$$\int e^{tx} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = \underbrace{\int \frac{e^{-\frac{(x-t)^2}{2}}}{\sqrt{2\pi}}}\, e^{\frac{t^2}{2}} = e^{\frac{t^2}{2}}$$

$$= \sum \frac{\left(\frac{t^2}{2}\right)^k}{k!} = \sum \frac{t^{2k}}{2^k\,k!}$$

$$\int e^{tx}\, x^{\alpha-1} e^{-\beta x} \frac{\beta^\alpha}{\Gamma(\alpha)}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} = \underbrace{\left(\frac{\beta}{\beta-t}\right)^\alpha}$$

NICE.

$$\frac{\beta}{\beta-t} = \frac{1}{1-t/\beta}$$

$$= 1 + t/\beta + (t/\beta)^2 + \dots$$

cela marchera quand même.

$$1/\beta = E[X] \qquad E(X^2) = \frac{2}{\beta^2} \qquad Var(X) = \frac{1}{\beta^2}$$

$$\overset{\mathrm{Exp}(1)}{Y} = \beta X$$

TD PROBA DU 22/10 Feuille de TD 5.

(4) $X \sim \mathrm{Exp}(\beta)$: densité $\beta e^{-\beta x} \mathbb{1}_{\{x>0\}}$.

1. $\varphi_X(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} \beta e^{-\beta x} \mathbb{1}_{\{x>0\}} dx = \int_0^{+\infty} \beta e^{-\overbrace{(\beta - t)}^{>0} x} dx$ $<+\infty$ ssi $t<\beta$

$$= \begin{cases} \frac{\beta}{\beta - t} & \text{si } t<\beta \\ +\infty & \text{si } t \geq \beta \end{cases}$$

2. $\varphi_X(t) = \sum_{k\geq 0} \frac{\mathbb{E}[X^k]}{k!} t^k = \frac{1}{1-\frac{t}{\beta}} = \sum_{k\geq 0} \left(\frac{t}{\beta}\right)^k$

si $t \in]-\beta, \beta[$

$\left(\frac{1}{1-t} = \sum_{k\geq 0} t^k \right)$.

$b^n - a^n = (b-a) \sum_{k=0}^{n-1} b^{n-1-k} a^k$

$$\frac{E[X^k]}{k!} = \frac{1}{\beta^k}$$

$$\boxed{E[X^k] = \frac{k!}{\beta^k}}$$

$$1 - t^n = (1-t)\sum_{k=0}^{n} t^k$$

$X \sim Exp(\beta)$

$$Var(X) = \frac{2}{\beta^2} - \left(\frac{1}{\beta}\right)^2 = \frac{1}{\beta^2}$$

3. $Y \sim \Gamma(\alpha, \beta)$: densité $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{\{x>0\}}$

$\forall \alpha, \beta > 0$

$$\int_0^{+\infty} x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

$$\varphi_Y(t) = E[e^{tY}] = \int_0^{+\infty} e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-(\beta-t)x} dx$$

① ②

$$= \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta-t}\right)^\alpha = \varphi_X(t)^\alpha$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{y}{\beta-t}\right)^{\alpha-1} e^{-y} \frac{dy}{\beta-t}$$

$$y = (\beta-t)x$$
$$\beta-t>0$$

$$= \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta-t}\right)^\alpha.$$

4. Deux premiers moments de la loi $\Gamma(\alpha,\beta)$?

$$\varphi_y(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha = \left(\frac{1}{1-\frac{t}{\beta}}\right)^\alpha = \boxed{\left(1+\frac{t}{\beta}+\frac{t^2}{\beta^2}+o(t^2)\right)^\alpha}$$

$$\boxed{(1+u)^\alpha = 1+\alpha u+\frac{\alpha(\alpha-1)}{2}u^2+o(u^2)}$$

$$= 1+\alpha\left(\frac{t}{\beta}+\frac{t^2}{\beta^2}+o(t^2)\right)+\frac{\alpha(\alpha-1)}{2}\left(\frac{t}{\beta}\right)^2+o(t^2)$$

$$= 1+\underbrace{\boxed{\frac{\alpha}{\beta}}}_{\mathbb{E}[y]}t+\underbrace{\boxed{\left(\frac{2\alpha}{\beta^2}+\frac{\alpha(\alpha-1)}{\beta^2}\right)}}_{\mathbb{E}[y^2]}\frac{t^2}{2}+o(t^2)$$

$$Var(Y) = E[Y^2] - E[Y]^2 = \frac{2\alpha}{\beta^2} + \frac{\alpha(\alpha-1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$= \frac{\alpha}{\beta^2} \quad \checkmark$$

$$Exp(\beta) \to \left(E = \frac{1}{\beta} \qquad Var = \frac{1}{\beta^2} \right)$$

$$X \sim Exp(\beta) \Rightarrow \beta X \sim \underbrace{Exp(1)}_{Y}$$

done

$$\beta E[X] = E[\beta X] = E[Y]$$

$$\beta^2 Var(X) = Var(\beta X) = Var(Y)$$

$$F_X(t) = \left(1 - e^{-\beta t}\right) \mathbb{1}_{\{t \geq 0\}} \Rightarrow F_{\beta X}(t) = F_X(t/\beta) = \left(1 - e^{-t}\right) \mathbb{1}_{\{t \geq 0\}}$$

(5) 1. $X \sim \mathcal{N}(0,1)$ densité $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\varphi_X(t) = \mathbb{E}[e^{tX}] = \boxed{\int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx} \quad ?$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2} + \frac{t^2}{2}} \, dx$$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy}_{=1} \qquad y = x - t$$

$$= e^{t^2/2}$$

2. $$= \sum_{k \geq 0} \left(\frac{t^2}{2}\right)^k \cdot \frac{1}{k!} = \sum_{k \geq 0} \frac{1}{2^k \cdot k!} t^{2k}$$

$$= \sum_{k \geq 0} \underbrace{\boxed{\frac{(2k)!}{2^k . k!}}}_{\mathbb{E}[X^{2k}]} \frac{t^{2k}}{(2k)!} \Rightarrow \begin{cases} \mathbb{E}[X^{2k}] = \underbrace{\frac{(2k)!}{2^k . k!}} \\ \mathbb{E}[X^{2k+1}] = 0 \end{cases}$$

⑥ $\mathbb{P}(X \in [ab]) = 1$. Mq $\underbrace{Var(X) \leq \frac{(b-a)^2}{4}}$

obtenue pour X tq

$\mathbb{P}(X = a) = \mathbb{P}(X = b) = \frac{1}{2}$

1. $[ab] = [0,1]$. Mq $Var(X) \leq 1$.

$0 \leq Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \underset{(1)}{\leq} \mathbb{E}[X^2] \underset{(2)}{\leq} 1.$$

car $X^2 \leq 1$

$X^2 \leq 1$
$\mathbb{E}[X^2] = 1$

? avec $Y = 1 - X^2 \geq 0$

$Y \geq 0$
$\mathbb{E}[Y] = 0$

$\Downarrow$

$\boxed{\mathbb{P}(Y = 0) = 1}$

avec égalité ssi on a égalité ds les 2 inégalités :

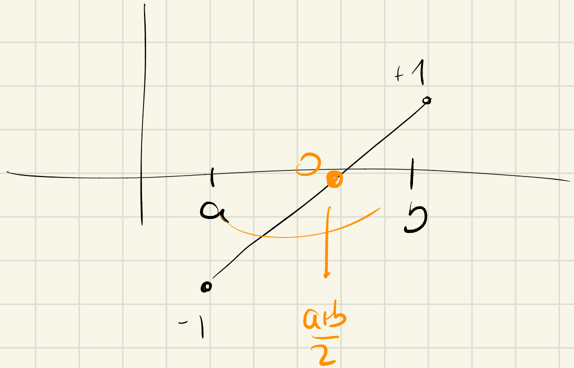
$$\begin{cases} \mathbb{E}[X] = 0 \\ \mathbb{P}(X \in \{-1, 1\}) = 1 \end{cases}$$

ssi $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1}{2}$.

Conclusion : Si $\mathbb{P}(X \in [-1, 1]) = 1$, alors

$\mathrm{Var}(X) \leq 1$ avec $\mathrm{Var}(X) = 1$ ssi $\mathbb{P}(X = -1) = \mathbb{P}(X = +1) = \frac{1}{2}$.

2. ϕ affine tq $\phi(a) = -1$, $\phi(b) = +1$.



$$\phi(x) = \frac{2}{b-a}\left(x - \frac{a+b}{2}\right)$$

$$= \frac{2}{b-a}x - \frac{a+b}{b-a}$$

$$\phi(x) = \alpha x + \beta.$$

$$\begin{cases} -1 = \alpha a + \beta \\ 1 = \alpha b + \beta \end{cases}$$

résoudre en (α, β) :

$$2 = \alpha(b-a)$$

$$\frac{2}{b-a} = \alpha$$

$$\beta = 1 - \frac{2}{b-a}b$$

$$= \frac{b-a-2b}{b-a}$$

$$= -\frac{a+b}{b-a}.$$

3. Etudier $\phi(X)$ et conclure.

si $a \leq X \leq b$ p.s alors $Var(X) \leq \frac{(b-a)^2}{4}$.

ce qu'on a : si $-1 \leq X \leq 1$ ps alors $Var(X) \leq 1$.

(+ ϕ affine tq $\phi(a) = -1$, $\phi(b) = 1$.)

Soit $X \in [a,b]$ ps. $\phi(X) \in [-1, 1]$ ps.

donc $Var(\phi(X)) \leq 1$ donc $\boxed{Var(X) \leq \frac{(b-a)^2}{4}}$

$$\begin{aligned} Var(\phi(X)) &= Var\left(\frac{2}{b-a}\left(X - \frac{a+b}{2}\right)\right) \\ &= \frac{4}{(b-a)^2} Var(X) \end{aligned}$$

avec égalité ssi

$P(X=a) = P(X=b) = 1/2$.

7

$N \in \mathbb{N}$

$$n = \sum_{i=1}^{+\infty} \mathbb{1}_{\{n \geq i\}} \qquad N = \sum_{i=1}^{+\infty} \mathbb{1}_{\{N \geq i\}}$$

$$E[N] = E\Big[\underbrace{\sum_{i=1}^{\infty} \mathbb{1}_{\{N \geq i\}}}_{\phi(N)}\Big]$$

$$= \sum_{n \in \mathbb{N}} \phi(n)\, \mathbb{P}(N=n) = \sum_{n \in \mathbb{N}} \left(\sum_{i=1}^{\infty} \mathbb{1}_{\{n \geq i\}} \right) \mathbb{P}(N=n).$$

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{n \in \mathbb{N}} \mathbb{1}_{\{n \geq i\}}\, \mathbb{P}(N=n)}_{\mathbb{P}(N \geq i)} = \sum_{i=1}^{\infty} \mathbb{P}(N \geq i).$$

$$Y = \begin{cases} +X & \text{avec proba } 1/2 \\ -X & \text{———— } 1/2 \end{cases} = BX \quad \text{où} \quad B \in \{-1, 1\}, \text{ indépendant de } X,$$

$$\mathbb{P}(B = +1) = \mathbb{P}(B = -1) = 1/2$$

$$\boxed{F_Y(t) = \frac{1}{2} F_X(t) + \frac{1}{2} F_{-X}(t)}$$

$$\mathbb{P}(Y \le t)$$

$$\parallel$$

$$\mathbb{P}(\cancel{B}X \le t, B = +1) + \mathbb{P}(\cancel{B}X \le t, B = -1)$$

$$\parallel$$

$$\mathbb{P}(X \le t)\frac{1}{2} + \mathbb{P}(-X \le t)\frac{1}{2}$$

$$X \sim \mathcal{E}xp(a)$$

or on sait que $F_X(t) = \left(1 - e^{-at}\right) \mathbb{1}_{\{t \ge 0\}}$

$$F_{-X}(t) = \mathbb{P}(-X \le t)$$

$$= \mathbb{P}(X \ge -t)$$

$$= \int_{-t}^{+\infty} a\, e^{-ax} \mathbb{1}_{\{x > 0\}}$$

$$= \boxed{\begin{cases} 1 & \text{si } -t < 0 \\ e^{+at} & \text{si } -t \ge 0 \end{cases}}$$

conclusion: $\frac{1}{2}\left(F_X(t) + F_X(-t)\right)$

$$= \begin{cases} \frac{1}{2}\left(1 - e^{-at} + 1\right) & \text{si } t \ge 0 \\ \frac{1}{2} e^{at} & \text{si } t \le 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{at} & \text{si } t \leq 0 \\ 1 - \frac{1}{2} e^{-at} & \text{si } t \geq 0 \end{cases}$$

$$= \int_{-\infty}^{t} \frac{1}{2} e^{-a|x|} \, dx$$

$$\mathbb{E}[\varphi(Y)] = \mathbb{E}[\varphi(BX)(\mathbb{1}_{\{B=-1\}} + \mathbb{1}_{\{B=+1\}})]$$

$$= \mathbb{E}[\varphi(\underset{-1}{\cancel{B}}X)\mathbb{1}_{B=-1}] + \mathbb{E}[\varphi(\underset{+1}{\cancel{B}}X)\mathbb{1}_{\{B=+1\}}]$$

$$= \mathbb{E}[\varphi(-X)\mathbb{1}_{\{B=-1\}}] + \mathbb{E}[\varphi(X)\mathbb{1}_{\{B=+1\}}]$$

$\frac{1}{2}$

$\mathbb{P}(B=+1) = \frac{1}{2}$