L^\infty(\Omega).$$

We define, for all $\varepsilon > 0$ and all $j \in [\![1, m]\!]$,

$$\overline{h}_\varepsilon(j) := \max \left\{ h_\varepsilon \, ; \, h_\varepsilon \, \big| [u(x_j)] \big| \, \frac{1}{\beta} \sqrt{\frac{a_0 \alpha}{2\kappa}} \right\}$$

where $[u] := u^+ - u^-$. Since $d := \min \{ |x_{j+1} - x_j| \, : \, j \in [\![1, m-1]\!] \} > 0$, one can find a subdivision of $\Omega = (0, L)$

$$0 = y_0^\varepsilon < \ldots < y_{n_\varepsilon}^\varepsilon = L$$

for some $n_\varepsilon \in \mathbb{N} \setminus \{0\}$, for $\varepsilon > 0$ small enough, such that

$$J_u \cap \bigcup_{i=0}^{n_\varepsilon} \{ y_i^\varepsilon \} = \emptyset,$$

each subinterval $(y_i^\varepsilon, y_{i+1}^\varepsilon)$ contains at most one point x_j of J_u and such subintervals are exactly of length $\overline{h}_\varepsilon(j)$, while all the remaining subintervals have length between h_ε and $6h_\varepsilon$. In other words, one can find $0 \leq i_1(\varepsilon) < \ldots < i_m(\varepsilon) \leq n_\varepsilon - 1$ such that

$$y_{i_j}^\varepsilon < x_j < y_{i_j+1}^\varepsilon \quad \text{and} \quad y_{i_j+1}^\varepsilon - y_{i_j}^\varepsilon = \overline{h}_\varepsilon(j) < \frac{d}{8} \quad \text{for all } j \in [\![1, m]\!]$$

and

$$h_\varepsilon \leq y_{i+1}^\varepsilon - y_i^\varepsilon \leq 6h_\varepsilon \leq \omega(h_\varepsilon) \leq \frac{d}{8} \quad \text{for all } i \in [\![0, n_\varepsilon - 1]\!] \setminus \bigcup_{j=1}^{m} \{ i_j \}.$$

Since $u \in C^\infty(\Omega \setminus J_u)$, we can consider the Lagrange interpolation $u_\varepsilon \in H^1(\Omega)$ of u at the points $\{ y_0^\varepsilon, \ldots, y_{n_\varepsilon}^\varepsilon \} \subset \Omega \setminus J_u$. Introducing the characteristic functions

$$\chi_\varepsilon = \mathbb{1}_{\bigcup_{j=1}^m \left(y_{i_j}^\varepsilon, y_{i_j+1}^\varepsilon \right)} \in L^\infty(\Omega; \{0, 1\}),$$

one can check that u_ε is affine and χ_ε is constant on each subinterval $(y_i^\varepsilon, y_{i+1}^\varepsilon)$ for $i \in [\![0, n_\varepsilon - 1]\!]$, so that

$$(u_\varepsilon, \chi_\varepsilon) \in X_{h_\varepsilon}(\Omega).$$

Indeed, since $\overline{h}_\varepsilon(j) \geq h_\varepsilon$, it is enough to further subdivide each interval $(y_{i_j}^\varepsilon, y_{i_j+1}^\varepsilon)$ into a finite number of subintervals of length between h_ε and $6h_\varepsilon$, in order for $(u_\varepsilon, \chi_\varepsilon)$ to be admissible. Note that

$$\frac{\kappa}{\varepsilon} \int_0^L \chi_\varepsilon \, dx = \sum_{j=1}^{m} \kappa \frac{\overline{h}_\varepsilon(j)}{\varepsilon} \leq m\kappa \max_{j \in [\![1, m]\!]} \left\{ \big| [u(x_j)] \big| \, \frac{1}{\beta} \sqrt{\frac{a_0 \alpha}{2\kappa}} \right\} \frac{h_\varepsilon}{\varepsilon}$$

is bounded, so that $\chi_\varepsilon \to 0$ in $L^1(\Omega)$ as $\varepsilon \searrow 0$. Moreover, using standard finite element estimates, one can check that there exists a constant $C > 0$ such that, for all $i \in [\![0, n_\varepsilon - 1]\!] \setminus \bigcup_{j=1}^m \{ i_j \}$,

$$\| u_\varepsilon - u \|_{H^1\left((y_i^\varepsilon, y_{i+1}^\varepsilon) \right)} \leq C \omega(h_\varepsilon) \| u'' \|_{L^\infty\left((y_i^\varepsilon, y_{i+1}^\varepsilon) \right)}.$$

In particular, since $\| u_\varepsilon \|_{L^\infty(\Omega)} \leq \| u \|_{L^\infty(\Omega)}$, we infer that

$$\| u_\varepsilon - u \|_{L^2(\Omega)} \leq 2 \| u \|_{L^\infty(\Omega)} \| \chi_\varepsilon \|_{L^1(\Omega)} + C \omega(h_\varepsilon) \| u'' \|_{L^\infty(\Omega \setminus J_u)} \to 0 \quad \text{as } \varepsilon \searrow 0$$

and

$$\|(1-\chi_\varepsilon)u'_\varepsilon - u'\|^2_{L^2(\Omega)} \le \int_{\{\chi_\varepsilon=1\}} \left|u'\right|^2\, dx + C^2\omega(h_\varepsilon)^2\|u''\|^2_{L^\infty(\Omega\setminus J_u)} \to 0 \quad \text{as } \varepsilon \searrow 0$$

where we used the facts that $u' \in L^2(\Omega)$ and $\mathcal{L}^1(\{\chi_\varepsilon = 1\}) \to 0$ when $\varepsilon \searrow 0$. Therefore,

$$(u_\varepsilon, \chi_\varepsilon) \to (u, 0) \quad \text{in } L^1(\Omega) \times L^1(\Omega) \quad \text{and} \quad (1-\chi_\varepsilon)u'_\varepsilon \to u' \quad \text{in } L^2(\Omega)$$

when $\varepsilon \searrow 0$, so that

$$\frac{1}{2}\int_\Omega a_1(1-\chi_\varepsilon)\left|u'_\varepsilon\right|^2\, dx \underset{\varepsilon\searrow 0}{\longrightarrow} \frac{1}{2}\int_\Omega a_1\left|u'\right|^2\, dx. \tag{3.5.3}$$

Next, let us prove that

$$\tilde{E}_\varepsilon(u_\varepsilon, \chi_\varepsilon) := \int_\Omega \left(\frac{1}{2}\eta_\varepsilon a_0\left|u'_\varepsilon\right|^2 + \frac{\kappa}{\varepsilon}\right)\chi_\varepsilon\, dx \underset{\varepsilon\searrow 0}{\longrightarrow} \int_{J_u} \phi_{\alpha,\beta}\left(u^+ - u^-\right)\, d\mathcal{H}^0. \tag{3.5.4}$$

First note that

$$\tilde{E}_\varepsilon(u_\varepsilon, \chi_\varepsilon) = \sum_{j=1}^m \left(\kappa\frac{\overline{h}_\varepsilon(j)}{\varepsilon} + \frac{a_0\eta_\varepsilon}{2\overline{h}_\varepsilon(j)}\left|u(y^\varepsilon_{i_j+1}) - u(y^\varepsilon_{i_j})\right|^2\right).$$

By definition of J_u, since $y^\varepsilon_{i_j} \nearrow x_j$ and $y^\varepsilon_{i_j+1} \searrow x_j$ when $\varepsilon \searrow 0$, we infer that there exists a subsequence (not relabeled), such that for all $j \in [\![1, m]\!]$:

$$u(y^\varepsilon_{i_j}) \to u^-(x_j) \quad \text{and} \quad u(y^\varepsilon_{i_j+1}) \to u^+(x_j) \quad \text{as } \varepsilon \searrow 0.$$

We now have to consider two cases, for each jump point $x_j \in J_u$.

- Either $\left|[u(x_j)]\right| \le \beta\sqrt{\frac{2\kappa}{a_0\alpha}}$, so that

$$\begin{cases} \overline{h}_\varepsilon(j) = h_\varepsilon \quad \text{for all } \varepsilon > 0, \\ \phi_{\alpha,\beta}\left([u(x_j)]\right) = \beta\kappa + \dfrac{a_0\alpha}{2\beta}\left|[u(x_j)]\right|^2. \end{cases}$$

  Therefore,

$$\kappa\frac{\overline{h}_\varepsilon(j)}{\varepsilon} + \frac{a_0\eta_\varepsilon}{2\overline{h}_\varepsilon(j)}\left|u(y^\varepsilon_{i_j+1}) - u(y^\varepsilon_{i_j})\right|^2 \underset{\varepsilon\searrow 0}{\longrightarrow} \beta\kappa + \frac{a_0\alpha}{2\beta}\left|[u(x_j)]\right|^2 = \phi_{\alpha,\beta}\left([u(x_j)]\right).$$

- Or $\left|[u(x_j)]\right| > \beta\sqrt{\frac{2\kappa}{a_0\alpha}}$, so that

$$\begin{cases} \overline{h}_\varepsilon(j) = h_\varepsilon\left|[u(x_j)]\right|\dfrac{1}{\beta}\sqrt{\dfrac{a_0\alpha}{2\kappa}} \quad \text{for all } \varepsilon > 0, \\ \phi_{\alpha,\beta}\left([u(x_j)]\right) = \sqrt{2\kappa a_0\alpha}\left|[u(x_j)]\right|. \end{cases}$$

  Therefore,

$$\kappa\frac{\overline{h}_\varepsilon(j)}{\varepsilon} + \frac{a_0\eta_\varepsilon}{2\overline{h}_\varepsilon(j)}\left|u(y^\varepsilon_{i_j+1}) - u(y^\varepsilon_{i_j})\right|^2 \underset{\varepsilon\searrow 0}{\longrightarrow} \frac{1}{2}\sqrt{2\kappa a_0\alpha}\left|[u(x_j)]\right| + \frac{1}{2}\sqrt{2\kappa a_0\alpha}\left|[u(x_j)]\right| = \phi_{\alpha,\beta}\left([u(x_j)]\right).$$

Taking the sum for $j \in [\![1, m]\!]$ then leads to (3.5.4). We conclude the upper bound inequality by gathering (3.5.3) and (3.5.4) which entail that

$$\lim_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, \chi_\varepsilon) = \Psi_{\alpha,\beta}(u, 0) \geq \mathcal{F}''_{\alpha,\beta}(u, 0).$$

$\square$

4 - Perfect plasticity versus Damage : an unstable interaction between irreversibility and Γ-convergence through variational evolutions.

This chapter addresses the question of the interplay between relaxation and irreversibility through quasi-static evolutions in damage mechanics, by inquiring the following question :

Can the quasi-static evolution of an elastic material undergoing a rate-independent process of plastic deformation be derived as the limit model of a sequence of quasi-static brittle damage evolutions ?

This question is motivated by the static analysis performed in [15], where the authors have shown how the brittle damage model introduced by Francfort and Marigo (see [64, 63]) can lead to a model of Hencky perfect plasticity. Problems of damage mechanics being rather described through evolution processes, it is natural to extend this analysis to quasi-static evolutions, where the inertia is neglected. We consider the case where the medium is subjected to time-dependent boundary conditions, in the one-dimensional setting. The idea is to combine the scaling law considered in [15] with the quasi-static brittle damage evolution introduced in [60] by Francfort and Garroni, and try to understand how the irreversibility of the damage process will be expressed in the limit evolution. Surprisingly, the interplay between relaxation and irreversibility is not stable through time evolutions. Indeed, depending on the choice of the prescribed Dirichlet boundary condition, the effective quasi-static damage evolution obtained may not be of perfect plasticity type.

4.1 . Introduction

4.1.1 . Interplay between Γ-convergence and variational evolutions

Rate-independent systems have proved to be very useful in many problems of continuum mechanics dealing with dissipative phenomena such as elastoplasticity, damage or fracture. These models share similar energetic formulations which put in competition a stored energy and a dissipated one which does not depend on the speed of the loading (see [79] and references therein). When the model involves a scaling parameter, say $\varepsilon > 0$, a natural attempt consists first in studying static models before treating the related evolutions. Such considerations can be dealt within a more general setting, such as in [80] where the authors derive a sufficient condition in order for a family of parametrized time-dependent energy functionals of the form $E_\varepsilon + D_\varepsilon$ to approximate the expected effective energy $E_0 + D_0$ in the limit, where E_ε and D_ε respectively stand for the stored and dissipated parts of the total energy and E_0 and D_0 are their corresponding Γ-limits. In a nutshell, for $\varepsilon \in [0, \infty)$, if E_ε stands for the stored energy and $D_\varepsilon(q, \tilde{q})$ stands for the minimal energy dissipated as the medium changes from the states q to $\tilde{q}$, an evolution q_ε is called an energetic solution during the time interval $[0, T]$ if it satisfies the following stability and energy balance conditions

$$E_\varepsilon(t, q_\varepsilon(t)) \leq E_\varepsilon(t, q) + D_\varepsilon(q_\varepsilon(t), q) \quad \text{for all admissible state } q$$

$$E_\varepsilon(t,q_\varepsilon(t))+\mathrm{Diss}_\varepsilon(q_\varepsilon;0,t)=E_\varepsilon(0,q_\varepsilon(0))+\int_0^t\partial_s E(s,q_\varepsilon(s))\,ds$$

for all time $t\in[0,T]$, where the cumulated dissipation $\mathrm{Diss}_\varepsilon(q_\varepsilon;0,t)$ is the total variation of q_ε with respect to the "distance" D_ε in the time interval $[0,t]$. Given energetic solutions $\{q_\varepsilon\}_{\varepsilon>0}$ converging to some q_0, the authors derive in [80, Theorem 3.1] a sufficient condition in order for q_0 to be an energetic solution of the limit problem associated to E_0 and D_0. In particular, a joint condition on the interplay between the stored and dissipated energies is needed. Unfortunately, even if we have separate Γ-convergence of E_ε and D_ε to E_0 and D_0 respectively, the Γ-limit of the total energies $E_\varepsilon+D_\varepsilon$ might differ from the sum of the Γ-limits. This particular issue is adressed in [25] where the authors consider a family of quasi-static evolutions involving internal oscillating energies E_ε and dissipations D_ε and show that the Γ-limit of the sum can still be additively split as the sum of a stored energy $\tilde{E}_0$ and a dissipated one $\tilde{D}_0$, even though they a priori differ from E_0 and D_0. More generally, the interaction between Γ-convergence and variational evolutions frequently involves unexpected and tedious non-commutability phenomena in various contexts. For instance, such considerations have attracted renewed interest in the derivation of lower dimensional models for thin structures in the evolutionary setting, in the context of elastoplasticity [55, 76], crack propagation [11] or delamination problems [81], without being exhaustive. Another case study concerns the stability of unilateral minimality properties through variational evolutions, as in fracture mechanics [70] or periodic homogenization in multi-phase elastoplasticity [62].

4.1.2 . Motivation and results

In the static analysis led in [15, Theorem 3.1], the authors consider a family of brittle damage energies (introduced in [64, 63]) within a specific scaling law, and show how an asymptotic analysis in a singular limit can lead to a model of Hencky perfect plasticity. More precisely, they introduce a small parameter $\varepsilon>0$ and consider a linearly elastic material which can only exist in one of two states : a damaged one whose elastic properties are described via a symmetric fourth-order Hooke Law εA_0 and a sound one with a stronger elasticity tensor A_1, satisfying $\varepsilon A_0<A_1$ in the sense of quadratic forms acting on $\mathbb{M}^{N\times N}_{\mathrm{sym}}$. Introducing the characteristic function of the damaged region, $\chi\in L^\infty(\Omega;\{0,1\})$, and following the model introduced by Francfort and Marigo, the total energy associated to a displacement $u\in H^1(\Omega;\mathbb{R}^N)$ and χ is given as the sum of the elastic energy stored inside the material and a dissipative cost, taken as proportional to the volume of the damaged zone :

$$\int_\Omega\frac{1}{2}\left(\chi\varepsilon A_0+(1-\chi)A_1\right)e(u):e(u)\,dx+\frac{\kappa}{\varepsilon}\int_\Omega\chi\,dx$$

where $\kappa/\varepsilon>0$ is the material toughness and the symmetric gradient $e(u)=\left(\nabla u+\nabla u^T\right)/2$ is the linearized elastic strain. As the parameter ε tends to 0, the elasticity coefficients of the weak material degenerate to zero while the diverging character of κ/ε forces the damaged region to concentrate on vanishingly small sets. It is by now well-known that for fixed $\varepsilon>0$, the minimization of the above energy with respect to the couple (u,χ) is ill-posed, so that the energy must be relaxed. By doing so, the brittle character of the damage is lost as minimizing sequences tend to develop microstructures and the class of admissible solutions is extended to the set of all possible homogenized elasticities, resulting from fine mixtures of strong and weak material (see [63, 60, 2, 3]). Given some displacement

u and minimizing first pointwise with respect to χ, one can check that the asymptotic analysis of these energies is equivalent to finding the Γ-limit of the family of functionals

$$\int_\Omega \underbrace{\min\left(\frac{1}{2}\varepsilon A_0 e(u) : e(u) + \frac{\kappa}{\varepsilon}\, ;\, \frac{1}{2}A_1 e(u) : e(u)\right)}_{=:W_\varepsilon(e(u))} dx$$

when $\varepsilon \searrow 0$, or still the Γ-limit of their lower semicontinuous envelopes, given by

$$\mathcal{E}_\varepsilon(u) := \int_\Omega SQW_\varepsilon(e(u))\, dx$$

where SQW_ε is the symmetric quasiconvex envelope of W_ε (see [3]). An explicit formula of SQW_ε is generally unknown, as its expression is obtained through a minimization among all attainable composite materials (the G-closure set, see [2]) and makes use of the Hashin-Shtrikman bounds. Slightly adapting the proof of [15, Theorem 3.1] (see Appendix 5 for a precise statement and its proof), the authors have shown that when A_0 and A_1 are isotropic Hooke Laws defined by

$$A_i\xi = \lambda_i \mathrm{tr}(\xi)\mathrm{Id} + 2\mu_i\xi \text{ for all } \xi \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$$

where $\lambda_1 > \lambda_0 > 0$ and $\mu_1 > \mu_0 > 0$ are the Lamé coefficients, the brittle damage energies $\mathcal{E}_\varepsilon$ Γ-converges in $L^1(\Omega;\mathbb{R}^N)$ as $\varepsilon \searrow 0$ to the functional

$$\mathcal{E} : u \in BD(\Omega) \mapsto \int_\Omega \overline{W}(e(u))\, dx + \int_\Omega I_K^*\left(\frac{dE^s u}{d|E^s u|}\right) d|E^s u| + \int_{\partial\Omega} I_K^*\left((w-u)\odot\nu\right) d\mathcal{H}^{N-1}$$

where $K = \{\tau \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \,:\, G(\tau) \le 2\kappa\}$ is a closed convex set, $G : \mathbb{M}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ is defined by

$$G(\tau) := \begin{cases} \frac{\tau_1^2}{\lambda_0+2\mu_0} & \text{if } \frac{\lambda_0+2\mu_0}{2(\lambda_0+\mu_0)}(\tau_1+\tau_N) < \tau_1, \\ \frac{(\tau_1-\tau_N)^2}{4\mu_0} + \frac{(\tau_1+\tau_N)^2}{4(\lambda_0+\mu_0)} & \text{if } \tau_1 \le \frac{\lambda_0+2\mu_0}{2(\lambda_0+\mu_0)}(\tau_1+\tau_N) \le \tau_N, \\ \frac{\tau_N^2}{\lambda_0+2\mu_0} & \text{if } \tau_N < \frac{\lambda_0+2\mu_0}{2(\lambda_0+\mu_0)}(\tau_1+\tau_N), \end{cases} \tag{4.1.1}$$

with $\tau_1 \le \ldots \le \tau_N$ the ordered eigenvalues of $\tau \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$, $\overline{W}$ is the infimal convolution

$$\overline{W} : \xi \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \mapsto \inf_{\tau\in\mathbb{M}^{N\times N}_{\mathrm{sym}}} \left\{\frac{1}{2}A_1\tau : \tau + I_K^*(\xi-\tau)\right\}$$

and

$$I_K^* : \xi \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \mapsto \sup_{\tau\in K}\{\tau : \xi\}$$

is the support function of K, standing for the plastic dissipation potential (see [92]). In particular, for all displacement $u \in BD(\Omega)$, writing the Radon-Nikodým decomposition of Eu with respect to Lebesgue

$$Eu = e(u)\mathcal{L}^N \llcorner \Omega + E^s u$$

and using the definition of the infimal convolution, we infer that the absolutely continuous linearized strain can be additively split as $e(u) = e + p^a$ with e and $p^a \in L^1(\Omega;\mathbb{M}^{N\times N}_{\mathrm{sym}})$ such that

$$W(e(u)) = \frac{1}{2}A_1 e : e + I_K^*(p^a) \quad \mathcal{L}^1\text{-a.e. in } \Omega.$$

Therefore, defining $p = E^s u + p^a \mathcal{L}^N \llcorner \Omega + (w-u) \odot \nu \mathcal{H}^{N-1} \llcorner \partial\Omega$, one can check that

$$Eu = e\mathcal{L}^N \llcorner \Omega + p \llcorner \Omega,$$

so that

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2} A_1 e : e\, dx + \int_{\overline{\Omega}} I_K^* \left(\frac{dp}{d\,|p|} \right) d\,|p|$$

is indeed the energy functional corresponding to Hencky perfect plasticity, as mentionned in [83].

The objective of the present paper is to extend this work to the quasi-static case in a one dimensional setting. More specifically, we consider a linearly elastic material whose reference configuration is $\Omega = (0, L)$, a bounded open interval, with toughness $\kappa > 0$ and stiffness $a_1 > 0$, subjected to a prescribed time-dependent displacement on $\partial\Omega = \{0, L\}$:

$$w \in AC([0,T]; H^1(\mathbb{R})).$$

Adapting the analysis led in [15] to the quasi-static setting, we consider a family of quasi-static brittle damage evolutions (introduced in [60]) within the same specific scaling law. More precisely, we introduce a small parameter $\varepsilon > 0$ and apply [60, Theorem 2] to a linearly elastic material which can only exist in a damaged state or in a sound state with respective stiffness $0 < \varepsilon a_0 < a_1$, subjected to the prescribed displacement w on $\partial\Omega$ and with toughness κ/ε. Thus, we recover a triple

$$(u_\varepsilon, \Theta_\varepsilon, a_\varepsilon) : [0,T] \to H^1(\Omega; \mathbb{R}) \times L^\infty(\Omega; [0,1]) \times L^\infty(\Omega; [0, a_1]) \tag{4.1.2}$$

discribing the quasi-static evolution of brittle damage undergone by the medium for a fixed $\varepsilon > 0$. In other words, the state of the damaged medium (for $\varepsilon > 0$ fixed) at time $t \in [0,T]$ is dictated by the displacement $u_\varepsilon(t)$ while its elastic properties are given by the stiffness $a_\varepsilon(t) \in \mathcal{G}_{\Theta_\varepsilon(t)}(\varepsilon a_0, a_1)$ (see Section 1.4), where $\Theta_\varepsilon(t)$ is the volume fraction of sound material a_1 (see Proposition 4.2.1 below). We next wish to perform the asymptotic analysis of these evolutions when taking the limit $\varepsilon \searrow 0$, in the hope of recovering a quasi-static evolution of perfect plasticity in the limit, of which we briefly recall the fundamentals now.

In [92], Suquet proposed the (first complete) mathematical kinematical framework adapted to evolutions of perfect plasticity for dissipative materials and proves the existence of solutions in terms of the displacement field, under the assumption of small deformations. Heuristically, let $\Omega = (0, L)$ be the configuration at rest of an elastoplastic medium with stiffness a_1, whose evolution is driven by a time-dependent boundary displacement $w : [0,T] \times \mathbb{R} \to \mathbb{R}$ prescribed on $\partial\Omega$. The behaviour of the material is described via three kinematic variables (u, e, p), where the displacement $u : [0,T] \times \Omega \to \mathbb{R}$ is such that the linearized strain $Du = e + p$ is additively decomposed in an elastic strain $e : [0,T] \times \Omega \to \mathbb{R}$ and a plastic strain $p : [0,T] \times \Omega \to \mathbb{R}$ accounting for the reversible and permanent deformations respectively. In the quasi-static setting, where inertia is neglected, the evolution satisfies the *Constitutive Equations*

$$\begin{cases} a_1 e(t) = \sigma(t) \\ \sigma(t) \in K \\ \dot{p}(t) : \sigma(t) = \sup_{\tau \in K} \dot{p}(t) : \tau \end{cases} \qquad \text{in } \Omega$$

and the *Equilibrium Equation*

$$\begin{cases}\sigma'(t) = 0 \text{ in } \Omega \\ u(t) = w(t) \text{ on } \partial\Omega\end{cases}$$

at all time $t \in [0,T]$. In other words, the *Constitutive Equations* mean that the elastic strain is proportional to the stress σ, which is constrained to lie in a given closed and convex set $K \subset \mathbb{R}$ standing for the elasticity domain and whose boundary ∂K is referred to as the yield surface. The last assertion is nothing but Hill's maximum work principle. More recently, quasi-static plastic evolutions have been revisited into a variational evolution formulation for rate-independent processes. The problem has been interpreted in an energetic form that does not require the solutions to be smooth in time nor in space, making use of modern tools of the calculus of variations instead (see [79, 49, 83] and references therein). Following [49, Definition 4.2], a quasi-static evolution of perfect plasticity is a triple

$$(u,e,p) : [0,T] \to BV(\Omega) \times L^2(\Omega;\mathbb{R}) \times \mathcal{M}(\overline{\Omega};\mathbb{R})$$

subjected to the relaxed boundary condition $p(t) \llcorner \partial\Omega = (w(t) - u(t))\,(\delta_L - \delta_0)$ and satisfying the additive decomposition $Du(t) = e(t)\mathcal{L}^1 \llcorner \Omega + p(t) \llcorner \Omega$, such that

$$\begin{gathered}\sigma(t) = a_1 e(t), \quad \sigma'(t) = 0 \text{ in } H^{-1}(\Omega), \quad \sigma(t) \in K\,\mathcal{L}^1\text{-a.e. in } \Omega, \\ p : [0,T] \to \mathcal{M}(\overline{\Omega};\mathbb{R}) \text{ has bounded variation}\end{gathered} \tag{4.1.3}$$

and the Energy Balance

$$\frac{1}{2}\int_\Omega a_1 e(t) : e(t)\,dx + \mathrm{Diss}_K(p;0,t) = \frac{1}{2}\int_\Omega a_1 e(0) : e(0)\,dx + \int_0^t \int_\Omega \sigma(s)\dot{w}'(s,x)\,dxds \tag{4.1.4}$$

holds, for all time $t \in [0,T]$. The dissipative plastic cost cumulated during the time interval $[0,t]$ associated to p, $\mathrm{Diss}_K(p;0,t)$, is defined as

$$\sup\left\{\sum_{i=1}^n \int_{\overline{\Omega}} I_K^*\left(\frac{d(p(s_i) - p(s_{i-1}))}{d\,|p(s_i) - p(s_{i-1})|}\right) d\,|p(s_i) - p(s_{i-1})| \;:\; n \in \mathbb{N},\, 0 = s_0 \le s_1 \le \dots \le s_n = t\right\}.$$

The existence of quasi-static evolutions is (by now classically) obtained by performing a time-discretization and solving incremental minimization problems inductively, before letting the time-step tend to 0 (see [79, 40, 41, 69, 49] for instance). The purpose of the present paper is not to prove the existence of quasi-static evolutions of perfect plasticity, but to establish whether such evolutions can be derived from the quasi-static brittle damage evolutions introduced above in (4.1.2). By analogy with the static analysis of [15], we expect to derive the same closed convex set of plasticity K which is given by the closed interval

$$K := \left[-\sqrt{2\kappa a_0}, \sqrt{2\kappa a_0}\,\right]$$

as one can check that $G(\tau) = \tau^2/a_0$ for all $\tau \in \mathbb{R}$ in this simplified setting. In particular, the support function of K is simply given by

$$I_K^* = \sqrt{2\kappa a_0}\,|\cdot|\,.$$

Therefore, following the variational framework of quasi-static perfect plasticity recalled above (see [49, 79]), the dissipative cost cumulated during a time interval $[s,t] \subset [0,T]$ due to a time dependent Radon measure $q : [0,T] \to \mathcal{M}([0,L])$ is defined as

$$\mathrm{Diss}_K(q;s,t) = \sqrt{2\kappa a_0}\,\mathcal{V}(q;s,t)$$

where

$$\mathcal{V}(q;s,t) = \sup\left\{\sum_{i=1}^{n} |q(s_i) - q(s_{i-1})|\left([0,L]\right) \,:\, n \in \mathbb{N},\, s = s_0 \leq s_1 \leq ... \leq s_n = t\right\}$$

is the total variation of q during the time interval $[s,t]$. The question inquired in the present work is then : when passing in the limit $\varepsilon \searrow 0$ (in some sense detailed in the next sections) in the above brittle damage evolutions (4.1.2), can we derive a quasi-static evolution of perfect plasticity

$$(u,e,p) : [0,T] \to BV((0,L)) \times L^2((0,L);\mathbb{R}) \times \mathcal{M}([0,L];\mathbb{R})$$

satisfying (4.1.3) and (4.1.4)? Contrary to the static analysis, the interplay between damage and Γ-convergence turns out to be unstable through the time evolution process. Indeed, as explained in Theorem 4.1.1 and Theorem 4.1.2, the effective quasi-static evolution derived in the subsequent sections might not be of perfect plasticity type. Instead, it can be interpreted as one of damage, characterised by means of the material's compliance as internal variable :

Theorem 4.1.1. *Let $\varepsilon > 0$ and $(u_\varepsilon, \Theta_\varepsilon, a_\varepsilon)$ be a quasi-static evolution of the homogenized brittle damage model given by Proposition 4.2.1. There exists a subsequence (not relabeled) and absolutely continuous functions*

$$(u,e,p,\sigma,\mu) : [0,T] \to BV((0,L)) \times \mathbb{R} \times \mathcal{M}([0,L]) \times K \times \mathcal{M}([0,L])$$

such that for all $t \in [0,T]$

$$\begin{cases} u_\varepsilon(t) \rightharpoonup u(t) \quad \textit{weakly-* in } BV((0,L)), \\ \mu_\varepsilon(t) := \dfrac{1-\Theta_\varepsilon(t)}{\varepsilon}\mathcal{L}^1 \llcorner (0,L) \rightharpoonup \mu(t) \quad \textit{weakly-* in } \mathcal{M}([0,L]), \\ \sigma_\varepsilon(t) \to \sigma(t) \quad \textit{in } \mathbb{R}, \\ e_\varepsilon(t) := \dfrac{\sigma_\varepsilon(t)}{a_1}\Theta_\varepsilon(t) \rightharpoonup e(t) \quad \textit{weakly in } L^2((0,L)), \\ p_\varepsilon(t) := \dfrac{\sigma_\varepsilon(t)}{a_0}\mu_\varepsilon(t) \rightharpoonup p(t) \quad \textit{weakly-* in } \mathcal{M}([0,L]), \end{cases}$$

when $\varepsilon \searrow 0$ and satisfying the following assertions for all $t \in [0,T]$:

i. Additive Decomposition : $Du(t) = e(t)\mathcal{L}^1 \llcorner (0,L) + p(t) \llcorner (0,L)$ *in* $\mathcal{M}((0,L))$

ii. Relaxed Dirichlet Condition : $p(t) \llcorner \{0,L\} = \big(w(t) - u(t)\big)\big(\delta_L - \delta_0\big)$ *in* $\mathcal{M}(\{0,L\})$

iii. Constitutive Equation : $\sigma(t) = a_1 e(t)$

iv. Equilibrium Equation : $\sigma'(t) = 0$ *in* $H^{-1}((0,L))$

v. Stress Constraint : $\sigma(t) \in K$.

Furthermore, the effective compliance defined by

$$c : t \in [0,T] \mapsto \frac{\mu(t)}{a_0} + \frac{1}{a_1}\mathcal{L}^1 \llcorner (0,L) \in \mathcal{M}([0,L];\mathbb{R}^+)$$

is non-decreasing in time and satisfies the following assertions :

vi. Constitutive Equation : $Du(t) = \sigma(t)c(t)\llcorner(0,L)$ *in* $\mathcal{M}((0,L))$ *for all* $t \in [0,T]$

vii. Griffith Evolution Law : $\dot{c}(t)\left(2\kappa a_0 - \sigma(t)^2\right) = 0$ *in* $\mathcal{M}([0,L];\mathbb{R}^+)$ *for* $\mathcal{L}^1$*-a.e.* $t \in [0,T]$.

As mentionned above, according to the choice of the Dirichlet condition, the medium's response to the loading might differ from a perfect plastic behaviour :

Theorem 4.1.2. *The quasi-static evolution* $(u,e,p) : [0,T] \to BV((0,L)) \times \mathbb{R} \times \mathcal{M}([0,L])$ *is a quasi-static evolution of the perfect plasticity model* (4.1.3) *and* (4.1.4) *if and only if the Dirichlet boundary condition* $w \in AC([0,T];H^1(\mathbb{R}))$ *satisfies :*

$$\text{For all } 0 \le s < t \le T, \quad \left|\left[w(t)\right]_0^L\right| < \left|\left[w(s)\right]_0^L\right| \Rightarrow \left|\left[w(t)\right]_0^L\right| \le \sqrt{2\kappa a_0}\frac{L}{a_1}. \tag{4.1.6}$$

The present study may be seen as an illustration of non-stability issues arising when dealing with problems of H-convergence in the $L^1(\Omega)$ framework where ellipticity is lost during the time process, even when working in the simplest evolution setting and in dimension one.

4.1.3 . Organization of the paper

In Section 1.4, we recall some notation and preliminary results. In Section 4.2, we introduce the family of quasi-static brittle damage evolutions derived in [60, Theorem 2] associated to a linearly elastic material with toughness κ/ε and stiffness tensors εa_0 and a_1, for $\varepsilon > 0$, given a prescribed boundary datum and no volume force load in the one-dimensional setting. Particularly, due to the explicit knowledge of the $\mathcal{G}$-closure set of all admissible homogenized composite materials in dimension one, we collect starting information of crucial interest for the subsequent sections. In Section 4.3, we derive the effective quasi-static evolution when passing to the limit $\varepsilon \searrow 0$. We first give uniform bounds in Proposition 4.3.1 and next analyse the behaviour and regularity properties of the effective evolution. Section 4.4 adresses the question of the nature of the quasi-static evolution and determines in Theorem 4.1.2 the necessary and sufficient condition ensuring the perfect plastic behaviour of the evolution. Finally, Section 4.5 discusses whether the present work could be improved in order to derive a quasi-static evolution of perfect plasticity.

4.2 . Francfort-Garroni's model of Quasi-Static Brittle Damage

For all $\varepsilon > 0$, we consider a linearly elastic material whose reference configuration is $\Omega = (0,L)$, with toughness κ/ε and stiffness tensors εa_0 and a_1, corresponding to its damaged and sound zones respectively. Applying Theorem 2 and Remark 5 of [60] to this linearly elastic material without volume force load and with a prescribed boundary condition $w \in AC([0,T];H^1(\mathbb{R}))$, it ensures the following existence result for a relaxed quasi-static damage evolution.

Proposition 4.2.1. *For all $\varepsilon > 0$, there exist a time-dependent density, a displacement and a stiffness tensor*

$$\begin{cases} \Theta_\varepsilon : [0,T] \to L^\infty((0,L);[0,1]) \\ u_\varepsilon : [0,T] \to H^1((0,L)) \\ a_\varepsilon = \left(\dfrac{1-\Theta_\varepsilon}{\varepsilon a_0} + \dfrac{\Theta_\varepsilon}{a_1}\right)^{-1} : [0,T] \to L^\infty((0,L)) \end{cases} \tag{4.2.1}$$

all weakly- measurable, such that*

Dirichlet Boundary Condition : $u_\varepsilon(t) \in w(t) + H^1_0((0,L))$ *for all* $t \in [0,T]$*;*

Initial Minimality : *for all* $v \in w(0) + H^1_0((0,L))$ *and* $\theta \in L^\infty((0,L);[0,1])$*,*

$$\int_0^L \left(\frac{1}{2} a_\varepsilon(0)|u'_\varepsilon(0)|^2 + \frac{\kappa}{\varepsilon}\big(1-\Theta_\varepsilon(0)\big)\right) dx \le \int_0^L \left(\frac{1}{2}\frac{\varepsilon a_0 a_1}{\theta a_1 + (1-\theta)\varepsilon a_0}|v'|^2 + \frac{\kappa}{\varepsilon}\theta\right) dx; \tag{4.2.2}$$

Monotonicity : *for all* $T \ge t \ge s \ge 0$*,* $a_\varepsilon(t) \le a_\varepsilon(s)$ *and* $\Theta_\varepsilon(t) \le \Theta_\varepsilon(s)$ $\mathcal{L}^1$*-a.e. in* $(0,L)$*;*

One-sided Minimality : *for all* $t \in [0,T]$*,* $v \in w(t) + H^1_0((0,L))$ *and* $\theta \in L^\infty((0,L);[0,1])$*,*

$$\frac{1}{2}\int_0^L a_\varepsilon(t)\left|u'_\varepsilon(t)\right|^2\, dx \le \frac{1}{2}\int_0^L \frac{\varepsilon a_0 a_\varepsilon(t)}{\theta a_\varepsilon(t) + (1-\theta)\varepsilon a_0}|v'|^2\, dx + \frac{\kappa}{\varepsilon}\int_0^L \theta\Theta_\varepsilon(t)\, dx; \tag{4.2.3}$$

Energy Balance : *for all* $t \in [0,T]$*, the total energy*

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)\left|u'_\varepsilon(t)\right|^2\, dx + \frac{\kappa}{\varepsilon}\int_0^L (1-\Theta_\varepsilon(t))\, dx$$

satisifes

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + \int_0^t \int_0^L a_\varepsilon(s) u'_\varepsilon(s)\dot{w}'(s)\, dx ds. \tag{4.2.4}$$

Proof. This is the direct application of [60, Theorem 2, Remark 5] together with [2, Lemma 1.3.32, Formula (1.109)] which stipulates that for all $0 < a < b$ and all $\theta \in L^\infty((0,L);[0,1])$,

$$\mathcal{G}_\theta(a;b) = \left\{\frac{ab}{\theta b + (1-\theta)a}\right\}.$$

In particular, one gets that $u_\varepsilon : [0,T] \to H^1((0,L))$ is strongly measurable (in the sense of [58, Definition 2.101]) as it is continuous outside of an at most countable subset of $[0,T]$. One also gets that $a_\varepsilon : [0,T] \to L^2((0,L))$ is strongly measurable (which is equivalent to the weak-* measurability according to Pettis' Theorem, see [58, Theorem 2.104]), since for all $\phi \in L^2((0,L))$

$$t \in [0,T] \mapsto \int_0^L \phi a_\varepsilon(t)\, dx = \int_0^L \max(0,\phi) a_\varepsilon(t)\, dx - \int_0^L \max(0,-\phi) a_\varepsilon(t)\, dx$$

is $\mathcal{L}^1$-measurable, as it is the difference between two non-increasing functions. For similar reasons, one also infers that $\Theta_\varepsilon : [0,T] \to L^2((0,L);[0,1])$ is strongly measurable. Note that $a_\varepsilon : [0,T] \to L^\infty((0,L))$ and $\Theta_\varepsilon : [0,T] \to L^\infty((0,L);[0,1])$ are (a priori) only weakly-* measurable. To see this, it suffices to take $\phi \in L^1((0,L))$ instead of $L^2((0,L))$ above. □

For all $\varepsilon > 0$, a naive first use of the One-sided Minimality (4.2.3) entails the following properties of the stress

$$\sigma_\varepsilon = a_\varepsilon u'_\varepsilon : [0,T] \to L^2((0,L)).$$

Proposition 4.2.2. *For all $\varepsilon > 0$ and all $t \in [0,T]$, the stress $\sigma_\varepsilon(t)$ is homogeneous in space and $\sigma_\varepsilon \in L^0\big([0,T];\mathbb{R}\big)$. Moreover,*

$$u'_\varepsilon = \sigma_\varepsilon \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_0} + \frac{\Theta_\varepsilon}{a_1} \right). \tag{4.2.5}$$

Proof. Indeed, for all $v \in H^1_0((0,L))$ and $\delta > 0$, applying (4.2.3) with $\theta = 0$ and $v^\pm := u_\varepsilon(t) \pm \delta v \in w(t) + H^1_0((0,L))$ ensures that

$$0 \le \int_0^L \sigma_\varepsilon(t) u'_\varepsilon(t)\, dx \le \int_0^L \sigma_\varepsilon(t) u'_\varepsilon(t)\, dx \pm 2\delta \int_0^L \sigma_\varepsilon(t) v' \, dx + \delta^2 \int_0^L a_\varepsilon(t) |v'|^2 \, dx.$$

Dividing by $\delta > 0$ then letting $\delta \searrow 0$ entails that $\int_0^L \sigma_\varepsilon(t) v'\, dx = 0$, which implies the space homogeneity of σ_ε. Formula (4.2.5) is a consequence of the expression of a_ε and the definition of σ_ε. □

For all $\varepsilon > 0$ and $t \in [0,T]$, we define the function

$$W^t_\varepsilon : (x,\xi) \in (0,L) \times \mathbb{R} \mapsto \min \left(\frac{\kappa \Theta_\varepsilon(t)(x)}{\varepsilon} + \frac{1}{2}\varepsilon a_0 |\xi|^2; \frac{1}{2} a_\varepsilon(t)(x) |\xi|^2 \right).$$

A second application of the One-Sided Minimality (4.2.3) implies that for $\mathcal{L}^1$-a.e. $x \in (0,L)$

$$\frac{1}{2} a_\varepsilon(t)(x) |u'_\varepsilon(t)(x)|^2 = \mathcal{C}\big(W^t_\varepsilon(x,\cdot)\big)\big(u'_\varepsilon(t)(x)\big) =: \mathcal{C}W^t_\varepsilon\big(x, u'_\varepsilon(t)(x)\big)$$

where $\mathcal{C}W^t_\varepsilon\big(x,\cdot\big)$ is the convex envelope of $W^t_\varepsilon(x,\cdot)$.

Proposition 4.2.3. *For $\mathcal{L}^1$-a.e. in $\{x \in (0,L),\ a_\varepsilon(t)(x) > \varepsilon a_0\}$,*

$$\frac{1}{2} a_\varepsilon(t) |u'_\varepsilon(t)|^2 = \begin{cases} \frac{1}{2} \sigma_\varepsilon(t) u'_\varepsilon(t) & \text{if } \frac{|\sigma_\varepsilon(t)|}{\sqrt{2\kappa a_0}} \le L_\varepsilon, \\ |\sigma_\varepsilon(t)| \sqrt{\frac{2\kappa a_0 \Theta_\varepsilon(t)}{a_\varepsilon(t)(a_\varepsilon(t) - \varepsilon a_0)}} - \frac{\kappa a_0 \Theta_\varepsilon(t)}{a_\varepsilon(t) - \varepsilon a_0} & \text{if } L_\varepsilon < \frac{|\sigma_\varepsilon(t)|}{\sqrt{2\kappa a_0}} \le \frac{a_\varepsilon(t)}{\varepsilon a_0} L_\varepsilon, \\ \frac{\kappa \Theta_\varepsilon(t)}{\varepsilon} + \frac{1}{2} \varepsilon a_0 |u'_\varepsilon(t)|^2 & \text{if } \frac{a_\varepsilon(t)}{\varepsilon a_0} L_\varepsilon < \frac{|\sigma_\varepsilon(t)|}{\sqrt{2\kappa a_0}}, \end{cases} \tag{4.2.6}$$

with

$$L_\varepsilon := \sqrt{\frac{a_1}{a_1 - \varepsilon a_0}} > 1. \tag{4.2.7}$$

Proof. Let $\varepsilon > 0$ and $t \in [0,T]$. As we are working in the scalar setting, symetric quasiconvex and convex envelopes coincide. According to [3, Lemma 3.1], we have that for all $\xi \in \mathbb{R}$ and $x \in (0,L)$

$$\begin{aligned} \mathcal{C}W^t_\varepsilon(x,\xi) &:= \inf \left\{ \int_0^1 W^t_\varepsilon(x, \xi + \phi'(y))\, dy \ : \ \phi \in H^1_0((0,1)) \right\} \\ &= \min_{\theta \in [0,1]} \left\{ \frac{\kappa \Theta_\varepsilon(t)(x)}{\varepsilon} \theta + \frac{1}{2} \left(\frac{1-\theta}{a_\varepsilon(t)(x)} + \frac{\theta}{\varepsilon a_0} \right)^{-1} |\xi|^2 \right\}. \end{aligned}$$

The above minimization being over a strictly convex function, the minimum is indeed reached at a unique minimizer in $[0,1]$. Since

$$g:(x,\theta)\in(0,L)\times[0,1]\mapsto \frac{\kappa\Theta_\varepsilon(t)(x)}{\varepsilon}\theta+\frac{1}{2}\frac{a_\varepsilon(t)(x)\varepsilon a_0}{\theta a_\varepsilon(t)(x)+(1-\theta)\varepsilon a_0}|u_\varepsilon'(t)(x)|^2$$

is a Carathéodory function, Aumann's measurable selection creterion (see [57, Theorem 1.2]) entails that the (unique) minimizer $\theta_\varepsilon(t)(x)\in[0,1]$ of $g(x,\cdot)$ is actually $\mathcal{L}^1$-measurable in $(0,L)$, *i.e.* $\theta_\varepsilon(t)\in L^\infty((0,L);[0,1])$ and for $\mathcal{L}^1$-a.e. $x\in(0,L)$

$$\mathcal{C}W_\varepsilon^t(x,u_\varepsilon'(t)(x))=\frac{\kappa\Theta_\varepsilon(t)(x)}{\varepsilon}\theta_\varepsilon(t)(x)+\frac{1}{2}\left(\frac{1-\theta_\varepsilon(t)(x)}{a_\varepsilon(t)(x)}+\frac{\theta_\varepsilon(t)(x)}{\varepsilon a_0}\right)^{-1}|u_\varepsilon'(t)(x)|^2.$$

Therefore, (4.2.3) implies that

$$\begin{aligned}\frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2\,dx&=\min_{\theta\in L^\infty((0,L);[0,1])}\int_0^L g(x,\theta(x))\,dx\\&\le\int_0^L g(x,\theta_\varepsilon(t)(x))\,dx=\int_0^L\mathcal{C}W_\varepsilon^t(t)(x,u_\varepsilon'(t)(x))\,dx.\end{aligned}$$

Since one simultaneously has $\mathcal{C}W_\varepsilon^t(t)(x,u_\varepsilon'(t)(x))\le\frac{1}{2}a_\varepsilon(t)(x)|u_\varepsilon'(t)(x)|^2$ for all $x\in(0,L)$, we infer that

$$\mathcal{C}W_\varepsilon^t(t)(x,u_\varepsilon'(t)(x))=\frac{1}{2}a_\varepsilon(t)(x)|u_\varepsilon'(t)(x)|^2\quad\text{for }\mathcal{L}^1\text{-a.e. }x\in(0,L).$$

Therefore, (4.2.6) is the consequence of Lemma 5.0.2 (see Appendix 5) applied to $W_\varepsilon^t(x,\cdot)$ at every point x in the set $\{a_\varepsilon(t)>\varepsilon a_0\}$. Note that the constant $L_\varepsilon=\sqrt{\frac{a_1}{a_1-\varepsilon a_0}}>1$ only depends on $\varepsilon>0$ and not on $x\in(0,L)$ nor $t\in[0,T]$. □

4.3 . The limit quasi-static evolution

As previously explained, the objective of this work is to derive an effective limit model by letting ε tend to 0. Since we expect a limit model of perfect plasticity type, one has to identify which quantities will play the role of the elastic and plastic strains at the scale $\varepsilon>0$. Meanwhile, in order to pass to the limit along converging subsequences in the brittle damage evolutions described in Proposition 4.2.1, we rely on uniform bounds computed in Proposition 4.3.1 below.

4.3.1 . Uniform bounds and compactness

Proposition 4.3.1. *There exists a constant $C(w,T)\in(0,+\infty)$ such that*

$$\sup_{t\in[0,T]}\sup_{\varepsilon>0}\mathcal{E}_\varepsilon(t)\le C\tag{4.3.1}$$

and

$$\sup_{t\in[0,T]}\sup_{\varepsilon>0}\left\{\frac{1}{\varepsilon}\|\Theta_\varepsilon(t)-1\|_{L^1((0,L))}+\|u_\varepsilon(t)\|_{BV((0,L))}+|\sigma_\varepsilon(t)|\right\}\le C.\tag{4.3.2}$$

Proof. Let us first prove (4.3.1). Note that (4.2.2) applied with $\theta = 0$ and $v = w(0)$ directly entails

$$\mathcal{E}_\varepsilon(0) \le \frac{1}{2}\int_0^L a_1|w'(0)|^2\,dx.$$

As for the subsequent times $t \in [0,T]$, (4.2.3) applied with $\theta = 0$ and $v = w(t)$ entails

$$\frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2\,dx \le \frac{1}{2}\int_0^L a_1|w'(t)|^2\,dx \le \frac{a_1}{2}\|w\|^2_{L^\infty([0,T];H^1(\mathbb{R}))}.$$

On the other hand, since w is absolutely continuous from $[0,T]$ into $H^1(\mathbb{R})$, we infer that

$$\dot{w}' \in L^1([0,T];L^2(\mathbb{R}))$$

is Bochner integrable. Therefore, gathering the uniform bound on the initial time energies together with Cauchy-Schwarz inequality for the scalar-product

$$(\xi,\eta) \in L^2((0,L)) \times L^2((0,L)) \mapsto \int_0^L a_\varepsilon(s)\xi\eta\,dx \in \mathbb{R}$$

(for all time $s \in [0,t]$ fixed) and the Energy Balance (4.2.4), we get that

$$\begin{aligned}\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + \int_0^t\int_0^L \sigma_\varepsilon(s)\dot{w}'(s)\,dxds \\ \le \mathcal{E}_\varepsilon(0) + \int_0^t \left(\int_0^L a_\varepsilon(s)|u_\varepsilon'(s)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^L a_\varepsilon(s)|\dot{w}'(s)|^2\,dx\right)^{\frac{1}{2}} ds \le C(w,T).\end{aligned}$$

We next show (4.3.2). Let $\varepsilon > 0$ and $t \in [0,T]$. First, as shown in [15, Lemma 2.3, Formula (2.6)], there exists a constant $c > 0$ (only depending on a_0, a_1 and κ) such that the function

$$W_\varepsilon : \xi \in \mathbb{R} \mapsto \min\left\{\frac{\kappa}{\varepsilon} + \frac{1}{2}\varepsilon a_0|\xi|^2; \frac{1}{2}a_1|\xi|^2\right\}$$

satisfies $\mathcal{C}W_\varepsilon(\xi) \ge c|\xi| - \frac{1}{c}$ for all $\xi \in \mathbb{R}$. Remembering the definition (4.2.1) of $a_\varepsilon(t)$ and the fact that

$$\mathcal{C}W_\varepsilon(\xi) = \min_{\theta\in[0,1]}\left\{\frac{\kappa\theta}{\varepsilon} + \frac{1}{2}\left(\frac{\theta}{\varepsilon a_0} + \frac{1-\theta}{a_1}\right)^{-1}|\xi|^2\right\}$$

for all $\xi \in \mathbb{R}$, we in particular get that

$$\mathcal{E}_\varepsilon(t) \ge \int_0^L \mathcal{C}W_\varepsilon\big(u_\varepsilon'(t)\big)\,dx \ge c\|u_\varepsilon'(t)\|_{L^1((0,L))} - \frac{L}{c}.$$

Thus, using the equivalent norm in $BV((0,L))$ recalled in (1.4.2) leads to

$$\|u_\varepsilon(t)\|_{BV((0,L))} \le \|u_\varepsilon(t)'\|_{L^1((0,L))} + |w(t)(0)| + |w(t)(L)| \le C(w,T).$$

Finally, the homogeneity in space of the stress $\sigma_\varepsilon(t) \in \mathbb{R}$ implies that

$$\mathcal{E}_\varepsilon(t) \ge \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2\,dx \ge \frac{T}{2a_1}|\sigma_\varepsilon(t)|^2$$

as well, thus concluding (4.3.2). □

From the uniform bounds (4.3.2), we obtain compactness properties.

Proposition 4.3.2. *There exists a subsequence (not relabelled and independent of t) and a non-negative Radon measure $\mu : [0,T] \to \mathcal{M}\big([0,L];\mathbb{R}^+\big)$, which is non-decreasing in time, such that*

$$\begin{cases} \Theta_\varepsilon(t) \to 1 \quad \textit{strongly in } L^1((0,L)) \textit{ for all } t \in [0,T] & (4.3.3a) \\ \mu_\varepsilon(t) := \dfrac{1-\Theta_\varepsilon(t)}{\varepsilon} \mathbb{1}_{(0,L)} \rightharpoonup \mu(t) \quad \textit{weakly-* in } \mathcal{M}\big([0,L]\big) \textit{ for all } t \in [0,T] & (4.3.3b) \end{cases}$$

as $\varepsilon \searrow 0$.

Remark 4.3.3. Henceforth, we will work along the subsequence (not relabeled) mentionned in Proposition 4.3.2.

Proof. One directly deduces from (4.3.2) that $\Theta_\varepsilon(t)$ strongly converges to 1 in $L^1((0,L))$ as $\varepsilon \searrow 0$ for all time $t \in [0,T]$. Next, using the non-decreasing character (in time) of the non-negative Radon measures

$$\mu_\varepsilon : t \in [0,T] \mapsto \frac{1-\Theta_\varepsilon(t)}{\varepsilon} \mathbb{1}_{(0,L)} \in \mathcal{M}\big([0,L];\mathbb{R}^+\big)$$

together with (4.3.2), one can apply the generalized version of Helly's Theorem recalled in [49, Lemma 7.2] for the topological duals of separable Banach spaces and find a subsequence and a limit time-dependent non-negative Radon measure $\mu : [0,T] \to \mathcal{M}\big([0,L];\mathbb{R}^+\big)$ such that $\mu_\varepsilon(t) \rightharpoonup \mu(t)$ weakly-* in $\mathcal{M}\big([0,L]\big)$ as ε tends to 0, for all $t \in [0,T]$. Note that the monotonicity in time of μ_ε is preserved by the pointwise weak-* convergence in $\mathcal{M}\big([0,L]\big)$. ☐

4.3.2 . Initial time of the evolution

We begin with a corollary of the analysis led in [15, Theorem 3.1] in the static setting, taking into account a prescribed Dirichlet boundary datum. We refer to Appendix 5 for a more general statement of this proposition and its proof.

Proposition 4.3.4. *The functional $\mathcal{F}_\varepsilon : L^1((0,L)) \to \overline{\mathbb{R}^+}$ defined for all $u \in L^1((0,L))$ by*

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_0^L \mathcal{C}W_\varepsilon(u')\,dx & \textit{if } u \in w(0) + H^1_0((0,L)) \\ +\infty & \textit{otherwise} \end{cases}$$

Γ-converges in L^1 as $\varepsilon \searrow 0$ to the functional

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_0^L \overline{W}(u')\,dx + \sqrt{2\kappa a_0}\,|D^s u|\big((0,L)\big) + \sqrt{2\kappa a_0}\big(|w(0)(L)-u(L)| + |w(0)(0)-u(0)|\big) \\ \hfill \textit{if } u \in BV((0,L)) \\ +\infty \quad \textit{otherwise} \end{cases}$$

where

$$\overline{W} : \xi \in \mathbb{R} \mapsto \inf_{\eta \in \mathbb{R}} \left\{ \frac{a_1}{2}|\xi-\eta|^2 + \sqrt{2\kappa a_0}\,|\eta| \right\}.$$

In particular, we infer that

$$\mathcal{E}_\varepsilon(0) = \mathcal{F}_\varepsilon(u_\varepsilon(0)) = \min \mathcal{F}_\varepsilon.$$

Indeed, according to [3, Lemma 3.1] we first remark that

$$\mathcal{C}W_\varepsilon(u_\varepsilon'(0)) \leq \frac{\kappa}{\varepsilon}(1-\Theta_\varepsilon(0)) + \frac{a_\varepsilon(0)}{2}\left|u_\varepsilon'(0)\right|^2$$

hence $\mathcal{F}_\varepsilon(u_\varepsilon(0)) \leq \mathcal{E}_\varepsilon(0)$. Besides, (4.2.2) implies that $\mathcal{E}_\varepsilon(0) \leq \mathcal{F}_\varepsilon(v)$ for all $v \in w(0)+H_0^1((0,L))$, where we exchanged the infimum and the integral thanks to Aumann's criterion.

Therefore, we deduce from the Fundamental Theorem of Γ-convergence together with the bounds (4.3.1) and (4.3.2) that there exist a further subsequence (still not relabeled) and a displacement $u(0) \in BV((0,L))$ such that

$$u_\varepsilon(0) \rightharpoonup u(0) \text{ weakly-* in } BV((0,L)) \quad \text{and} \quad \mathcal{E}_\varepsilon(0) \to \mathcal{E}(0) := \mathcal{F}(u(0)) = \min \mathcal{F} \tag{4.3.4}$$

when $\varepsilon \searrow 0$.

4.3.3 . Time independence of the subsequences

We first show that, along the whole subsequence introduced in Proposition 4.3.2 (not relabeled and independent of t), $\{\sigma_\varepsilon(t)\}_\varepsilon$ pointwise converges to some limit stress $\sigma(t)$ for all time in $[0,T]$.

Proposition 4.3.5. *Along the subsequence introduced in Proposition 4.3.2 (independent of t, not relabeled), we have that for all time $t \in [0,T]$,*

$$\mu_\varepsilon(t)\big([0,L]\big) = \int_0^L \frac{1-\Theta_\varepsilon(t)}{\varepsilon}\,dx \to l(t) := \mu(t)\big([0,L]\big) \geq 0$$

and

$$\sigma_\varepsilon(t) \to \sigma(t) := \frac{\big[w(t)\big]_0^L}{\frac{l(t)}{a_0} + \frac{L}{a_1}} \tag{4.3.5}$$

when $\varepsilon \searrow 0$, where $\big[w(t)\big]_0^L := w(t)(L) - w(t)(0)$. Moreover,

$$\sigma(t) \in K = [-\sqrt{2\kappa a_0}, \sqrt{2\kappa a_0}] \quad \textit{for all } t \in [0,T]. \tag{4.3.6}$$

In particular, we infer that $\sigma \in L^\infty([0,T];K)$,

$$\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma \quad \textit{weakly-* in } L^\infty([0,T];\mathbb{R}) \tag{4.3.7}$$

and for all $t \in [0,T]$,

$$\mathcal{E}_\varepsilon(t) \underset{\varepsilon\searrow 0}{\longrightarrow} \mathcal{E}(0) + \int_0^t\int_0^L \sigma(s)\dot{w}'(s)(x)\,dxds = \frac{\big[w(t)\big]_0^L}{2}\sigma(t) + \kappa l(t) =: \mathcal{E}(t). \tag{4.3.8}$$

Proof. We work along the subsequence introduced in Proposition 4.3.2. Let $t \in [0,T]$. Note that by the weak-* convergence of $\mu_\varepsilon(t)$ to $\mu(t)$ in $\mathcal{M}([0,L])$, one gets that

$$\mu_\varepsilon(t)\big([0,L]\big) = \int_0^L \frac{1-\Theta_\varepsilon(t)(x)}{\varepsilon}\,dx \underset{\varepsilon\searrow 0}{\longrightarrow} l(t) := \mu(t)\big([0,L]\big) \geq 0.$$

Since $u_\varepsilon(t) \in w(t) + H^1_0((0,L))$ satisfies $u'_\varepsilon(t) = \sigma_\varepsilon(t)a_\varepsilon(t)^{-1}$ and $\sigma_\varepsilon(t)$ is homogeneous in space, we infer by the Integration by Parts Formula in $H^1((0,L))$ that

$$\big[w(t)\big]_0^L = \int_0^L u'_\varepsilon(t)(x)\,dx = \sigma_\varepsilon(t)\int_0^L \left(\frac{1-\Theta_\varepsilon(t)(x)}{\varepsilon a_0} + \frac{\Theta_\varepsilon(t)(x)}{a_1}\right)dx.$$

Since

$$\int_0^L \left(\frac{1-\Theta_\varepsilon(t)(x)}{\varepsilon a_0} + \frac{\Theta_\varepsilon(t)(x)}{a_1}\right)dx \underset{\varepsilon\searrow 0}{\longrightarrow} \frac{l(t)}{a_0} + \frac{L}{a_1} > 0,$$

we obtain

$$\sigma_\varepsilon(t) \underset{\varepsilon\searrow 0}{\longrightarrow} \sigma(t) := \frac{\big[w(t)\big]_0^L}{\frac{l(t)}{a_0}+\frac{L}{a_1}},$$

which proves (4.3.5) and (4.3.7). Next, let us remark that for all $\varepsilon > 0$ and $\mathcal{L}^1$-a.e. in $\{a_\varepsilon(t) > \varepsilon a_0\}$,

$$\frac{1}{2}\sigma_\varepsilon(t)u'_\varepsilon(t) > |\sigma_\varepsilon(t)|\sqrt{\frac{2\kappa a_0\Theta_\varepsilon(t)}{a_\varepsilon(t)(a_\varepsilon(t)-\varepsilon a_0)}} - \frac{\kappa a_0\Theta_\varepsilon(t)}{a_\varepsilon(t)-\varepsilon a_0} \quad \text{if } \frac{|\sigma_\varepsilon(t)|}{\sqrt{2\kappa a_0}} > L_\varepsilon.$$

Indeed,

$$\frac{1}{2}\sigma_\varepsilon(t)u'_\varepsilon(t) - |\sigma_\varepsilon(t)|\sqrt{\frac{2\kappa a_0\Theta_\varepsilon(t)}{a_\varepsilon(t)(a_\varepsilon(t)-\varepsilon a_0)}} + \frac{\kappa a_0\Theta_\varepsilon(t)}{a_\varepsilon(t)-\varepsilon a_0} = \left(\sqrt{\frac{a_\varepsilon(t)}{2}}\,|u'_\varepsilon(t)| - \sqrt{\frac{\kappa a_0\Theta_\varepsilon(t)}{a_\varepsilon(t)-\varepsilon a_0}}\right)^2 \geq 0$$

and the equality holds if and only if $|\sigma_\varepsilon(t)| = \sqrt{2\kappa a_0}L_\varepsilon$. Thus, by homogeneity in space of σ_ε and (4.2.6) we infer that $|\sigma_\varepsilon(t)| \leq \sqrt{2\kappa a_0}L_\varepsilon$ for all $\varepsilon > 0$ such that $\{a_\varepsilon(t) > \varepsilon a_0\} \neq \emptyset$.

- Either $\limsup_{\varepsilon\searrow 0}\mathcal{L}^1\left(\{a_\varepsilon(t) > \varepsilon a_0\}\right) = 0$, hence

  $$L\,|\sigma(t)| = \lim_\varepsilon L\,|\sigma_\varepsilon(t)| = \lim_\varepsilon\left(\int_{\{a_\varepsilon(t)=\varepsilon a_0\}} \varepsilon a_0\,\big|u'_\varepsilon(t)\big|\,dx + |\sigma_\varepsilon(t)|\,\mathcal{L}^1\left(\{a_\varepsilon(t) > \varepsilon a_0\}\right)\right) = 0$$

  and $\sigma(t) = 0 \in K$.
- Or $\limsup_{\varepsilon\searrow 0}\mathcal{L}^1\left(\{a_\varepsilon(t) > \varepsilon a_0\}\right) > 0$. Up to another subsequence (still not relabeled), we can assume that $\mathcal{L}^1\left(\{a_\varepsilon(t) > \varepsilon a_0\}\right) > 0$ for all $\varepsilon > 0$. In particular, it implies that $\{a_\varepsilon(t) > \varepsilon a_0\} \neq \emptyset$ hence $|\sigma_\varepsilon(t)| \leq \sqrt{2\kappa a_0}L_\varepsilon$ for all $\varepsilon > 0$. Taking the limit when $\varepsilon \searrow 0$, we obtain that $|\sigma(t)| \leq \sqrt{2\kappa a_0}$ hence $\sigma(t) \in K$.

In any cases, we get that the stress constraint (4.3.6) is satisfied. Thus, by homogeneity in space of $\sigma_\varepsilon(t)$, the Dominated Convergence Theorem entails that

$$\int_0^t\int_0^L \sigma_\varepsilon(s)\dot{w}'(s)(x)\,dxds = \int_0^t \sigma_\varepsilon(s)\left(\int_0^L \dot{w}'(s)(x)\,dx\right)ds \underset{\varepsilon\searrow 0}{\longrightarrow} \int_0^t\int_0^L \sigma(s)\dot{w}'(s)(x)\,dxds.$$

Therefore, using the convergence of the energies at the initial time (4.3.4) together with the Energy Balance (4.2.4), one gets that

$$\mathcal{E}_\varepsilon(t) \underset{\varepsilon\searrow 0}{\longrightarrow} \mathcal{E}(0) + \int_0^t \int_0^L \sigma(s)\dot{w}'(s)(x)\,dxds =: \mathcal{E}(t) \in \mathbb{R}^+.$$

By homogeneity in space of $\sigma_\varepsilon(t)$ again, since

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\sigma_\varepsilon(t)\int_0^L w'(t)(x)\,dx + \kappa\int_0^L \frac{1-\Theta_\varepsilon(t)(x)}{\varepsilon}\,dx \underset{\varepsilon\searrow 0}{\longrightarrow} \frac{\big[w(t)\big]_0^L}{2}\sigma(t) + \kappa l(t)$$

we deduce that

$$\mathcal{E}(t) = \frac{\big[w(t)\big]_0^L}{2}\sigma(t) + \kappa l(t)$$

which completes the proof of (4.3.8) and Proposition 4.3.5. □

We now define what will play the role of the elastic and plastic strains at the scale $\varepsilon > 0$, by setting

$$\begin{cases} e_\varepsilon = \sigma_\varepsilon \dfrac{\Theta_\varepsilon}{a_1} \in L^2((0,L)) & \text{(4.3.9a)} \\ p_\varepsilon = \dfrac{\sigma_\varepsilon}{a_0}\mu_\varepsilon = \sigma_\varepsilon \dfrac{1-\Theta_\varepsilon}{\varepsilon a_0}\mathbb{1}_{(0,L)} \in L^2((0,L)) & \text{(4.3.9b)} \end{cases}$$

which, by (4.2.5), satisfy the additive decomposition $u'_\varepsilon = e_\varepsilon + p_\varepsilon$ at all time. Using Proposition 4.3.2 together with the homogeneity in space of σ_ε and σ, we infer that for all $t \in [0,T]$

$$\begin{cases} e_\varepsilon(t) \to e(t) := \dfrac{\sigma(t)}{a_1} \quad \text{strongly in } L^2((0,L)) & \text{(4.3.10a)} \\ p_\varepsilon(t) \rightharpoonup p(t) := \dfrac{\sigma(t)}{a_0}\mu(t) \quad \text{weakly-* in } \mathcal{M}\big([0,L]\big) & \text{(4.3.10b)} \\ u'_\varepsilon(t) \rightharpoonup \sigma(t)\left(\dfrac{\mu(t)\llcorner(0,L)}{a_0} + \dfrac{\mathcal{L}^1}{a_1}\right) = p(t)\llcorner(0,L) + e(t)\mathcal{L}^1 \quad \text{weakly-* in } \mathcal{M}((0,L)) & \text{(4.3.10c)} \end{cases}$$

when $\varepsilon \searrow 0$. Let us recall that

$$\mu(t)([0,L]) = l(t) \quad \text{and} \quad p(t)([0,L]) = \frac{\sigma(t)}{a_0}l(t). \tag{4.3.11}$$

The uniform bound (4.3.2) ensures that for all $t \in [0,T]$, there exist a further subsequence (depending on t, not relabeled) and a displacement $u(t) \in BV((0,L))$ such that

$$u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly-* in } BV((0,L))$$

when $\varepsilon \searrow 0$. In particular, we deduce from (4.3.10c) that

$$Du(t) = p(t)\llcorner(0,L) + e(t)\mathcal{L}^1 \quad \text{in } \mathcal{M}((0,L);\mathbb{R})$$

is independent of the subsequence defining $u(t)$. Moreover, one can check that

$$p(t)\llcorner\{0,L\} = \big(w(t)-u(t)\big)\big(\delta_L - \delta_0\big). \tag{4.3.12}$$

Indeed, extending the problem on a larger open interval $[0, L] \subset \Omega'$ and setting

$$\begin{cases} \overline{u}_\varepsilon(t) = u_\varepsilon(t)\mathbb{1}_{(0,L)} + w(t)\mathbb{1}_{\Omega'\setminus(0,L)} \\ \overline{p}_\varepsilon(t) = p_\varepsilon(t)\mathbb{1}_{[0,L]} \\ \overline{e}_\varepsilon(t) = e_\varepsilon(t)\mathbb{1}_{(0,L)} + w'(t)\mathbb{1}_{\Omega'\setminus(0,L)} \end{cases}$$

one can check that

$$\begin{cases} \overline{u}_\varepsilon(t) \rightharpoonup u(t)\mathbb{1}_{(0,L)} + w(t)\mathbb{1}_{\Omega'\setminus(0,L)} \quad \text{weakly-* in } BV(\Omega') \\ \overline{p}_\varepsilon(t) \rightharpoonup \overline{p}(t) := p(t)\mathbb{1}_{[0,L]} \quad \text{weakly-* in } \mathcal{M}(\Omega') \\ \overline{e}_\varepsilon(t) \rightharpoonup \overline{e}(t) := e(t)\mathbb{1}_{(0,L)} + w'(t)\mathbb{1}_{\Omega'\setminus(0,L)} \quad \text{weakly in } L^2(\Omega') \end{cases}$$

when $\varepsilon \searrow 0$. Therefore, using [94, Remark 2.3 (i)], we get that

$$D\overline{u}_\varepsilon(t) = \overline{e}_\varepsilon(t) + \overline{p}_\varepsilon(t) \rightharpoonup \overline{e}(t) + \overline{p}(t) = Du(t)\mathbb{1}_{(0,L)} + w'(t)\mathbb{1}_{\Omega'\setminus(0,L)}\mathcal{L}^1 + \big(w(t) - u(t)\big)\big(\delta_L - \delta_0\big)$$

which implies (4.3.12). In particular, we infer that the limit displacement $u(t)$ is actually independent of the subsequence. Indeed, let $u_1(t)$ and $u_2(t) \in BV((0, L))$ be two weak limits of $\{u_\varepsilon(t)\}_{\varepsilon>0}$ in $BV((0, L))$. On the one hand, (4.3.10c) entails that $D(u_1(t) - u_2(t)) = 0$ in $\mathcal{M}\big((0, L)\big)$ hence $u_1(t) - u_2(t) = C(t) \in \mathbb{R}$ is homogeneous in space. On the other hand, the internal traces of $u_1(t)$ and $u_2(t)$ being prescribed on $\{0, L\}$ by (4.3.12), we infer that $C(t) = \big(u_1(t) - u_2(t)\big)(L) = 0$ hence $u_1(t) = u_2(t)$ in $BV((0, L))$. Therefore, we infer that the whole sequence converges and there exists

$$u : [0, T] \to BV((0, L))$$

such that

$$u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly-* in } BV((0, L)) \text{ when } \varepsilon \searrow 0, \text{ for all } t \in [0, T]. \tag{4.3.13}$$

Note that $u(0) \in BV((0, L))$ was already given by (4.3.4) and the static analysis led in [15], entailing the following Constitutive Equation at the initial time.

Proposition 4.3.6.

$$\mathcal{E}(0) = \frac{La_1}{2} e(0)^2 + \sqrt{2\kappa a_0}\, |p(0)| \big([0, L]\big) \tag{4.3.14}$$

and

$$\big(2\kappa a_0 - \sigma(0)^2\big)\, l(0) = 0. \tag{4.3.15}$$

Proof. Gathering (4.3.10c) and (4.3.12), we can identify the absolutely continuous and singular parts (with respect to $\mathcal{L}^1$) in the Radon-Nikodým decompositions of $Du(0)$ and $p(0)$:

$$\begin{cases} D^a u(0) = e(0)\mathcal{L}^1 + p(0)^a \\ p(0)^s = D^s u(0)\mathbb{1}_{(0,L)} + (w(0) - u(0))\,(\delta_L - \delta_0)\,. \end{cases}$$

By definition of the inf-convolution and (4.3.4), we get that $\overline{W}(D^a u(0)) \leq \frac{a_1}{2} e(0)^2 + \sqrt{2\kappa a_0}\, |p(0)^a|$ $\mathcal{L}^1$-a.e. in $(0, L)$ and $\mathcal{E}(0) = \mathcal{F}(u(0)) \leq \frac{La_1}{2} e(0)^2 + \sqrt{2\kappa a_0}\, |p(0)| \big([0, L]\big)$, where we identified absolutely continuous measures with their densities. Besides, combining (4.3.8), (4.3.5) and (4.3.11), we also have that

$$\begin{aligned}\mathcal{E}(0) = \frac{\sigma(0)^2}{2}\left(\frac{l(0)}{a_0}+\frac{L}{a_1}\right)+\kappa l(0)\\ = \frac{La_1}{2}e(0)^2+\sqrt{2\kappa a_0}\,|p(0)|\big([0,L]\big)+\big(\sqrt{2\kappa a_0}-|\sigma(0)|\big)^2\frac{l(0)}{2a_0}\\ \geq \frac{La_1}{2}e(0)^2+\sqrt{2\kappa a_0}\,|p(0)|\big([0,L]\big).\end{aligned}$$

Therefore, we obtain that $\mathcal{E}(0)=\frac{La_1}{2}e(0)^2+\sqrt{2\kappa a_0}\,|p(0)|\big([0,L]\big)$ and $\big(2\kappa a_0-\sigma(0)^2\big)\,l(0)=0$. □

4.3.4 . Regularity of the evolution

Looking at the proof of Proposition 4.3.5 and defining, for all time $t\in[0,T]$, the function

$$\Delta(t)=\left(\frac{\mathcal{E}(t)}{a_0}+\frac{\kappa L}{a_1}\right)^2-\frac{2\kappa}{a_0}\left|\big[w(t)\big]_0^L\right|^2,$$

one can check that

$$\Delta(t)=\left(\frac{\sigma(t)^2}{2a_0}-\kappa\right)^2\left(\frac{l(t)}{a_0}+\frac{L}{a_1}\right)^2\geq 0 \quad \text{and} \quad l(t)=\frac{a_0}{2\kappa}\left(\frac{\mathcal{E}(t)}{a_0}-\frac{\kappa L}{a_1}+\sqrt{\Delta(t)}\right).$$

On the one hand, since

$$\mathcal{E}(t)=\mathcal{E}(0)+\int_0^t\int_0^L\sigma(s)\dot{w}'(s,x)\,dxds$$

and

$$s\mapsto\int_0^L\sigma(s)\dot{w}'(s,x)\,dx \quad \text{is integrable on } [0,T],$$

we infer that $\mathcal{E}\in AC([0,T];\mathbb{R})$. On the other hand, since $w\in AC\big([0,T];C^0([0,L])\big)$ and the product of absolutely continuous functions remains absolutely continuous, we infer that $\Delta\in AC([0,T];\mathbb{R})$. In particular, we deduce that l, σ, e and p are continuous on the whole interval $[0,T]$. By non-negativity and monotonicity in time of μ, we also infer that μ is continuous from $[0,T]$ to $\mathcal{M}([0,L])$.

Using the Energy Balance in Proposition 4.2.1 together with the non-decreasing character of l, (4.3.8) and (4.3.5), we can actually show that $\sigma\in AC([0,T];\mathbb{R})$. From this, we will deduce that l, u, p and μ inherit the same regularity. This is a strong result because it is usually obtained a posteriori, once an Energy Balance of the type (4.3.19) is proved to be satisfied (see [49]). Remarkably, here we do not rest on such an Energy Balance in order to prove the regularity of the quasi-static evolution. This is the content of the following proposition.

Proposition 4.3.7. *The mapping* $(\sigma,e,l,\mu,p,u):[0,T]\to K\times\mathbb{R}\times\mathbb{R}^+\times\mathcal{M}([0,L])\times\mathcal{M}([0,L])\times BV((0,L))$ *is absolutely continuous.*

Proof. One can check that for all $0 \le s \le t \le T$,

$$\frac{L}{2a_1}\left(\sigma(t)-\sigma(s)\right)^2 = \int_s^t (\sigma(r)-\sigma(s))\big[\dot w(r)\big]_0^L\,dr \\ - \frac{l(s)}{2a_0}(\sigma(t)-\sigma(s))^2 - \frac{(l(t)-l(s))}{2a_0}\left(\sigma(t)^2 - 2\sigma(t)\sigma(s) + 2\kappa a_0\right). \quad (4.3.16)$$

Indeed, the convergences in Proposition 4.3.5 together with the homogeneity in space of σ and σ_ε imply that

$$\frac{L}{2a_1}(\sigma(t)-\sigma(s))^2 = L\left(\frac{\sigma(t)e(t)}{2} - \frac{\sigma(s)e(s)}{2} - \sigma(s)(e(t)-e(s))\right) \\ = \lim_{\varepsilon\searrow 0}\int_0^L \left(\frac{\sigma_\varepsilon(t)e_\varepsilon(t)}{2} - \frac{\sigma_\varepsilon(s)e_\varepsilon(s)}{2} - \sigma_\varepsilon(s)(e_\varepsilon(t)-e_\varepsilon(s))\right)dx.$$

Using (4.3.10a), (4.3.9a), (4.3.9b) and writing $l_\varepsilon := \int_0^L \frac{1-\Theta_\varepsilon}{\varepsilon}\,dx$, we get that

$$\frac{L}{2a_1}(\sigma(t)-\sigma(s))^2 = \lim_{\varepsilon\searrow 0}\Bigg(\mathcal{E}_\varepsilon(t) - \int_0^L \frac{\sigma_\varepsilon(t)p_\varepsilon(t)}{2}\,dx - \kappa l_\varepsilon(t) - \mathcal{E}_\varepsilon(s) + \int_0^L \frac{\sigma_\varepsilon(s)p_\varepsilon(s)}{2}\,dx + \kappa l_\varepsilon(s) \\ - \sigma_\varepsilon(s)\int_s^t \big[\dot w(r)\big]_0^L\,dr + \sigma_\varepsilon(s)\int_0^L (p_\varepsilon(t)-p_\varepsilon(s))\,dx\Bigg) \\ = \mathcal{E}(t) - \mathcal{E}(s) - \kappa(l(t)-l(s)) - \sigma(s)\int_s^t \big[\dot w(r)\big]_0^L\,dr \\ - \frac{\sigma(t)^2}{2a_0}l(t) + \frac{\sigma(s)^2}{2a_0}l(s) + \sigma(s)\left(\frac{\sigma(t)l(t)}{a_0} - \frac{\sigma(s)l(s)}{a_0}\right)$$

so that (4.3.8) entails (4.3.16). In particular, since $2\kappa a_0 \ge \sigma(s)^2$, we obtain

$$\frac{L}{2a_1}\left|\sigma(t)-\sigma(s)\right|^2 \le \int_s^t |\sigma(r)-\sigma(s)|\left|\big[\dot w(r)\big]_0^L\right|dr =: V_s(t). \quad (4.3.17)$$

Applying a Gronwall type Lemma (in the spirit of [49, Lemma 5.3.]), we infer that σ is absolutely continuous on $[0,T]$. Indeed, fixing $T \ge s \ge 0$ momentarily, one can remark that $V_s \in AC([s,T];\mathbb{R})$ and

$$\dot V_s(t) = |\sigma(t)-\sigma(s)|\left|\big[\dot w(t)\big]_0^L\right| \le \sqrt{\frac{2a_1}{L}V_s(t)+\eta}\left|\big[\dot w(t)\big]_0^L\right|$$

for $\mathcal{L}^1$-a.e. $t \in [s,T]$ and any $\eta > 0$. As $\sqrt{\cdot}$ is locally lipschitz on $[\eta,+\infty)$, we infer that $\sqrt{\frac{2a_1}{L}V_s+\eta}$ is absolutely continuous on $[s,T]$ and

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\sqrt{\frac{2a_1}{L}V_s+\eta}\right)(t) = \frac{a_1}{L}\frac{\dot V_s(t)}{\sqrt{\frac{2a_1}{L}V_s(t)+\eta}} \le \frac{a_1}{L}\left|\big[\dot w(t)\big]_0^L\right| \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [s,T].$$

Thus, integrating between s and t for any $T \ge t \ge s \ge 0$ entails that

$$\sqrt{\frac{2a_1}{L}V_s(t)+\eta} - \sqrt{\eta} \le \frac{a_1}{L}\int_s^t \left|\big[\dot w(r)\big]_0^L\right|dr.$$

After taking the limit $\eta \searrow 0$, (4.3.17) yields that

$$|\sigma(t)-\sigma(s)| \leq \sqrt{\frac{2a_1}{L}V_s(t)} \leq \frac{a_1}{L}\int_s^t \left|\big[\dot{w}(r)\big]_0^L\right| dr \quad \text{for all } 0 \leq s \leq t \leq T,$$

thus proving the absolute continuity of σ and $e = \sigma a_1^{-1}$ from $[0,T]$ to $\mathbb{R}$. In particular, (4.3.8) entails that

$$l = \frac{1}{\kappa}\Big(\mathcal{E} - \frac{\sigma}{2}\big[w\big]_0^L\Big) \in AC([0,T];\mathbb{R}).$$

Since $\mu : [0,T] \to \mathcal{M}([0,L];\mathbb{R}^+)$ is non-decreasing in time, (4.3.11) implies

$$0 \leq (\mu(t)-\mu(s))\big([0,L]\big) = l(t) - l(s)$$

for all $0 \leq s \leq t \leq T$, so that

$$\mu \in AC\big([0,T];\mathcal{M}([0,L])\big) \quad \text{and} \quad p = \frac{\sigma}{a_0}\mu \in AC\big([0,T];\mathcal{M}([0,L])\big).$$

Finally, the Trace Theorem in $BV((0,L))$, (4.3.13) and (4.3.12) imply that for all $0 \leq s \leq t \leq T$,

$$\|u(t)-u(s)\|_{BV((0,L))} \leq L\,|e(t)-e(s)| + |p(t)-p(s)|\,([0,L]) + |w(t)-w(s)|\,(L) + |w(t)-w(s)|\,(0),$$

thus proving the absolute continuity of u from $[0,T]$ to $BV((0,L))$ as well. □

At this point, we have identified good candidates for the limit evolution :

$$(u,e,p,\sigma,\mu) : [0,T] \to BV((0,L)) \times \mathbb{R} \times \mathcal{M}([0,L]) \times K \times \mathcal{M}([0,L]),$$

which are all absolutely continuous on $[0,T]$ and satisfy the following assertions for all $t \in [0,T]$:

i. Additive Decomposition : $Du(t) = e(t)\mathcal{L}^1 \llcorner (0,L) + p(t) \llcorner (0,L)$ in $\mathcal{M}((0,L))$
ii. Relaxed Dirichlet Condition : $p(t)\llcorner\{0,L\} = \big(w(t)-u(t)\big)\big(\delta_L - \delta_0\big)$ in $\mathcal{M}(\{0,L\})$
iii. Constitutive Equation : $\sigma(t) = a_1 e(t)$
iv. Equilibrium Equation : $\sigma'(t) = 0$ in $H^{-1}((0,L))$
v. Stress Constraint : $\sigma(t) \in K$.

The absolute continuity of (u,σ,p,μ,l) guarantees that (u,σ) describes a quasi-static damage evolution, whose internal variable is the effective compliance (inverse effective rigidity)

$$c : t \in [0,T] \mapsto \frac{\mu(t)}{a_0} + \frac{1}{a_1}\mathcal{L}^1 \llcorner (0,L) \in \mathcal{M}([0,L];\mathbb{R}^+)$$

satisfying the Constitutive Equation and Griffith Evolution Law stated in Theorem 4.1.1.

Proof of Theorem 4.1.1 : Using (4.3.8) and (4.3.5), one can check that for $\mathcal{L}^1$-a.e. $t \in [0,T]$ the following quantities are well defined and satisfy

$$\big[\dot{w}(t)\big]_0^L = \dot{\sigma}(t)\left(\frac{l(t)}{a_0} + \frac{L}{a_1}\right) + \sigma(t)\frac{\dot{l}(t)}{a_0}$$

and

$$\dot{\mathcal{E}}(t) = \sigma(t)\big[\dot{w}(t)\big]_0^L = \frac{\dot{\sigma}(t)\big[w(t)\big]_0^L}{2} + \frac{\sigma(t)\big[\dot{w}(t)\big]_0^L}{2} + \kappa\dot{l}(t).$$

Hence

$$\kappa\dot{l}(t) + \frac{\dot{\sigma}(t)\big[w(t)\big]_0^L}{2} = \frac{\sigma(t)\big[\dot{w}(t)\big]_0^L}{2} = \frac{\dot{\sigma}(t)\big[w(t)\big]_0^L}{2} + \frac{\sigma(t)^2\dot{l}(t)}{2a_0},$$

entailing that

$$\dot{l}(t)\left(\kappa - \frac{\sigma(t)^2}{2a_0}\right) = 0.$$

Besides, using that μ is non-decreasing in time together with [49, Theorem 7.1, Formula (7.4)] and (4.3.11) ensures that

$$\dot{\mu}(t)\big([0,L]\big) = \lim_{h\searrow 0}\frac{\mu(t+h)\big([0,L]\big) - \mu(t)\big([0,L]\big)}{h} = \lim_{h\searrow 0}\frac{l(t+h)-l(t)}{h} = \dot{l}(t) \tag{4.3.18}$$

for $\mathcal{L}^1$-a.e. $t \in [0,T]$. Thus, by non-negativity of the Radon measure $\dot{\mu}(t)$, we infer that

$$\dot{\mu}(t)\left(2\kappa a_0 - \sigma(t)^2\right) = 0 \quad \text{in } \mathcal{M}([0,L];\mathbb{R}^+) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T].$$

□

Remark 4.3.8. The effective limit model obtained here is a different type of damage model where the dissipative phenomena is described by means of an internal variable, the effective compliance $c : [0,T] \to \mathcal{M}([0,L];\mathbb{R}^+)$, whose non-decreasing character in time accounts for the irreversibility of damage. This is a threshold stress model, based on the conjecture that damage propagates if and only if the stress saturates the constraint.

- Formally inverting the compliance $c(t)$, the Constitutive Equation allows us to interpret

  $$\frac{1}{2}\sigma(t)Du(t) = \frac{1}{2}c(t)^{-1}Du(t)^2$$

  as the stored (elastic) energy density of the effective damaged medium at time $t \in [0,T]$. In other words, one can interpret

  $$\mathcal{Q}(t) := \frac{1}{2}\sigma(t)Du(t)((0,L)) = \frac{\sigma(t)^2}{2}\left(\frac{\mu(t)((0,L))}{a_0} + \frac{L}{a_1}\right)$$

  as the elastic energy in the body at time t. Therefore, since

  $$\mathcal{E}(t) = \frac{\sigma(t)^2}{2}\left(\frac{l(t)}{a_0} + \frac{L}{a_1}\right) + \kappa l(t),$$

  we can write the following energy balance

  $$\mathcal{Q}(t) + \int_0^t \frac{d}{ds}\left(\frac{\sigma(s)p(s)(\{0,L\})}{2} + \kappa l(s)\right)\,ds = \mathcal{Q}(0) + \int_0^t \sigma(s)\big[\dot{w}(s)\big]_0^L\,ds$$

  where the left hand side is the sum of the elastic energy and a dissipative cost due to damage.

- As explained above, the Griffith type Evolution Law states that damage can only grow when the stress σ saturates the constraint. This threshold condition generalizes the Initial Constituve Law (4.3.15) to the quasi-static setting. As will be explained in the Section 4.4, this Constitutive Law (4.3.15) a priori does not propagate to subsequent times through the evolution process, unless the prescribed boundary condition satisfies (4.1.6), corresponding to the case of perfect plasticity. In this case, one can check that $\big(2\kappa a_0 - \sigma(t)^2\big)l(t) = 0$ for all $t \in [0,T]$.

Motivated by the static analysis led in [15] where the authors have shown how brittle damage can lead to Hencky perfect plasticity, it is natural to extend this analysis to quasi-static evolutions and inquire whether (u,e,p,σ) is of perfect plasticity type or not. Following [49, Definition 4.2], in order for this quasi-static evolution to be of perfect plasticity type, it only remains to prove the Energy Balance :

$$\frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\int_0^t |\dot p(s)|\,([0,L])\,ds = \frac{La_1}{2}e(0)^2 + \int_0^t\int_0^L \sigma(s)\dot w'(s)(x)\,dxds \tag{4.3.19}$$

for all time $t \in [0,T]$, where we used that $\int_0^t |\dot p(s)|\,([0,L])\,ds = \mathcal{V}(p;0,t)$ according to [49, Theorem 7.1]. As u, σ, p, μ and l are all absolutely continuous on $[0,T]$, we have the following Proposition :

Proposition 4.3.9. *For all* $t \in [0,T]$*,*

$$\frac{La_1}{2}e(t)^2 + \int_0^t \sigma(s)\dot p(s)([0,L])\,ds = \frac{La_1}{2}e(0)^2 + \int_0^t \sigma(s)\big[\dot w(s)\big]_0^L\,ds. \tag{4.3.20}$$

Proof. Indeed, for all $t \in [0,T]$, (4.3.8) and (4.3.5) entail that

$$\mathcal{E}(t) = \frac{La_1}{2}e(t)^2 + l(t)\left(\frac{\sigma(t)^2}{2a_0}+\kappa\right)$$
$$= \frac{La_1}{2}e(t)^2 + l(0)\left(\frac{\sigma(0)^2}{2a_0}+\kappa\right) + \int_0^t \frac{\mathrm{d}}{\mathrm{ds}}\left(l\left(\frac{\sigma^2}{2a_0}+\kappa\right)\right)(s)\,ds.$$

Using that $\dot{\widehat{l\sigma^2}} = \dot l\sigma^2 + 2l\sigma\dot\sigma$ together with (4.3.15) and Theorem 4.1.1, we infer that

$$\mathcal{E}(t) = \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|\sigma(0)|\,\frac{l(0)}{a_0} + \int_0^t\left(\frac{\dot l}{a_0}\sqrt{2\kappa a_0}\,|\sigma| + \frac{l\sigma\dot\sigma}{a_0}\right)ds.$$

Thus (4.3.11), Theorem 4.1.1 and (4.3.18) entail that

$$\mathcal{E}(t) = \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|p(0)|\,([0,L]) + \int_0^t \sigma\left(\frac{\dot l\sigma + l\dot\sigma}{a_0}\right)ds$$
$$= \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|p(0)|\,([0,L]) + \int_0^t \sigma(s)\dot p(s)([0,L])\,ds.$$

Then, (4.3.8), (4.3.4) and (4.3.15) complete the proof of (4.3.20). □

Therefore, due to the homogeneity in space of σ together with the stress constraint, (4.3.20) ensures that the upper bound inequality of (4.3.19) is always satisfied :

Proposition 4.3.10. *For all $t \in [0,T]$,*

$$\frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\int_0^t |\dot p(s)|\,([0,L])\,ds \geq \frac{La_1}{2}e(0)^2 + \int_0^t\int_0^L \sigma(s)\dot w'(s)(x)\,dxds.$$

Having all the previous results in mind, one would naturally be tempted to intuit the validity of the Energy Balance (4.3.19), hence proving that the quasi-static damage evolution (u,e,p,σ) is indeed one of perfect plasticity. Surprisingly, the interplay between relaxation and irreversibility of the damage is not stable through time evolutions. Indeed, depending on the choice of the prescribed Dirichlet boundary condition $w \in AC([0,T];H^1(\mathbb{R}))$, the effective quasi-static damage evolution may not be of perfect plasticity type, as illustrated in the example of Figure 4.3. Understanding on which condition the effective quasi-static evolution is of perfect plasticity type is the content of the next section.

4.4 . Energy Balance

We can prove that the Energy Balance (4.3.19) is satisfied if and only if σ saturates the constraint once l is non-zero, until the end of the process (see Figure 4.2).

Lemma 4.4.1. *Let $t_0 = \sup\{t\in[0,T]\,:\,\mu(t)\big([0,L]\big)=0\}$. The Energy Balance* (4.3.19) *is satisfied if and only if σ saturates the constraint during the whole time interval $[t_0,T]$, i.e.*

$$|\sigma(t)| = \sqrt{2\kappa a_0} \quad \textit{for all } t\in[t_0,T]. \tag{4.4.1}$$

Proof. Let us heuristically explain the argument. On the one hand, let us assume that (4.4.1) holds. Since $\mu(t)=0=p(t)$ in $\mathcal{M}\big([0,L]\big)$ for all $t\in[0,t_0]$ and $|\sigma(t)|=\sqrt{2\kappa a_0}$ for all $t\in[t_0,T]$, the Flow-Rule holds :

$$\sigma(t)\dot p(t)\big([0,L]\big) = \sqrt{2\kappa a_0}\,|\dot p(t)|\,\big([0,L]\big) \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in[0,T]. \tag{4.4.2}$$

This is immediate during the time interval $[0,t_0]$, while during the time interval $[t_0,T]$, by continuity we infer that σ is constant and either $\sigma\equiv\sqrt{2\kappa a_0}$ or $\sigma\equiv-\sqrt{2\kappa a_0}$. Especially, since $p=\frac{\sigma}{a_0}\mu$ and μ is non-decreasing in time, we infer that p is either non-decreasing or non-increasing in time on $[t_0,T]$, according to the sign of σ. Hence, $\sigma\dot p = \frac{\sigma^2}{a_0}\dot\mu = \sqrt{2\kappa a_0}\,|\dot p|$ in $\mathcal{M}\big([0,L];\mathbb{R}^+\big)$. Therefore, the validity of the Energy Balance (4.3.19) follows from (4.3.20) and (4.4.2).

On the other hand, (4.4.1) is also a necessary condition. Let us assume that the Energy Balance (4.3.19) is satisfied. From (4.3.20), we infer that

$$\int_0^t \sigma(s)\dot p(s)\big([0,L]\big)\,ds = \sqrt{2\kappa a_0}\int_0^t |\dot p(s)|\,\big([0,L]\big)\,ds$$

for all $t\in[0,T]$. As $\sigma\dot p\big([0,L]\big)\leq\sqrt{2\kappa a_0}\,|\dot p|\,\big([0,L]\big)$ always holds, we deduce that equality (4.4.2) is satisfied $\mathcal{L}^1$-a.e. on $[0,T]$. In particular, (4.4.1) must hold, otherwise the Flow-Rule will not be satisfied during a non $\mathcal{L}^1$-negligible set of times in $[t_0,T]$. Indeed, if $|\sigma(t)|<\sqrt{2\kappa a_0}$ for some $t\in(t_0,T)$, considering the maximal time interval $t\in I\subset(t_0,T]$ during which σ never saturates the constraint, we get by continuity of σ and Theorem 4.1.1 that $\mathrm{Int}(I)$ is a non empty interval and μ is a constant non-zero measure on I. Moreover, there exists $E\subset I$ such that $\mathcal{L}^1(E)>0$ and $\dot\sigma\neq0$ on E. If such was not the case, σ would be constant on I, which is impossible by maximality of the interval. Thus, one simultaneously has $\dot{p}\big([0,L]\big) = \dot{\sigma}/a_0 l \neq 0$ and $|\sigma| < \sqrt{2\kappa a_0}$ on E, so that $|\sigma|\,|\dot{\sigma}/a_0|\,l < \sqrt{2\kappa a_0}\,|\dot{\sigma}/a_0|\,l$ which is in contradiction with the Flow-Rule (4.4.2). □

Yet, this sufficient and necessary condition (4.4.1) relies on the definition of the time t_0. It remains to find an equivalent condition which can be expressed only in terms of the data of the setting. This is the content of Theorem 4.1.2, illustrated in Figure 4.1.

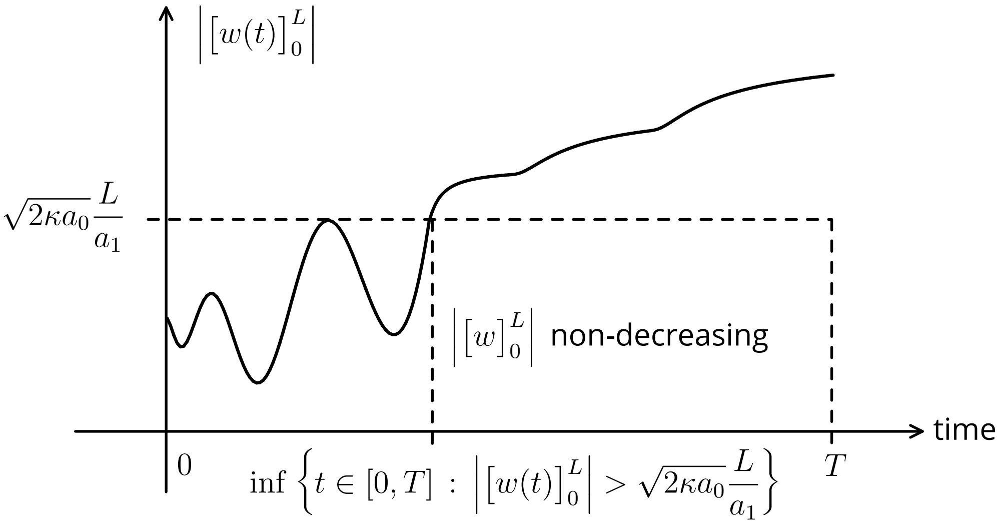

Figure 4.1

Proof of Theorem 4.1.2 : Necessary condition. Assume that the Energy Balance (4.3.19) is satisfied at all time $t \in [0,T]$. Then, Theorem 5.2 and Theorem 6.1 of [49] ensure that

$$(u,e,p,\sigma) \in AC\big([0,T]; BV((0,L)) \times \mathbb{R} \times \mathcal{M}([0,L]) \times K\big)$$

satisfies the Flow Rule :

$$\sigma(t)\dot{p}(t)\big([0,L]\big) = \sqrt{2\kappa a_0}\,|\dot{p}(t)|\,\big([0,L]\big) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T]. \tag{4.4.3}$$

On the one hand, (4.3.5) together with the trace theorem in $BV((0,L))$ entail that, at all time,

$$Du((0,L)) = \big[u\big]_0^L = Le + p((0,L)) = \big[w\big]_0^L - \frac{\sigma l}{a_0} + p((0,L))$$

so that

$$p([0,L]) = p((0,L)) + \big[w-u\big]_0^L = \frac{\sigma l}{a_0} = \frac{\sigma\mu([0,L])}{a_0}.$$

On the other hand, by (4.3.18) we infer that for $\mathcal{L}^1$-a.e. time in $[0,T]$, the following time derivatives exist and satisfy

$$\dot{p} = \frac{\dot{\sigma}\mu}{a_0} + \frac{\sigma\dot{\mu}}{a_0} \text{ in } \mathcal{M}([0,L]) \quad \text{hence} \quad \dot{p}([0,L]) = \frac{\dot{\sigma}l}{a_0} + \frac{\sigma\dot{\mu}([0,L])}{a_0} = \frac{\dot{\widehat{\sigma l}}}{a_0} = \dot{\widehat{p([0,L])}}. \tag{4.4.4}$$

Let $t_0 := \sup\{t \in [0,T] \,:\, l(t) = 0\}$, with the convention $t_0 = 0$ if $l(0) > 0$ (see Figure 4.2). As l is non-decreasing on $[0,T]$, if $t_0 = T$ then $l \equiv 0$ at all time and (4.3.5) entails that $\big[w\big]_0^L \equiv \sigma\frac{L}{a_1}$ at all time as well, so that (4.1.6) is always true since $\sigma \in K$. We can thus assume that $t_0 < T$. Suppose that $0 \le a < b \le T$ are such that

$$\left|\big[w(b)\big]_0^L\right| < \left|\big[w(a)\big]_0^L\right|.$$

We have to prove that

$$\left|\big[w(b)\big]_0^L\right| \le \sqrt{2\kappa a_0}\frac{L}{a_1}. \tag{4.4.5}$$

Using again the non-decreasing character of l and (4.3.5), since

$$\left|\big[w\big]_0^L\right| = |\sigma|\left(\frac{l}{a_0} + \frac{L}{a_1}\right),$$

we necessarily have $|\sigma(b)| < |\sigma(a)| \le \sqrt{2\kappa a_0}$ hence $|\sigma(b)| < \sqrt{2\kappa a_0}$ in particular. By continuity of σ and because $0 < b \le T$, there exists $\eta > 0$ such that for all $s \in [b-\eta, b] \subset [0,T]$, $|\sigma(s)| < \sqrt{2\kappa a_0}$. Let us consider

$$s_0 := \inf\{s < b \,:\, |\sigma(t)| < \sqrt{2\kappa a_0} \quad \text{for all } s < t \le b\}$$

and

$$I := (s_0, b].$$

Recalling Theorem 4.1.1, by continuity we know that l is constant on the whole segment

$$\overline{I} := [s_0, b].$$

Let us prove that $l \equiv 0$ on $\overline{I}$ (hence on $[0,b]$), which will prove (4.4.5) due to (4.3.5) once more. By contradiction, assume that $l \equiv c > 0$ on $\overline{I}$. Since σ does not saturate the constraint during all the time interval $(s_0, b]$, the Flow Rule (4.4.3) implies that $\dot{p}(s)([0,L]) = \frac{\dot{\sigma}(s)l}{a_0} = 0$ for $\mathcal{L}^1$-a.e. $s \in (s_0, b]$. Therefore, σ is constant during the whole time interval $\overline{I}$ and in particular

$$|\sigma(s_0)| < \sqrt{2\kappa a_0}.$$

Next, let us show that $s_0 > 0$. Indeed,

- either $l(0) = 0$. Hence, one can check that $l(t_0) = 0$ by definition of t_0 and continuity of l. This implies that $s_0 > t_0$ since $l(s_0) = c > 0$.
- Or $l(0) > 0$ thus $t_0 = 0$. Then, (4.3.15) entails that $|\sigma(0)| = \sqrt{2\kappa a_0}$ so that $s_0 > 0$ since $|\sigma(s_0)| < \sqrt{2\kappa a_0}$.

Thus, since $s_0 > 0$, $|\sigma(s_0)| < \sqrt{2\kappa a_0}$ and σ is continuous on $[0,T]$, there exists $\eta > 0$ small enough such that $|\sigma(s)| < \sqrt{2\kappa a_0}$ for all $s \in (s_0 - \eta, b] \subset [0,T]$, which is impossible by definition of I.

Sufficient condition. Assume (4.1.6). Let us show that the Energy Balance (4.3.19) is satisfied. As before, we consider the time $t_0 := \sup\{t \in [0,T] \,:\, l(t) = 0\}$, with the convention $t_0 = 0$ if $l(0) > 0$. If $t_0 = T$ then $l \equiv 0$ at all time and (4.3.11) entails that $\mu \equiv 0 \equiv p$ in $\mathcal{M}([0,L])$ at all time as well. Hence, the Energy Balance is obviously satisfied as it simply states that

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_0^L \sigma\dot{w}'\,dxds = \frac{\sigma(t)^2 L}{2a_1},$$

which is true thanks to (4.3.8) and (4.3.5). Therefore, we can assume that $t_0 < T$.

Let us first note that we always have

$$|\sigma(t_0)| = \sqrt{2\kappa a_0}. \tag{4.4.6}$$

Indeed,

- either $t_0 > 0$. In this case, by continuity of l and definition of t_0, we must have $l(t_0) = 0$. By contradiction, assume that $|\sigma(t_0)| < \sqrt{2\kappa a_0}$. Then, by continuity of σ and l again, there exists $\eta > 0$ small enough such that $|\sigma(s)| < \sqrt{2\kappa a_0}$ for all $s \in (t_0 - \eta, t_0 + \eta) \subset [0, T]$. Once more, Theorem 4.1.1 entails that l is constant on the whole segment $[t_0 - \eta, t_0 + \eta]$. In particular, $0 = l(t_0) = l(t_0 + \eta)$ which is impossible by definition of t_0.
- Or $t_0 = 0$. We must again consider two cases.
  - Either $l(0) = 0$. By contradiction, assume that $|\sigma(0)| < \sqrt{2\kappa a_0}$. By continuity of l and σ, as before there exists $\eta > 0$ small enough such that $|\sigma(s)| < \sqrt{2\kappa a_0}$ for all $s \in [0, \eta) \subset [0, T]$, entailing that $0 = l(0) = l(\eta)$ which is again impossible by definition of t_0.
  - Or $l(0) > 0$ and (4.3.15) ensures that $|\sigma(0)| = \sqrt{2\kappa a_0}$.

Next, let us show that, as illustrated in Figure 4.2,

$$|\sigma(t)| = \sqrt{2\kappa a_0} \quad \text{for all subsequent time } t \in [t_0, T]. \tag{4.4.7}$$

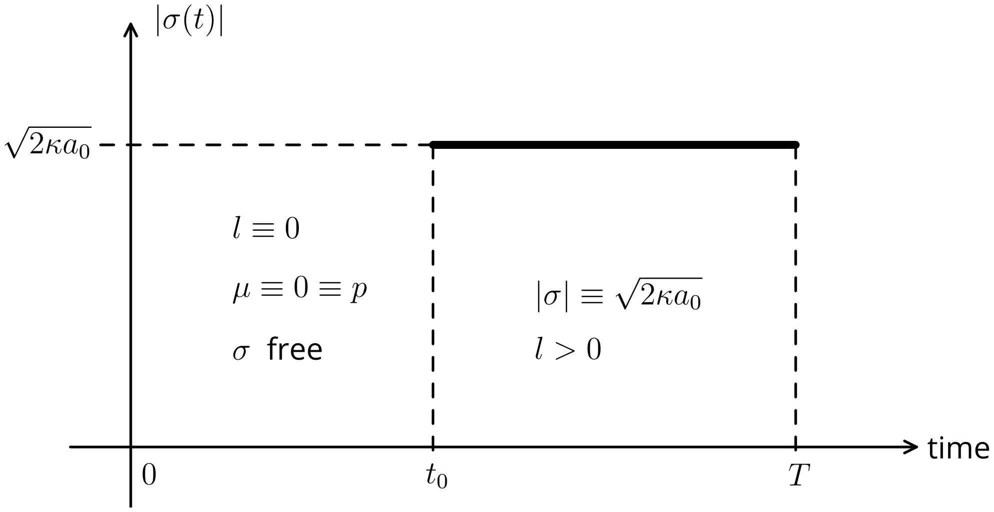

Figure 4.2

By contradiction, assume there exists $t_0 < t \leq T$ such that $|\sigma(t)| < \sqrt{2\kappa a_0}$ and consider the maximal time interval where σ does not saturate the constraint, by defining :

$$t_1 = \inf\{s \in [0, t] \ : \ |\sigma| < \sqrt{2\kappa a_0} \text{ for all time in } (s, t]\}$$

and

$$t_2 = \sup\{s \in [t,T] \,:\, |\sigma| < \sqrt{2\kappa a_0} \text{ for all time in } [t,s)\}.$$

By continuity of σ and (4.4.6), we have that $t_0 \leq t_1 < t \leq t_2 \leq T$ and $|\sigma(s)| < \sqrt{2\kappa a_0}$ for all $s \in (t_1, t_2)$. Since $t_1 \geq t_0$, we infer that $l \equiv l(t_1) > 0$ is a positive constant on the whole segment $[t_1, t_2]$ by Theorem 4.1.1. One can also check that $|\sigma(t_1)| = \sqrt{2\kappa a_0}$. Indeed, either $t_1 = t_0$ and (4.4.6) concludes, or $t_1 > t_0$. In particular $t_1 > 0$ and the continuity of σ together with the definition of t_1 imply that $|\sigma(t_1)| = \sqrt{2\kappa a_0}$. Therefore, (4.3.5) yields that for all $s \in (t_1, t_2)$

$$\left|\big[w(t_1)\big]_0^L\right| = \sqrt{2\kappa a_0}\left(\frac{l}{a_0} + \frac{L}{a_1}\right) > \left|\big[w(s)\big]_0^L\right| = |\sigma(s)|\left(\frac{l}{a_0} + \frac{L}{a_1}\right).$$

By (4.1.6), we get that $\left|\big[w(s)\big]_0^L\right| \leq \sqrt{2\kappa a_0}\dfrac{L}{a_1}$ for all $s \in (t_1, t_2)$. The continuity of $\big[w\big]_0^L$ and (4.3.5) imply that

$$\left|\big[w(t_1)\big]_0^L\right| = \sqrt{2\kappa a_0}\left(\frac{l}{a_0} + \frac{L}{a_1}\right) \leq \sqrt{2\kappa a_0}\frac{L}{a_1},$$

which is impossible by positivity of $l > 0$. We have thus proven the validity of (4.4.7).

Therefore, we are in the configuration of Figure 4.2. Whether $l(0) > 0$ or not, one can check that

$$p(t) = \frac{\sigma(t_0)}{a_0}\mu(t) \quad \text{and} \quad \left(\sqrt{2\kappa a_0} - |\sigma(t)|\right)^2 \frac{l(t)}{2a_0} = 0 \quad \text{for all } t \in [0,T]. \tag{4.4.8}$$

Indeed, either $l(t_0) > 0$ hence $t_0 = 0$ and $\sigma = \sigma(0) = \pm\sqrt{2\kappa a_0}$ is constant on the whole segment $[0,T]$. Or $l(t_0) = 0$, so that $\mu(t) = 0 = p(t)$ for all $t \in [0, t_0]$ and $\sigma = \sigma(t_0) = \pm\sqrt{2\kappa a_0}$ is constant on the whole segment $[t_0, T]$. In particular (4.4.8) is satisfied. Consequently, the monotonicity of μ entails that

$$\mathcal{V}(p; 0, t) = \left|\frac{\sigma(t_0)}{a_0}\right| \big(\mu(t)\big([0,L]\big) - \mu(0)\big([0,L]\big)\big) = |p(t)|\,\big([0,L]\big) - |p(0)|\,\big([0,L]\big)$$

for all $t \in [0,T]$. Using (4.3.4), (4.3.14), (4.3.15) and (4.3.8), we infer that

$$\begin{aligned}
\mathcal{E}(t) = \frac{La_1}{2}e(0)^2 + \sqrt{2\kappa a_0}\,|p(0)|\,\big([0,L]\big) + \int_0^t\int_0^L \sigma\dot{w}'\,dxds \\
= \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|p(t)|\,\big([0,L]\big) + \left(\sqrt{2\kappa a_0} - |\sigma(t)|\right)^2\frac{l(t)}{2a_0} \\
= \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|p(t)|\,\big([0,L]\big)
\end{aligned}$$

for all $t \in [0,T]$, which concludes the proof of the Energy Balance (4.3.19) since

$$\begin{aligned}
\frac{La_1}{2}e(0)^2 + \int_0^t\int_0^L \sigma\dot{w}'\,dxds = \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\,|p(t)|\,\big([0,L]\big) - \sqrt{2\kappa a_0}\,|p(0)|\,\big([0,L]\big) \\
= \frac{La_1}{2}e(t)^2 + \sqrt{2\kappa a_0}\mathcal{V}(p;0,t).
\end{aligned}$$

□

Remark 4.4.2. Condition (4.1.6) is equivalent to the non-decreasing character of $\left|\left[w\right]_0^L\right|$ during the time interval $[t_0^*, T]$ where

$$t_0^* = \inf\left\{t \in [0,T] \,:\, \left|\left[w(t)\right]_0^L\right| > \sqrt{2\kappa a_0} L/a_1\right\},$$

with the convention $t_0^* = T$ if $\left|\left[w\right]_0^L\right|$ remains smaller or equal to $\sqrt{2\kappa a_0}L/a_1$ during the whole time interval $[0,T]$. Note that

$$t_0^* = t_0. \tag{4.4.9}$$

Indeed, on the one hand, either $t_0 = 0 \leq t_0^*$, or $t_0 > 0$. In this case, we infer that

$$\left|\left[w(t)\right]_0^L\right| = |\sigma(t)| \frac{L}{a_1} \leq \sqrt{2\kappa a_0}\frac{L}{a_1}$$

for all previous time $0 \leq t < t_0$ according to (4.3.5), Proposition 4.3.2 and the fact that $l(t) = 0$. Therefore, $t \leq t_0^*$ which leads to $t_0 \leq t_0^*$ when t tends to t_0. On the other hand, assume by contradiction that $t_0^* > t_0$. By definition of t_0 and t_0^*, we deduce that for all $t \in (t_0, t_0^*)$,

$$\left|\left[w(t)\right]_0^L\right| = |\sigma(t)| \left(\frac{l(t)}{a_0} + \frac{L}{a_1}\right) \leq \sqrt{2\kappa a_0}\frac{L}{a_1} \quad \text{and} \quad l(t) > 0.$$

In particular, we infer that $|\sigma(t)| < \sqrt{2\kappa a_0}$ for all $t \in (t_0, t_0^*)$, entailing that $l \equiv l(t_0^*) > 0$ on $[t_0, t_0^*]$ by (4.1.1) and continuity of l. Then, (4.4.6) implies that

$$\left|\left[w(t_0)\right]_0^L\right| = \sqrt{2\kappa a_0}\left(\frac{l(t_0^*)}{a_0} + \frac{L}{a_1}\right) > \sqrt{2\kappa a_0}\frac{L}{a_1}$$

which is impossible.

4.5 . Concluding remarks

In spite of the conjecture motivated by the static analysis of [15], Theorem 4.1.2 determines the exact conditions on which the quasi-static evolution (u, e, p, σ) is of perfect plasticity type or not. In particular, when the prescribed boundary datum $w \in AC([0,T]; H^1(\mathbb{R}))$ is such that $|[w]_0^L|$ is decreasing and remains larger than $\sqrt{2\kappa a_0}\frac{L}{a_1}$, the Energy Balance (4.3.19) is never satisfied (see Figure 4.3).

This suggests to interpret (u, c) rather as a quasi-static evolution of damage as stated in Theorem 4.1.1, even when the prescribed boundary datum satisfies (4.1.6). In this case, the very specific nature of the plastic evolution illustrated in Figure 4.2 seems to confirm the interpretation of the evolution as one of damage. Indeed, the only configuration of perfect plasticity we obtain is very restrictive as the evolution remains purely elastic until a threshold time t_0, after which the stress σ always saturates the constraint and the damage keeps on increasing, so that the elastic strain remains constant until the end of the process which is rather specific to damage than plasticity. Besides, when choosing $T > 2\sqrt{2\kappa a_0}/a_1$ and applying a loading-unloading Dirichlet condition

$$w : (t,x) \in [0,T] \times [0,L] \mapsto x\left(t\mathbb{1}_{[0,\frac{T}{2}]}(t) + (T-t)\mathbb{1}_{(\frac{T}{2},T)}(t)\right),$$

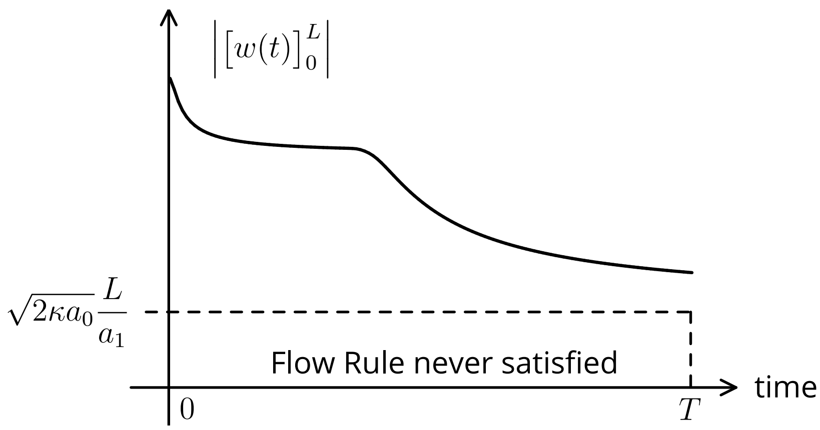

Figure 4.3

the response of the limit model is indeed typical of damage, as illustrated in Figure 4.4. Using (4.4.9), one can check that $t_0 = \sqrt{2\kappa a_0}/a_1 > 0$, hence

$$\mu \equiv 0 \quad \text{and} \quad \sigma \equiv \frac{a_1}{L}\left[w\right]_0^L \geq 0 \quad \text{on } [0, t_0].$$

By (4.4.6), (4.1.1) and the increasing character of $\left[w\right]_0^L$ during the time interval $\left[t_0, \frac{T}{2}\right]$, we infer that

$$\sigma \equiv \sqrt{2\kappa a_0} \quad \text{on } \left[t_0, \frac{T}{2}\right],$$

which corresponds to the hardening phase. Then, since l is non-decreasing and $\left[w\right]_0^L$ is decreasing during the unloading phase, (4.3.8) entails that σ decreases during the time interval $\left[\frac{T}{2}, T\right]$. In particular, $0 \leq \sigma(t) < \sqrt{2\kappa a_0}$ for all $t \in \left(\frac{T}{2}, T\right]$, so that $\mu \equiv \mu\left(\frac{T}{2}\right)$ is constant during the unloading phase and

$$\sigma(t) \equiv \frac{\left[w(t)\right]}{\frac{l(\frac{T}{2})}{a_0} + \frac{L}{a_1}} = 2\sqrt{2\kappa a_0}\frac{T-t}{T} \quad \text{for all } t \in \left[\frac{T}{2}, T\right].$$

In particular, at the end of the loading-unloading process, the medium goes back to its reference configuration, whereas in perfect plasticity one expects a residual plastic strain (see Figure 4.5). On the other hand, interpreting the evolution as one of damage also fails to be completely satisfactory, as $|\sigma|$ never exceeds the damage yield threshold $\sqrt{2\kappa a_0}$, including during the hardening phase, which is specific to perfect plasticity and therefore consists in a painful lack of generality for the description of a damage evolution as well (see Figure 4.5). One could wonder if the effective model we obtained lies somehow in between damage and plasticity. Without being able to answer completely this question, let us remark that the effective evolution obtained here does not fit in the large class of elastoplasticity-damage models introduced in [4, 40, 41] where we expect the Energy Balance to hold and the plastic yield surface to shrink as damage increases, whereas here (4.3.19) is not always satisfied and $\partial K = \{\pm\sqrt{2\kappa a_0}\}$ is fixed.

Looking at the constructive proof of [60, Theorem 2], one could argue that passing to the limit $\varepsilon \searrow 0$ in the time-continuous quasi-static evolutions might not have been the right approach, as

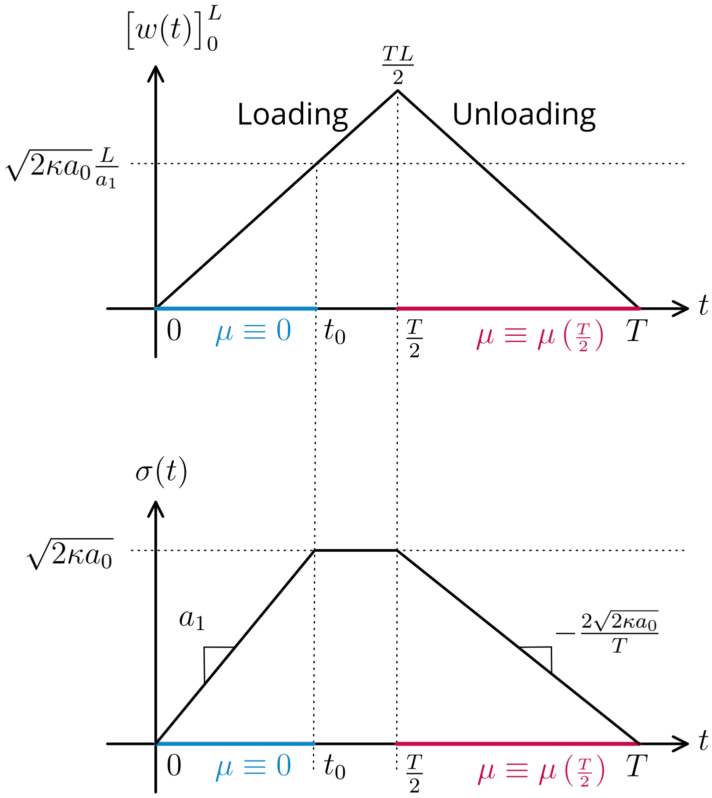

Figure 4.4

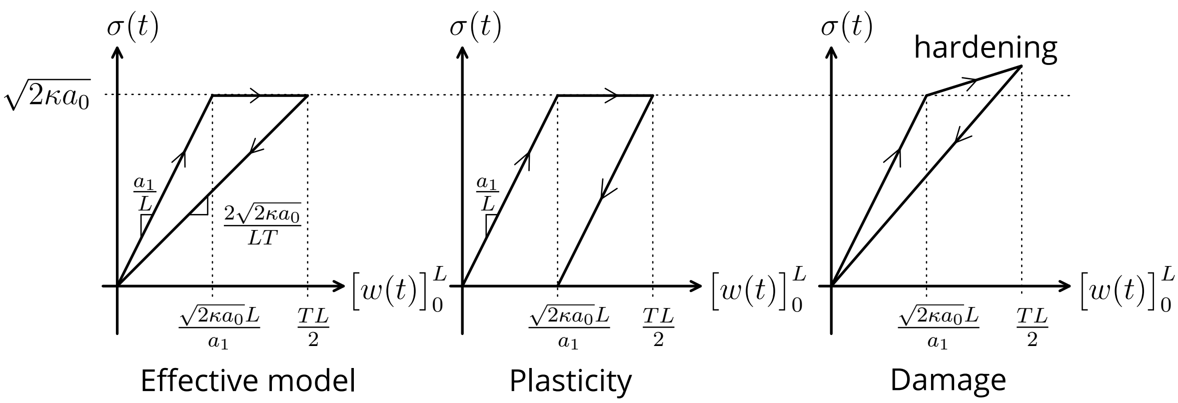

Figure 4.5

some incremental information (see the minimality formulae and track of the history of damage [60, Formulae (15), (16), (21)]) available at the stage of time discretizations is lost once the time step has been sent to 0 (see [25] for a related case study of non-commutability). Indeed, let us fix a time subdivision $\mathcal{T}_N = \{0 = t_0^N, ..., t_N^N = T\}$ of $[0,T]$ with

$$\delta_N := \sup_{i \in [\![0,N-1]\!]} \left| t_{i+1}^N - t_i^N \right|.$$

Following minor adaptations of the present work and passing first to the limit $\varepsilon \searrow 0$, we infer the existence of a piecewise constant in time evolution

$$(u_N, e_N, p_N, \sigma_N, \mu_N) : [0,T] \to BV((0,L)) \times L^2((0,L)) \times \mathcal{M}([0,L]) \times \mathbb{R} \times \mathcal{M}([0,L])$$

with uniformly bounded variation in N such that

$$\sigma_N \in L^\infty([0,T];K) \quad \text{is homogeneous in space,}$$

$$p_N = \frac{\sigma_N}{a_0}\mu_N, \quad e_N = \frac{\sigma_N}{a_1}, \quad [w]_0^L = \sigma_N \left(\frac{\mu_N\big([0,L]\big)}{a_0} + \frac{L}{a_1} \right),$$

and for all $i \in [\![0,N]\!]$

$$\begin{cases} \big(u_N^i, e_N^i, p_N^i\big) \in \mathcal{A}(w(t_i^N)), \\ \mathcal{E}_N^i := \dfrac{\sigma_N^i \left[w(t_i^N)\right]_0^L}{2} + \kappa \mu_N^i\big([0,L]\big) = \mathcal{E}(0) + \displaystyle\int_0^{t_i^N} \sigma_N \left[\dot{w}\right]_0^L ds + o_{\delta_N \searrow 0}(1), \\ \mu_N^i\big([0,L]\big) = \dfrac{a_0}{2\kappa} \left(\dfrac{\mathcal{E}_N^i}{a_0} - \dfrac{\kappa L}{a_1} + \sqrt{\Delta_N^i} \right), \end{cases} \tag{4.5.1}$$

where

$$\Delta_N^i = \left(\frac{\mathcal{E}_N^i}{a_0} + \frac{\kappa L}{a_1} \right)^2 - \frac{2\kappa}{a_0} \left| \left[w(t_i^N)\right]_0^L \right|^2 \geq 0$$

and $(v, \eta, q) \in \mathcal{A}(w(t_i^N))$ means that $v \in BV((0,L))$, $\eta \in L^2((0,L))$, $q \in \mathcal{M}\big([0,L]\big)$, $Dv = \eta + q \llcorner (0,L)$ and $q \llcorner \{0,L\} = (w(t_i^N) - v)\big(\delta_L - \delta_0\big)$. Moreover, as we pass to the limit $\varepsilon \searrow 0$ in the incremental minimality [60, Formulae (15) and (16)]

$$(u_{N,\varepsilon}^i, 1 - \Theta_{N,\varepsilon}^i, a_{N,\varepsilon}^i) \in \underset{\substack{u \in w(t_i^N) + H_0^1((0,L)) \\ \theta \in L^\infty((0,L);[0,1]) \\ a \in \mathcal{G}_\theta(\varepsilon a_0, a_{N,\varepsilon}^{i-1})}}{\operatorname{argmin}} \left\{ \int_0^L \left(\frac{1}{2} a \left| u' \right|^2 + \frac{\kappa}{\varepsilon} \theta \Theta_{N,\varepsilon}^{i-1} \right) dx \right\}$$

of the discrete evolution of [60], one could hope that (u_N, e_N, p_N) satisfies a stronger incremental minimality as in [49, Formula (4.12)] :

$$(u_N^i, e_N^i, p_N^i) \in \underset{(u,e,p) \in \mathcal{A}(w(t_i^N))}{\operatorname{argmin}} \left\{ \int_0^L \frac{1}{2} a_1 e^2 \, dx + \sqrt{2\kappa a_0} \left| p - p_N^{i-1} \right| \big([0,L]\big) \right\}. \tag{4.5.2}$$

Assume that (4.5.2) holds. Then, following exactly the proof of [49, Theorem 4.5] ensures the existence of a subsequence (independent of t, not relabeled) and a quasi-static evolution of perfect plasticity

$$(u, e, p) : [0, T] \to BV((0, L)) \times L^2((0, L)) \times \mathcal{M}([0, L])$$

satisfying (4.1.3) and (4.1.4), such that for all $t \in [0, T]$

$$\begin{cases} u_N(t) \rightharpoonup u(t) \text{ weakly-* in } BV((0, L)), \\ e_N(t) \rightharpoonup e(t) \text{ weakly in } L^2((0, L)), \\ p_N(t) \rightharpoonup p(t) \text{ weakly-* in } \mathcal{M}([0, L]), \end{cases}$$

when passing to the limit $\delta_N \to 0$. In particular, all the above quantities in (4.5.1) pass to the limit as $\delta_N \to 0$. Therefore, Theorem 4.1.2 holds true for the quasi-static evolution (u, e, p) as well. Consequently, for any prescribed boundary datum $w \in AC([0, T]; H^1(\mathbb{R}))$ not satisfying (4.1.6), we come to a contradiction. It unfortunately proves that commuting the limits in ε and N leads to no better statement.

One could then argue that the One-sided Minimality of [60, Theorem 2] might be too weak and that it might have been preferable to pass to the limit $\varepsilon \searrow 0$ in the time-continuous quasi-static evolution of [68, Theorem 7] instead, since the Minimality condition considered in [68, Definition 3] is stronger as stated in [68, Remark 4]. Unfortunately again, as we are working in the one-dimensional setting, the two minimality conditions are equivalent. Indeed, let us fix $\varepsilon = 1$ for the present discussion. By optimality of the evolution [60, Proposition 1], there exists $\chi_n : [0, T] \to L^\infty((0, L); \{0, 1\})$ non-decreasing in time such that $\chi_n(t) \rightharpoonup 1 - \Theta(t)$ weakly-* in $L^\infty((0, L))$ for all time $t \in [0, T]$. Setting $D_n := \{\chi_n = 1\}$, we have that for all (A, θ) such that $A \in \widehat{\mathcal{G}}_\theta(\{D_n(t)\}, a_0, a_1) = \mathcal{G}_\theta(a_0, a_1)$, $A \in G_{\tilde{\theta}}(a_0, a(t))$ with

$$\tilde{\theta} := \frac{\theta - (1 - \Theta(t))}{\Theta(t)} \mathbb{1}_{\{\Theta(t) > 0\}} \in L^\infty((0, L); [0, 1]).$$

Therefore, A is an admissible competitor for the One-sided minimality of [60, Theorem 2] too.

Eventually, we conclude with a general remark by noticing that this one-dimensional analysis seems to raise the question whether Hencky perfect plasticity is distinguishable from damage or not in a static setting, as mentionned in the recent survey [78, Section 1, p10].

5 - Appendix

Proposition 5.0.1. *Let $N, m \in \mathbb{N} \setminus \{0\}$, $p \in (1,2]$, $k \in \mathbb{N} \setminus \{0,1\}$, $w \in W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^m)$ and $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. For all $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that $u = w$ in an open bounded neighborhood of $\partial\Omega$, there exist a sequence $\{u_h\}_{h\in\mathbb{N}}$ in $SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ as well as N_h disjoint closed $(N-1)$-dimensional simplexes $\Sigma_1^h, \ldots, \Sigma_{N_h}^h \subset \Omega$ satisfying :*

$$\overline{J_{u_h}} = \bigcup_{i=1}^{N_h} \Sigma_i^h, \quad \mathcal{H}^{N-1}(\overline{J_{u_h}} \setminus J_{u_h}) = 0, \quad u_h \in W^{k,\infty}(\Omega \setminus \overline{J_{u_h}};\mathbb{R}^m),$$

$$\begin{cases} u_h = w \text{ in an open bounded neighborhood of } \partial\Omega, \\ u_h \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^m), \\ \nabla u_h \to \nabla u \quad \text{strongly in } L^p(\Omega;\mathbb{M}^{m\times N}), \\ \limsup_{h\to\infty} \mathcal{H}^{N-1}(J_{u_h}) \le \mathcal{H}^{N-1}(J_u). \end{cases} \tag{5.0.1}$$

We do not detail the proof of this result which follows the steps of the constructive proofs of [26, Lemma 5.2], [46, Theorem 3.1] and [45, Theorem 3.9, Corollary 3.11] with minor adaptations to the Dirichlet setting. The key point here is that, due to our definition of $V_\varepsilon^{\mathrm{Dir}}(\Omega')$, we need the approximating sequence to coincide with w in an open neighborhood of the boundary and not only on $\partial\Omega$ (as in Theorem 4.2, Remark 4.3 and formula (4.1) in [38]).

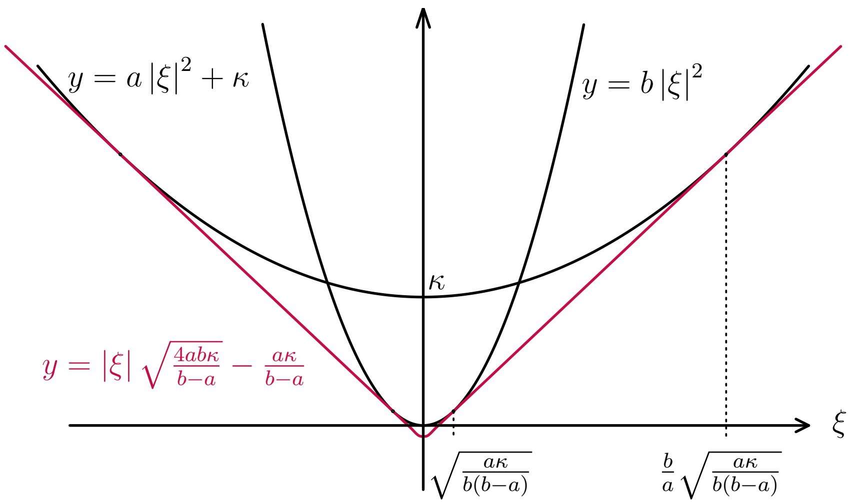

Figure 5.1

Lemma 5.0.2. *Let $a, b, K > 0$ with $a < b$ and $f : \xi \in \mathbb{R} \mapsto \min\{K + a|\xi|^2; b|\xi|^2\}$. Then, for all $\xi \in \mathbb{R}$,*

$$\mathcal{C}f(\xi) = \begin{cases} b|\xi|^2 & \text{if } |\xi| \le \sqrt{\frac{aK}{b(b-a)}}, \\ |\xi|\sqrt{\frac{4abK}{b-a}} - \frac{aK}{b-a} & \text{if } \sqrt{\frac{aK}{b(b-a)}} < |\xi| \le \frac{b}{a}\sqrt{\frac{aK}{b(b-a)}}, \\ K + a|\xi|^2 & \text{if } |\xi| > \frac{b}{a}\sqrt{\frac{aK}{b(b-a)}}. \end{cases}$$

Proposition 5.0.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and A_0 and A_1 be two symmetric fourth order isotropic tensors, i.e.*

$$A_i\xi = \lambda_i \mathrm{tr}(\xi)\mathrm{Id} + 2\mu_i \xi$$

for all $\xi \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$ where $\lambda_1 > \lambda_0 > 0$ and $\mu_1 > \mu_0 > 0$ are the Lamé coefficients. For a fixed toughness $\kappa > 0$, we consider the closed convex set

$$K = \{\tau \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \ : \ G(\tau) \leq 2\kappa\}$$

with G defined by (4.1.1) *in the Introduction. We define*

$$\overline{W}(\xi) = \left(\frac{1}{2}A_1 \mathrm{id} : \mathrm{id}\right) \,\square\, I_K^*(\xi)$$

and

$$W_\varepsilon(\xi) = \min\left\{\frac{\kappa}{\varepsilon} + \frac{1}{2}\varepsilon A_0\xi : \xi; \frac{1}{2}A_1\xi : \xi\right\}$$

for all $\varepsilon > 0$ and all $\xi \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$. Then, given a boundary datum $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$, the functional $\mathcal{F}_\varepsilon : L^1(\Omega;\mathbb{R}^N) \to \overline{\mathbb{R}^+}$ defined for all $u \in L^1(\Omega;\mathbb{R}^N)$ by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega SQW_\varepsilon(e(u))\,dx & \text{if } u \in w + H^1_0(\Omega;\mathbb{R}^N) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converges in $L^1(\Omega;\mathbb{R}^N)$ as $\varepsilon \searrow 0$ to the functional

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_\Omega \overline{W}(e(u))\,dx + \int_\Omega I_K^*\left(\frac{dE^s u}{d|E^s u|}\right)\,d|E^s u| + \int_{\partial\Omega} I_K^*\left((w-u)\odot\nu\right)\,d\mathcal{H}^{N-1} \\ \hfill \text{if } u \in BD(\Omega) \\ +\infty \quad \text{otherwise.} \end{cases}$$

Proof. The proof of the Γ-lim inf inequality presents no particular difficulty and we do not give the details of its proof. The key point is to extend approximating sequences $u_k \to u$ by w in a larger open bounded set $\Omega \subset\subset \Omega'$ and rely on [15, Theorem 3.1] in Ω' to conclude.

The proof of the Γ-lim sup inequality relies on the approximation result [83, Theorem 3.5] in BD, which allows us to reduce to the case where $u \in w + C^\infty_c(\Omega;\mathbb{R}^N)$. Indeed, let $u \in BD(\Omega)$. Using the Radon-Nikodým decomposition of Eu with respect to Lebesgue and owing to a measurable selection criterion together with the definition of the inf-convolution, there exist e and p^a in $L^2(\Omega;\mathbb{M}^{N\times N}_{\mathrm{sym}})$ such that

$$Eu = e(u)\mathcal{L}^N \llcorner \Omega + E^s u, \quad e(u) = e + p^a$$
$$\text{and} \quad \overline{W}(e(u)) = \frac{1}{2}A_1 e : e + I_K^*(p^a) \quad \mathcal{L}^N\text{-a.e. in } \Omega.$$

We define $p = E^s u \mathbb{1}_\Omega + p^a\mathcal{L}^N \llcorner \Omega + (w-u)\odot\nu\mathcal{H}^{N-1}\llcorner\partial\Omega \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{N\times N}_{\mathrm{sym}})$. Following the proof of [83, Theorem 3.5] with minor adaptations, one can find a sequence

$$(u_k, e_k, p_k) \in LD(\Omega) \times L^2(\Omega;\mathbb{M}^{N\times N}_{\mathrm{sym}}) \times L^1(\Omega;\mathbb{M}^{N\times N}_{\mathrm{sym}})$$

such that

$$e(u_k) = e_k + p_k \quad \text{for all } k \in \mathbb{N},$$
$$(u_k - w, e_k - e(w), p_k) \in C_c^\infty(\Omega; \mathbb{R}^N) \times C_c^\infty(\Omega; \mathbb{M}^{N\times N}_{\text{sym}}) \times C_c^\infty(\Omega; \mathbb{M}^{N\times N}_{\text{sym}})$$

and

$$u_k \to u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^N), \quad e_k \to e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{N\times N}_{\text{sym}}),$$
$$p_k \rightharpoonup p \quad \text{weakly-* in } \mathcal{M}\big(\overline{\Omega}; \mathbb{M}^{N\times N}_{\text{sym}}\big) \quad \text{and} \quad \int_\Omega |p_k| \, dx \to |p| \left(\overline{\Omega}\right)$$

when $k \nearrow \infty$. By Reshetnyak's Continuity Theorem (see [6, Theorem 2.39]), we get that

$$\int_\Omega I_K^*(p_k)\, dx \to \int_{\overline{\Omega}} I_K^*\left(\frac{dp}{d\,|p|}\right) d\,|p|$$
$$= \int_\Omega I_K^*(p^a)\, dx + \int_\Omega I_K^*\left(\frac{dE^s u}{d\,|E^s u|}\right) d\,|E^s u| + \int_{\partial\Omega} I_K^*\left((w-u)\odot\nu\right) d\mathcal{H}^{N-1}$$

when $k \nearrow \infty$. Therefore, by definition of the inf-convolution and by lower semi-continuity of the Γ-upper limit, denoted by $\mathcal{F}''$, if $\mathcal{F}''(u_k) \leq \mathcal{F}(u_k)$ we would get that

$$\limsup_{k\nearrow\infty} \int_\Omega \overline{W}\big(e(u_k)\big)\, dx$$
$$\leq \int_\Omega \frac{1}{2} A_1 e : e\, dx + \int_\Omega I_K^*(p^a)\, dx + \int_\Omega I_K^*\left(\frac{dE^s u}{d\,|E^s u|}\right) d\,|E^s u|$$
$$+ \int_{\partial\Omega} I_K^*\left((w-u)\odot\nu\right) d\mathcal{H}^{N-1} = \mathcal{F}(u).$$

Therefore, we can assume without loss of generality that $u \in w + C_c^\infty(\Omega; \mathbb{R}^N)$ and conclude the proof of the upper bound arguing as in [15, Proposition 3.3]. □

Proof of the density result. Let $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{N\times N}_{\text{sym}}) \times \mathcal{M}\big(\overline{\Omega}; \mathbb{M}^{N\times N}_{\text{sym}}\big)$ be such that $Eu = e\mathcal{L}^N \llcorner \Omega + p \llcorner \Omega$ and $p \llcorner \partial\Omega = (w-u)\odot\nu\mathcal{H}^{N-1} \llcorner \partial\Omega$. Up to replacing the triplet (u, e, p) by $(u-w, e-e(w), p)$, we can assume that $w = 0$. Let $\Omega' \supset\supset \Omega$ be an open bounded set. We extend u and e by 0 in $\Omega' \setminus \Omega$ and p by 0 in $\Omega' \setminus \overline{\Omega}$. Since Ω is bounded and has a Lipschitz boundary, there exists a finite open cover $\{A_j\}_{j\in[\![1,J]\!]}$ of $\partial\Omega$, made of open cubes included in Ω' and centred at points on $\partial\Omega$ with a face orthogonal to some vector $\xi_j \in \mathbb{S}^{N-1}$ and such that $A_j \cap \Omega$ is a Lipschitz subgraph in the direction ξ_j. We set $A_0 = \Omega$ and $\xi_0 = 0$. For every $j \in [\![0, J]\!]$ and every $k \in \mathbb{N}$, we consider the translation

$$\tau_{j,k} : x \in \mathbb{R}^N \mapsto x + \frac{1}{k}\xi_j.$$

Let $\{\phi_j\}_{j\in[\![0,J]\!]}$ be a partition of unity subordinated to the covering $\{A_j\}_j$ of $\overline{\Omega}$ (*i.e.* $\sum_{j=0}^J \phi_j \equiv 1$ in $\overline{\Omega}$) and $\{\varrho_n\}_n$ be a sequence of mollifiers. We introduce the functions

$$u_k = \sum_{j=0}^J \left(\phi_j u\right) \circ \tau_{j,k} \in BD(\Omega), \quad e_k = \sum_{j=0}^J \left(\phi_j e\right) \circ \tau_{j,k} \in L^2(\Omega; \mathbb{M}^{N\times N}_{\text{sym}})$$

and

$$p_k = \sum_{j=0}^{J} \tau_{j,k}^{\#}\big(\phi_j p\big) + \sum_{j=0}^{J} \big(\nabla\phi_j \odot u\big) \circ \tau_{j,k} \in \mathcal{M}\big(\Omega; \mathbb{M}^{N\times N}_{\mathrm{sym}}\big)$$

where $\tau_{j,k}^{\#}\big(\phi_j p\big)\big(E\big) = \int_{\tau_{j,k}\left(E\right)} \phi_j \, dp$. Note that $\phi_j u \equiv 0$ in $A_j \cap \big(\Omega' \setminus \Omega\big)$ and $\phi_0 u$ is compactly supported in Ω, so that for all $k \in \mathbb{N} \setminus \{0\}$, u_k is compactly supported in Ω as well and $u_k \equiv 0$ in a neighbourhood of $\partial\Omega$. The same holds for e_k and p_k, hence we directly consider their restrictions to Ω. Moreover, $Eu_k = e_k \mathcal{L}^N \llcorner \Omega + p_k$ and

$$u_k \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^N), \quad e_k \to e \text{ strongly in } L^2(\Omega; \mathbb{M}^{N\times N}_{\mathrm{sym}})$$
$$\sum_{j=0}^{J} \big(\nabla\phi_j \odot u\big) \circ \tau_{j,k} \to 0 \text{ strongly in } L^1(\Omega; \mathbb{M}^{N\times N}_{\mathrm{sym}})$$

when $k \nearrow \infty$ by continuity of the translation in $L^p(\mathbb{R}^N; \mathbb{M}^{N\times N}_{\mathrm{sym}})$, where we also used that $u = \sum_{j=0}^{J} \phi_j u$ and $\sum_{j=0}^{J} \nabla\phi_j \odot u = \nabla\left(\sum_{j=0}^{J} \phi_j\right) \odot u = 0$. Also note that $\sum_{j=0}^{J} \left|\tau_{j,k}^{\#}(\phi_j p)\right|(\Omega) \le \sum_{j=0}^{J} \int_{\Omega'} \phi_j \, d\,|p| = |p|\,(\overline{\Omega})$, so that

$$\limsup_k |p_k|\,(\Omega) \le |p|\,(\overline{\Omega}).$$

We now set $\tilde{u}_k = \varrho_{n_k} * u_k$, $\tilde{e}_k = \varrho_{n_k} * e_k$ and $\tilde{p}_k = \varrho_{n_k} * p_k$ where n_k is chosen such that $\tilde{u}_k \in C^\infty(\Omega; \mathbb{R}^N)$, $\tilde{e}_k$, $\tilde{p}_k \in C_c^\infty(\Omega; \mathbb{M}^{N\times N}_{\mathrm{sym}})$ and

$$\|\tilde{u}_k - u_k\|_{L^1} + \|\tilde{e}_k - e_k\|_{L^2} + \left|\int_\Omega |\tilde{p}_k| \, dx - |p_k|\,(\Omega)\right| \le \frac{1}{k},$$

where the last inequality follows from the fact that $|p_k|\,(\partial\Omega) = 0$, hence $\tilde{p}_k \rightharpoonup p_k$ strictly in $\mathcal{M}\big(\Omega; \mathbb{M}^{N\times N}_{\mathrm{sym}}\big)$ when $k \nearrow \infty$ (see [94, Formula (2.7)]). Note that we still have $e(\tilde{u}_k) = \tilde{e}_k + \tilde{p}_k$ in Ω. Finally, we gather that

$$\limsup_k |\tilde{p}_k|\,(\Omega) = \limsup_k |p_k|\,(\Omega) \le |p|\,(\overline{\Omega})$$

so that, up to a subsequence, $\tilde{p}_k$ weakly-* converges to some $q \in \mathcal{M}\big(\overline{\Omega}; \mathbb{M}^{N\times N}_{\mathrm{sym}}\big)$. Identifying the limits in the sens of distributions, we infer that $q = Eu - e = p$ (hence the whole sequence converges) and by l.s.c. of the norm for the weak-* convergence of measures, we deduce that

$$|p|\,(\overline{\Omega}) \le \liminf_k \le \limsup_k |\tilde{p}_k| \, dx \le |p|\,(\overline{\Omega})$$

which concludes the proof of the density result. □

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