CHAPTER 1

Basics of the Ricci flow equation

1. Basic notions from Riemannian geometry

1.1. Different notions of curvature. The **curvature tensor** $\mathrm{Rm}(g)$ interpreted here as a $(1,3)$-tensor is defined in this set of notes as:

$$\mathrm{Rm}(g)(X,Y)Z := \left[\nabla^g_X, \nabla^g_Y\right] Z - \nabla^g_{[X,Y]} Z, \quad X, Y, Z \in TM.$$

The curvature tensor $\mathrm{Rm}(g)$ interpreted as a $(0,4)$-tensor is then defined by:

$$\mathrm{Rm}(g)(X,Y,Z,W) := g(\mathrm{Rm}(g)(X,Y)Z, W), \quad X, Y, Z, W \in TM.$$

It satisfies the following symmetry properties:

$$\mathrm{Rm}(g)(X,Y,Z,W) = \mathrm{Rm}(g)(Z,W,X,Y) = -\mathrm{Rm}(Y,X,Z,W),$$

The **Ricci curvature** is then defined as the trace of the curvature tensor with respect to g:

$$\mathrm{Ric}(g)(U,V) := \mathrm{tr}_g\left((Y,Z) \to \mathrm{Rm}(g)(U,Y,Z,V)\right), \quad U, V \in TM.$$

If $(e_k)_k$ denotes an orthonormal basis with respect to the metric g at a given point:

$$\mathrm{Ric}(g)(U,V) = \sum_{k=1}^n \mathrm{Rm}(g)(U,e_k,e_k,V) = \sum_{k=1}^n \mathrm{Rm}(g)(e_k,U,V,e_k).$$

In coordinates, we get: $\mathrm{Ric}(g)_{ij} = g^{kl}\,\mathrm{Rm}(g)_{kijl}$ where g^{kl} denotes the (k,l) components of g^{-1}. The Ricci curvature is a *symmetric* $(0,2)$ tensor.

Finally, we define the **scalar curvature** associated to a metric g as the following function:

$$\mathrm{R}_g = \mathrm{tr}_g((U,V) \to \mathrm{Ric}(g)(U,V)).$$

If $(e_k)_k$ denotes an orthonormal basis with respect to the metric g at a given point:

$$\mathrm{R}_g = \sum_{k=1}^n \mathrm{Ric}(g)(e_k,e_k).$$

Recall the first Bianchi identity that states that for all vector fields X, Y, Z and W,

$$\mathrm{Rm}(g)(X,Y,Z,W) + \mathrm{Rm}(g)(Y,Z,X,W) + \mathrm{Rm}(g)(Z,X,Y,W) = 0.$$

In order to state the second Bianchi identity, we take the opportunity to define the covariant derivative of a given $(0,p)$ tensor T as follows:

$$\nabla^g_X T(Y_1,...,Y_p) = X \cdot (T(Y_1,...,Y_p)) - \sum_{k=1}^p T(Y_1,...,\nabla^g_X Y_k,...,Y_p), \quad Y_k \in TM.$$

It is immediate that $\nabla^g T$ defines a $(0,p+1)$ tensor on M.

The second Bianchi identity states that for all vector fields X, Y, Z, U and W,

$$\nabla^g_U \mathrm{Rm}(g)(X,Y,Z,V) + \nabla^g_X \mathrm{Rm}(g)(Y,U,Z,V) + \nabla^g_Y \mathrm{Rm}(g)(U,X,Z,V) = 0. \tag{1.1}$$

In order to state the contracted Bianchi identity, we take the opportunity to define the divergence of a tensor T say of type $(0,p+1)$ as the following $(0,p)$ tensor:

$$\operatorname{div}_g T(Y_1,...,Y_p) := \operatorname{tr}_g\left((U,V)\to\nabla^g_U T(V,Y_1,...,Y_p)\right).$$

By contracting (1.1) twice, we end up with the contracted Bianchi identity:

$$2\operatorname{div}_g \operatorname{Ric}(g) = g(\nabla^g \mathrm{R}_g,\cdot). \tag{1.2}$$

Identity (1.2) shows in particular that if $n\geq 3$, a Riemannian metric satisfying $\operatorname{Ric}(g)=\rho g$ where ρ is a function on M must be *Einstein*, i.e. ρ must be constant on M (if connected). This is known as Schur's lemma.

1.2. Laplacian on tensors. If T denotes a $(0,p)$ tensor, we define its second covariant derivative as:

$$\nabla^{g,2}_{U,V}T := \nabla^g_U\left(\nabla^g_V T\right) - \nabla^g_{\nabla^g_U V}T.$$

This defines a $(0,p+2)$ tensor. It equals the ∇^g composed twice to the tensor T. We can speak of the kth iterate of ∇^g applied to T and we write $\nabla^{g,k}T$.

The **rough Laplacian** of a $(0,p)$ tensor T is then defined by considering the trace of its second covariant derivatives:

$$\Delta_g T := \operatorname{tr}_g\left((U,V)\to\nabla^{g,2}_{U,V}T\right).$$

The following lemma recalls commutation formulae for any tensor:

Lemma 1.1. *If T denotes a $(0,p)$ tensor then:*

$$\nabla^{g,2}_{U,V}T(Y_1,...,Y_p) - \nabla^{g,2}_{V,U}T(Y_1,...,Y_p) = -\sum_{k=1}^{p} T\left(Y_1,...,\operatorname{Rm}(g)(U,V)Y_k,...,Y_p\right).$$

Equivalently, in coordinates,

$$\nabla^{g,2}_{ij}T_{k_1\dots k_p} = \nabla^{g,2}_{ji}T_{k_1\dots k_p} - \sum_{l=1}^{p}\operatorname{Rm}(g)^l_{ijk_l}T_{k_1,\dots,k_{l-1},l,k_{l+1},\dots,k_p}.$$

We have then the crucial fact that will enable us to integrate by parts. Recall the scalar product on $(0,p)$ tensors induced by a Riemannian metric: if S and T are $(0,p)$ tensors then,

$$\langle S,T\rangle_g := g^{i_1j_1}...g^{i_pj_p}S_{i_1,\dots,i_p}T_{j_1,\dots,j_p},\quad |T|_g := \sqrt{\langle T,T\rangle_g}.$$

Lemma 1.2. *If T is a tensor,*

$$\Delta_g|T|^2_g = 2|\nabla^g T|^2_g + 2\left\langle \Delta_g T, T\right\rangle_g.$$

2. First definitions and remarks

Definition 1.3. A smooth one-parameter family of Riemannian metrics $(g(t))_{t\in(a,b)}$ on a manifold M is a solution to the Ricci flow if it satisfies:

$$\frac{\partial}{\partial t}g(t) = -2\operatorname{Ric}(g(t)),\quad \text{on } M\times(a,b). \tag{2.1}$$

Equation (2.1) can be supplemented with an initial condition to turn this equation into a Cauchy problem. Whether this Cauchy problem is well-posed or not is a subtle question that will be addressed in a later Chapter.

Basic examples are:

(i) (**Shrinking spheres**) If $(\mathbb{S}^n, g_{\mathbb{S}^n})$ denotes the standard metric on the unit sphere in Euclidean space $\mathbb{R}^{n+1}$ of curvature $1/(2(n-1))$ then the one-parameter family of metrics defined by

$$g(t) := (-t)g_{\mathbb{S}^n},\quad t<0,$$

is a solution to the Ricci flow living *in the past*. Such solutions are said to be ancient. Indeed, on the one hand, $\partial_t g(t) = -g_{\mathbb{S}^n}$ for all $t < 0$. On the other hand, scaling properties of the Ricci tensor gives $-2\,\mathrm{Ric}(g(t)) = -2\,\mathrm{Ric}(g_{\mathbb{S}^n}) = -g_{\mathbb{S}^n}$ by assumption on the curvature of $g_{\mathbb{S}^n}$.

(ii) (**Steady flat metrics**) If $(\mathbb{R}^n, g_{\mathbb{R}^n})$ denotes the standard metric on Euclidean space $\mathbb{R}^n$ of curvature 0 then the one-parameter family of metrics defined by

$$g(t) := g_{\mathbb{R}^n}, \quad t \in \mathbb{R},$$

is a solution to the Ricci flow living *eternally*. Such solutions are said to be eternal.

(iii) (**Expanding hyperbolic space**) If $(\mathbb{H}^n, g_{\mathbb{H}^n})$ denotes the standard hyperbolic metric on $\mathbb{H}^n$ of curvature $-1/(2(n-1))$ then the one-parameter family of metrics defined by

$$g(t) := t\, g_{\mathbb{H}^n}, \quad t > 0,$$

is a solution to the Ricci flow living *in the future*. Such solutions are said to be immortal. Indeed, on the one hand, $\partial_t g(t) = g_{\mathbb{H}^n}$ for all $t > 0$. On the other hand, $-2\,\mathrm{Ric}(g(t)) = -2\,\mathrm{Ric}(g_{\mathbb{H}^n}) = g_{\mathbb{H}^n}$ by assumption.

(iv) (**Einstein metrics**) More generally, if (M^n, g_0) is an Einstein metric with constant $1/2$, i.e. if $2\,\mathrm{Ric}(g_0) = g_0$ then $g(t) := (-t)g_0$ defines a Ricci flow living on $(-\infty, 0)$ that shrinks the Einstein metric as t goes to 0^-. Similarly, if $\mathrm{Ric}(g_0) = 0$ (including flat manifolds) then the static $g(t) := g_0$ defines a Ricci flow living on $\mathbb{R}$. Finally, if $2\,\mathrm{Ric}(g_0) = -g_0$ then $g(t) := t\, g_0$ defines a Ricci flow living on $(0, +\infty)$ that expands the Einstein metric as t goes to $+\infty$.

(v) (**Product Ricci flows**) If $(M_i^{n_i}, g_i(t))_{t\in(a,b)}$ denotes two solutions to the Ricci flow defined on a common interval of time (a, b) then the product metric $g(t) := g_1(t) + g_2(t)$ defined on the product manifold $M_1^{n_1} \times M_2^{n_2}$ defines a solution to the Ricci flow.

(vi) (**Round shrinking cylinders**) This is a special case from the previous example which is of considerable importance. These are the Ricci flow products $(\mathbb{S}^{n-k} \times \mathbb{R}^k, (-t)g_{\mathbb{S}^{n-k}} + g_{\mathbb{R}^k})$, living on $(-\infty, 0)$. As t goes to 0^-, observe that the solution collapses to a lower dimensional Euclidean space $\mathbb{R}^k$.

Next, we describe special features in low dimensions together with the basic symmetries of the Ricci flow.

- Invariance under the action of diffeomorphisms

Recall first that for a C^2_{loc} function on a Riemannian manifold, the **Hessian** of f with respect to the metric g is:

$$\nabla^{g,2} f(X, Y) := g(\nabla^g_X \nabla^g f, Y), \quad X, Y \in TM.$$

It is a basic result that the Hessian of a function defines a *symmetric* $(0, 2)$ tensor. Then the **Laplacian** of f is the *trace* of its hessian:

$$\Delta_g f := \mathrm{tr}_g(\nabla^{g,2} f).$$

In coordinates,

$$\Delta_g f = g^{ij} \left(\frac{\partial^2}{\partial x_i \partial x_j} f - \Gamma(g)^k_{ij} \frac{\partial}{\partial x_k} \right).$$

Recall the fundamental properties of the Levi-Civita connection and the different notions of curvature under the action of diffeomorphisms on the space of metrics:

Proposition 1.4. *Let (M, g) be a Riemannian manifold and let φ be a diffeomorphism of M.*

(i) $\varphi^* \left(\nabla^g_X Y\right) = \nabla^{\varphi^* g}_{\varphi^* X} \varphi^* Y$, *for any vector fields X and Y on M.*

(ii) *If $f : M \to \mathbb{R}$ is a C^2 function,* $\varphi^*(\nabla^g f) = \nabla^{\varphi^* g}(\varphi^* f)$, *and* $\Delta_{\varphi^* g}(\varphi^* f) = \varphi^* \left(\Delta_g f\right)$.

(iii) *(Scalar curvature)* $\varphi^* \mathrm{R}_g = \mathrm{R}_{\varphi^* g}$.

(iv) *(Ricci curvature)* $\mathrm{Ric}(\varphi^* g) = \varphi^* \mathrm{Ric}(g)$ *as symmetric 2-tensors.*

(v) *(Curvature operator as a $(1, 3)$ tensor)* $\varphi^* \mathrm{Rm}(g) = \mathrm{Rm}(\varphi^* g)$ *as $(1, 3)$ tensors.*

Corollary 1.5. *Let $(M, g(t))_{t\in(t_1,t_2)}$, $t_1 < t_2$, be a solution to the Ricci flow and let φ be a diffeomorphism of M. Then the family of metrics $(\varphi^* g(t))_{t\in(0,T)}$ is a solution to the Ricci flow.*

- Invariance by scalings

Recall the following scaling properties of the Levi-Civita connection and the different notions of curvature:

Proposition 1.6. *If $\lambda > 0$ and g is a Riemannian metric on a manifold M,*
(i) $\nabla^{\lambda g}_X Y = \nabla^g_X Y$, *for any vector fields X and Y.*
(ii) *If $f : M \to \mathbb{R}$ is a C^1 function, $\nabla^{\lambda g} f = \lambda^{-1}\nabla^g f$, and $\Delta_{\lambda g} f = \lambda^{-1}\Delta_g f$.*
(iii) *(Scalar curvature)* $\mathrm{R}_{\lambda g} = \lambda^{-1}\,\mathrm{R}_g$.
(iv) *(Ricci curvature)* $\mathrm{Ric}(\lambda g) = \mathrm{Ric}(g)$ *as symmetric* 2*-tensors.*
(v) *(Curvature operator as a* $(1,3)$ *tensor)* $\mathrm{Rm}(\lambda g) = \mathrm{Rm}(g)$ *as* $(1,3)$ *tensors.*
(vi) *(Sectional curvature) if P denotes a plane of TM then:* $K_{\lambda g}(P) = \lambda^{-1} K_g(P)$.

Corollary 1.7. *If $\lambda > 0$ and if $(M^n, g(t))_{(t_1,t_2)}$ is a solution to the Ricci flow with $-\infty \le t_1 < t_2 \le +\infty$ then the family of metrics $g_\lambda(t) := \lambda g(t/\lambda)$ defined on $(\lambda t_1, \lambda t_2)$ is a solution to the Ricci flow.*

- Special expression in dimension 2

In real dimension 2, if (M^2, g) is a Riemannian surface then the Ricci curvature of g is related to the scalar curvature of g as follows:
$$\mathrm{Ric}(g) = \frac{\mathrm{R}_g}{2} g.$$
In particular, this makes the Ricci flow conformal, i.e. it preserves the conformal class of a given metric, say the initial metric the flow starts from.

- Terminology for solutions to the Ricci flow:

A solution to the Ricci flow is said to be *ancient* (respectively *immortal*) if it is defined on $(-\infty, T)$, $T \le +\infty$ (respectively on $(T, +\infty)$, $T \ge -\infty$). Finally, a solution to the Ricci flow is *eternal* if it is both ancient and immortal, i.e. if it is defined on $(-\infty, +\infty)$.

- Different singularities types of solutions to the Ricci flow:

A solution $(M, g(t))_{t\in[0,T)}$, $T < +\infty$, to the Ricci flow forms a
(i) **Type I** singularity if $\limsup_{t\to T^-}(T-t)\sup_M |\mathrm{Rm}(g(t))|_{g(t)} < +\infty$,

(ii) **Type IIa** singularity if it is not Type I, i.e., $\limsup_{t\to T^-}(T-t)\sup_M |\mathrm{Rm}(g(t))|_{g(t)} = +\infty$.

If a solution $(M, g(t))_{t\in[-\infty,T)}$ to the Ricci flow is ancient, we also define Type I ancient solutions at $t = -\infty$, if $\limsup_{t\to-\infty} |t|\sup_M |\mathrm{Rm}(g(t))|_{g(t)} < +\infty$. Similarly, one can define ancient Type II solutions at $t = -\infty$ to be ancient solutions which are not Type I at $t = -\infty$.

By analogy, one defines singularity types of immortal solutions $(M, g(t))_{t\in(T,+\infty)}$ to the Ricci flow:
(i) **Type III** singularity if $\limsup_{t\to+\infty} t\sup_M |\mathrm{Rm}(g(t))|_{g(t)} < +\infty$,

(ii) **Type IIb** singularity if it is not Type III, i.e., $\limsup_{t\to+\infty} t\sup_M |\mathrm{Rm}(g(t))|_{g(t)} = +\infty$.

Solutions $(M,g(t))_{t\in(0,T)}$ to the Ricci flow satisfying $\limsup_{t\to 0^+} t\sup_M|\mathrm{Rm}(g(t))|_{g(t)}<+\infty$ can also be considered but do not have a specific terminology to avoid any confusion with the previous definitions. Depending on the context, Type IIa or Type IIb singularities are simply called Type II singularities.

Let us explain the choice of the linear scaling in time to understand singularities of the Ricci flow. Let us stick for instance to a Type III solution, i.e. a solution $(M,g(t))_{t\in(0,+\infty)}$ (the general case $(T,+\infty)$ can be easily deduced by translating the solution in time) that satisfies

$$|\mathrm{Rm}(g(t))|_{g(t)}\leq\frac{C}{t},\quad t>0,$$

for some time-independent positive constant C.

Thanks to Corollary 1.7, the rescaled solution $g_\lambda(t):=\lambda g(t/\lambda)$, for $t>0$ and $\lambda>0$ still satisfies

$$|\mathrm{Rm}(g_\lambda(t))|_{g_\lambda(t)}\leq\frac{C}{t},\quad t>0,$$

for the same uniform positive constant C. This estimate follows from the definition of the norm of a tensor together with Proposition 1.6. This fact alone suggests that one can perform parabolic dilations by letting λ either go to $+\infty$ (a blow-up) or to 0^+ (a blow-down). The limit in some suitable topology (Gromov-Hausdorff, Cheeger-Gromov are key words here), if it exists (up to a subsequence) thanks to an adequate compactness result, is likely to carry additional structure (e.g. a fixed point of the Ricci flow). Classifying fixed points of the Ricci flow is then a crucial task to understand the formation of singularities.

3. Evolutions of zeroth order geometric quantities

3.1. Evolution of distances. The easiest way to compare distances between two points at two different times is through estimating metrics first.

Proposition 1.8. *Let $(M,g(t))_{t\in(t_1,t_2)}$ be a complete solution to the Ricci flow. Assume*

$$r_-(t)g(t)\leq\mathrm{Ric}(g(t))\leq r_+(t)g(t),$$

where $r_\pm(t)$ is bounded on subintervals of (t_1,t_2). Then,

$$e^{-2\int_s^t r_+(\tau)\,d\tau}g(s)\leq g(t)\leq e^{-2\int_s^t r_-(\tau)\,d\tau}g(s),\quad t_1<s<t<t_2,$$

in the sense of quadratic forms. In particular,

$$e^{-\int_s^t r_+(\tau)\,d\tau}d_{g(s)}(x,y)\leq d_{g(t)}(x,y)\leq e^{-\int_s^t r_-(\tau)\,d\tau}d_{g(s)}(x,y),\quad t_1<s<t<t_2,\quad x,y\in M.$$

The drawback of this proposition is when the curvature is allowed to blow up at a linear rate, for instance, if $r_+(t)$ behaves like t^{-1} close to $t=0$.

Hamilton has circumvented this issue as we now explain in a series of lemmata:

Lemma 1.9. *If $\gamma:[0,L]\to M$ is a minimizing unit speed geodesic in a Riemannian manifold with $\mathrm{Ric}(g)\leq r^2g$ for some constant $r\geq 0$ then,*

$$\int_\gamma \mathrm{Ric}(g)(\dot\gamma,\dot\gamma)\leq 4(n-1)r.$$

Observe that the righthand side of the previous estimate does not depend on the geodesic γ.

Proof. Recall the following second variation along the minimizing geodesic γ:

$$0\leq\int_0^L\left((n-1)\dot\varphi^2(s)-\varphi^2(s)\,\mathrm{Ric}(g)(\dot\gamma(s),\dot\gamma(s))\right)\,ds,$$

for any Lipschitz function with compact support in $[0,L]$.

If $L\leq 2r^{-1}$, there is nothing to prove. If $L>2r^{-1}$ then let φ be a piecewise linear function such that $\varphi(s)=1$ on $[r^{-1},L-r^{-1}]$ and such that $|\dot\varphi(s)|=r$ for $s\in(0,r^{-1})\cup(L-r^{-1},L)$ with

$\varphi(0) = \varphi(L) = 0$. Then,

$$\begin{aligned}\int_0^L \operatorname{Ric}(g)(\dot\gamma(s),\dot\gamma(s))\,ds &\leq 2(n-1)r^1 + \int_{[0,r^{-1}]\cup[L-r^{-1},L]} (1-\varphi^2(s))\operatorname{Ric}(g)(\dot\gamma(s),\dot\gamma(s))\,ds\\ &\leq 2(n-1)r + r^2\int_{[0,r^{-1}]\cup[L-r^{-1},L]}(1-\varphi^2(s))\,ds\\ &\leq 2(n-1)r + \frac{4}{3}r \leq 4(n-1)r.\end{aligned}$$

□

Remark 1.10. The proof of Lemma 1.9 only requires that the upper bound on the Ricci curvature holds on $B_g(\gamma(0), 1/\sqrt{r}) \cup B_g(\gamma(L), 1/\sqrt{r})$. Do we need the geodesic γ to be minimizing ?

The following lemma is just stating the fact that $d_{g(t)}(x,y) = l_{g(t)}(\gamma)$ and $d_{g(t-s)}(x,y) \leq l_{g(t-s)}(\gamma)$ for all minimizing geodesics γ joining x to y, i.e. $t \to l_{g(t)}(\gamma)$ is a barrier for $d_{g(t)}(x,y)$:

Lemma 1.11.
$$\liminf_{s\to 0^+} \frac{d_{g(t)}(x,y) - d_{g(t-s)}(x,y)}{s} \geq -\sup_\gamma \int_\gamma \operatorname{Ric}(g)(\dot\gamma,\dot\gamma),$$
where γ runs all minimizing geodesics joining x to y.

We are finally in a position to prove the main estimate due to Hamilton:

Proposition 1.12. *Let $(M^n, g(t))_{t\in(0,T)}$ be a complete solution to the Ricci flow such that*
$$\operatorname{Ric}(g(t)) \leq \frac{C}{t}, \quad t \in (0,T),$$
for some uniform positive constant C. Then,
$$d_{g(t_2)}(x,y) \geq d_{g(t_1)}(x,y) - K(n,C)(\sqrt{t_2} - \sqrt{t_1}), \quad 0 < t_1 < t_2 < T,$$
where $K(n,C)$ is a positive constant depending on n and C only. Moreover, if $\operatorname{Ric}(g(t)) \geq 0$ *then the following distortion estimates on the distance hold:*
$$d_{g(t_1)}(x,y) - K(n,C)(\sqrt{t_2}-\sqrt{t_1}) \leq d_{g(t_2)}(x,y) \leq d_{g(t_1)}(x,y), \quad 0 < t_1 < t_2 < T.$$

Proof. The combination of Lemma 1.9 applied to $r^2 := Ct^{-1}$ and Lemma 1.11 shows that:
$$\liminf_{s\to 0^+} \frac{d_{g(t)}(x,y) - d_{g(t-s)}(x,y)}{s} \geq -4(n-1)\sqrt{C}t^{-1/2}, \quad t \in (0,T).$$
By integrating this differential inequality (invoking Lebesgue's integral theorem), one gets the desired estimate:
$$d_{g(t_2)}(x,y) \geq d_{g(t_1)}(x,y) - 8(n-1)\sqrt{C}(\sqrt{t_2}-\sqrt{t_1}), \quad 0 < t_1 < t_2 < T.$$

□

3.2. Evolution of the volume form. Since in local coordinates, the volume form associated to a Riemannian metric g denoted by $d\mu_g$ is expressed by $\sqrt{\det g_{ij}}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, one easily obtains that:
$$\frac{\partial}{\partial t} d\mu_{g(t)} = \frac{1}{2}\operatorname{tr}_{g(t)}(\partial_t g(t))\,d\mu_{g(t)} = -\mathrm{R}_{g(t)}\ d\mu_{g(t)}.$$
In particular, if $(M, g(t))_{t\in(0,T)}$ is solution to the Ricci flow on a closed manifold, then:
$$\frac{d}{dt}\operatorname{vol}_{g(t)} M = -\int_M \mathrm{R}_{g(t)}\ d\mu_{g(t)}.$$
It might be more convenient to keep the volume fixed along the Ricci flow. This is the normalized Ricci flow and it is defined as follows:
$$\partial_t \tilde g(t) = -2\operatorname{Ric}(\tilde g(t)) + \frac{2}{n}\tilde r(t)\tilde g(t), \quad \tilde r(t) := (\operatorname{vol}_{\tilde g(t)} M)^{-1}\int_M \mathrm{R}_{\tilde g(t)}\ d\mu_{\tilde g(t)}.$$

Going back and forth from the Ricci flow to the normalized Ricci flow is done as follows: if $(M, g(t))_t$ is a solution to the Ricci flow on a closed manifold M then

$$\tilde{g}(\tilde{t}) := c(t)g(t), \qquad \text{where } c(t) := \exp\left(\frac{2}{n}\int_0^t r(\tau)\,d\tau\right) \text{ and } \tilde{t}(t) := \int_0^t c(\tau)\,d\tau,$$

is a solution to the normalized Ricci flow (exercise).

4. Evolutions of first order geometric quantities

We start with a general formula that computes the variation of the Levi-Civita connection in an arbitrary direction:

Lemma 1.13. *If $(g(t))_{-\varepsilon<t<\varepsilon}$ is a smooth one-parameter family of Riemannian metrics with $g(0) =: g$ and $\partial_t g(t)\big|_{t=0} =: h$ a symmetric 2-tensor then:*

$$g\left(\left(\frac{\partial}{\partial t}\Big|_{t=0}\nabla^{g(t)}\right)_X Y, Z\right) = \frac{1}{2}\left((\nabla^g_X h)(Y,Z) + (\nabla^g_Y h)(X,Z) - (\nabla^g_Z h)(X,Y)\right).$$

In coordinates, this gives:

$$\frac{\partial}{\partial t}\Big|_{t=0}\Gamma(g)^k_{ij} = \frac{1}{2}g^{kl}\left(\nabla^g_i h_{jl} + \nabla^g_j h_{il} - \nabla^g_l h_{ij}\right).$$

Proof. Recall the fundamental formula relating the Levi-Civita connection of a Riemannian metric g to the Lie bracket and the metric g:

$$2g(\nabla^g_X Y, Z) = X\cdot g(Y,Z) + Y\cdot g(X,Z) - Z\cdot g(X,Y) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).$$

for any vector field X, Y and Z on M.

Differentiating this formula applied to each metric $g(t)$ and time-independent vector fields X, Y and Z, gives the expected result if one recalls the fact that ∇^g is torsion free (i.e. $[X,Y] = \nabla^g_X Y - \nabla^g_Y X$) together with the definition of the covariant derivative of a tensor h:

$$(\nabla^g_X h)(Y,Z) := X\cdot(h(Y,Z)) - h(\nabla^g_X Y, Z) - h(Y, \nabla^g_X Z).$$

□

As a corollary, one easily obtains the variation of the Levi-Civita connection along the Ricci flow:

Proposition 1.14. *If $(M, g(t))_t$ is a solution to the Ricci flow then:*

$$g(t)\left(\left(\frac{\partial}{\partial t}\nabla^{g(t)}\right)_X Y, Z\right) = (\nabla^g_Z \operatorname{Ric}(g(t)))(X,Y) - (\nabla^g_X \operatorname{Ric}(g(t))(Y,Z) - (\nabla^g_Y \operatorname{Ric}(g(t)))(X,Z).$$

In coordinates, this gives:

$$\frac{\partial}{\partial t}\Gamma(g(t))^k_{ij} = g^{kl}\left(\nabla^{g(t)}_l \operatorname{Ric}(g)_{ij} - \nabla^{g(t)}_i \operatorname{Ric}(g)_{jl} - \nabla^{g(t)}_j \operatorname{Ric}(g)_{il}\right).$$

We also take the opportunity to derive the variation of the Laplacian acting on functions along the Ricci flow:

Proposition 1.15. *If $(M, g(t))_t$ is a solution to the Ricci flow and if $f(t) : M \to \mathbb{R}$ is a smooth one-parameter family of smooth functions then:*

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)} f(t)\right) = \Delta_{g(t)}\left(\frac{\partial}{\partial t}f(t)\right) + 2\left\langle \operatorname{Ric}(g(t)), \nabla^{g(t),2} f(t)\right\rangle_{g(t)}.$$

In particular, if $n = 2$,

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)} f(t)\right) = \Delta_{g(t)}\left(\frac{\partial}{\partial t}f(t)\right) + \mathrm{R}_{g(t)}\,\Delta_{g(t)} f(t).$$

PROOF. Recall that the Laplacian associated to a metric g acting on a function f is defined in coordinates as follows:

$$\Delta_g f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma(g)_{ij}^k \frac{\partial f}{\partial x_k}\right).$$

Then Lemma 1.13 ensures that:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\Delta_{g(t)} f(t)\right) &= \Delta_{g(t)}\left(\frac{\partial}{\partial t} f(t)\right) + 2\left\langle \mathrm{Ric}(g(t)), \nabla^{g(t),2} f(t)\right\rangle_{g(t)} - g(t)^{ij}\frac{\partial}{\partial t}\Gamma(g(t))_{ij}^k \frac{\partial f(t)}{\partial x_k}\\
&= \Delta_{g(t)}\left(\frac{\partial}{\partial t} f(t)\right) + 2\left\langle \mathrm{Ric}(g(t)), \nabla^{g(t),2} f(t)\right\rangle_{g(t)}\\
&\quad - g(t)^{ij}\left(\nabla_i^{g(t)}\mathrm{Ric}(g(t))_{jk} + \nabla_j^{g(t)}\mathrm{Ric}(g(t))_{ik} - \nabla_k^{g(t)}\mathrm{Ric}(g(t))_{ij}\right)\frac{\partial f(t)}{\partial x_k}\\
&= \Delta_{g(t)}\left(\frac{\partial}{\partial t} f(t)\right) + 2\left\langle \mathrm{Ric}(g(t)), \nabla^{g(t),2} f(t)\right\rangle_{g(t)}\\
&\quad + 2\,\mathrm{div}_{g(t)}\mathrm{Ric}(g(t))(\nabla^{g(t)} f(t)) - g(t)(\nabla^{g(t)}\mathrm{R}_{g(t)}, \nabla^{g(t)} f(t)),
\end{aligned}$$

which implies the desired result thanks to the contracted Bianchi identity. □

Remark 1.16. Recall that for a vector field X, $\mathrm{div}_g(X) := \sum_{p=1}^n g(\nabla^g_{e_p} X, e_p)$ where $(e_p)_p$ is an orthonormal basis with respect to g. Check that $\Delta_g f = \mathrm{div}_g(\nabla^g f)$.

5. Evolutions of second order geometric quantities

- First variation of the Riemann tensor
  In local coordinates, recall that:

$$\mathrm{Rm}(g)_{ijk}^l = \partial_i\Gamma(g)_{jk}^l - \partial_j\Gamma(g)_{ik}^l + \Gamma(g)_{jk}^p\Gamma(g)_{ip}^l - \Gamma(g)_{ik}^p\Gamma(g)_{jp}^l.$$

This formula is simply obtained by the definition of the Christoffel symbols and that of the Riemann tensor. In particular, in the perspective of computing the first variation of the Riemann tensor, we consider geodesic coordinates centered at a point, then $\Gamma(g) \equiv 0$ at that point. If $(g(t))_{-\varepsilon<t<\varepsilon}$ is a smooth one-parameter family of Riemannian metrics with $g(0) =: g$ and $\partial_t g(t)\big|_{t=0} =: h$ a symmetric 2-tensor then we are left with:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0}\mathrm{Rm}(g)_{ijk}^l &= \nabla_i^g\left(\frac{\partial}{\partial t}\Big|_{t=0}\Gamma(g)_{jk}^l\right) - \nabla_j^g\left(\frac{\partial}{\partial t}\Big|_{t=0}\Gamma(g)_{ik}^l\right)\\
&= \nabla_i^g\left(\frac{1}{2}g^{pl}\left(\nabla_j^g h_{kp} + \nabla_k^g h_{jp} - \nabla_p^g h_{jk}\right)\right) - \nabla_j^g\left(\frac{1}{2}g^{pl}\left(\nabla_i^g h_{kp} + \nabla_k^g h_{ip} - \nabla_p^g h_{ik}\right)\right)
\end{aligned}$$

Therefore, by rearranging terms together, we have proved the:

Proposition 1.17. *If $(g(t))_{-\varepsilon<t<\varepsilon}$ is a smooth one-parameter family of Riemannian metrics with $g(0) =: g$ and $\partial_t g(t)\big|_{t=0} =: h$ a symmetric 2-tensor then:*

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathrm{Rm}(g)_{ijk}^l = \frac{1}{2}g^{lp}\left(\nabla_i^g\nabla_j^g h_{kp} - \nabla_j^g\nabla_i^g h_{kp} + \nabla_i^g\nabla_k^g h_{jp} - \nabla_j^g\nabla_k^g h_{ip} - \nabla_i^g\nabla_p^g h_{jk} + \nabla_j^g\nabla_p^g h_{ik}\right).$$

- Linearized equation

Definition 1.18. Let (M, g) be a Riemannian manifold. Let h be symmetric 2-tensor. Then we define:

– the **Lichnerowicz Laplacian** as the following second order operator:

$$\Delta_{L,g} h_{ij} := \Delta_g h_{ij} + 2g^{pk}g^{ql}\,\mathrm{Rm}(g)_{ipqj}h_{kl} - g^{pq}\,\mathrm{Ric}(g)_{ip}h_{jq} - g^{pq}h_{ip}\,\mathrm{Ric}(g)_{jq}.$$

Equivalently, if

$$\overset{\circ}{\mathrm{Rm}}(g)(h)(X,Y) := -\sum_{p=1}^{n} h(\mathrm{Rm}(g)(X,e_p)Y,e_p),$$
$$(h\circ k)(X,Y) := \sum_{p=1}^{n} h(X,e_p)k(Y,e_p),$$

for $(e_p)_p$ an orthonormal basis with respect to g,

$$\Delta_{L,g}h := \Delta_g h + 2\overset{\circ}{\mathrm{Rm}}(g)(h) - \mathrm{Ric}(g)\circ h - h\circ \mathrm{Ric}(g).$$

– the **Bianchi one-form** denoted by $B_g(h)$ as follows:

$$B_g(h) := \mathrm{div}_g\, h - \frac{1}{2}d(\mathrm{tr}_g h),$$

Remark 1.19. Check that the operator $\overset{\circ}{\mathrm{Rm}}(g) : S^2T^*M \to S^2T^*M$ is well-defined and that $\mathrm{tr}_g(\overset{\circ}{\mathrm{Rm}}(g)(h)) = \langle \mathrm{Ric}(g), h\rangle_g$. Deduce from this that $\mathrm{tr}_g(\Delta_{L,g}h) = \Delta_g(\mathrm{tr}_g h)$.

Lemma 1.20. *The differential of the tensor $-2\,\mathrm{Ric}$ at a Riemannian metric g along a variation h is:*

$$D_g(-2\,\mathrm{Ric})(h) = \Delta_{L,g}h - \mathcal{L}_{B_g(h)}(g),$$
$$\mathcal{L}_{B_g(h)}(g)_{ij} := \nabla^g_i B_g(h)_j + \nabla^g_j B_g(h)_i.$$

In particular, the differential of the scalar curvature at a Riemannian metric g along a variation h is:

$$D_g\,\mathrm{R}(h) = -\Delta_g(\mathrm{tr}_g h) + \mathrm{div}_g(\mathrm{div}_g\, h) - \langle h, \mathrm{Ric}(g)\rangle_g.$$

Remark 1.21. The definition of $\mathcal{L}_{B_g(h)}(g)$ is consistent with the more general definition of the Lie derivative of a tensor T with respect to a vector field X:

$$\mathcal{L}_X T(Y_1,...,Y_p) := X\cdot(T(Y_1,...,Y_p)) - T(\mathcal{L}_X Y_1,...,Y_p) - ... - T(Y_1,...,\mathcal{L}_X Y_p).$$

PROOF. Tracing Proposition 1.17 gives:

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0}\mathrm{Ric}(g)_{jk} &= \frac{1}{2}g^{pq}\left(\nabla^g_q\nabla^g_j h_{kp} - \nabla^g_j\nabla^g_q h_{kp} + \nabla^g_q\nabla^g_k h_{jp} - \nabla^g_j\nabla^g_k h_{qp} - \nabla^g_q\nabla^g_p h_{jk} + \nabla^g_j\nabla^g_p h_{qk}\right)\\
&= \frac{1}{2}\left(-\Delta_g h_{jk} - \nabla^g_j\nabla^g_k \mathrm{tr}_g h + g^{pq}\nabla^g_q\nabla^g_j h_{kp} - g^{pq}\nabla^g_j\nabla^g_q h_{kp}\right)\\
&\quad + \frac{1}{2}\left(\nabla^g_k(\mathrm{div}_g\, h)_j + \nabla^g_j(\mathrm{div}_g\, h)_k + g^{pq}\left(\nabla^g_q\nabla^g_k h_{jp} - \nabla^g_k\nabla^g_q h_{jp}\right)\right)\\
&= \frac{1}{2}\left(-\Delta_g h_{jk} + \mathcal{L}_{B_g(h)}(g)_{jk}\right)\\
&\quad + \frac{1}{2}g^{pq}\left(\nabla^g_q\nabla^g_j h_{kp} - \nabla^g_j\nabla^g_q h_{kp}\right) + \frac{1}{2}g^{pq}\left(\nabla^g_q\nabla^g_k h_{jp} - \nabla^g_k\nabla^g_q h_{jp}\right).
\end{aligned}$$

It is now a matter of commuting covariant derivatives:

$$\begin{aligned}
g^{pq}\left(\nabla^g_q\nabla^g_j h_{kp} - \nabla^g_j\nabla^g_q h_{kp}\right) &= -g^{pq}\,\mathrm{Rm}(g)^l_{qjk}h_{lp} - g^{pq}\,\mathrm{Rm}(g)^l_{qjp}h_{kl}\\
&= -\overset{\circ}{\mathrm{Rm}}(g)(h)_{jk} + (\mathrm{Ric}(g)\circ h)_{jk}\\
g^{pq}\left(\nabla^g_q\nabla^g_k h_{jp} - \nabla^g_k\nabla^g_q h_{jp}\right) &= -g^{pq}\,\mathrm{Rm}(g)^l_{qkj}h_{lp} - g^{pq}\,\mathrm{Rm}(g)^l_{qkp}h_{jl}\\
&= -\overset{\circ}{\mathrm{Rm}}(g)(h)_{jk} + (h\circ \mathrm{Ric}(g))_{jk},
\end{aligned}$$

as expected. $\square$

- Evolution equation of the Ricci tensor

Proposition 1.22. *Let $(M, g(t))_t$ be a solution to the Ricci flow. Then,*

$$\frac{\partial}{\partial t} \operatorname{Ric}(g(t)) = \Delta_{L,g(t)} \operatorname{Ric}(g(t)).$$

Proof. According to Lemma 1.20 applied to $h := -2\operatorname{Ric}(g(t))$,

$$-2\frac{\partial}{\partial t} \operatorname{Ric}(g(t)) = \Delta_{L,g(t)}(-2\operatorname{Ric}(g(t))) - \mathcal{L}_{B_{g(t)}(-2\operatorname{Ric}(g(t)))}(g(t)). \tag{5.1}$$

By the contracted Bianchi identity, $d(\operatorname{tr}_g(\operatorname{Ric}(g))) = 2\operatorname{div}_g(\operatorname{Ric}(g))$ for any Riemannian metric g, therefore, $B_{g(t)}(-2\operatorname{Ric}(g(t))) = 0$. This fact together with (1.22) implies the expected result.

$\square$

- Evolution equation of scalar curvature

The scalar curvature satisfies a reaction-diffusion equation, more precisely:

Proposition 1.23. *Let $(M, g(t))_t$ be a solution to the Ricci flow. Then,*

$$\frac{\partial}{\partial t} \mathrm{R}_{g(t)} = \Delta_{g(t)} \mathrm{R}_{g(t)} + 2|\operatorname{Ric}(g(t))|^2_{g(t)}.$$

In particular, if $n = 2$,

$$\frac{\partial}{\partial t} \mathrm{R}_{g(t)} = \Delta_{g(t)} \mathrm{R}_{g(t)} + \mathrm{R}^2_{g(t)}.$$

Proof. According to Lemma 1.20 applied to $h := -2\operatorname{Ric}(g(t))$,

$$\begin{aligned}
\frac{\partial}{\partial t} \mathrm{R}_{g(t)} &= 2\Delta_{g(t)}(\operatorname{tr}_{g(t)} \operatorname{Ric}(g(t))) + \operatorname{div}_{g(t)}(\operatorname{div}_{g(t)}(-2\operatorname{Ric}(g(t))) - \langle -2\operatorname{Ric}(g(t)), \operatorname{Ric}(g(t))\rangle_{g(t)} \\
&= 2\Delta_{g(t)} \mathrm{R}_{g(t)} - \operatorname{div}_{g(t)}(\nabla^{g(t)} \mathrm{R}_{g(t)}) + 2|\operatorname{Ric}(g(t))|^2_{g(t)} \\
&= \Delta_{g(t)} \mathrm{R}_{g(t)} + 2|\operatorname{Ric}(g(t))|^2_{g(t)},
\end{aligned}$$

where we have used the contracted Bianchi identity in the penultimate line.

$\square$

- Evolution equation of the curvature tensor

Proposition 1.24. *Let $(M^n, g(t))_{t\in(0,T)}$ be a solution to the Ricci flow. Then,*

$$\frac{\partial}{\partial t} \operatorname{Rm}(g(t)) = \Delta_{g(t)} \operatorname{Rm}(g(t)) + Q(\operatorname{Rm}(g(t))), \quad t \in (0, T],$$

where $Q(\operatorname{Rm}(g(t)) = \operatorname{Rm}(g(t))) * \operatorname{Rm}(g(t))$. Here the symbol $S_1 * S_2$ means linear combinations of contractions of the tensorial product of two tensors S_i, $i = 1, 2$.

See the exercices section.

6. Exercises

Exercise 1.25. *In dimension* 3*, show that*

$$\mathrm{Rm}(g)_{ijkl} = \mathrm{Ric}(g)_{il}g_{jk} + \mathrm{Ric}(g)_{jk}g_{il} - \mathrm{Ric}(g)_{ik}g_{jl} - \mathrm{Ric}(g)_{jl}g_{ik} - \frac{\mathrm{R}_g}{2}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right).$$

Deduce from this that along a solution to the Ricci flow,

$$\frac{\partial}{\partial t}\mathrm{Ric}(g)_{jk} = \Delta_g \mathrm{Ric}(g)_{jk} + 3\,\mathrm{R}_g \mathrm{Ric}(g)_{jk} - 6\,\mathrm{Ric}(g)\circ\mathrm{Ric}(g)_{jk} + (2|\mathrm{Ric}(g)|_g^2 - \mathrm{R}_g^2)g_{jk}.$$

Exercise 1.26. *Under the assumptions of Proposition 1.12, show that the pointwise limit $(M, d_{g(t)})$ as t goes to 0 exists, there is a unique distance d_0 on M such that d_0 induces the same topology as that induced by the distances $d_{g(t)}$ on M.*

Exercise 1.27. *The aim of this exercise is to derive the evolution equation of the full Riemann tensor along the Ricci flow.*

(i) *Show that on M,*

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Rm}(g(t))_{ijkl} &= g(t)_{lp}\frac{\partial}{\partial t}\mathrm{Rm}(g(t))_{ijk}^{l} - 2\,\mathrm{Ric}(g(t))_{lp}\,\mathrm{Rm}(g(t))_{ijk}^{l}\\ &= -\nabla_i^{g(t)}\nabla_j^{g(t)}\mathrm{Ric}(g(t))_{kl} + \nabla_j^{g(t)}\nabla_i^{g(t)}\mathrm{Ric}(g(t))_{kl} - \nabla_i^{g(t)}\nabla_k^{g(t)}\mathrm{Ric}(g(t))_{jl}\\ &\quad + \nabla_j^{g(t)}\nabla_k^{g(t)}\mathrm{Ric}(g(t))_{il} + \nabla_i^{g(t)}\nabla_l^{g(t)}\mathrm{Ric}(g(t))_{jk} - \nabla_j^{g(t)}\nabla_l^{g(t)}\mathrm{Ric}(g(t))_{ik}.\end{aligned}$$

(ii) *Show that for a static Riemannian manifold (M^n, g):*

$$\begin{aligned}&\Delta_g \mathrm{Rm}(g)_{ijkl} = -\nabla_j^g\nabla_p^g\mathrm{Rm}(g)_{pikl} - \nabla_i^g\nabla_p^g\mathrm{Rm}(g)_{jpkl}\\ &+ \mathrm{Rm}(g)_{pij}^q\,\mathrm{Rm}(g)_{qpkl} + \mathrm{Ric}(g)_{iq}\,\mathrm{Rm}(g)_{qjkl} + \mathrm{Rm}(g)_{pik}^q\,\mathrm{Rm}(g)_{jpql} + \mathrm{Rm}(g)_{pil}^q\,\mathrm{Rm}(g)_{jpkq}\\ &+ \mathrm{Ric}(g)_{jq}\,\mathrm{Rm}(g)_{iqkl} - \mathrm{Rm}(g)_{jpi}^q\,\mathrm{Rm}(g)_{pqkl} - \mathrm{Rm}(g)_{jpk}^q\,\mathrm{Rm}(g)_{piql} - \mathrm{Rm}(g)_{jpl}^q\,\mathrm{Rm}(g)_{pikq},\end{aligned}$$

(Hint: use the second Bianchi identity)

(iii) *Show that:*

$$\nabla_i^g\nabla_j^g\mathrm{Ric}(g)_{kl} - \nabla_j^g\nabla_i^g\mathrm{Ric}(g)_{kl} = -\mathrm{Rm}(g)_{ijk}^p\,\mathrm{Ric}(g)_{pl} - \mathrm{Rm}(g)_{ijl}^p\,\mathrm{Ric}(g)_{kp}.$$

(iv) *Show that:*

$$\begin{aligned}&\Delta_g \mathrm{Rm}(g)_{ijkl} = \nabla_j^g\nabla_k^g\mathrm{Ric}(g)_{li} - \nabla_j^g\nabla_l^g\mathrm{Ric}(g)_{ik} - \nabla_i^g\nabla_k^g\mathrm{Ric}(g)_{lj} + \nabla_i^g\nabla_l^g\mathrm{Ric}(g)_{jk}\\ &+ \mathrm{Rm}(g)_{pij}^q\,\mathrm{Rm}(g)_{qpkl} + \mathrm{Ric}(g)_{iq}\,\mathrm{Rm}(g)_{qjkl} + \mathrm{Rm}(g)_{pil}^q\,\mathrm{Rm}(g)_{jpkq}\\ &+ \mathrm{Ric}(g)_{jq}\,\mathrm{Rm}(g)_{iqkl} - \mathrm{Rm}(g)_{jpi}^q\,\mathrm{Rm}(g)_{pqkl} - \mathrm{Rm}(g)_{jpk}^q\,\mathrm{Rm}(g)_{piql} - \mathrm{Rm}(g)_{jpl}^q\,\mathrm{Rm}(g)_{pikq}.\end{aligned}$$

(v) *Conclude that:*

$$\frac{\partial}{\partial t}\mathrm{Rm}(g(t)) = \Delta_{g(t)}\mathrm{Rm}(g(t)) + Q(\mathrm{Rm}(g(t))) - \mathrm{Ric}(g(t)) * \mathrm{Rm}(g(t)),$$

where:

$$\begin{aligned}Q(\mathrm{Rm}(g(t)))_{ijkl} &:= -\mathrm{Rm}(g(t))_{ijpq}\,\mathrm{Rm}(g(t))_{pqkl} - 2\,\mathrm{Rm}(g(t))_{ipkq}\,\mathrm{Rm}(g(t))_{jpql}\\ &\quad + 2\,\mathrm{Rm}(g(t))_{iplq}\,\mathrm{Rm}(g(t))_{jpkq}\\ \mathrm{Ric}(g(t)) * \mathrm{Rm}(g(t))_{ijkl} &:= \mathrm{Ric}(g(t))_{iq}\,\mathrm{Rm}(g(t))_{qjkl} + \mathrm{Ric}(g(t))_{jq}\,\mathrm{Rm}(g(t))_{iqkl}\\ &\quad + \mathrm{Ric}(g(t))_{kq}\,\mathrm{Rm}(g(t))_{ijql} + \mathrm{Ric}(g(t))_{lq}\,\mathrm{Rm}(g(t))_{ijkq}.\end{aligned}$$