Integrable systems on (multiplicative) quiver varieties

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Calogero-Moser space (1)

$$\mathcal{M} := \{X, Y \in \mathfrak{gl}_n(\mathbb{C}), V \in \mathrm{Mat}_{1\times n}(\mathbb{C}), W \in \mathrm{Mat}_{n\times 1}(\mathbb{C})\}$$

Action of $\mathrm{GL}_n(\mathbb{C})$, $g \cdot (X, Y, V, W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$

Calogero-Moser space $\mathcal{C}_n := \{[X, Y] - WV = \mathrm{Id}_n\} /\!/ \mathrm{GL}_n$ [Wilson,'98]

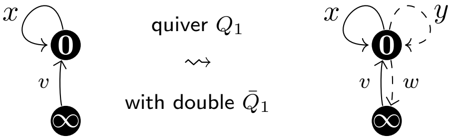
On open dense subset : $X = \mathrm{diag}(q_1, \ldots, q_n)\,,\quad V = (1, \ldots, 1)$

from which $W = -(1, \ldots, 1)^T$ $\quad Y_{ij} = \delta_{ij} p_j - \delta_{(i\neq j)} \frac{1}{q_i - q_j}$

Calogero-Moser Hamiltonian

$$\frac{1}{2} \operatorname{tr} Y^2 = \frac{1}{2} \sum_j p_j^2 - \sum_{i\neq j} \frac{1}{(q_i - q_j)^2}$$

Calogero-Moser Space (2)



$$\mathrm{Rep}\left(\mathbb{C}\bar{Q}_1, (1, n)\right) = \{X, Y \in \mathfrak{gl}_n, V \in \mathrm{Mat}_{1\times n}, W \in \mathrm{Mat}_{n\times 1}\} = \mathcal{M}$$

$\rightsquigarrow$ The Calogero-Moser space $\mathcal{C}_n$ is a *quiver variety* obtained by reduction from the representation space of a quiver

The Poisson bracket can also be understood on $\bar{Q}_1$!
Same for the moment map $[X, Y] - WV$ (cf. part 3 if enough time)

Plan for the talk

1. **Integrable systems from quiver varieties**
2. Integrable systems from multiplicative quiver varieties
3. Method : Noncommutative Poisson geometry

How to read a quiver ? (1)

Quiver Q : directed graph (vertices $s \in I$, arrows $a \in Q$)

Double quiver $\bar{Q}$: for each $s \xrightarrow{a} t \in Q$, add $t \xrightarrow{a^*} s$

Fix a dimension vector $\mathbf{n} := (n_s) \in \mathbb{N}^I$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \mathrm{Mat}(n_s \times n_t, \mathbb{C})$$

Such matrices obtained from the arrows parametrise $\mathrm{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

How to read a quiver ? (2)

Given Q, $\mathbf{n} = (n_s) \in \mathbb{N}^I$, we got $\mathrm{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \mathrm{Mat}(n_s \times n_t, \mathbb{C})$$

Embed matrices as elements of $\mathrm{End}(\oplus_{s\in I}\mathbb{C}^{n_s})$
$\rightsquigarrow$ natural $\mathrm{GL}(\mathbf{n}) := \prod_{s\in I} \mathrm{GL}(n_s)$ action by conjugation

There is a simple* Poisson structure with moment map $\sum_{a\in Q}[A, A^*]$

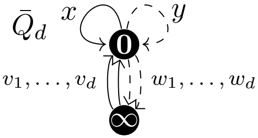
Fix parameters $(\lambda_s) \in \mathbb{C}^I$.

Quiver variety : $\{\sum_{a\in Q}[A, A^*] = \prod_{s\in I} \lambda_s \, \mathrm{Id}_{\mathbb{C}^{n_s}}\} /\!/ \, \mathrm{GL}(\mathbf{n})$

* Nonzero Poisson bracket only between the matrices A, A^* of an arrow a and its double a^*

Calogero-Moser space with spin (1)

[Wilson,∼'98 ;Bielawski-Pidstrygach,'10 ; Tacchella,'15 ; Chalykh-Silantyev,'17]



$d \geq 2$. $\mathcal{M} = \mathrm{Rep}\left(\mathbb{C}\bar{Q}_d, (1,n)\right)$

space parametrised by :

$X, Y \in \mathfrak{gl}_n$

$V_\alpha \in \mathrm{Mat}_{1\times n}, W_\alpha \in \mathrm{Mat}_{n\times 1}$

$$\mathcal{C}_{n,d} := \left\{ [X,Y] - \sum_{1\leq\alpha\leq d} W_\alpha V_\alpha = \lambda_0 \, \mathrm{Id}_n \right\} // \, \mathrm{GL}_n \qquad (\lambda_0 \neq 0)$$

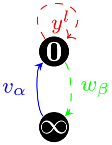
Calogero-Moser space with d spins/degrees of freedom [Gibbons-Hermsen,'84]

On open dense subset where $X = \mathrm{diag}(q_1, \dots, q_n)$:

$Y_{ij} = \delta_{ij} p_j - \delta_{(i\neq j)} \frac{f_{ij}}{q_i - q_j}$, $f_{ij} := \sum_{\alpha=1}^d (W_\alpha V_\alpha)_{ij}$, Lax matrix for spin CM

Calogero-Moser space with spin (2)

We can compute on $\mathcal{C}_{n,d}$ the Poisson brackets of $\operatorname{tr} Y^k$ and $t^l_{\alpha\beta} = V_\alpha Y^l W_\beta$.



$$\{\operatorname{tr} Y^k, \operatorname{tr} Y^l\}_{\mathrm{P}} = 0 = \{\operatorname{tr} Y^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

$$\{t^k_{\alpha\beta}, t^l_{\gamma\epsilon}\}_{\mathrm{P}} = \delta_{\beta\gamma} t^{k+l}_{\alpha\epsilon} - \delta_{\alpha\epsilon} t^{k+l}_{\gamma\beta}$$

Proposition

1. The commutative algebra generated by $(\operatorname{tr} Y^k, t^k_{\alpha\alpha})$, $1 \leq \alpha \leq d$, is an abelian Poisson algebra of dimension nd.
2. The commutative algebra generated by $(\operatorname{tr} Y^k, t^k_{\alpha\beta})$ (any indices), is a Poisson algebra of dimension $2nd - n$, with centre of dimension n containing the $(\operatorname{tr} Y^k)$.

1. $\Rightarrow$ Liouville integrability of the $(\operatorname{tr} Y^k)$.
2. $\Rightarrow$ Degenerate integrability of the $(\operatorname{tr} Y^k)$.

Calogero-Moser space with spin (3)

Each $\operatorname{tr} Y^k$ (fixed k) is *maximally* degenerately integrable

Idea of [Wojciechowski,'83] : the following are first integrals

$$\operatorname{tr}(XY^j)\operatorname{tr}(Y^{i+k-1}) - \operatorname{tr}(XY^i)\operatorname{tr}(Y^{j+k-1})$$

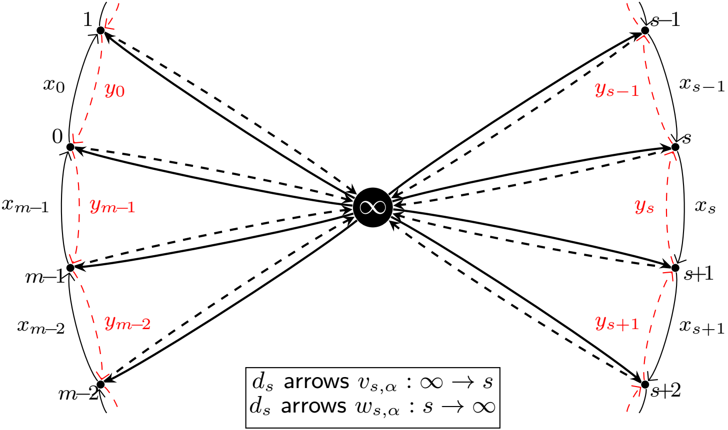
because $\{\operatorname{tr} Y^k, \operatorname{tr}(XY^j)\} = k \operatorname{tr}(Y^{j+k-1})$ and this nice lemma :

Lemma

If $(g_k)_k$ such that $\frac{d^2}{dt^2} g_k = \mu g_k$ for some derivation d/dt and constant μ, then the $C_{j,k} := g_k \frac{dg_j}{dt} - g_j \frac{dg_k}{dt}$ are first integrals.

CM spaces from cyclic quivers (1)

Extended cyclic quiver ⇒ system of CM type [Chalykh-Silantyev,'17]
arbitrary extension in [F.-Görbe,'21] for spin vector $\mathbf{d} \in \mathbb{N}^{\mathbb{Z}_m}$



We are interested in $\operatorname{Rep}\left(\mathbb{C}\bar{Q}_{\mathbf{d}}, (1, n, \dots, n)\right)$ and $y_\bullet = \sum_s y_s$

CM spaces from cyclic quivers (2)

$\mathcal{M} = \operatorname{Rep}\left(\mathbb{C}\bar{Q}_{\mathbf{d}}, (1, n, \ldots, n)\right)$ parametrised by

$$X_s \in \mathfrak{gl}_n, \;\; Y_s \in \mathfrak{gl}_n, \quad V_{s,\alpha} \in \operatorname{Mat}_{1\times n}, \;\; W_{s,\alpha} \in \operatorname{Mat}_{n\times 1}\,,$$
$$s = 0, \ldots, m-1, \quad (s,\alpha) \text{ with } 1 \leq \alpha \leq d_s$$

Action of $g = (g_s) \in \prod_s \mathrm{GL}_n$:
$g \cdot (X_s, Y_s, W_{s,\alpha}, V_{s,\alpha}) = (g_s X_s g_{s+1}^{-1}, g_{s+1} Y_s g_s^{-1}, g_s W_{s,\alpha}, V_{s,\alpha} g_s^{-1})\,,$

Calogero-Moser space $\mathcal{C}_{n,\mathbf{d}}$: $\left(\prod_s \mathrm{GL}_n\right)$-orbits of [$(\lambda_s) \in \mathbb{C}^m$ generic]

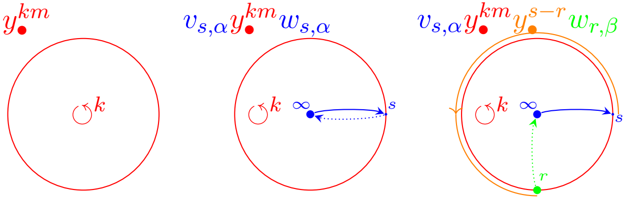
$$X_s Y_s - Y_{s-1} X_{s-1} - \sum_{1\leq \alpha \leq d_s} W_{s,\alpha} V_{s,\alpha} = \lambda_s \operatorname{Id}_{n_s}\,, \quad \forall 0 \leq s \leq m-1$$

On open dense subset : $\operatorname{tr}(Y_\bullet^{km})$ generalise spin CM system

$\rightsquigarrow \quad \frac{1}{m}\operatorname{tr}(Y_\bullet^{km}) = \sum_{i=1}^n p_i^{mk} + \mathrm{O}(p_i^{mk-1})$

CM spaces from cyclic quivers (3)

By adapting [Chalykh-Silantyev,'17], we can "visualise" the functions that guarantee integrability of the generalised CM systems :



$\mathrm{tr}(Y_{\bullet}^{km})$ $\quad$ $t^{k}_{s\alpha;s\alpha}$ $\quad$ $t^{k}_{r\beta;s\alpha} = \mathrm{tr}(W_{r,\beta} V_{s,\alpha} Y_{\bullet}^{km+s-r})$

Proposition ([F.-Görbe,'21 / 2101.05520])

1. The commutative algebra generated by $(\mathrm{tr}(Y_{\bullet}^{km}), t^{k}_{s\alpha;s\alpha})$*, is an abelian Poisson algebra of dimension* $n|\mathbf{d}|$*.*
2. The commutative algebra generated by $(\mathrm{tr}(Y_{\bullet}^{km}), t^{k}_{r\beta;s\alpha})$*, is a Poisson algebra of dimension* $2n|\mathbf{d}| - n$*, with centre of dimension* n *containing the* $\mathrm{tr}(Y_{\bullet}^{km})$*.*

In fact, max. deg. integrability of each $\mathrm{tr}(Y_{\bullet}^{km})$ from the "nice lemma"

Plan for the talk

1. Integrable systems from quiver varieties
2. **Integrable systems from multiplicative quiver varieties**
3. Method : Noncommutative Poisson geometry

How to read a quiver multiplicatively ?

Given Q, $\mathbf{n} = (n_s) \in \mathbb{N}^I$, we got $\mathrm{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \mathrm{Mat}(n_s \times n_t, \mathbb{C})$$

Also : natural $\mathrm{GL}(\mathbf{n}) := \prod_{s \in I} \mathrm{GL}(n_s)$ action by conjugation

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There is a cumbersome* quasi-Poisson structure [Van den Bergh,'08]
+ a multiplicative moment map $\Phi_{Q,<} := \underbrace{\prod_{a \in Q} (\mathrm{Id} + AA^*)(\mathrm{Id} + A^*A)^{-1}}_{\text{inverted elements + order } <}$

Need quasi-Poisson geometry [Alekseev – Kosmann-Schwarzbach – Meinrenken,'02]

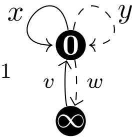
Fix parameters $(q_s) \in (\mathbb{C}^\times)^I$.

Multiplicative quiver variety : $\{\Phi_{Q,<} = \prod_{s \in I} q_s \, \mathrm{Id}_{\mathbb{C}^{n_s}}\} // \mathrm{GL}(\mathbf{n})$

* Bracket with quadratic terms between matrices A, B of arrows a, b sharing a vertex
Extra constant term for matrices A, A^* of an arrow+double a, a^*

Ruijsenaars-Schneider space (1)

$\bar{Q}_1$



$\bar{Q}_1 \rightsquigarrow$ add inverses for $1+xy, 1+yx, 1+vw, 1+wv$

e.g. $(1+xy)^{-1}$ in $\mathbb{C}\bar{Q}_1^{\mathrm{loc}}$ such that

$$(1+xy)(1+xy)^{-1} = 1 = (1+xy)^{-1}(1+xy)$$

$$\mathrm{Rep}\left(\mathbb{C}\bar{Q}_1, (1,n)\right) = \{X, Y \in \mathfrak{gl}_n, V \in \mathrm{Mat}_{1\times n}, W \in \mathrm{Mat}_{n\times 1}\} = \mathcal{M}$$

$$\cup$$

$$\mathrm{Rep}\left(\mathbb{C}\bar{Q}_1^{\mathrm{loc}}, (1,n)\right) = \{\det(\mathrm{Id}_n + XY) \neq 0,\ 1 + VW \neq 0\} =: \mathcal{M}^\circ$$

<u>Same</u> action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{M}^\circ \subset \mathcal{M}$ which preserves

- a *quasi-Poisson bracket* on $\mathcal{M}^\circ$;
- a $\mathrm{GL}_n(\mathbb{C})$-valued moment map on $\mathcal{M}^\circ$.

Ruijsenaars-Schneider space (2)

$$\operatorname{Rep}\left(\mathbb{C}\bar{Q}_1,(1,n)\right)=\{X,Y\in\mathfrak{gl}_n, V\in\operatorname{Mat}_{1\times n}, W\in\operatorname{Mat}_{n\times 1}\}=\mathcal{M}$$
$$\operatorname{Rep}\left(\mathbb{C}\bar{Q}_1^{\mathrm{loc}},(1,n)\right)=\{\det(\operatorname{Id}_n+XY)\neq 0,\ 1+VW\neq 0\}=:\mathcal{M}^\circ$$

Ruijsenaars-Schneider space : [Chalykh-F.,'17 /1704.05814] (also [Oblomkov,'04])

$$\mathcal{C}_{n,q,1}:=\{(\operatorname{Id}_n+XY)(\operatorname{Id}_n+YX)^{-1}(\operatorname{Id}_n+WV)^{-1}=q\operatorname{Id}_n\}//\operatorname{GL}_n\ (q^k\neq 1)$$

Parametrised on dense subset with X invertible by :

$$X=\operatorname{diag}(x_1,\ldots,x_n)\,,\quad V=(1,\ldots,1)$$

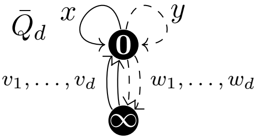
from which $(Y+X^{-1})_{ij}=\frac{(1-q)x_j}{x_i-qx_j}\sigma_j\prod_{k\neq j}\frac{x_k-qx_j}{x_k-x_j}$

Hamiltonian of Ruijsenaars-Schneider $(x_i=e^{q_i},\sigma_i=e^{p_i})$

$$\operatorname{tr}(Y+X^{-1})=\sum_{1\leq j\leq n}\sigma_j\prod_{k\neq j}\frac{x_k-qx_j}{x_k-x_j}$$

Spin RS space (1)

Case $d \geq 2$ with spin [Chalykh-F.,'20 / 1811.08727]



"add inverses" to $\bar{Q}_d \rightsquigarrow \bar{Q}_d^{\mathrm{loc}}$

$\mathcal{M}^\circ = \mathrm{Rep}\left(\bar{Q}_d^{\mathrm{loc}}, (1,n)\right)$
parametrised by : $X, Y \in \mathfrak{gl}_n$
$V_\alpha \in \mathrm{Mat}_{1\times n}, W_\alpha \in \mathrm{Mat}_{n\times 1}$

$$\mathcal{C}_{n,q,d} := \left\{ (\mathrm{Id}_n + XY)(\mathrm{Id}_n + YX)^{-1} \overrightarrow{\prod_{1\leq\alpha\leq d}} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\, \mathrm{Id}_n \right\} // \mathrm{GL}_n$$

Ruijsenaars-Schneider space with d spin/degrees of freedom

Spin RS space (2)

$$\mathcal{C}_{n,q,d} := \left\{ (\mathrm{Id}_n + XY)(\mathrm{Id}_n + YX)^{-1} \overrightarrow{\prod_{1\leq\alpha\leq d}} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n \right\} /\!/ \mathrm{GL}_n$$

We set $Z := Y + X^{-1}$ $\rightsquigarrow$ (what we needed for $d = 1$)

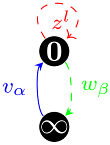
Proposition ([Chalykh-F.,'20])

On open subset of $\mathcal{C}_{n,q,d}$, there exists local coordinates such that

- *the equations of motion associated with $\mathrm{tr}(Z)$ reproduce the eqs of trigonometric RS system with spin from [Krichever-Zabrodin,'95] ;*
- *the matrix Z is the Lax matrix of that system ;*
- *the Poisson bracket written in those coordinates allows to prove a conjecture from [Arutyunov-Frolov,'98].*

Spin RS space (3)

We write $t^l_{\alpha\beta} = V_\alpha Z^l W_\beta$, $\quad Z = Y + X^{-1}$



$$\{\mathrm{tr}\, Z^k, \mathrm{tr}\, Z^l\}_{\mathrm{P}} = 0 = \{\mathrm{tr}\, Z^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

$$\{t^k_{\alpha\beta}, t^l_{\gamma\epsilon}\}_{\mathrm{P}} = \text{very complicated !}$$

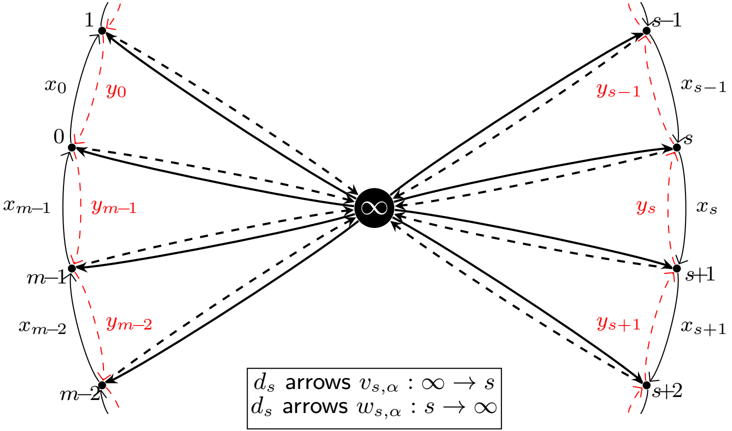
Proposition ([Chalykh-F.,'20])

The commutative algebra generated by $(\mathrm{tr}\, Z^k, t^k_{\alpha\beta})$, $1 \leq \alpha, \beta \leq d$, *is a Poisson algebra of dimension* $2nd - n$, *with centre of dimension* n *containing the* $(\mathrm{tr}\, Z^k)$.

$\Rightarrow$ degenerate integrability of the $(\mathrm{tr}\, Z^k)$.

RS spaces from cyclic quivers (1)

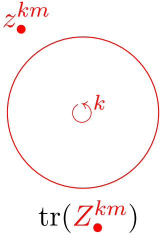
General case : [F.,21' /2108.02496] for spin vector $\mathbf{d} \in \mathbb{N}^{\mathbb{Z}_m}$
See [F.-Chalykh,'17 ;F.,'19] for $\mathbf{d} = (1, 0, \ldots, 0)$ and $\mathbf{d} = (d, 0, \ldots, 0)$

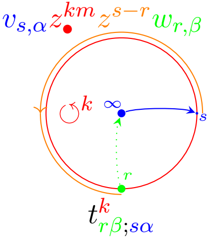


We are interested in $\mathrm{Rep}\left(\mathbb{C}\bar{Q}^{\mathrm{loc}}_{\mathbf{d}}, (1, n, \ldots, n)\right)$ and $z_{\bullet} = \sum_s (y_s + x_s^{-1})$

RS spaces from cyclic quivers (2)

As in CM case, can "visualise" interesting functions : $\quad (Z_\bullet = \sum_s (Y_s + X_s^{-1}))$





Proposition ([F.,21' /2108.02496])

The commutative algebra generated by $(\mathrm{tr}(Z_\bullet^{km}), t^k_{r\beta;s\alpha})$, *is a Poisson algebra of dimension* $2n|\mathbf{d}| - n$, *with centre of dimension* n *containing the* $\mathrm{tr}(Z_\bullet^{km})$.

$\Rightarrow$ degenerate integrability of the $(\mathrm{tr}\, Z_\bullet^{km})$.

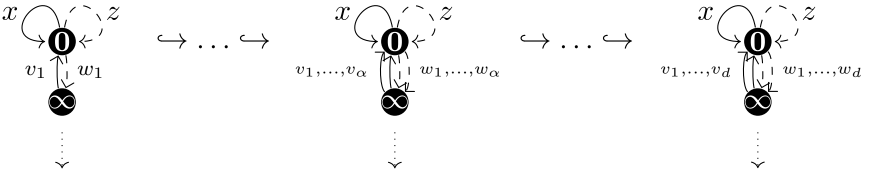
RS spaces from cyclic quivers (3)

In fact, 3 distinct families of degenerately integrable systems for
$(\operatorname{tr} Z_\bullet^{km} = \operatorname{tr}(Y_\bullet + X_\bullet^{-1})^{km})$; $(\operatorname{tr} Y_\bullet^{km})$; $(\operatorname{tr}(\mathrm{Id} + X_\bullet Y_\bullet)^k)$

In local coord., eq. of motion for the 3^{rd} are a *many-spin generalisation* of the eqs of trigonometric RS system with spin from [Krichever-Zabrodin,'95] ;

Remark (Liouville integrability through embeddings of quivers)

How it works in the simplest case for $(\operatorname{tr} Z^k)$ (1-loop quiver)



$$\mathcal{C}_{n,q,1} \hookrightarrow \dots \hookrightarrow \mathcal{C}_{n,q,\alpha} \hookrightarrow \dots \hookrightarrow \mathcal{C}_{n,q,d}$$

$S_\alpha := (\mathrm{Id}_n + W_\alpha V_\alpha) \dots (\mathrm{Id}_n + W_1 V_1) Z$ ($= q^{-1} X Z X^{-1}$ on $\mathcal{C}_{n,q,\alpha}$)
$\Rightarrow \operatorname{tr} S_\alpha^k \in \mathbb{C}[\mathcal{C}_{n,q,d}]$ Poisson commute for all indices

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3. **Method : Noncommutative Poisson geometry**

Noncommutative Poisson geometry (1)

Algebra $\mathcal{A}$ over $\mathbb{C}$ $\dashrightarrow$ comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$
space $\rightsquigarrow$ $\mathrm{Rep}(\mathcal{A},n)$ (above)

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\mathcal{A}^\lambda := \mathcal{A}/(\mu-\lambda)$ $\dashrightarrow$ comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A}^\lambda,n)]$

$\mu \in \mathcal{A}$, $\lambda \in \mathbb{C}$ $\qquad$ space $\rightsquigarrow$ $\{(\mu_{ij}) = \lambda\,\mathrm{Id}_n\}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

vector space $\mathcal{A}^\lambda/[\mathcal{A}^\lambda,\mathcal{A}^\lambda]$ $\dashrightarrow$ comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A}^\lambda,n)]^{\mathrm{GL}_n}$
space $\rightsquigarrow$ $\mathrm{Rep}(\mathcal{A}^\lambda,n)/\!/\,\mathrm{GL}_n$

$\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$ generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \le i,j \le n$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik}b_{kj}$.

$\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]^{\mathrm{GL}_n}$ generated by $\mathrm{tr}(a)$, $a \in \mathcal{A}$

Noncommutative Poisson geometry (2)

Algebra $\mathcal{A}$ ⇢ comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A}, n)]$

double Poisson bracket $\{\!\{-,-\}\!\}$ — Poisson bracket $\{-,-\}_{\mathrm{P}}$

NC moment map μ — moment map (μ_{ij})

↓ ↓

vector space $\mathcal{A}^\lambda/[\mathcal{A}^\lambda, \mathcal{A}^\lambda]$ ⇢ comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A}^\lambda, n)]^{\mathrm{GL}_n}$

Lie bracket $m \circ \{\!\{-,-\}\!\}$ — Poisson bracket $\{-,-\}_{\mathrm{P}}$

A double bracket is a $\mathbb{C}$-bilinear map $\{\!\{-,-\}\!\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}^{\otimes 2}$ s.t.

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\ \{\!\{b,a\}\!\}$ (skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1)\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}(1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a)\{\!\{d,b\}\!\} + \{\!\{a,b\}\!\}(d \otimes 1)$ (inner derivation)

A double bracket is Poisson if "double Jacobi" [Van den Bergh,'08]

Induced Poisson bracket : $\{a_{ij}, b_{kl}\}_{\mathrm{P}} = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}$ [Van den Bergh,'08]

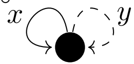
How does it relate to the rest ?

[Van den Bergh,'08] : double Poisson bracket on $\mathcal{A} = \mathbb{C}\bar{Q}$, $\forall$ quiver Q

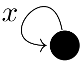
Example

$$\mathcal{A} = \mathbb{C}\langle x, y\rangle = \mathbb{C}\bar{Q}_\circ$$

$\bar{Q}_\circ$

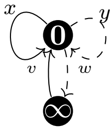


$Q_\circ$

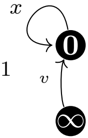


The data $\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\}$, $\{\!\{x, y\}\!\} = 1 \otimes 1$, $\mu_{\mathcal{A}} = xy - yx$ are defined on $\mathbb{C}\bar{Q}_\circ$ (from $Q_\circ$) and induce the Poisson bracket on associated quiver varieties.

$\bar{Q}_1$



Q_1



Original example from [Wilson,'98]...
Attach $\mathbb{C}^n$ to 0, $\mathbb{C}$ to ∞
$\Rightarrow$ Calogero-Moser space
$\left(\mathrm{tr}(Y^k) := \mathrm{tr}(y_{ij})^k\right)_{k=1}^n$ commute

Noncommutative quasi-Poisson geometry

Algebra $\mathcal{A}$ $\dashrightarrow$	comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$
double quasi-Poisson bracket $\{\!\{-,-\}\!\}$	quasi-Poisson bracket $\{-,-\}_{\mathrm{P}}$
NC moment map $\Phi \in \mathcal{A}^{\times}$	moment map (Φ_{ij}) (in GL_n)
$\downarrow$	$\downarrow$
vector space $\mathcal{A}^q/[\mathcal{A}^q,\mathcal{A}^q]$ $\dashrightarrow$	comm. algebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A}^q,n)]^{\mathrm{GL}_n}$
Lie bracket $m \circ \{\!\{-,-\}\!\}$	Poisson bracket $\{-,-\}_{\mathrm{P}}$

$\mathcal{A}^q = \mathcal{A}/(\Phi - q)$ for $q \in \mathbb{C}^{\times}$

[Van den Bergh,'08] : double quasi-Poisson bracket on $\mathcal{A} = \mathbb{C}\bar{Q}_{(1+aa^*)_{a\in\bar{Q}}}$

Thank you for your attention !

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