A triangular mathematical recreation from scratch

When complexes drive out complexity

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Beta version, comments/suggestions welcome

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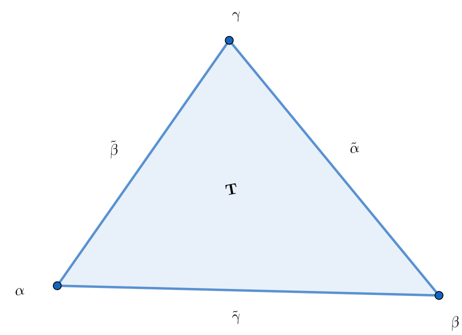
This note is of pedagogical nature, nothing else. It does not contain any original result. The targeted readers are undergraduate students and high school teachers.

Contents

Introduction

Let $T = \{\alpha, \beta, \gamma\}$ with edge lengths $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$ be a (non flat and usually non oriented[1]) triangle in the Euclidean (oriented affine) plane[2] $\mathbf{C}$.



Our approach is to focus on algebraic relations after reducing the circumcircle to the unit complex circle $U = \{z \in \mathbf{C} | |z| = 1\}$. Indeed, up to similarity[3] (see below) the circumcircle can be assumed to be the unit circle $U \subset \mathbf{C}$. The first advantage of working on the unit circle is that conjugation becomes the algebraic map $z \mapsto 1/z$. The second one is that most of the relevant lines for triangle geometry have very nice equations.

Lemma 0.1 *For $z_1, z_2 \in U$ we will denote by $\langle z_1, z_2 \rangle$ the line passing through z_1, z_2 if $z_1 \neq z_2$ and the tangent line $\langle z_1, z_1 + \mathbf{i} z_1 \rangle$ to U at $z_1 = z_2$ otherwise. If $z_1, z_2 \in U$, the equation $t + z_1 z_2 \bar{t} = z_1 + z_2$ in the complex variable t defines the line[4] $\langle z_1, z_2 \rangle$.*

[1]It will be time to time convenient to look at the "oriented triangle T" thanks to the alphabetical order for instance.

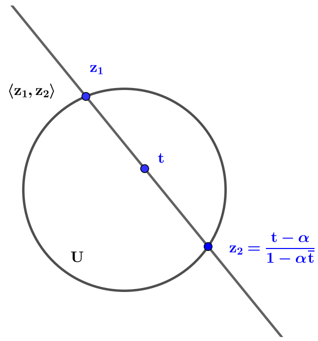
[2] The scalar product is given by $(z_1, z_2) = \mathrm{Re}(z_1 \bar{z}_2)$ and $\det(z_1, z_2) = -\mathrm{Im}(z_1 \bar{z}_2)$, that is twice the algebraic area of the oriented triangle $0, z_1, z_2$. In particular, the (real) barycentric coordinates $x_\alpha, x_\beta, x_\gamma$ of $z \in \mathbf{C}$ are characterized by

$$2x_\alpha S = \det(z - \gamma, \gamma - \beta), 2x_\beta S = \det(z - \alpha, \alpha - \gamma), 2x_\gamma = \det(z - \beta, \beta - \alpha)$$

where $S = \frac{1}{2} \det(\beta - \alpha, \gamma - \alpha)$ is the algebraic area of the oriented triangle (α, β, γ). This observation allows to understand the position of z inside our outside of T as we will see.

[3]Translate the circumcenter to 0 and then use the homothety of ratio $1/R$ where R is the radius of circumcircle.

[4]Observe that z_1, z_2 are always solutions and that $z_1 + \mathbf{i} z_1$ is solution if $z_1 = z_2$. Conversely, The relation $t \in \langle \alpha, t \rangle$ characterizes the line $\langle \alpha, t \rangle$ for any complex $t \neq \alpha$ and gives the formula $z_\alpha = \dfrac{t - \alpha}{1 - \alpha \bar{t}}$.



Therefore, we have two very tractable ways to describe lines made up from T, both of which keep the cyclic symmetry: barycentric equations and the previous method.

In Part I, we will illustrate how these elementary facts, which will be used repeatedly, lead to simple, natural, and very short algebraic proofs[5] (and correct!) of classical results in classical triangular geometry, even those often considered difficult[6]. The complement part II is devoted to the counting of Morley triangles (without any genericity assumption ofT).

There is a (mild) price to pay: neither an half plane nor the interior of a triangle is algebraically definable. In particular, as we will see, there is no algebraic or simply continuous way (with respect of the vertices) to distinguish internal bisector or trisector from the other one without to make non algebraic choices (see below. This will force to extract roots of the vertices to study such notions (the algebraic notion of base change). This is the only place where angle considerations are somehow hidden.

This does not mean that "usual geometric intuition coming from pictures" is useless, in particular because the existence of the classical results on triangle geometry come from this intuition. But algebra give powerful tool to (re)prove these results in full generality in a quite systematic way. More important, modern algebro-geometric methods are essential to go beyond the simple geometry of the triangle and consider geometric situations of greater degree. Finally, as the reader will see, provided we do not break any symmetries, the computations are very simple and, in a sense, aesthetically pleasing.

We are astonished by the strange vitality of triangle geometry — a passion that we do not share. We believe that mathematics should primarily contribute to the broader advancement of science, including its own evolution, rather than offering puzzles that, clever or challenging as they may be, focus on subjects that have been known for centuries, or even millennia.

However, paradoxically, we hope that the following facts could provide a path from these rather outdated results to modern, powerful algebro-geometric methods (including the use of formal computation software to verify algebraic identities) in this elementary setup. This approach combines geometric, analytic and algebraic intuitions and could lead to profound and novel results. It is certainly non unique neither optimal on some place, whatever it means. May this note encourage us to move away from this triangular scientific impasse.

[5]Hence which can be generalized to more or less arbitrary fields.

[6]Such as Feuerbach or Morley theorems

Part I

An express line towards classical results

All the items of this part are independent. They give comprehensive and easy proofs of classical theorems due to Euclide, Euler, Feuerbach, Morley[7].

1 Explicit reduction of the circumcircle to the unit circle

The basic well known fact is that T has a unique non trivial circumcircle whose center is by construction the intersection point ω of the mediatrices of T. It is is the unique solution of $|t-\gamma|^2 = |t-\alpha|^2 = |t-\beta|^2$, hence of the system

$$\begin{cases} (\overline{\gamma}-\overline{\alpha})t + (\gamma-\alpha)\overline{t} = & |\gamma|^2 - |\alpha|^2 \\ (\overline{\alpha}-\overline{\beta})t + (\alpha-\beta)\overline{t} = & |\alpha|^2 - |\beta|^2 \end{cases}$$

we get

$$\omega = \frac{(|\gamma|^2-|\beta|^2)\alpha + (|\alpha|^2-|\gamma|^2)\beta + (|\beta|^2-|\alpha|^2)\gamma}{4\mathbf{i}S} \tag{i}$$

where $S = \frac{-1}{4i}((\beta-\alpha)(\overline{\gamma}-\overline{\alpha}) - (\overline{\beta}-\overline{\alpha})(\gamma-\alpha)) = -\frac{1}{2}\mathrm{Im}(\beta-\alpha)(\overline{\gamma}-\overline{\alpha}) = \frac{1}{2}\det(\overrightarrow{\beta\alpha}, \overrightarrow{\gamma\alpha})$ is the algebraic area of the oriented triangle (α, β, γ) (see note 2). This gives existence and uniqueness of the circumcircle.

To give a closed formula for S (hence of ω and accordingly of the circumradius R), we can proceed as follows. Up to the affine (positive) isometry $\iota: \ t \mapsto \frac{|\beta-\alpha|}{\beta-\alpha}(t-\alpha)$, one can assume $T = \{0, \tilde{\gamma} = |\beta-\alpha|, z\}$ with $y = \mathrm{Im}(z = x+iy) \neq 0$. We have ($\iota$ isometry)

$$\tilde{\beta}^2 = |z|^2 = x^2+y^2, \tilde{\alpha}^2 = |z-\tilde{\gamma}|^2 = (x-\tilde{\gamma})^2 + y^2 \text{ hence } x = \frac{-\tilde{\alpha}^2+\tilde{\beta}^2+\tilde{\gamma}^2}{2\tilde{\gamma}}$$

and

$$\begin{aligned} y^2 =& \frac{4\tilde{\beta}^2\tilde{\gamma}^2 - (-\tilde{\alpha}^2+\tilde{\beta}^2+\tilde{\gamma}^2)}{4\tilde{\gamma}^2} \\ =& \frac{(\tilde{\alpha}^2-(\tilde{\beta}-\tilde{\gamma})^2)(-\tilde{\alpha}^2+(\tilde{\beta}+\tilde{\gamma})^2)}{4\tilde{\gamma}^2} \\ =& \frac{(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})}{4\tilde{\gamma}^2} \end{aligned}$$

from which we get the Heron formula for the (geometric) area $|S| = \frac{1}{2}|\det(\overrightarrow{\beta\alpha}, \overrightarrow{\gamma\alpha})| = \frac{1}{2}|-\tilde{\gamma}y|$ of T

$$|S| = \frac{\sqrt{(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})}}{4} \tag{ii}$$

[7] Who owes the most recent result, discovered in…1899 (Taylor FG, Marr WL. The six trisectors of each of the angles of a triangle. Proceedings of the Edinburgh Mathematical Society. 1913;32:119-131)

Using (i), we get

$$R^2 = |\omega|^2 = \frac{|0 - \tilde{\beta}^2\tilde{\gamma} + \tilde{\gamma}^2 z|^2}{16|S|^2} = \frac{(x\tilde{\gamma}^2 - \tilde{\beta}^2\tilde{\gamma})^2 + y^2\tilde{\gamma}^2}{16|S|^2} = \frac{\tilde{\alpha}^2\tilde{\beta}^2\tilde{\gamma}^2}{16|S|^2}$$

hence

$$R = \frac{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{4|S|} \overset{\text{Heron}}{=} \frac{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{\sqrt{(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})}} \tag{iii}$$

Remark The usual pictures-based proofs using the Pythagorean Theorem can be made to work correctly by starting with an acute vertex angle, which guarantees that its orthogonal projection onto the basis lies inside the edge. However, our goal is to demonstrate that simple computations can also achieve the desired result.

All properties we are interested in are (quasi)-invariants, meaning invariant by translation and rotations and suitably homogeneous by similarities.

From now, we do assume that U is the circumcircle of T.

Our goal is to show that most of the classical properties become easy (and nice) calculations in this case.

2 Special triangles

Lemma 2.1

1. T *is isosceles*[8] *in* α *if and only if* $\alpha^2 = \beta\gamma$.

2. T *is rectangle in* α *if and only if* $\beta + \gamma = 0$.

3. T *is equilateral if and only if either* $\alpha + j\beta + j^2\gamma = 0$ *or* $\alpha + j^2\beta + j\gamma = 0$ *or equivalently*

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha = 0.$$

Proof.

1. Just compute $|(\alpha - \beta)|^2 = (\alpha - \beta)(1/\alpha - 1/\beta) = -(\alpha - \beta)^2\gamma$ and therefore for $|(\alpha - \gamma)|^2 = -(\alpha - \gamma)^2\beta$. The difference between these terms is therefore $-(\alpha - \beta)^2\gamma + (\alpha - \gamma)^2\beta = (\beta - \gamma)(\alpha^2 - \beta)$.

2. By definition of the scalar product, two vectors $x, y \in \mathbf{C}$ are orthogonal if and only if $\mathrm{Im}(x\overline{y}) = 0$. We compute $(\beta-\alpha)\overline{(\gamma-\alpha)} = -(\beta-\alpha)(\gamma-\alpha)\beta$ and $(\gamma-\alpha)\overline{(\beta-\alpha)} = -(\gamma-\alpha)(\beta-\alpha)\gamma$ hence $2\mathrm{Im}(\gamma-\alpha)(\beta-\alpha) = -(\gamma-\alpha)(\beta-\alpha)(\beta+\gamma)$.

3. Assume for instance $\alpha + j\beta + j^2\gamma = 0$ (with $j = \exp(\frac{2i\pi}{3})$. Because $\alpha(1+j+j^2) = 0$, we get $j(\beta-\alpha) = j^2(\gamma-\alpha)$ hence T isosceles at α and by symmetry T is equilateral. Conversely, if T is equilateral, the (oriented) angle $(\widehat{\alpha, 0, \beta})$ is $\pm 2\pi/3$ hence $\beta = j^{\pm 1}\alpha$. Analogously, we get $\gamma = j^{\pm 1}\beta$ hence $\alpha + j^{\pm 1}\beta + j^{\pm 2}\gamma = (1 + j^{\pm 2} + j^{\pm 4})\alpha = 0$. The last point is the formula

$$(\alpha + j\beta + j^2\gamma)(\alpha + j^2\beta + j\gamma) = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha.$$

$\square$

Observe that (2) is nothing but the classical fact that a triangle is rectangle at some vertex if and only if the opposite edge is a diameter of the circumcircle. Observe also that we cannot hope a formula with a, b, c but without j even for the historical case of "adjacent trisectors" simply because the sum of oriented angles of a triangle is π (exercise).

3 Circumcircle and tangent circles

Let us chose square roots a, b, c of α, β, γ. As a warm-up, let us start with the barycentric coordinates of the circumcenter $\Omega = 0 \in \mathbf{C}$. We will use the length computation

$$\tilde{\gamma}^2 = |\alpha - \beta|^2 = -\frac{(\alpha-\beta)^2}{\alpha\beta} \tag{iv}$$

1. The formula $|a^2 - ab| = |a - b| = |b^2 - ab|$ shows that the line $\langle -ab, ab\rangle = \langle -\sqrt{\alpha\beta}, \sqrt{\alpha\beta}\rangle$ of equation $t + \alpha\beta\overline{t} = 0$ is the mediatrix of T (passing through $(\alpha+\beta)/2$).

2. By direct calculation using (iv), we get

$$\tilde{\alpha}^2(-\tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{\gamma}^2)\alpha + \tilde{\beta}^2(\tilde{\alpha}^2 - \tilde{\beta}^2 + \tilde{\gamma}^2)\beta + \tilde{\gamma}^2(\tilde{\alpha}^2 + \tilde{\beta}^2 - \tilde{\gamma}^2)\gamma = 0$$

proving that the barycentric coordinates of the circumcenter Ω are

$$\big(\tilde{\alpha}^2(-\tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{\gamma}^2), \tilde{\beta}^2(\tilde{\alpha}^2 - \tilde{\beta}^2 + \tilde{\gamma}^2)\beta, \tilde{\gamma}^2(\tilde{\alpha}^2 + \tilde{\beta}^2 - \tilde{\gamma}^2)\big)$$

unless T is equilateral (where the coordinates[9] are $(1, 1, 1)$).

[8]With our normalization, this means that up to the diagonal action of μ_3 on normalized triangles, the triangle T is $\{1, \beta, 1/\beta\}$ which is invariant by the complex conjugation.

[9]Of course, if we like (oriented) angles, we have also $\det(\alpha - \Omega, \beta - \Omega) = R^2 \sin(\widehat{\overrightarrow{\Omega\alpha}, \overrightarrow{\Omega\beta}}) = R^2 \sin(2\widehat{\gamma})$ by the inscribed angle theorem giving the other classical system of barycentric coordinates $\big(\sin(2\widehat{\alpha}), \sin(2\widehat{\beta}), \sin(2\widehat{\gamma})\big)$ where $\widehat{\alpha} \in \mathbf{R}/\pi\mathbf{Z}$ is the angle (of lines) of the edges through α.

Let us turn to the tangent circles to T (circles tangent to the three lines $\langle\alpha,\beta\rangle, \langle\beta,\gamma\rangle, \langle\gamma,\alpha\rangle$ defined by T).

1. Recall[10] that a line ℓ through α is a bisector of T if there exists a rotation ρ such that

$$\rho(\langle\alpha,\beta\rangle) = \ell \text{ and } \rho(\ell) = \langle\gamma,\alpha\rangle.$$

   Observe that the image o of the orthogonal projection of $\omega \in \ell$ on $\langle\alpha,\beta\rangle$ by ρ^2 is the the orthogonal projection o' of $\omega \in \ell$ on $\langle\gamma,\beta\rangle$ (If u a unit vector of ℓ, $o = \alpha + (\rho^{-1}(u), \omega - \alpha) = (\rho^(u), \omega - \alpha)$ because ρ is a rotation of center α.) hence $|o - \omega| = |o' - \omega|$. In particular, if ω is the intersection of two bisectors, its orthogonal projections on the three edges are cocyclic.

2. The coloured lines of type $\langle c^2, \pm ab\rangle$ are bisectors[11] (inscribed angle theorem or direct calculation). The bisector[12] $\ell^{ab} = \langle c^2, ab\rangle$ from $\gamma = c^2$ has equation $t + c^2 ab\bar{t} = c^2 + ab$. Changing a, c to their opposite if necessary, one can assume that these bisectors are $\ell^{ab}, \ell^{bc}, \ell^{\varepsilon ac}, \varepsilon = \pm 1$. The intersection point of $\ell^{ab} \cap \ell^{bc}$ is therefore $t^b = ab + bc - ca$ and the one $\ell^{bc} \cap \ell^{\varepsilon ac}$ is $t^c = -\varepsilon ab + bc + \varepsilon ca$ hence $t_a - t_c = (\varepsilon + 1)a(b - c)$. One deduces that the three bisectors meet if and only if $\varepsilon = -1$ and that the corresponding incenter is $ab + bc - ca$ with $ab = \sqrt{\alpha\beta}, bc = \sqrt{\beta\gamma}, -ca = \sqrt{\gamma\alpha}$. Letting $\{\pm 1\}^3$ act, we get (at most at this point) four tangent circles to T with centers $\sqrt{\alpha\beta} + \sqrt{\beta\gamma} + \sqrt{\gamma\alpha}$ with the normalization of the square roots $\sqrt{\alpha\beta}\sqrt{\beta\gamma}\sqrt{\gamma\alpha} = -\alpha\beta\gamma$.

   Conversely, such a normalized oriented[13] choice $\sqrt{T} \overset{\text{def}}{=} (\sqrt{\alpha\beta}\sqrt{\beta\gamma}\sqrt{\gamma\alpha})$ -a square root of T for short) comes from an ordered triple (a, b, c) unique up to multiplication by ± 1.

   Let $\sqrt{T}$ be a square root of T and $\omega_{\sqrt{T}} = \sqrt{\alpha\beta} + \sqrt{\beta\gamma} + \sqrt{\gamma\alpha}$ the corresponding center of the tangent circle (with the normalization of the square roots $\sqrt{\alpha\beta}\sqrt{\beta\gamma}\sqrt{\gamma\alpha} = -\alpha\beta\gamma$) and $r_{\sqrt{T}}$ the corresponding radius. We define the triple[14] of "algebraic lengths" of $\sqrt{T}$ by $\check{\gamma} = \mathbf{i}\frac{\alpha-\beta}{\sqrt{\alpha\beta}} \in \mathbf{R}, \ldots$ (as already observed (iv), we have $|\check{\gamma}| = \tilde{\gamma}$). With the above notations, we have

$$\check{\alpha} = \mathbf{i}\frac{b^2 - c^2}{bc},\ \check{\beta} = -\mathbf{i}\frac{c^2 - a^2}{ca},\ \check{\gamma} = \mathbf{i}\frac{a^2 - b^2}{ab} \tag{v}$$

3. From (v), a direct computation shows

$$(\check{\alpha} + \check{\beta} + \check{\gamma})\omega_{\sqrt{T}} = \check{\alpha}\alpha + \check{\beta}\beta + \check{\gamma}\gamma \tag{vi}$$

[10] In terms of angles of lines, this means exactly $2(\widehat{\langle\alpha,\beta\rangle, \ell}) = (\widehat{\langle\alpha,\beta\rangle, \langle\gamma,\alpha\rangle})$.

[11] By (2.1), $c^2 = \pm ab$ means T isosceles at γ and the (external) bisector of T at γ is tangent to U at γ which is coherent with our definitions.

[12] At least when $c^2 \neq ab$. If we have equality, this line is the tangent line of U at α which is an (external) bisector either by a continuity argument or using that T is isosceles at γ in this case (2.1).

[13] Alphabetical order on $\sqrt{T}_\gamma = \sqrt{\alpha\beta}\ldots$

[14] Defined up to sign by we do not choose an orientation of $\sqrt{T}$.

with

$$\check{\alpha}+\check{\beta}+\check{\gamma} = \frac{\mathbf{i}(a^2b+b^2c+ab^2-ac^2-b^2c-bc^2}{abc}$$
$$= \frac{\mathbf{i}(b+c)(a-c)(a+b)}{abc} \neq 0$$

proving that $\mathbf{i}(\check{\alpha},\check{\beta},\check{\gamma})$ are barycentric coordinates of $\omega_{\sqrt{T}}$. Taking modules, we get moreover

$$|\check{\alpha}+\check{\beta}+\check{\gamma}|^2 = (b+c)(a-c)(a+b)(1/b+1/c)(1/a-1/c)(1/a+1/b) = -\frac{(b+c)^2(b+a)^2(c-a)^2}{\alpha\beta\gamma} \quad \text{(vii)}$$

Changing the signs of the triple of square roots does not change the circle hence we can assume $\mathbf{i}(\check{\alpha}+\check{\beta}+\check{\gamma}>0)$. Using triangle inequality, the possible values are therefore the four positive numbers

$$\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}, -\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}, \tilde{\alpha}-\tilde{\beta}+\tilde{\gamma}, \tilde{\alpha}+\tilde{\beta}-\tilde{\gamma}$$

from which we get four possible values for the triple $\mathbf{i}(\check{\alpha},\check{\beta},\check{\gamma})$ namely

$$(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}), (-\tilde{\alpha},\tilde{\beta},\tilde{\gamma}), (\tilde{\alpha},-\tilde{\beta},\tilde{\gamma}), (\tilde{\alpha},\tilde{\beta},-\tilde{\gamma})$$

Therefore there is one tangent circle (the "incircle") inside T (positive normalized coordinates) and three circles not inside T (the "excircles") -with only one positive normalized coordinate- which is inside to the pair of lines through the vertex with positive coordinate and outside the two others.

4. We have $2x_\alpha S = \det(\omega_{\sqrt{T}}-\gamma, \gamma-\beta)$ where $x_\alpha = \frac{\check{\alpha}}{\check{\alpha}+\check{\beta}+\check{\gamma}}$ is he normalized barycentric coordinate of the incenter (see note 2). By construction, the height from $\omega_{\sqrt{T}}$ of the triangle $\{\beta,\omega,\gamma\}$ is a ray of the incircle (because it is tangent to T) showing that the its area[15] $\frac{1}{2}|\det(\omega_{\sqrt{T}}-\gamma,\gamma-\beta)|$ is $\frac{1}{2}\tilde{\alpha} r_{\sqrt{T}}$ hence using $|\check{\alpha}| = \tilde{\alpha} \neq 0$, we get

$$r_{\sqrt{T}} = \frac{2|S|}{|\check{\alpha}+\check{\beta}+\check{\gamma}|} \overset{\text{Heron}}{=} \frac{\sqrt{(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})}}{2|\check{\alpha}+\check{\beta}+\check{\gamma}|}. \quad \text{(viii)}$$

5. By the discussion of item 3, the four real numbers

$$\mathbf{i}(\check{\alpha}+\check{\beta}+\check{\gamma}), \mathbf{i}(-\check{\alpha}+\check{\beta}+\check{\gamma}), \mathbf{i}(\check{\alpha}-\check{\beta}+\check{\gamma}), \mathbf{i}(\check{\alpha}+\check{\beta}-\check{\gamma})$$

have the same sign and their product is the positive real number

$$(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})(\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma}).$$

By direct computation, it is also equal to

$$\frac{(-1)(b-c)^2(b+c)^2(a-b)^2(a-c)^2(a+c)^2(a+b)^2}{a^3b^3c^3} = \frac{(\beta-\gamma)^2(\alpha-\beta)^2(\alpha-\gamma)^2}{\alpha\beta\gamma abc}$$

[15] This is just a determinant computation and has nothing to do with advanced area-measure theory.

6. By (vi), we have

$$
\begin{aligned}
(\breve{\alpha}+\breve{\beta}+\breve{\gamma})^2|\omega_{\sqrt{T}}|^2 =&(\breve{\alpha}\alpha+\breve{\beta}\beta+\breve{\gamma}\gamma)(\breve{\alpha}\frac{1}{\alpha}+\breve{\beta}\frac{1}{\beta}+\breve{\gamma}\frac{1}{\gamma})\\
=&\breve{\alpha}^2+\breve{\beta}^2+\breve{\gamma}^2+\breve{\alpha}\breve{\beta}(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})+\breve{\beta}\breve{\gamma}(\frac{\beta}{\gamma}+\frac{\gamma}{\beta})+\breve{\gamma}\breve{\alpha}(\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma})\\
=&(\breve{\alpha}+\breve{\beta}+\breve{\gamma})^2+\breve{\alpha}\breve{\beta}(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}-2)+\breve{\beta}\breve{\gamma}(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}-2)+\breve{\gamma}\breve{\alpha}(\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}-2)
\end{aligned}
$$

But

$$
\begin{aligned}
\breve{\alpha}\breve{\beta}(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}-2) =&\breve{\alpha}\breve{\beta}(\frac{\alpha}{\sqrt{\alpha\beta}}-\frac{\sqrt{\alpha\beta}}{\alpha})^2\\
=&\breve{\alpha}\breve{\beta}(\frac{\alpha-\beta}{\sqrt{\alpha\beta}})^2\\
=&-\breve{\alpha}\breve{\beta}\breve{\gamma}^2\\
\overset{(iii)}{=}&-4\varepsilon|S|\breve{\gamma}\\
\overset{(vi)}{=}&-2\varepsilon r_{\sqrt{T}}\breve{\gamma}|\breve{\alpha}+\breve{\beta}+\breve{\gamma}|
\end{aligned}
$$

where $\varepsilon=\mathrm{sign}(\breve{\alpha}\breve{\beta}\breve{\gamma})=\pm 1$. Summarizing, we have

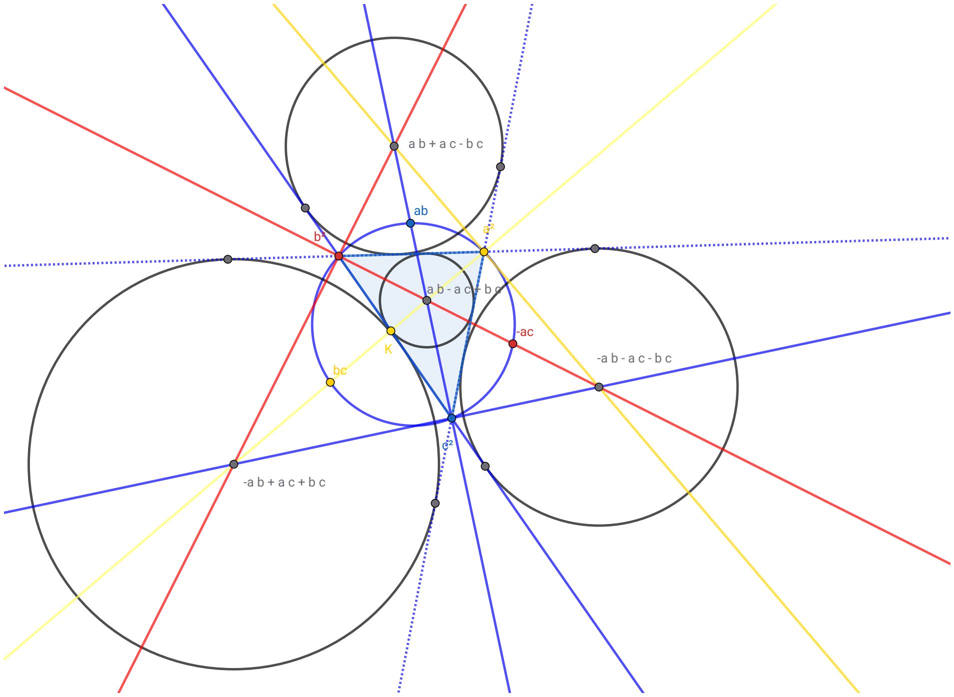
$$(\breve{\alpha}+\breve{\beta}+\breve{\gamma})^2|\omega_{\sqrt{T}}|^2=(\breve{\alpha}+\breve{\beta}+\breve{\gamma})^2-2\varepsilon r_{\sqrt{T}}(\breve{\alpha}+\breve{\beta}+\breve{\gamma})|\breve{\alpha}+\breve{\beta}+\breve{\gamma}|$$

hence

$$|\omega_{\sqrt{T}}|^2=1-2\varepsilon_{\sqrt{T}}r_{\sqrt{T}}\overset{\text{homogeneity}}{=}R^2-2\varepsilon_{\sqrt{T}}r_{\sqrt{T}}R \qquad \text{(ix)}$$

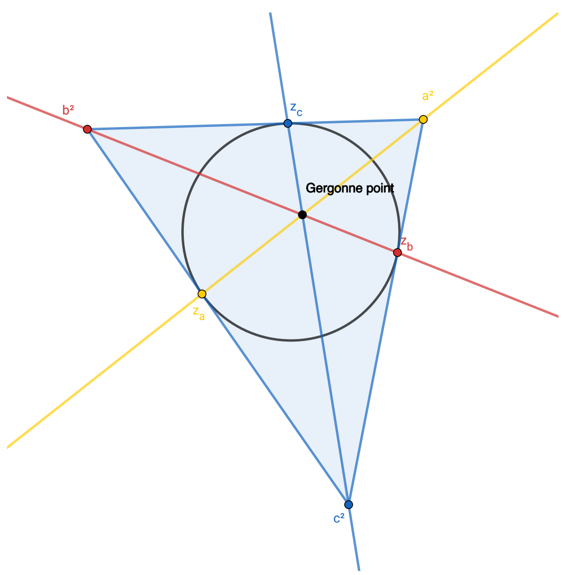
with $\varepsilon_{\sqrt{T}}=\mathrm{sign}(\breve{\alpha}\breve{\beta}\breve{\gamma}(\breve{\alpha}+\breve{\beta}+\breve{\gamma}))=\mathrm{sign}(\breve{\alpha}\breve{\beta}\breve{\gamma}(\breve{\alpha}+\breve{\beta}+\breve{\gamma})^3)$. By (4), $\varepsilon_{\sqrt{T}}$ is the product of the signs of the normalized barycentric coordinates of $\omega_{\sqrt{T}}$ and is equal to $+1$ for the incircle and -1 for the other three excircles. With this relation, (ix) is called the Euler formula.

Remark 3.1 *There is now way to choose continuously the roots to get the incenter because there is no continuous square root defined on* U. *In other words, the choice depends of the values of the arguments of* α,β,γ. *One way to do that is to define* $\alpha=\exp(\mathbf{i}2A),\dots$ *with* $0\leqslant A<B<C<\pi$ *and to define* $a=\exp(\mathbf{i}A),\dots$. *Then, the incircle is* $ab+bc-ca$*: indeed,* ab, bc *belong to the oriented arc* $\overset{\frown}{\alpha\beta}$, $\overset{\frown}{\beta\gamma}$ *because* $2A<A+B<2B<B+C<2C<2\pi$. *But* ca *does not belong to the arc* $\overset{\frown}{\gamma\alpha}$ *because* $2\pi>C+A>2A$, *hence the choice of* $-ca$ *of argument* $A+C-\pi<A$. *Its radius is denoted by* r. *Moreover, the formula (see iv)* $\tilde{\gamma}^2=-\frac{(a^2-b^2)^2}{a^2b^2}$ *shows* $\alpha-\beta=\pm\mathbf{i}\tilde{\gamma}ab$. *By note 2, if we change the sign of* $ab=\sqrt{\alpha\beta}$, *the corresponding, the barycentric coordinates* x_γ *is changed to its opposite. By the normalization condition, two barycentric coordinates change in their opposite : the four incircles are either inside* T *(radius* r*) or inside exactly one pair of edges from a given vertex* $\tau\in T$ *with radius* r_τ *as in the picture above.*



4 Gergonne-Nagel points

Let $\omega = \omega_{\sqrt{T}} = ab + bc - ca$ be the center of one of the tangent circle $C = C_{\sqrt{T}}$ and let $\ell_c = \langle z_1, z_2 \rangle$ the line passing through ω and the touchpoint z_c of $C \cap [\alpha, \beta]$.



Lemma 4.1 *We have* $z_c = \frac{1}{2}(a^2 + b^2 + s_c)$ *with* $s_c = (-a^2b + ab^2 - ac^2 + bc^2)/c$.

Proof. Let us compute the equation of ℓ_c which is $t+p_c\bar{t}=s_c$ with $p_c=z_1z_2$ and $s_c=z_1+z_2$ of ℓ_c. But ℓ_c is characterized by passing through ω and being orthogonal to (α,β). By (2) of (2.1), the normal line of $\langle\alpha,\beta\rangle$ passing through α say is $\langle\alpha,-\beta\rangle$ and therefore has equation $t-\alpha\beta\bar{t}=\alpha-\beta$. Because it is parallel to $\langle z_1,z_2\rangle$, we get $p_c=-\alpha\beta=-a^2b^2$. Because ω belongs to this line, we get[16] $s_c=(-a^2b+ab^2-ac^2+bc^2)/c$. But z_c is the solution of

$$\begin{cases} t-a^2b^2\bar{t} &= s_c \\ t+a^2b^2\bar{t} &= a^2+b^2 \end{cases}$$

hence[17] $z_c=\frac{1}{2}(a^2+b^2+s_c)$. □

The equation of the "Ceva" line L_c through c^2 and z_c is

$$t+c^2Z_c\bar{t}=c^2+Z_c \text{ with } Z_c=\frac{c^2-z_c}{c^2\bar{z}_c-1}$$

remembering

$$\begin{aligned}
\bar{z}_a &=(-a^2b+a^2c+ab^2+ac^2-b^2c+bc^2)/(2ab^2c^2)\\
\bar{z}_b &=(a^2b+a^2c-ab^2+ac^2-b^2c+bc^2)/(2a^2bc^2)\\
\bar{z}_c &=(a^2b+a^2c-ab^2+ac^2+b^2c-bc^2)/(2a^2b^2c)
\end{aligned}$$

are just obtained from z_a,z_b,z_c by inverting a,b,c. The intersection point of $L_c\cap L_b$ and $L_c\cap L_a$ are

$$\xi=\frac{(Z_cc^2)(Z_b+b^2)-(Z_bb^2)(Z_c+c^2)}{Z_cc^2-Z_bb^2},\ \xi'=\frac{(Z_cc^2)(Z_a+a^2)-(Z_aa^2)(Z_c+c^2)}{Z_cc^2-Z_aa^2}$$

.

Proposition 4.2 (Gergonne-Nagel points) *The three lines between the touchpoint of the tangent circle $C_{\sqrt{T}}$ with an edge of T and its opposite vertex meet in the Gergonne-Nagel point $\Gamma_{\sqrt{T}}$.*

Proof. Just check[18] $\xi=\xi'$ by a (very) tedious straightforward inspection or better using a computer. Or see the discussion below. □

[16] And analogously for the equations of ℓ_a,ℓ_b with

$$p_a=-b^2c^2, p_b=-c^2a^2 \text{ and } s_a=(a^2b-a^2c+b^2c-bc^2)/a, s_b=(-a^2c+ab^2-ac^2+b^2c)/b$$

[17] And analogously $z_a=\frac{1}{2}(b^2+c^2+s_a)$ and $z_b=\frac{1}{2}(c^2+a^2+s_b)$

[18] We could certainly simplify the computation but our goal was elsewhere. We would like to convince the reader that he method itself is crystal clear and, at least if we accept to use XXIth century methods, there is no mystery to prove the result.

[1]:
```
from sage.all import *
R.<a,b,c> = PolynomialRing(QQ, order='lex')
#center
m=a*b+b*c-c*a
#conjugate of the center
n=m(1/a,1/b,1/c)
#touchpoints
sc=m-a^2*b^2*n
sa=m-b^2*c^2*n
sb=m-c^2*a^2*n
zc=(sc+a^2+b^2)/2
zb=(sb+a^2+c^2)/2
za=(sa+b^2+c^2)/2
#equation of the lines beween contact point and opposite vertex
Zc=(c^2-zc)/((c^2)*zc(1/a,1/b,1/c)-1)
Za=(a^2-za)/((a^2)*za(1/a,1/b,1/c)-1)
Zb=(b^2-zb)/((b^2)*zb(1/a,1/b,1/c)-1)
#intersection between L_a an L_c
N=(Zc*c^2)*(Za+a^2)-(Za*a^2)*(Zc+c^2)
D=Zc*c^2-Za*a^2
#intersection between L_b an L_c
N1=(Zc*c^2)*(Zb+b^2)-(Zb*b^2)*(Zc+c^2)
D1=Zc*c^2-Zb*b^2
#comparison of the intersection points
N/D==N1/D1
```

[1]: True

It is easy to be more precise by cyclically defining the six *real* numbers

$$\lambda_{a,c} = \frac{z_a - b^2}{ibc}, \lambda_{a,b} = \frac{z_a - c^2}{ibc}, \lambda_{b,a} = \frac{z_b - c^2}{ica}, \lambda_{b,c} = \frac{z_b - a^2}{ica}, \lambda_{c,b} = \frac{z_c - a^2}{iab}, \lambda_{c,a} = \frac{z_c - b^2}{iab}$$

One has

$$(\lambda_{a,c} - \lambda_{a,b})z_a = \lambda_{a,c}c^2 - \lambda_{a,b}b^2 \text{ with } \lambda_{a,c} - \lambda_{a,b} = \frac{c^2 - b^2}{-ibc} \neq 0$$

showing that[19]

$$(0, -\lambda_{a,b}, \lambda_{a,c}), (\lambda_{b,a}, 0, -\lambda_{b,c}), (-\lambda_{c,a}, \lambda_{c,b}, 0)$$

are barycentric coordinates of z_a, z_b, z_c respectively. The barycentric equation[20] of the Ceva line L_a is therefore $\lambda_{a,c}B + C\lambda_{a,b}C = 0$ and analogously for L_b, L_c by cyclic symmetry. A direct computation shows

$$\det \begin{pmatrix} 0 & \lambda_{a,b} & \lambda_{a,c} \\ \lambda_{b,a} & 0 & \lambda_{b,c} \\ \lambda_{c,a} & \lambda_{c,b} & 0 \end{pmatrix} = \lambda_{a,b}\lambda_{b,c}\lambda_{c,a} + \lambda_{a,c}\lambda_{b,a}\lambda_{c,b} = 0$$

[19] Using cyclic symmetry

[20] (A, B, C) being the barycentric coordinates of a generic point.

giving another proof[21] of the existence of the Gergonne-Nagel point. Moreover the formula

$$\mathbf{R}.\mathbf{i}(\frac{b+c}{b-c}, \frac{c-a}{c+a}, \frac{a+b}{a-b}) \subset \mathbf{R}^3$$

for the kernel of the above (real) matrix gives

$$(\mathbf{i}\frac{b+c}{b-c}, \mathbf{i}\frac{c-a}{c+a}, \mathbf{i}\frac{a+b}{a-b})$$

for the barycentric coordinates[22] of $\Gamma_{\sqrt{T}}$ hence the classical formula in terms of the values of the tangent function at the half triangle angles.

5 Simpson line

The altitude $\langle \alpha, z_\alpha \rangle$ is characterized by the vanishing of the scalar product[23] $z_\alpha - a$ and $\beta - \alpha$. We get the equation $(z_\alpha - \alpha)(1\beta - 1/\alpha) + (1/z_\alpha - 1_\alpha)(\beta - \alpha) = 0$ hence[24] $z_\alpha = -\beta\gamma/\alpha \in U$. The equation of the altitude is therefore

$$t - \beta\gamma\bar{t} = \alpha - \beta\gamma/\alpha$$

and the intersection h of two altidudes verifies

$$\begin{cases} h - \beta\gamma\overline{h} & = & \alpha - \beta\gamma/\alpha \\ h - \gamma\alpha\overline{h} & = & \beta - \gamma\alpha/\beta \end{cases}$$

hence

$$h = \alpha + \beta + \gamma$$

and by symmetry the three altitudes meet in the so called orthocenter h. We recover that circumcenter 0, the orthocenter h and the centroid $g = \frac{1}{3}h$ are collinear points with the relation $3\overrightarrow{0g} = \overrightarrow{0, h}$. The corresponding line is called the Simpson line.

6 Euler circle

The foot h_α of the altitude is the intersection altitude $\langle \alpha, z_\alpha \rangle$ and the base $\langle \beta\gamma \rangle$ hence satisfies

$$\begin{cases} h_\alpha + \beta\gamma\overline{h}_\alpha & = & \beta + \gamma \\ h_\alpha - \beta\gamma\overline{h}_\alpha & = & \alpha - \beta\gamma/\alpha \end{cases}$$

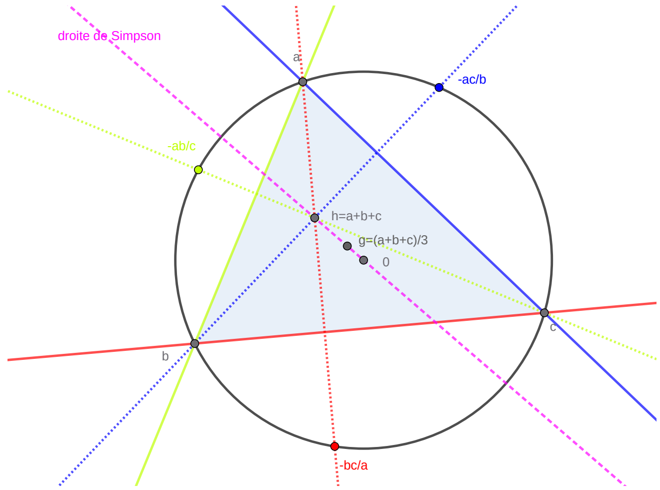
hence $h_\alpha = \frac{1}{2}h - \frac{1}{2}\beta\gamma/\alpha/$

[21]This is one form of the so called Ceva's theorem in classical geometry.

[22]The sum of these coefficients is nonzero because α, β, γ is an affine frame

[23](at least if $z_\alpha \neq \alpha$ geometrically meaning that the tangent line $T_\alpha = \langle \alpha, \alpha + \mathbf{i}\alpha \rangle$ is not orthogonal to the base line $\langle \beta\gamma \rangle$.

[24]Proving also that the condition $\alpha = -\beta\gamma/\alpha$ means $T_\alpha \perp \langle \beta\gamma \rangle$.

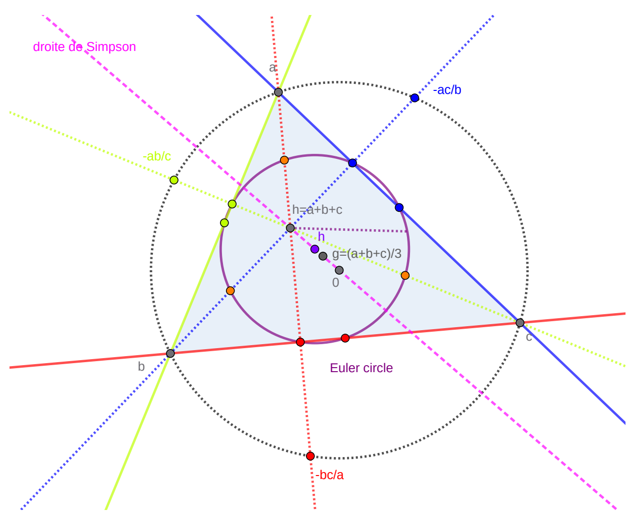


The foot of the median from α is $(\beta+\gamma)/2$.

The midpoint of $[h, \alpha]$ is $(h+\alpha)/2 = \alpha + \beta/2 + \gamma/2$.

Proposition 6.1 (Euler nine points circle) *The circle of center $h/2$ (half the orthocenter) and radius $\frac{1}{2}$ passes through the nine preceding points: the three median foots, the three altitudes foots, the three midpoints between the orthocenter and the vertices of* T.

Proof. Just write $(\beta+\gamma)/2 - h/2 = \alpha/2,\ h_\alpha - h/2 = -\frac{1}{2}\beta\gamma/\alpha,\ (h+\alpha)/2 - h/2 = \alpha/2$ and $|\gamma/2| = |-\frac{1}{2}\beta\gamma/\alpha| = |\alpha/2| = \frac{1}{2}$. □



7 Feuerbach theorem

We use notations and computations of §3 and §6.

Proposition 7.1 (Feuerbach) *The Euler circle is tangent to the four tangent circles of* T.

The algebraic tool is this high school elementary lemma (intersection of two circles).

Lemma 7.2 *Let* C_1, C_2 *two circles of radius* R_1, R_2 *distance* d *between their centers. Then* C_1 *and* C_2 *are tangent if and only if* d *is equal to* $R_1 + R_2$ *or to* $|R_1 + R_2|$ *or equivalently*[25] *if*

$$(d^2 - (R_1 + R_1)^2)(d^2 - (R_1 - R_2)^2) = R_1^4 + R_2^4 + d^4 - 2R_1^2R_2^2 - 2R_1^2d^2d - 2R_2^2d^2 + = 0$$

.

Proof. Our circles are tangent if and only if they meet in a unique point. In a suitable orthonormal frame, the equations of the circles are $x^2 + y^2 = R_1^2$ and $(x - d)^2 + y^2 = R_2^2$. An intersection point (x, y) satisfies

$$x = \frac{R_1^2 - R_2^2}{2d} \text{and } y^2 = -\frac{(d^2 - (R_1 - R_2)^2)(d^2 - (R_1 + R_2)^2)}{4d^2}$$

from which the lemma follow (and also the usual condition for a nonempty intersection of two circles: $|R_1 - R_2| \leqslant d \leqslant R_1 + R_2$). $\square$

By (viii) and (iii), we have

$$r_{\sqrt{T}}^2 = \frac{\tilde{\alpha}^2\tilde{\beta}^2\tilde{\gamma}^2}{4|\check{\alpha} + \check{\beta} + \check{\gamma}|^2}$$

Using

$$|\check{\alpha} + \check{\beta} + \check{\gamma}|^2 \overset{(vii)}{=} -\frac{(b+c)^2(c-a)^2(a+b)^2}{\alpha\beta\gamma} \text{ and } \tilde{\alpha}^2\tilde{\beta}^2\tilde{\gamma}^2 \overset{(iv)}{=} -\frac{(\beta-\gamma)^2}{\beta\gamma}\frac{(\gamma-\alpha)^2}{\gamma\alpha}\frac{(\alpha-\beta)^2}{\alpha\beta}$$

we get

$$r_{\sqrt{T}}^2 = \frac{(b-c)^2(c+a)^2(a-b)^2}{4a^2b^2c^2} \tag{x}$$

Moreover, we have $\omega_{\sqrt{T}} - h = -\frac{1}{2}(a - b + c)^2$ hence

$$|\omega_{\sqrt{T}} - h|^2 = \frac{1}{4}(a - b + c)^2(1/a - 1/b + 1/c)^2 \tag{xi}$$

Using the above lemma with $R_1 = r_{\sqrt{T}}, R_2 = \frac{1}{2}$ and $d = |\omega_{\sqrt{T}} - h|$, the Feuerbach theorem follows from a straightforward computation[26] using (x) and (xi).

[25] In terms of the algebraically computable squares of radius or distance.

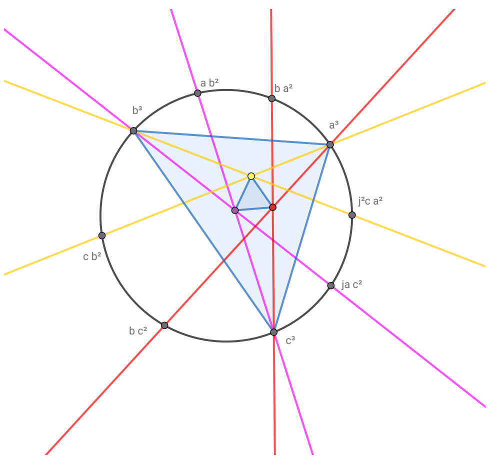
[26] Just expending the relevant polynomial expression in a, b, c or in a more modern way using a formal computation software like SAGEMATH as in above picture.

```
[13]: from sage.all import *
      R.<x,y,z> = PolynomialRing(QQ, order='lex')
      d=(((x-y+z)^2)*((1/x-1/y+1/z)^2))/4
      S=(((x-y)*(y-z)*(z+x))^2)/(4*(x^2)*(y^2)*(z^2))
      r=1/4
      #d,S,r are the square of the distances, radius
      S^2 - 2*S*r - 2*S*d + r^2 - 2*r*d + d^2==0
```

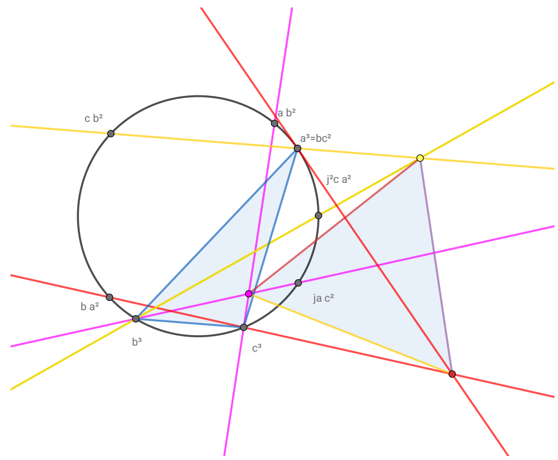
```
[13]: True
```

8 Weak Morley theorem

Let $\varepsilon \in \{0, 1\}$ and $j = \exp(\pm\frac{2i\pi}{3})$ a primitive third root of 1. We choose third roots a, b, c of the vertices. Morley theorem (as shown in the picture below) follows from the following simple observations



Non tangent case $a^3 \neq bc^2$



Tangent case $a^3 = bc^2$

1. The coloured lines are trisectors (inscribed angle theorem) where a, b, c are arbitrary third roots of α, β, γ. In particular, the intersection point between the two trisectors $\langle b^3, ac^2\rangle$ and $\langle c^3, ab^2\rangle$ is

$$z_a = -bc(b + c) + a(b^2 + bc + c^2)$$

   and an analogous formulas for the two others.

2. From (1), we get $z_a + jz_b + j^2 z_c = 0$ with the vertices as in the picture above by direct calculation (12.2).

finishing the proof[27]. We now would like to take this approach a step further to shed light on the algebro-geometric situation.

[27]Of course, this argument is very close to the one of Connes (A. Connes, A new proof of Morley's theorem, in *Les relations entre les mathématiques et la physique théorique*, 43–46, Inst. Hautes Études Sci.) but seems to me much more natural and perhaps more elementary

Remark 8.1 *In we want internal trisectors, we proceed in the same way as for bisectors (3.1) by choosing in this case* $0 \leqslant A < B < C < 2\pi/3$ *defining cubic roots* $a = \exp(iA), \dots$ *of* $\alpha, \dots$. *The internal trisectors are* $a^2b, ab^2, b^2c, bc^2, jc^2a, j^2ca^2$ *for the same argument reasons. But as in the bisector case, there is no continuous choice with respect to the vertices to do that.*

Part II

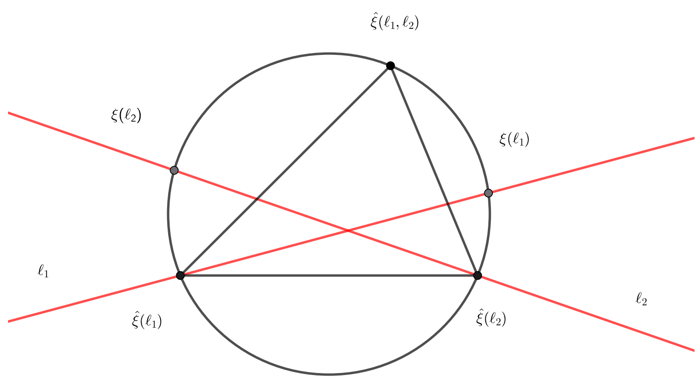
Complement: counting Morley triangles

9 Admissible pair of lines (from vertices of T)

Rotating the triangle around 0 if necessary[28]

$$\boxed{\text{we now normalize}^{29}\ T \text{ by assuming } \alpha\beta\gamma = 1.}$$

A line ℓ of T is a line $\ell = \langle \widehat{\xi}(\ell), \xi(\ell) \rangle$ where $\widehat{\xi}(\ell) \in T$ and $\xi(\ell)$ is the unique point of $T \cap U - \{\widehat{\xi}(\ell)\}$.



Definition 9.1

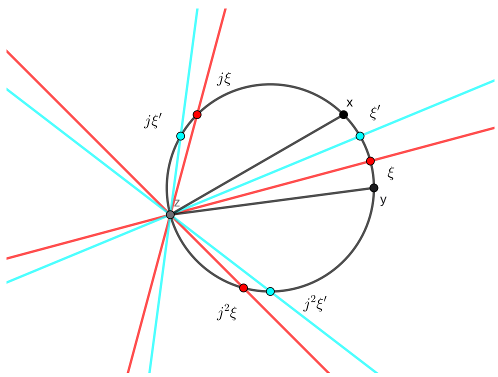
- *We say that a pair*[30] *of two such lines* $\lambda = (\ell_1, \ell_2)$ *is an admissible pair if* $\widehat{\xi}(\ell_1) \neq \widehat{\xi}(\ell_2)$ *and we define* $\widehat{\xi}(\lambda)$ *as the missing point:* $\widehat{\xi}(\lambda)$ *is the unique point of* $T - \{\widehat{\xi}(\ell_1), \widehat{\xi}(\ell_2)\}$.
- *For any such admissible pair, we define* $[\lambda]_\varepsilon = \dfrac{\xi(\ell_1)\xi(\ell_2)}{\widehat{\xi}(\lambda)^\varepsilon}$.

[28] By the inverse of a third root of $\alpha\beta\gamma$.

[29] This is certainly not essential. But this avoid powers of $\alpha\beta\gamma$ in the coming formulas.

10 Trisectors

We classically define the three trisector points of the oriented angle $(\widehat{(y,z,x)})$ as one of the 3 points $\xi \in U - \{z\}$ such that we have the equality of oriented angles $3(\widehat{y,z,\xi}) = (\widehat{y,z,x}) \mod 2\pi$ which defines 3 points of U well defined up to μ_3. If we look at the opposite angle $(\widehat{x,z,y})$, we get 3 other points ξ': the trisector points of $(\widehat{x,z,y})$ or the 6 trisector points[31] of z for short[32].



Three facts about trisectors:

Lemma 10.1 *Let* $T = \{x^3, y^3, z^3\}$.

1. *The trisectors* ℓ *of* T *are the lines* $\ell_{x,y} = \langle z^3, xy^2 \rangle$ *and* $\widehat{\xi}(\ell_{x,y}) = z^3$, $\xi(\ell_{x,y}) = xy^2$.
2. $\ell_{x,y} = \ell_{X,Y}$ *if and only if* $X = ux$, $Y = uy$ *for some* $u \in \mu_3$.
3. *They are* 6 *distinct trisectors* ℓ *passing through* $z^3 = \xi(\ell)$ *characterized by*

$$\widehat{\xi}(\ell) \in \{xy^2, jxy^2, j^2xy^2, x^2y, jx^2y, j^2x^2y\}$$

and $3 * 6 = 18$ *distinct trisectors in total.*

Proof.

1. Observe that the cube of the rotation $t \mapsto \frac{x}{y}t$ maps y^3 to x^3, we have $3(\widehat{y^3,0,xy^2}) = (\widehat{y^3,0,x^3}) \mod 2\pi$.

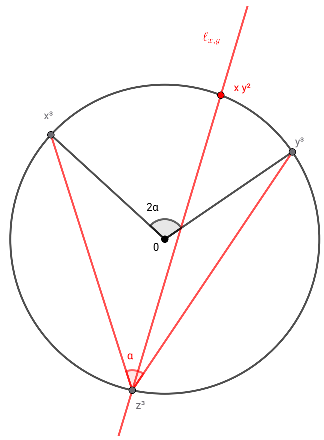
   If $z^3 \neq xy^2$, the inscribed angle theorem (or a direct complex computation) shows that xy^2 is one of the three trisector points of the oriented angle $(\widehat{y^3,z^3,x^3})$. Using the natural action of μ_3, we conversely

[30] By pair we mean unordered pair, or equivalently a cardinal two set.

[31] Unless in case of a right angle in which case we have only 2 trisectors.

[32] Another way to define a trisector point is to say line ℓ is a trisector of is to say that the different angles have to be understood as geometric angle of lines -which are defined up to sign and up to $\mod \pi$.

obtain that we get this way all the trisectors. If $z^3 \neq xy^2$ a continuity argument shows that the tangent line $\langle a^3, bc^2 \rangle$ is still a trisector.



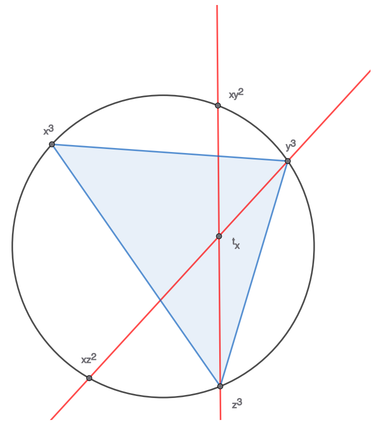
2. We have $z^3 = \widehat{\xi}(\ell_{x,y}) = \widehat{\xi}(\ell_{X,Y}) = Z^3$ hence $\{x^3, y^3\} = \{X^3, Y^3\}$. Moreover, we have $xy^2 = \xi(\ell_{x,y}) = \xi(\ell_{X,Y}) = XY^2$ hence $x^3y^6 = X^3Y^6$.
   - If $x^3 = Y^3$ we get $y^3 = X^3$ and $x^3y^6 = X^3Y^6 = y^3x^6$ hence $y^3 = x^3$, a contradiction.
   - If $x^3 = X^3$, we have $x = uX$ for some $u \in \mu^3$ and $y^3 = Y^3$. We get $XY^2 = xy^2 = uXy^2$ hence $Y^2 = uy^2$ and $Y^2/Y^3 = uy^2/y^3$ hence $y = uY$.
3. Immediate consequence of (1) and (2).

□

11 Admissible pairs (of trisectors) of type ε

Definition 11.1 *Let $\lambda = (\ell_1, \ell_2)$ be a pair of trisectors, $\widehat{\xi}(\lambda) = \widehat{\xi}(\ell_1, \ell_2)$ the "missing vertex" and $x, y, z \in U$ such that $\{x^3, y^3, z^3\} = T$.We say that*

- *We say that λ is an* admissible pair *of type ε (of trisectors)*[33] *if $[\lambda]_\varepsilon = [\ell_1, \ell_2]_\varepsilon \in \mu_3$.*
- *Assume $\varepsilon = 0$. We define $\lambda = \lambda^{\varepsilon}_{x,y,z} = (\ell_{x,y}, \ell_{x,z})$.*
- *Assume $\varepsilon = 1$. We define $\lambda = \lambda^{\varepsilon}_{x,y,z} = (\ell_{x,y}, \ell_{z,y})$.*



$\lambda^0_{x,y,z}$

Lemma 11.2 *Let $x, y, z \in U$ such that $\{x^3, y^3, z^3\} = T$ and $\lambda = \lambda^\varepsilon_{x,y,z}$.*

1. *We have $[\lambda_{x,y,z}]_\varepsilon = (xyz)^{2-\varepsilon}$ hence $\lambda^\varepsilon_{x,y,z}$ is an admissible pair. Moreover we have $\widehat{\xi}(\lambda^\varepsilon_{x,y,z}) = x^3$ if $\varepsilon = 0$ and $\widehat{\xi}(\lambda^\varepsilon_{x,y,z}) = y^3$ if $\varepsilon = 1$.*

2. *Let X, Y, Z such that $\{X^3, Y^3, Z^3\} = T$ and $\lambda^\varepsilon_{X,Y,Z} = \lambda^\varepsilon_{x,y,z}$. Then, there exists $u \in \mu_3$ such that $(X, Y, Z) = u(x, y, z)$.*

3. *The stabilizer of λ under $(\mu_3)^2$ is μ_3 hence the orbit $(\mu_3)^3/\mu_3$ of $\lambda^\varepsilon_{x,y,z}$ has 9 elements.*

4. *(*Complement*) Assume that T is neither isosceles nor rectangle and let λ be an* admissible pair *of type ε. Then, there exists $x, y, z \in U$ such that $\{x^3, y^3, z^3\} = T$ and $\lambda = \lambda^\varepsilon_{x,y,z}$.*

Proof.

1. Direct computation.

2. We have

   - $\{x^3, y^3, z^3\} = \{X^3, Y^3, Z^3\}$.
   - $z^3 = \widehat{\xi}(\lambda) = Z^3$ hence $Z = uz$ for some $u \in \mu_3$.
   - $\{x^3, y^3\} = \{X^3, Y^3\}$ hence $x^3y^3 = X^3Y^3$
   - $xy^2 = [\lambda]_\varepsilon = XY^2$ hence $x^3y^6 = X^3Y^6$.

   From the last two equalities we get $y^3 = Y^3$ and finally $x^3 = Z^3$ or equivalently

$$X = ux, Y = vy, Z = wz \text{ with } u, v, w \in \mu_3.$$

[33]For $\varepsilon = 0$, this is the algebraic notion coming from the adjacent trisectors of a given edge, see picture below.

From $xy^2 = XY^2$, we get $uv^2 = 1$ and λ being admissible, we get $x^2y^2z^2 = [\lambda]_\varepsilon = X^2Y^2Z^2$ hence $u^2v^2w^2 = 1$. Because u, v, w are of exponent 3, we get hence $u = v = w$.

3. Direct consequence of (2).

4. We make the (tedious but straightforward) computation for $\varepsilon = 0$, the case $\varepsilon = 1$ being analogous. By the above lemma 2.1, our hypothesis means that for any distinct vertices of T, one has $\alpha + \beta \neq 0$ and $\alpha^2 \neq \beta\gamma$.

   - Let $\lambda = (\ell_1, \ell_2)$ be an admissible pair of type $\varepsilon = 0$ and choose thanks to the above lemma $x, y, z \in U$ such that $z^3 = \widehat{\xi}(\ell_1)$ and $\ell_1 = \ell_{x,y}$. We have $\xi(\ell_1) = xy^2$ and $T = \{x^3, y^3, z^3\}$ and $x^3y^3z^3 = 1$.
   - If $\widehat{\xi}(\ell_2) = z^3$, we have $\xi(\ell_2) = uxy^2$ or $\xi(\ell_2) = ux^2y$ for some $u \in \mu_3$. This implies $[\lambda]_\varepsilon = ux^2y^4$ or $[\lambda]_\varepsilon = ux^3y^3$ whose cube are x^6y^{12} or x^9y^9 respectively. If $x^9y^9 = x^6y^6z^6$, we get $x^3y^3 = z^6$, a contradiction. If $x^6y^{12} = x^6y^6z^6$, we get $y^6 = z^6$ hence $y^3 = -z^3$, a contradiction.
   - If $\widehat{\xi}(\ell_2) = x^3$, we have analogously $\xi(\ell_2) = uyz^2$ or $\xi(\ell_2) = uy^2z$ for some $u \in \mu_3$. This would imply $[\lambda]_\varepsilon = uxy^3z^2$ or $[\lambda]_\varepsilon = uxy^4z$ whose cube are $x^3y^9z^6$ and $x^3y^{12}z^3$. If $x^3y^9z^6 = x^6y^6z^6$, we get $y^3 = x^3$ a contradiction. If $x^3y^{12}z^3 = x^6y^6z^6$, we get $y^6 = x^3z^3$, a contradiction.
   - We have therefore $\widehat{\xi}(\ell_2) = y^3$ and $\xi(\ell_2) = uxz^2$ or $\xi(\ell_2) = ux^2z$ for some $u \in \mu_3$. In the second case, $[\lambda]_\varepsilon = ux^3y^2z$ whose cube is $x^9y^6z^3$ which cannot be equal to $x^6y^6z^6$ because $x^3 \neq z^3$. Therefore, $\xi(\ell_2) = uxz^2$ and changing z to uz finishes the proof.

$\square$

Corollary 11.3 *Let x, y, z such that $\{x^3, y^3, z^3\} = T$ and $\lambda = \lambda^\varepsilon_{x,y,z} = (\ell_1, \ell_2)$ be an admissible pair of type ε.*

1. *If $\varepsilon = 0$, the intersection point of $\ell_1 \cap \ell_2$ is*
$$t_\varepsilon(\lambda) = -yz(y + z) + x(y^2 + yz + z^2) = xy^2 + xyz + xz^2 - y^2z - xz^2.$$
2. *If $\varepsilon = 1$ and T not rectangle at y^3, the unique intersection point of $\ell_1 \cap \ell_2$ is*
$$t_\varepsilon(\lambda) = \frac{-x^2z^2 + y^2(x^2 + xz + z^2)}{x + z}.$$
3. *If $\varepsilon = 1$ and T rectangle at y^3, for every $u \in \mu_3$, the two lines of $\lambda^\varepsilon_{x,y,\omega ux}$ are parallel and distinct and the 6 other pairs missing y^3 meets.*

Proof.

1. Just solve $\begin{cases} t + xy^2z^3\bar{t} &= xy^2 + z^3 \\ t + xz^2y^3\bar{t} &= xz^2 + y^3 \end{cases}$

2. Just solve $\begin{cases} t + xy^2z^3\bar{t} &= xy^2 + z^3 \\ t + xz^2y^3\bar{t} &= xz^2 + y^3 \end{cases}$ using $x + z \neq 0$ by 2.1.

3. We have $x^3 + z^3 = 0$ by (2.1) hence $z = ux, u \in \mu_3$. We get

$$-x^2z^2 + y^2(x^2 + xz + z^2) = -u^2x^4 + 3u^2y^2z^2 \neq 0$$

because both the modules of u^2x^4 and $u^2y^2z^2$ are equal to 1.

$\square$

12 Morley triangles of type ε

Definition 12.1 *Let* x, y, z *such that* $\{x^3, y^3, z^3\} = T$.

- *If* $\varepsilon = 0$, *we define*

$$\underline{\lambda}^{\varepsilon}_{x,y,z} = (\lambda^{\varepsilon}_{x,y,z}, \lambda^{\varepsilon}_{j^2z,j^2x,y}, \lambda^{\varepsilon}_{jy,jx,z}) \text{ and } \underline{t}^{\varepsilon}_{x,y,z} = (t^{\varepsilon}_{x,y,z}, t^{\varepsilon}_{j^2z,j^2x,y}, t^{\varepsilon}_{jy,jx,z}).$$

- *If* $\varepsilon = 1$ *and moreover* $x + z \neq 0$ if T rectangle at y^3, *we define*

$$\underline{\lambda}^{\varepsilon}_{x,y,z} = (\lambda^{\varepsilon}_{x,y,z}, \lambda^{\varepsilon}_{j^2x,y,j^2z}, \lambda^{\varepsilon}_{jx,y,jz}) \text{ and } \underline{t}^{\varepsilon}_{x,y,z} = (t^{\varepsilon}_{x,y,z}, t^{\varepsilon}_{j^2x,y,j^2z}, t^{\varepsilon}_{jx,y,jz}).$$

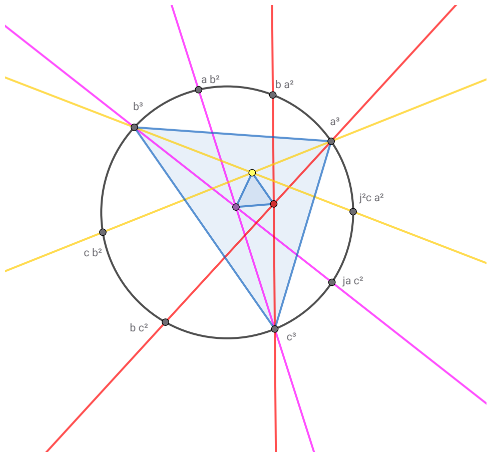
We say that $\underline{t}^{\varepsilon}_{x,y,z}$ *is a Morley triangle of type*[34] (ε, j).

The main property of the triples $(\lambda_0, \lambda_1, \lambda_2)$ associated to a Morley triple is the equality

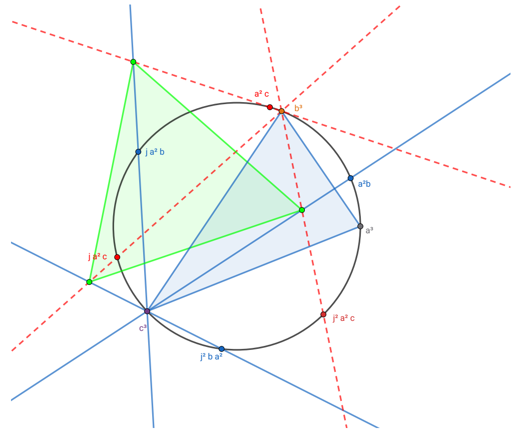
$$\{[\lambda_0]_\varepsilon, [\lambda_1]_\varepsilon, [\lambda_2]_\varepsilon\} = \mu_3.$$

By construction, the 6 trisectors of a type one triple miss the vertex y^3 and none point for a type zero triple, hence the notation.

[34] Of course, all the preceding constructions depend on the choice of j as one of the two primitive third root of 1.



Morley triangle of type $\varepsilon = 0$



Morley triangle of type $\varepsilon = 1$

Proposition 12.2 (The 27 Morley triangles) *Let* X, Y, Z *such that* $\{X, Y, Z\} = T$ *and*

$$\text{Morley}_{\varepsilon,j} = \{t^{\varepsilon}_{x,y,z},\ x^3 = X, y^3 = Y, z^3 = Z \text{ and } x + z \neq 0\}$$

be the associated set of Morley triangles.

1. *Any Morley triangle is non degenerate and equilateral.*
2. *The cardinality of* $\text{Morley}_{\varepsilon,j}$.
   (a) 3 *or* 2 *whether* T *is rectangle at* y^3 *or not and* $\varepsilon = 1$.
   (b) 9 *if* $\varepsilon = 0$.
3. *Dependence on* j.
   (a) *If* $\varepsilon = 0$, *we have* $\text{Morley}_{0,j} \cap \text{Morley}_{0,j^2} = \varnothing$ *(*9 *new triangles).*
   (b) *If* $\varepsilon = 1$ *we have* $\text{Morley}_{1,j} = \text{Morley}_{\varepsilon,j^2}$ *(no new triangles).*
4. *There is no triangle of both type* 0 *and* 1.
5. *There is in total 2*9+9=27 Morley triangles if* T *is not rectangle and 2*9+6=24 triangle otherwise.*

Proof.

1. We use the formulas 11.3.

(a) Assume first $\varepsilon = 0$.

$$\begin{aligned} t^{\varepsilon}_{x,y,z} + j t^{\varepsilon}_{j^2z,j^2x,y} + j^2 t^{\varepsilon}_{jy,jx,z} &= xy^2 + xyz + xz^2 - y^2z - yz^2 \\ &\quad \underline{-j^2x^2y} + \underline{jx^2z} - xy^2 + j^2xyz + y^2z \\ &\quad \underline{+j^2x^2y} - \underline{jx^2z} + jxyz - xz^2 + yz^2 \\ &= (1+j+j^2)xyz \\ &= 0 \end{aligned}$$

hence is equilateral. If this triangle is reduced to a single point τ, we have

$$\tau \in \ell_{x,z} \subset \lambda^{\varepsilon}_{x,y,z} \text{ and } \tau \in \ell_{j^2z,j^2x} \subset \lambda^{\varepsilon}_{j^2z,j^2x,y},$$

both lines being distinct (10.1) trisectors passing through y^3. This implies $\tau = y^3$ and by symmetry $T = \{\tau\}$, a contradiction.

(b) Assume now $\varepsilon = 1$ and $x + z \neq 0$ if T rectangle at y^3.

$$\begin{aligned} (x+z)t^{\varepsilon}_{x,y,z} &= x^2y^2 - x^2z^2 + xy^2z + y^2z^2 \\ (x+z)t^{\varepsilon}_{j^2x,y,j^2z} &= jx^2y^2 - x^2z^2 + jxy^2z + jy^2z^2 \\ (x+z)jt^{\varepsilon}_{jx,y,jz} &= j^2x^2y^2 - x^2z^2 + j^2xy^2z + j^2y^2z^2 \end{aligned}$$

gives again

$$t^{\varepsilon}_{x,y,z} + j t^{\varepsilon}_{jx,y,jz} + j^2 t^{\varepsilon}_{j^2x,jx,j^2z} = 0$$

More precisely $\underline{t}^{\varepsilon}(x,y,z)$ is the equilateral triangle

$$\underline{t}^{\varepsilon}(x,y,z) = -\frac{x^2z^2}{x+z} + \rho\{1,j,j^2\} \text{ with } \rho = \frac{x^2y^2 + xy^2z + y^2z^2}{x+z} = y^2\frac{x^3-z^3}{x^2-z^2} \neq 0 \qquad \text{(xii)}$$

hence is non degenerate.

2.

(a) Assume $\varepsilon = 1$ and let $d = 3$ (if T not rectangle at Y) and $d = 2$ (if T rectangle at y). The centroids of Morley triangles are of the form $-\frac{-x^2z^2}{x+z}$ with $x + z \neq 0$ and $x^3 = X, z^3 = Z$. Fixing such x, y, under the action of $(\mu_3)^2$ on x, y, we get at most d values for the Morley triangles

$$-\frac{x^2z^2}{x+z}, -\frac{j^2x^2z^2}{x+jz}, -\frac{jx^2z^2}{x+j^2z}.$$

But

$$\frac{x^2z^2}{x+z} \neq \frac{j^2x^2z^2}{x+jz}$$

for $x, z \in U$ because $x + jz = j^2(x+z)$ if and only if $(1-j)(1+j)x = j(j-1)z$ which is impossible ($|1+j| \neq 1$). This argument says also that $(ux, vy, wz), w \in (\mu_3)^3$ define the same centroid if and only if $u = w$. The associated ρ is therefore well defined up to μ_3 when the centroid is fixed and therefore the Morley triangle is well defined by its centroid.

(b) Assume $\varepsilon = 0$ and let $\tau = \tau^\varepsilon(x, y, z)$ be one of the vertices of $\underline{t}^\varepsilon_{x,y,z} = \underline{t}^\varepsilon_{X,Y,Z}$. By construction, we have

$$\lambda^\varepsilon_{x,y,z} = \langle \tau, y^3 \rangle \cup \langle \tau, z^3 \rangle. \tag{xiii}$$

But $\tau \in \{t^\varepsilon_{X,Y,Z}, t^\varepsilon_{j^2Z,j^2X,Z}, t^\varepsilon_{jY,jX,Z}\}$.

- If $\tau = t^\varepsilon_{j^2Z,j^2X,Y}$, we get

$$\lambda^\varepsilon_{j^2Z,j^2X,Y} = \langle \tau, X^3 \rangle \cup \langle \tau, Y^3 \rangle \tag{xiv}$$

  hence $\ell_{x,y} = \ell_{j^2Z,Y}$ which would imply $x^3 = Z^3$ by (10.1), a contradiction.

- If $\tau = t^\varepsilon_{jY,jX,Z}$, changing j to j^2 and Z to Y, we get from the preceding point $x^3 = y^3$, a contradiction.

- We have therefore $\tau = t^\varepsilon_{X,Y,Z}$ hence

$$\lambda^\varepsilon_{X,Y,Z} = \langle \tau, Y^3 \rangle \cup \langle \tau, Z^3 \rangle = \langle \tau, y^3 \rangle \cup \langle \tau, z^3 \rangle = \lambda^\varepsilon_{x,y,z}$$

  and we conclude by (11.2) $X = ux, Y = uy, Z = vz$ for some $u \in \mu_3$.

3.

(a) Assume $\varepsilon = 0$ and $\underline{t}^{\varepsilon,j}(x, y, z) = \underline{t}^{\varepsilon,j^2}(X, Y, Z)$. Using exactly the same argument before, we get $X = ux, Y = uy, Z = vz$ and $\lambda^{\varepsilon,j}(x, y, z) = \lambda^{\varepsilon,j^2}(X, Y, Z)$. Using (xiv), we obtain more generally $\lambda^{\varepsilon,j}(j^2z, j^2x, y) = \lambda^{\varepsilon,j^2}(j^2Z, j^2X, Y)$ the equality of ordered pairs $\underline{\lambda}^{\varepsilon,j}(x, y, z) = \underline{\lambda}^{\varepsilon,j^2}(X, Y, Z)$. Applying the invariant $\lambda \mapsto [\lambda_\varepsilon]$, we get the equality of ordered pairs $(1, j, j^2) = (1, j^2, j)$, a contradiction.

(b) Use for instance (xii).

4. Because triangle of type $(1, j)$ are also of type $(1, j^2)$, one has to look at the equation

$$t^{0,j}(x, y, z) = t^{1,j}(a, b, c) = \{t_1, t_2, t_3\}$$

(with $\{x^3, y^3, z^3\} = \{a^3, b^3, c^3\}$).

Because the trisectors of $\lambda^{1,j}(a, b, c)$ miss b^3, we have

$$\underline{\lambda}^1(a, b, c) = \{\langle a^3, t_1 \rangle \cup \langle c^3, t_1 \rangle, \langle a^3, t_2 \rangle \cup \langle c^3, t_2 \rangle, \langle a^3, t_3 \rangle \cup \langle c^3, t_3 \rangle\}$$

Using (xiii) again, there exists i such that $\lambda^0(x, y, z) = \langle a^3, t_i\rangle \cup \langle c^3, t_i\rangle = \ell_1 \cup \ell_2$ hence there exists $u \in \mu_3$ such that $\ell_1 \cup \ell_2 = \lambda^1(ua, b, uc)$. Looking at at $[\lambda]_\varepsilon$, we get $b^3 = \xi(\ell_1, \ell_2) \in \mu_3$ hence T is isosceles at $b^3 = y^3$ and $\{a^3, c^3\} = \{x^3, z^3\}$.

We are reduced to the isosceles situation and, changing j to j^2 if necessary, we can assume $\underline{t}^1 = t^1(x, 1, 1/x)$ (centroid equal to $\frac{x}{x^2+1}$) and $\underline{t}^0 = t^0(a, j, 1/a)$ (centroid equal to

$$\xi = \frac{(2j+1)a^4 + (-j+1)a^3 + (j+2)a - (2j+1)}{3a^2}$$

The first centroid is real but $\xi - \overline{\xi} = \frac{(j-1)(x+1)(x-1)(x^2+x+1)}{x^2}$ which is nonzero because T is not flat.

5. The remaining point is to prove that when $\varepsilon = 1$ the vertex y^3 of $t^\varepsilon_{x,y,z}$ is determined by the triangle itself. By direct computation, the centroid of $t^\varepsilon_{x,y,z}$ is $\frac{-x^2z^2}{x+z}$ and the centroid of its conjugate triangle $t^\varepsilon_{1/x,1/y,1/z}$ is $\frac{-1}{x^2z+xz^2}$. The quotient of these centroids is $x^3z^3 = y^3$. In particular, the missed point y^3 by the trisectors is determined by the triangle $t^\varepsilon_{x,y,z}$ independently showing that we get 9 or 6 distinct equilateral triangles in total when $\varepsilon = 1$.

$\square$

Remark 12.3 *Using for instance a computer, if* T *is generic enough (precisely two vertices are algebraically independent over* $\mathbf{Q}[j]$*), then one can check that Morley triangles are the only equilateral triangles that can be obtained by intersecting trisectors.*