PENCILS OF PROJECTIVE HYPERSURFACES, GRIFFITHS HEIGHTS AND GEOMETRIC INVARIANT THEORY. II
HYPERSURFACES WITH SEMIHOMOGENEOUS SINGULARITIES

THOMAS MORDANT

Abstract. This paper establishes the formula for the stable Griffiths height of the middle-dimensional cohomology of a pencil of projective hypersurfaces H, with semihomogeneous singularities, over some smooth projective curve C, that appears as Theorem 5.1 in the first part of this paper [Mor25].

The proof of this formula relies on the strategy developed in [Mor22] to derive an expression for these Griffiths heights when the only singularities of the fibers of H over C are ordinary double points. To deal with general semihomogeneous singularities, we complement this strategy by the construction of a finite covering C' of C such that the pencil $H' = H \times_C C'$ over C' admits a smooth model $\widetilde{H}'$ with semistable fibers with smooth components. This allows us to circumvent the delicate issue of the determination of the elementary exponents attached to the singular fibers of H/C.

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1. Introduction

1.1. **The stable Griffiths height of the middle-dimensional cohomology of a pencil of projective hypersurfaces with semihomogeneous singularities.** This paper is a sequel to [Mor25], and is devoted to the proof of [Mor25, Theorem 5.1], which provides a closed formula for the stable Griffiths height of the middle-dimensional cohomology of a pencil of projective hypersurfaces with semihomogeneous singularities.

Let us recall the statement of this theorem:

Theorem 1.1. *Let C be a connected smooth projective complex curve with generic point η, E a vector bundle of rank $N+1 \geq 2$ over C, and $H \subset \mathbb{P}(E)$ an horizontal hypersurface of relative degree $d \geq 2$, smooth over $\mathbb{C}$. Let us assume that the set of critical points Σ of the restriction*

$$f := \pi_{|H} : H \longrightarrow C$$

is finite, and that the restriction

$$\pi_{|\Sigma} : \Sigma \longrightarrow C$$

is injective. For every point P in H, let δ_P be the multiplicity at P of the projective hypersurface $H_{\pi(P)}$ in $\mathbb{P}(E_{\pi(P)})$.[1] *Let us further assume that for every point P of H, the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ of $H_{\pi(P)}$ at P is smooth.*

Then the following equality of integers holds:

$$\sum_{P \in \Sigma} (\delta_P - 1)^N = (N+1)(d-1)^N \mathrm{ht}_{int}(H/C). \tag{1.1}$$

Moreover the following equality of rational numbers holds:

$$\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) = -(N+1)w_{N,d}\, \mathrm{ht}_{int}(H/C) + \sum_{P \in \Sigma} w_{N,\delta_P}, \tag{1.2}$$

where for every positive integer δ, $w_{N,\delta}$ is the rational number in $(1/12)\mathbb{Z}$ defined by:

$$w_{N,\delta} = (\delta - 1)\big[(N\delta + 1)(\delta - 1)^{N-1} + (-1)^N(\delta + 1)\big]/(12\delta^2). \tag{1.3}$$

The stable Griffiths height $\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big)$ of the variation of Hodge structures

$$\mathbb{H}^{N-1}(H_\eta/C_\eta)$$

attached to the middle-dimensional relative cohomology of the pencil H/C is defined in [Mor25, 2.2], and the intersection theoretic height $\mathrm{ht}_{int}(H/C)$ of the hypersurface H in the projective bundle $\mathbb{P}(E)$ in [Mor25, 3.2.1].

The requirement in Theorem 1.1 on the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ of $H_{\pi(P)}$ at any point P of Σ to be smooth precisely means that P is a semihomogeneous singularity of $H_{\pi(P)}$. We refer the reader to [AGZV85, Part II, 12 and 13] for a general presentation and references concerning semihomogeneous (and more generally semiquasihomogeneous) singularities.

[1] Observe that the positive integer δ_P is at least 2 if and only if P is in Σ.

In the special case where the multiplicities δ_P of the points P of Σ are all equal to 2 — or equivalently when the singularities of the fibers of H/C are ordinary double points — then the equality (1.2) in Theorem 1.1 is easily seen to become the equality:

$$\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) = F_{stab}(d,N)\,\mathrm{ht}_{int}(H/C),$$

established in [Mor22, Theorem 1.4.2], and used in the first part of this paper (see [Mor25, 4.2]).

1.2. **Main Theorem.** Theorem 1.1 will be a consequence of a more general theorem, concerning hypersurfaces with semihomogeous singularities in arbitrary smooth pencils.

1.2.1. *Notation.* Consider a connected smooth projective complex curve C, with generic point η, and:

$$\pi : X \longrightarrow C$$

a smooth projective morphism of (necessarily smooth) complex projective varieties, of fibers of pure dimension N.

Consider also a non-singular hypersurface H in X such that the following conditions are satisfied:

(i) the set Σ of critical points of the restriction $\pi_{|H} : H \to C$ is finite;[2]
(ii) the restriction $\pi_{|\Sigma} : \Sigma \to C$ is injective;
(iii) for every point P in Σ, the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ is non-singular.

We shall denote by:

$$\Delta := \pi(\Sigma)$$

the "locus of bad reduction" of $\pi_{|H} : H \to C$.

For every $x \in \Delta$, we shall denote by P_x the unique point in $\pi_{|\Sigma}^{-1}(x)$, and by δ_{P_x} the multiplicity at P_x of the fiber $H_x := \pi_{|H}^{-1}(x)$. Observe that the hyperplanes $T_{P_x}X_x$ and $T_{P_x}H$ in $T_{P_x}X$ coincide, and that δ_{P_x} is precisely the degree of the projective tangent cone $\mathbb{P}(C_{P_x}H_x)$ seen as an hypersurface in the projective space $\mathbb{P}(T_{P_x}X_x) = \mathbb{P}(T_{P_x}H) \simeq \mathbb{P}^{N-1}$.

We shall also denote by L the line bundle $\mathcal{O}_X(H)$ on X.

1.2.2. The following theorem is the main result of this paper.

Theorem 1.2. *With the above notation, the following equality holds in* $\mathrm{CH}_0(X)$*:*

$$\sum_{P\in\Sigma}(\delta_P - 1)^N\,[P] = [(1-c_1(L))^{-1}c(\Omega^1_{X/C})]^{(N+1)}. \tag{1.4}$$

Furthermore, if the line bundle L on X is ample relatively to the morphism π, then the following equality of rational numbers holds:

$$\begin{aligned}\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) &= \mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(X/C)\big) - \mathrm{ht}_{GK}\big(\mathbb{H}^{N}(X/C)\big) + \mathrm{ht}_{GK}\big(\mathbb{H}^{N+1}(X/C)\big)\\ &+ 1/12\int_X (1-c_1(L))^{-1}c_1(\Omega^1_{X/C})\,c(\Omega^1_{X/C}) - 1/12\int_X c_1(L)\,c_N(\Omega^1_{X/C}) + \sum_{P\in\Sigma} w_{N,\delta_P},\end{aligned} \tag{1.5}$$

where $w_{N,\delta}$ is defined by (1.3)*.*

In the right-hand sides of (1.4) and (1.5), we denote by $c(\Omega^1_{X/C})$ the total Chern class of the vector bundle $\Omega^1_{X/C}$.

This theorem extends our results in [Mor22, 6.1], where we consider pencils of hypersurfaces whose only singularities are ordinary double points. Indeed Proposition 6.1.1 in *loc. cit.*, with $\pi_{|\Sigma}$

[2] This implies that $\pi_{|H}$ is a flat morphism.

assumed to be injective, is the variant concerning lower and upper Griffiths heights of the special case of Theorem 1.2 where all the δ_P are equal to 2.[3]

1.2.3. The proof of Theorem 1.2 will be completed at the end of Section 5, devoted to the geometry of the pencil of hypersurfaces H in the smooth pencil X over C. It will rely on the auxiliary results from [Mor22] gathered in Section 2, on the construction of a semistable model of H over some finite covering C' of the curve C developed in Section 3, and on the computations of characteristic classes of Kähler and logarithmic relative differentials in Section 4, which themselves are minor variations on computations in [Mor22].

Section 6 provides the proof of a technical result stated in Section 3 (Proposition 4.1). It follows from standard arguments, and could have been left as an exercise for the reader. Considering the length of its derivation, we preferred to give some details.

1.2.4. A major difference between the proof of Theorem 1.2 and the proof of its "special case" Proposition 6.1.1 in [Mor22] is the introduction in Section 3 of the covering C' of C such that, base changed to C', the pencil of hypersurfaces H over C admits a model $\widetilde{H}'$ with semistable fibers with smooth components. This construction allows us to compute directly the stable Griffiths height $\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big)$ with no knowledge of the elementary exponents attached to the singular fibers of H/C (compare [Mor22, 2.2-5]).

1.3. **Proof of Theorem 1.1 from Theorem 1.2.** Before turning to the proof of Theorem 1.2, we explain how it implies Theorem 1.1.

We adopt the notation introduced in 1.2.1, and we introduce some further notation, similar to the one in [Mor22, 6.2.1].

1.3.1. The structure of the Chow groups of $\mathbb{P}(E)$ (see [Ful98, Th. 3.3 (b)], with $k = \dim(\mathbb{P}(E)) - 1$), implies that the line bundle $L := \mathcal{O}_{\mathbb{P}(E)}(H)$ can be written:

$$L \simeq \mathcal{O}_E(d) \otimes \pi^* M,$$

where M is some line bundle on C.

Moreover we define classes in $\mathrm{CH}^1(C)$ and $\mathrm{CH}^1(\mathbb{P}(E))$ by:

$$e := c_1(E), \quad m := c_1(M), \quad \text{and} \quad h := c_1(\mathcal{O}_E(1)).$$

With this notation, the height $\mathrm{ht}_{int}(H/C)$ is given by (see [Mor22, (6.2.13)]):

$$\mathrm{ht}_{int}(H/C) = \int_C \big(m - d/(N+1)\, e\big). \tag{1.6}$$

Moreover, according to the definition of the Segre classes of E and to their relation to the Chern classes of E (see for instance [Ful98, 3.1, 3.2]), the following equalities hold in $\mathrm{CH}^0(C)$ and $\mathrm{CH}^1(C)$ respectively:

$$\pi_* h^N = [C] \quad \text{and} \quad \pi_* h^{N+1} = -e. \tag{1.7}$$

The proof of Theorem 1.1 shall rely on the following generalization of [Mor22, Proposition 6.2.5].

Proposition 1.3. *With the above notation, and denoting by L the line bundle $\mathcal{O}_{\mathbb{P}(E)}(H)$ on $\mathbb{P}(E)$, the following equalities hold in $\mathrm{CH}_0(\mathbb{P}(E))$:*

$$\sum_{P \in \Sigma} (\delta_P - 1)^N [P] = (d-1)^N h^N [(d-1)h + (N+1)\pi^* m - \pi^* e], \tag{1.8}$$

[3] The interested reader may actually check that the statement of Theorem 1.2 and its proof in the next sections remain valid when $\pi_{|\Sigma}$ is not assumed to be injective anymore, but when the multiplicity δ_P of a point P in Σ only depends on $\pi(P)$. This covers the variant of Proposition 6.1.1 in *loc. cit.* concerning stable Griffiths heights.

$$c_1(L)c_N(\Omega^1_{\mathbb{P}(E)/C}) = (-1)^N h^N[d(N+1)h + (N+1)\pi^*m + dN\pi^*e], \tag{1.9}$$

and:

$$[(1-c_1(L))^{-1}c_1(\Omega^1_{\mathbb{P}(E)/C})c(\Omega^1_{\mathbb{P}(E)/C})]^{(N+1)} = h^N(a_{N,d}h + b_{N,d}\pi^*m + c_{N,d}\pi^*e), \tag{1.10}$$

where $a_{N,d}$, $b_{N,d}$ and $c_{N,d}$ are the integers defined by:

$$a_{N,d} := \frac{N+1}{d}\big(-(d-1)^{N+1} + (-1)^{N+1}\big),$$

$$b_{N,d} := \frac{N+1}{d^2}\big(-(d-1)^N(dN+1) + (-1)^N\big),$$

and:

$$c_{N,d} := \frac{1}{d}\big(-(d-1)^N(d-N-2) + (-1)^{N+1}(N+2)\big).$$

Proof. The proof follows from the computations of Chern classes on H and X in the proof of [Mor22, Proposition 6.2.5], by replacing the 0-cycle $i_*[\Sigma]$ in *loc. cit.* by $\sum_{P\in\Sigma}(\delta_P - 1)^N[P]$ and using the expression of this sum given by equality (1.4). □

1.3.2. As in the proof of [Mor22, Theorem 6.2.4], pushing forward equality (1.8) by the morphism π, then using equalities (1.7), and finally taking the degree and using equality (1.6) establishes equality (1.1).

Reasoning similarly with equalities (1.9) and (1.10) yields the following equalities:

$$\int_{\mathbb{P}(E)} c_1(L)c_N(\Omega^1_{\mathbb{P}(E)/C}) = (-1)^N(N+1)\mathrm{ht}_{int}(H/C) \tag{1.11}$$

and:

$$\begin{aligned}\int_{\mathbb{P}(E)} (1-c_1(L))^{-1}c_1(\Omega^1_{\mathbb{P}(E)/C})c(\Omega^1_{\mathbb{P}(E)/C}) &= \int_C (b_{N,d}m + (c_{N,d} - a_{N,d})e)\\ &= (N+1)/d^2\ \big(-(Nd+1)(d-1)^N + (-1)^N\big)\,\mathrm{ht}_{int}(H/C).\end{aligned} \tag{1.12}$$

Applying equality (1.5) from Theorem 1.2 to the hypersurface H in the smooth pencil $X := \mathbb{P}(E)$ over C, and using that the Griffiths height of the relative cohomology in any degree of $\mathbb{P}(E)$ over C vanishes, we obtain the following equality:

$$\begin{aligned}\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) = 1/12 \int_X &(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})\,c(\Omega^1_{X/C})\\ &- 1/12 \int_X c_1(L)\,c_N(\Omega^1_{X/C}) + \sum_{P\in\Sigma} w_{N,\delta_P}.\end{aligned} \tag{1.13}$$

Combining this equality with equalities (1.11) and (1.12) yields the following equalities:

$$\begin{aligned}\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) &= \left[\frac{N+1}{12d^2}(-(Nd+1)(d-1)^N + (-1)^N) - \frac{(-1)^N}{12}(N+1)\right].\mathrm{ht}_{int}(H/C)\\ &\quad + \sum_{P\in\Sigma} w_{N,\delta_P}\\ &= -(N+1)w_{N,d}\,\mathrm{ht}_{int}(H/C) + \sum_{P\in\Sigma} w_{N,\delta_P}.\end{aligned}$$

This concludes the proof of Theorem 1.1.

2. Some results from [Mor22]

The remainder of this paper is devoted to the proof of Theorem 1.2. It will rely on various auxiliary results established in [Mor22], which we recall in this section.

2.1. **Alternating sums of Griffiths heights.** Firstly it will use the expression for the alternating sum of Griffiths heights associated to a pencil of projective varieties whose singular fibers are divisors with strict normal crossings established in [Mor22, Theorem 4.2.3].

For the convenience of the reader, we recall this expression.

Let C be a connected smooth projective complex curve with generic point η, Y be a connected smooth projective N-dimensional complex scheme, and let

$$g : Y \longrightarrow C$$

be a surjective morphism of complex schemes. Let us assume that there exists a finite subset Δ in C such that g is smooth over $C \setminus \Delta$, and such that the divisor $D := Y_\Delta$ is a divisor with strict normal crossings in Y.

We write this divisor as:

$$D = \sum_{i \in I} m_i D_i,$$

where I is a finite set, for every i in I, $m_i \geq 1$ is an integer, and D_i is a smooth connected divisor, such that the $(D_i)_{i\in I}$ intersect each other transversally.

The set I may be written as the disjoint union:

$$I = \bigcup_{x \in \Delta} I_x,$$

where, for every $x \in \Delta$, I_x denotes the non-empty subset of I defined by:

$$I_x := \big\{ i \in I \mid g(D_i) = \{x\} \big\}.$$

For every subset J of I, let us denote:

$$D_J := \bigcap_{i \in J} D_i,$$

it is a smooth subscheme of codimension $|J|$. As in [Del70, II, 3.4], for every integer $r \geq 1$, we denote by D^r the subscheme of codimension r of Y defined by the union of all the intersections of r different components (D_i):

$$D^r := \bigcup_{J \subset I, |J| = r} D_J.$$

Let us choose a total order $\preceq$ on I. For every element i in I, we define an open subset $\mathring{D}_i$ of the divisor D_i by:

$$\mathring{D}_i := D_i \setminus D_i \cap D^2.$$

Similarly, for every pair (i, j) in I^2 such that $i \prec j$, we define an open subset $\mathring{D}_{ij}$ of the subscheme

$$D_{ij} := D_{\{i,j\}}$$

by:

$$\mathring{D}_{ij} := D_{ij} \setminus D_{ij} \cap D^3.$$

Finally we denote by χ_{top} the topological Euler characteristic.

Theorem 2.1 ([Mor22, Theorem 4.2.3]). *With the above notation, we have:*

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}\big(\mathbb{H}^n(Y_\eta/C_\eta)\big) = \frac{1}{12} \int_Y (c_1 c_{N-1})(\omega^{1\vee}_{Y/C}) + \sum_{x \in \Delta} \alpha_x, \tag{2.1}$$

where for every x in Δ, α_x is the rational number given by:

$$\alpha_x = \frac{N-1}{4} \sum_{i \in I_x} (m_i - 1) \chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12} \sum_{\substack{(i,j) \in I_x^2, \\ i \prec j}} (3 - m_i/m_j - m_j/m_i)\, \chi_{\mathrm{top}}(\mathring{D}_{ij}), \tag{2.2}$$

In the proof of Theorem 1.2, we shall only use the case of Theorem 2.1 where the divisor Y_Δ is reduced, i.e. the multiplicities m_i are all equal to 1. In this case equality (2.1) becomes the following equality (also given in [Mor22, (1.3.8)]):

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK}\big(\mathbb{H}^n(Y_\eta/C_\eta)\big) = \frac{1}{12}\int_Y c_1(\omega^{1\vee}_{Y/C}) c_{N-1}(\omega^{1\vee}_{Y/C}) + \frac{1}{12}\chi_{\mathrm{top}}(D^2 \setminus D^3). \tag{2.3}$$

2.2. **Some combinatorial formulas.** The following combinatorial formulas will also be used in the proof of Theorem 1.2.

Proposition 2.2 ([Mor22, Proposition 5.3.1])**.** *For every a in $\mathbb{C}^*$, and every n and r in $\mathbb{N}$, the following equalities hold:*

(2.4)
$$\left[\frac{(1+y)^n}{1+ay}\right]^{[n-r]} = (-1)^r \sum_{r\leq k\leq n} \binom{n}{k} (-1)^k a^{k-r} = \frac{(-1)^{n+r}}{a^r}\Big[(a-1)^n - \sum_{0\leq k\leq r-1}\binom{n}{k}(-1)^{n-k}a^k\Big],$$

and:

(2.5)
$$\left[\frac{(1+y)^{n+1}}{(1+ay)^2}\right]^{[n-r]} = (-1)^{r+1} \sum_{r+1\leq k\leq n+1} (k-r)\binom{n+1}{k}(-1)^k a^{k-r-1}$$

$$= \frac{(-1)^{n+r}}{a^{r+1}}\Big[\big(r+(n+1-r)a\big)(a-1)^n - \sum_{0\leq k\leq r}(k-r)\binom{n+1}{k}(-1)^{n+1-k}a^k\Big], \tag{2.6}$$

where $f(y)^{[p]}$ denotes the coefficient of y^p in some formal series $f(y) \in \mathbb{C}[[y]]$.

3. Construction of a semistable model of H

In this section, we introduce an auxiliary construction which will allow us to compute the stable Griffiths height in the left-hand side of (1.5) as the Griffiths height attached to some family of projective varieties with semistable reduction, hence with unipotent monodromy.

3.1. **A local description of the divisor H in X.** Let P be a point in Σ, and let t be a local coordinate of C near $\pi(P)$, namely a uniformizer of the discrete valuation ring $\mathcal{O}_{C,\pi(P)}$.

The pullback π^*t can be completed into a local coordinate system $(\pi^*t, x_1, \ldots, x_N)$ of X near P. In other words, $x_1, \ldots, x_N$ belong to the local ring $\mathcal{O}_{X,P}$ and $(\pi^*t, x_1, \ldots, x_N)$ generate its maximal ideal $\mathfrak{m}_{X,P}$. Since P belongs to Σ, the hyperplanes $T_P X_{\pi(P)}$ and $T_P H$ in $T_P X$ coincide, and therefore the restrictions $(x_{1|H}, \ldots, x_{N|H})$ induce a local coordinate system on H near P.

Let $F_{\delta_P}(T_1, \ldots, T_N)$ be an equation in $\mathbb{C}[T_1, \ldots, T_N]$ of the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ seen as a projective hypersurface in $\mathbb{P}(T_P H)$ that we identify with $\mathbb{P}^{N-1}_{\mathbb{C}}$ through the (differential at P of the) coordinate system $(x_{1|H}, \ldots, x_{N|H})$. Since by hypothesis $\mathbb{P}(C_P H_{\pi(P)})$ is a smooth hypersurface of degree δ_P, the homogeneous polynomial F_{δ_P} has degree δ_P, and has nonzero discriminant.

Proposition 3.1. *After possibly replacing F_{δ_P} by λF_{δ_P} for some λ in $\mathbb{C}^*$, the germ at P of the divisor H in X is defined by an equation of the form:*

$$\pi^* t = F_{\delta_P}(x_1, \ldots, x_N) + F_{>\delta_P}, \tag{3.1}$$

where $F_{>\delta_P}$ belongs to the ideal $\mathfrak{m}^{\delta_P+1}_{X,P}$ of $\mathcal{O}_{X,P}$.

Proof. The divisor $H_{\pi(P)}$ in H is defined by the equation $(\pi^*_{|H} t = 0)$. Therefore there exist $\lambda \in \mathbb{C}^*$ and $G_{>\delta_P} \in \mathfrak{m}^{\delta_P+1}_{H,P}$ such that the following equality holds in $\mathcal{O}_{H,P}$:

$$\pi^*_{|H} t = \lambda F_{\delta_P}(x_{1|H}, \ldots, x_{N|H}) + G_{>\delta_P}.$$

We may assume $\lambda = 1$ and choose an element $F_{>\delta_P}$ of $\mathfrak{m}_{X,P}^{\delta_P+1}$ whose restriction to (the germ at P of) H is $G_{>\delta_P}$. Then the equation (3.1) defines a germ of smooth hypersurface in X which contains, hence coincides with, the germ of H at P. $\square$

3.2. Construction of the blow-ups $\widetilde{H}$ and $\widetilde{H}'$.

3.2.1. Let C' be a connected smooth projective complex curve, and let

$$\sigma : C' \longrightarrow C$$

be a finite morphism such that, for every point x' in the finite set

$$\Delta' := \sigma^{-1}(\Delta),$$

the index of ramification of σ at x' is precisely $\delta_{\sigma(x')}$. Such a finite covering C' of C is easily constructed, for instance as a cyclic covering.

Consider the fiber product of X and C' over C:

$$X' := X \times_C C',$$

and denote by:

$$\pi' : X' \longrightarrow C' \quad \text{and} \quad \tau : X' \longrightarrow X$$

the two projections.

Moreover let us denote the set theoretic inverse images of H and Σ by τ by:

$$H' := \tau^{-1}(H) \quad \text{and} \quad \Sigma' := \tau^{-1}(\Sigma),$$

which we shall also see as reduced[4] subschemes of the smooth scheme X'.

Let:

$$\chi' : \widetilde{X}' \longrightarrow X' \quad (\text{resp. } \chi : \widetilde{X} \longrightarrow X)$$

be the blow-up of Σ' (resp. Σ) seen as a reduced subscheme in X' (resp. in X), and let:

$$Z' := \chi'^{-1}(\Sigma') \quad (\text{resp. } Z := \chi^{-1}(\Sigma))$$

be the exceptional divisor of χ' (resp. of χ). Its connected components are the inverse images:

$$Z'_{P'} := \chi'^{-1}(P'), \quad P' \in \Sigma' \quad (\text{resp. } Z_P := \chi^{-1}(P), \quad P \in \Sigma).$$

Finally let $\widetilde{H}'$ (resp. $\widetilde{H}$) be the strict transform of H' (resp. H) in $\widetilde{X}'$ (resp. in $\widetilde{X}$).

The following morphism:

$$\nu' := \chi'_{|\widetilde{H}'} : \widetilde{H}' \longrightarrow H' \quad (\text{resp. } \nu := \chi_{|\widetilde{H}} : \widetilde{H} \longrightarrow H)$$

can be identified with the blow-up of Σ' (resp. Σ) in the complex scheme H' (resp. in the smooth complex scheme H).

[4]The divisor H' coincides with the scheme theoretic inverse image $\tau^*(H)$. However, if Σ is not empty, then its scheme theoretic inverse image by τ is not reduced.

3.2.2. These constructions are summarized by the following diagram where Z'_δ is the Cartier divisor in $\widetilde{X}'$ defined as:

$$Z'_\delta := \sum_{P' \in \Sigma'} \delta_{\tau(P')} Z'_{P'},$$

and where we identify the effective Cartier divisors $\widetilde{H}' + Z'_\delta$ and $\widetilde{H} + Z$ to the codimension one subschemes they define:

$$\begin{tikzcd}
 & & Z' \arrow[dl] \arrow[ddd] & & & Z \arrow[dl] \arrow[ddd] \\
 & \widetilde{H}' + Z'_\delta \arrow[dl] \arrow[ddd] & & & \widetilde{H} + Z \arrow[dl] \arrow[ddd] & \\
\widetilde{X}' \arrow[ddd, "\chi'"'] & & & \widetilde{X} \arrow[ddd, "\chi"] & & \\
 & & \Sigma' \arrow[dl] \arrow[rrr, dotted] & & & \Sigma \arrow[dl] \\
 & H' \arrow[dl] \arrow[rrr] & & & H \arrow[dl] & \\
X' \arrow[rrr, "\tau"] \arrow[d, "\pi'"'] & & & X \arrow[d, "\pi"] & & \\
C' \arrow[rrr, "\sigma"'] & & & C & &
\end{tikzcd}$$

Observe that, in this diagram, all rectangles with non-dotted arrows are cartesian.

The divisor $\widetilde{H}'$ in $\widetilde{X}'$ is non-singular and intersects transversally the divisor $Z' := \sqcup_{P' \in \Sigma'} Z'_{P'}$. Moreover, the following equality of divisors in $\widetilde{X}'$ holds:

$$\chi'^*(H') = \widetilde{H}' + Z'_\delta. \tag{3.2}$$

Similarly the divisor $\widetilde{H}$ in $\widetilde{X}$ is non-singular and intersects transversally the divisor $Z := \sqcup_{P \in \Sigma} Z_P$, and the following equality of divisors in $\widetilde{X}$ holds:

$$\chi^*(H) = \widetilde{H} + Z. \tag{3.3}$$

Moreover we denote by $E'_{P'}$ (resp. E_P) the intersection $Z'_{P'} \cap \widetilde{H}'$ (resp. $Z_P \cap \widetilde{H}$) for every point P' in Σ' (resp. P in Σ), by $H'_{\Delta'}$ and $\widetilde{H}'_{\Delta'}$ (resp. by H_Δ and $\widetilde{H}_\Delta$) the inverse images of the reduced divisor Δ' in C' (resp. of the reduced divisor Δ in C) by $\pi'_{|H'}$ and $\pi'_{|H'} \circ \nu'$ respectively (resp. by $\pi_{|H}$ and $\pi_{|H} \circ \nu$), and by $\widetilde{H'_{\Delta'}}$ (resp. by $\widetilde{H_\Delta}$) the strict transform in $\widetilde{H}'$ (resp. in $\widetilde{H}$) of the divisor $H'_{\Delta'}$ (resp. H_Δ).

The divisor $\widetilde{H'_{\Delta'}}$ is non-singular and intersects transversally the non-singular divisor $E' := \sqcup_{P' \in \Sigma'} E'_{P'}$, and the following equality of divisors in $\widetilde{H}'$ holds:

$$\widetilde{H}'_{\Delta'} = \widetilde{H'_{\Delta'}} + \sum_{P' \in \Sigma'} E'_{P'}. \tag{3.4}$$

Similarly the divisor $\widetilde{H_\Delta}$ is non-singular and intersects transversally the non-singular divisor $E := \sqcup_{P\in\Sigma} E_P$, and the following equality of divisors in $\widetilde{H}$ holds:

$$\widetilde{H}_\Delta = \widetilde{H_\Delta} + \sum_{P\in\Sigma} \delta_P E_P. \tag{3.5}$$

In particular, the divisor $\widetilde{H}'_{\Delta'}$ in $\widetilde{H}'$ is a reduced divisor with strict normal crossings, and the divisor $\widetilde{H}_\Delta$ in $\widetilde{H}$ is a divisor with strict normal crossings, non-reduced when Σ is not empty.

3.2.3. *Computations of topological Euler characteristics.*

Proposition 3.2. *For every point P' in Σ' with image $P := \tau(P')$ in Σ, the following equality of integers holds:*

$$\chi_{\mathrm{top}}(E'_{P'} \cap \widetilde{H'_{\Delta'}}) = (-1)^N/\delta_P \, [(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)], \tag{3.6}$$

where χ_{top} denotes the topological Euler characteristic.

Proof. Consider P' and P as above. As over some open neighborhood of P' the complex scheme $\widetilde{X}'$ is the blow-up at P' of the smooth $(N+1)$-dimensional complex scheme X', there is a canonical isomorphism:

$$\psi : Z'_{P'} \xrightarrow{\sim} \mathbb{P}(T_{P'}X').$$

Under this isomorphism, the divisor $E'_{P'} = Z'_{P'} \cap \widetilde{H}'$ in $Z'_{P'}$ is mapped precisely onto the projective tangent cone $\mathbb{P}(C_{P'}H')$ at P' of the hypersurface H' in X', of which P' is an isolated singular point. By using Proposition 3.1, this projective tangent cone is easily seen to be smooth of degree δ_P.

Moreover, under the isomorphism ψ, the subscheme $E'_{P'} \cap \widetilde{H'_{\Delta'}}$ of $Z'_{P'}$ is mapped onto the projective tangent cone $\mathbb{P}(C_{P'}H'_{\Delta'})$ at P' of the subscheme $H'_{\Delta'} = H' \cap X'_{\Delta'}$ of X', or equivalently onto the intersection of the projective tangent cone $\mathbb{P}(C_{P'}H')$ and the projective tangent space $\mathbb{P}(T_{P'}X'_{\Delta'})$, which is well-defined because π' is smooth.

If we let:

$$h := c_1(\mathcal{O}_{T_{P'}X'_{\Delta'}}(1)) \quad \big(\in \mathrm{CH}^1(\mathbb{P}(T_{P'}X'_{\Delta'}))\big),$$

and:

$$I := \mathbb{P}(C_{P'}H') \cap \mathbb{P}(T_{P'}X'_{\Delta'}) = \psi(E'_{P'} \cap \widetilde{H'_{\Delta'}}),$$

which is a smooth hypersurface of degree δ_P in the projective space $\mathbb{P}(T_{P'}X'_{\Delta'})$, a standard computation of Chern classes yields the following equalities:

$$\begin{aligned}
\chi_{\mathrm{top}}(E'_{P'} \cap \widetilde{H'_{\Delta'}}) &= \int_{E'_{P'}\cap\widetilde{H'_{\Delta'}}} c_{N-2}(T_{E'_{P'}\cap\widetilde{H'_{\Delta'}}}) \\
&= \int_I c_{N-2}(T_I) \\
&= \int_I c_{N-2}\big([T_{\mathbb{P}(T_{P'}X'_{\Delta'})|I}] - [N_I\mathbb{P}(T_{P'}X'_{\Delta'})]\big) \\
&= \int_I c_{N-2}\big([T_{P'}X'_{\Delta'} \otimes \mathcal{O}_{T_{P'}X'_{\Delta'}}(1)_{|I}] - [\mathcal{O}_I] - [\mathcal{O}_{T_{P'}X'_{\Delta'}}(\delta_P)_{|I}]\big) \\
&= \int_I (1+h)^N(1+\delta_P h)^{-1} \\
&= \delta_P \int_{\mathbb{P}(T_{P'}X'_{\Delta'})} h(1+h)^N(1+\delta_P h)^{-1}
\end{aligned}$$

Finally, using equality (2.4) applied to $n := N$, $r := 2$ and $a := \delta_P$, we get:

$$\chi_{\mathrm{top}}(E'_{P'} \cap \widetilde{H'_{\Delta'}}) = \delta_P \left[\frac{(1+y)^N}{1+\delta_P y}\right]^{[N-2]} = (-1)^N/\delta_P \left[(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)\right]. \tag{3.7}$$

This concludes the proof of Proposition 3.2. See also [Mor22, Proposition 5.2.1] for a similar computation. □

4. Characteristic classes of Kähler and logarithmic relative differentials on the hypersurfaces H, $\widetilde{H}$, and $\widetilde{H}'$

4.1. **Comparing $\omega^1_{\widetilde{H}/C}$ and $\omega^1_{\widetilde{H}'/C'}$.** The following proposition is arguably the point where the proof of Theorem 1.2 differs the most significantly from the proof of [Mor22, Proposition 6.1.1]. This result concerning the geometry of the morphisms:

$$\pi_{|H} \circ \nu : \widetilde{H} \longrightarrow C$$

and:

$$\pi'_{|H'} \circ \nu' : \widetilde{H}' \longrightarrow C'$$

will allow us to compute the stable Griffiths height $\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big)$ without controlling the elementary exponents of the degeneration $\pi_{|H} : H \to C$.

Proposition 4.1. *With the notation introduced in 3.2.1 and 3.2.2, the inverse image $(\tau_{|H'} \circ \nu')^*(\Sigma)$, seen as a subscheme of $\widetilde{H}'$, is precisely the exceptional divisor $E' = \sqcup_{P' \in \Sigma'} E'_{P'}$; in particular, it is a (Cartier) divisor in $\widetilde{H}'$. Moreover there exists a unique morphism $\upsilon : \widetilde{H}' \to \widetilde{H}$ such that the following diagram is commutative:*

$$\begin{tikzcd} \widetilde{H}' \arrow[r, "\upsilon"] \arrow[d, "\nu'"'] & \widetilde{H} \arrow[d, "\nu"] \\ H' \arrow[r, "\tau_{|H'}"'] & H. \end{tikzcd} \tag{4.1}$$

If we denote by:

$$\varphi : \tau^*_{|H'}\Omega^1_{H/C} \xrightarrow{\sim} \Omega^1_{H'/C'}$$

the isomorphism of coherent sheaves on H' induced by the cartesian diagram:

$$\begin{tikzcd} H' \arrow[r, "\tau_{|H'}"] \arrow[d, "\pi'_{|H'}"'] & H \arrow[d, "\pi_{|H}"] \\ C' \arrow[r, "\sigma"'] & C, \end{tikzcd}$$

then the isomorphism of locally free sheaves on $\widetilde{H}' \setminus \nu'^{-1}(\Sigma') \simeq H' \setminus \Sigma'$ defined by the restriction of φ extends to an isomorphism of locally free sheaves on $\widetilde{H}'$:

$$\widetilde{\varphi} : \upsilon^*\omega^1_{\widetilde{H}/C} \xrightarrow{\sim} \omega^1_{\widetilde{H}'/C'}.$$

As explained in 1.2.2 above, we defer the proof of Proposition 4.1 to Section 6.

4.2. **Characteristic classes of $\omega^1_{\widetilde{H}/C}$.**

4.2.1. The following proposition is a generalization of [Mor22, Lemma 5.1.1].

Proposition 4.2. *The following equality holds in* $\mathrm{CH}^N(H)$*:*

$$\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\widetilde{H}/C}) = (c_1c_{N-1})([T_{\pi_{|H}}]) + (-1)^{N-1}\sum_{P\in\Sigma}(1-N/\delta_P)\left((\delta_P-1)^N+(-1)^{N+1}\right)[P]. \tag{4.2}$$

The remainder of this Subsection is devoted to the proof of Proposition 4.2. The method we shall use for this proof is the same as the one used in the derivation of [Mor22, Lemma 5.1.1], and we shall simply state without proof the intermediate results generalizing the ones in [Mor22, Sections 5.2 and 5.3] which were used to establish [Mor22, Lemma 5.1.1].

4.2.2. Let us first introduce some notation and recall several results of [Mor22, 5.2], which did not assume any condition about the singularities of the morphism $\pi_{|H} : H \to C$, and therefore hold in the present setting.

As in [Mor22], for every P in Σ, we introduce the following characteristic class:

$$\eta_P := c_1(\mathcal{O}_{\widetilde{H}}(E_P)) \in \mathrm{CH}^1(\widetilde{H}).$$

As in [Mor22, Proposition 5.2.1], for every point P in Σ, the following equality holds in $\mathrm{CH}^N(H)$:

$$\nu_*\eta_P^N = (-1)^{N-1}[P], \tag{4.3}$$

and for every non-negative integer r, the following equality holds in $\mathrm{CH}^r(E_P)$:

$$c_r(T_{E_P}) = (-1)^r\binom{N}{r}\eta^r_{P|E_P}. \tag{4.4}$$

For every P in Σ and for every non-negative integer r, we may also directly apply equality [Mor22, (5.2.5)] to the smooth hypersurface $Q_P := \widetilde{H_\Delta}\cap E_P$ of degree $m_P := \delta_P$ in the projective space E_P, and get the following equality in $\mathrm{CH}^r(E_P)$:

$$i_{\widetilde{H_\Delta}\cap E_P,E_P*}c_r(T_{\widetilde{H_\Delta}\cap E_P}) = (-1)^{r+1}\delta_P\left[\frac{(1+y)^N}{1+\delta_P y}\right]^{[r]}\eta^{r+1}_{P|E_P}, \tag{4.5}$$

where $i_{\widetilde{H_\Delta}\cap E_P,E_P}$ denotes the inclusion of $\widetilde{H_\Delta}\cap E_P$ in E_P and where, as above, $f(y)^{[r]}$ denotes the coefficient of y^r in some formal series $f(y)\in\mathbb{C}[[y]]$.

Moreover, as in [Mor22, Corollary 5.2.4], for every positive integer r, the following equality holds in $\mathrm{CH}^r(\widetilde{H})$:

$$c_r([T_{\pi_{|H}\circ\nu}]) = \nu^*c_r([T_{\pi_{|H}}]) + (-1)^r\left[\binom{N}{r}-\binom{N}{r-1}\right]\sum_{P\in\Sigma}\eta_P^r. \tag{4.6}$$

4.2.3. Applying the comparison in [Mor22, Proposition 3.1.7, (3.1.12)] of the Chern classes of the vector bundle $\omega^{1\vee}_{\widetilde{H}/C}$ and the relative tangent class $[T_{\pi_{|H}\circ\nu}]$ in the case of a degeneration whose singular fibers are divisors with strict normal crossings with empty 3-codimensional strata[5], we obtain the following equality in $\mathrm{CH}^r(\widetilde{H})$ for every non-negative integer r:

$$c_r(\omega^{1\vee}_{\widetilde{H}/C}) = c_r([T_{\pi_{|H}\circ\nu}]) + \sum_{P\in\Sigma}(\delta_P-1)\,i_{E_P*}c_{r-1}(T_{E_P}) - \sum_{P\in\Sigma}\delta_P\,i_{E_P\cap\widetilde{H_\Delta}*}c_{r-2}(T_{E_P\cap\widetilde{H_\Delta}}). \tag{4.7}$$

[5]This is the case here because of the expression (3.5) for the divisor $\widetilde{H}_\Delta$.

Combining equality (4.7) with equalities (4.4) applied to $r' := r-1$ and (4.5) applied to $r' := r-2$ yields the following equality in $\mathrm{CH}^r(\widetilde{H})$ generalizing [Mor22, Proposition 5.3.2], valid for every non-negative integer r:

$$c_r(\omega^{1\vee}_{\widetilde{H}/C}) = c_r([T_{\pi_{|H}\circ\nu}]) + \sum_{P\in\Sigma} \alpha(N, r, \delta_P)\, \eta_P^r, \tag{4.8}$$

where $\alpha(N, r, \delta)$ is the integer defined by:

$$\alpha(N, r, \delta) = (-1)^{r-1}\left[(\delta-1)\binom{N}{r-1} - \delta^2\left[\frac{(1+y)^N}{1+\delta y}\right]^{[r-2]}\right].$$

When r is positive, combining equality (4.8) with (4.6) yields the following equality in $\mathrm{CH}^r(\widetilde{H})$ generalizing [Mor22, Corollary 5.3.3]:

$$c_r(\omega^{1\vee}_{\widetilde{H}/C}) = \nu^* c_r([T_{\pi_{|H}}]) + \sum_{P\in\Sigma} \beta(N, r, \delta_P)\, \eta_P^r, \tag{4.9}$$

where $\beta(N, r, \delta)$ is the integer defined by:

$$\beta(N, r, \delta) = (-1)^r\left[\binom{N}{r} - \delta\binom{N}{r-1} + \delta^2\left[\frac{(1+y)^N}{1+\delta y}\right]^{[r-2]}\right].$$

Multiplying equality (4.9) applied to $r = 1$ with itself applied to $r = N-1$, and using (2.4) applied to $n := N$, $r := 3$ and $a := \delta_P$, and the fact that the restriction to E_P of the class $\nu^*[T_{\pi_{|H}}]$ vanishes, we obtain the following equality in $\mathrm{CH}^N(H)$ generalizing [Mor22, Corollary 5.3.4]:

$$(c_1 c_{N-1})(\omega^{1\vee}_{\widetilde{H}/C}) = \nu^*(c_1 c_{N-1})([T_{\pi_{|H}}]) + \sum_{P\in\Sigma} (1 - N/\delta_P)\left((\delta_P - 1)^N + (-1)^{N+1}\right)\eta_P^N. \tag{4.10}$$

Pushing forward by ν equality (4.10) and using equality (4.3) concludes the proof.

5. The geometry of the hypersurface H in the smooth pencil X over C

5.1. **The top Chern class of $\Omega^1_{X/C|H} \otimes L_{|H}$.** Let us define a morphism s of vector bundles on H by the composition:

$$s : L^\vee_{|H} \simeq \mathcal{N}_H X^\vee \longrightarrow \Omega^1_{X|H} \longrightarrow \Omega^1_{X/C|H}.$$

We will also see s as a section of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$.

The following proposition is a generalization of the second statement of [Mor22, Lemma 6.1.3].

Proposition 5.1. *The section s is a regular section*[6] *of $\Omega^1_{X/C|H} \otimes L_{|H}$, and the 0-cycle in H defined by its vanishing is given by:*

$$[(s = 0)] = \sum_{P\in\Sigma} (\delta_P - 1)^N [P].$$

Proof. By definition, the subset of X defined by the vanishing of s is precisely the set of critical points of the morphism $\pi_{|H} : H \to C$, namely Σ. The regularity of the section s follows from the fact that Σ has codimension N in the smooth N-dimensional manifold H, while the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$ is of rank N.

Now let us compute the multiplicity of the 0-cycle $[(s = 0)]$ at some point P in Σ.

As in Subsection 3.1 and Proposition 3.1 above, we denote by t a local coordinate of C near $\pi(P)$, and we complete its pullback $\pi^* t$ into a local coordinate system $(\pi^* t, x_1, \ldots, x_N)$ of X near P. We still denote by $F_{\delta_P}(T_1, \ldots, T_N)$ an equation in $\mathbb{C}[T_1, \ldots, T_N]$ of the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ seen as a projective hypersurface in $\mathbb{P}(T_P H)$ that we identify with $\mathbb{P}^{N-1}_{\mathbb{C}}$ through the

[6] in the sense of [Ful98, B.3.4].

(differential at P of the) coordinate system $(x_{1|H}, \ldots, x_{N|H})$. It is a homogeneous polynomial of degree δ_P, with nonzero discriminant.

According to Proposition 3.1, we may assume that the germ at P of the divisor H in X is defined by the equation:

$$\pi^* t = F_{\delta_P}(x_1, \ldots, x_N) + F_{>\delta_P}, \tag{5.1}$$

for some $F_{>\delta_P}$ in $\mathfrak{m}_{X,P}^{\delta_P+1}$.

The local trivialization of the line bundle $L := \mathcal{O}_X(H)$ on X given by this equation and the local frame of the vector bundle $\Omega^1_{X/C}$ on X defined by the relative differentials $([dx_1], \ldots, [dx_N])$ induce by restriction and tensor product a local frame of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$ on H. In this local frame, the section s is given by:

$$s = \big(\partial_{x_{1|H}}(F_{\delta_P}(x_{1|H}, \ldots, x_{N|H}) + F_{>\delta_P|H}), \ldots, \partial_{x_{N|H}}(F_{\delta_P}(x_{1|H}, \ldots, x_{N|H}) + F_{>\delta_P|H})\big).$$

Consequently, the multiplicity at P of the 0-cycle $(s = 0)$ is precisely the multiplicity at P of the function on H given by the restriction to H of either side of equality (5.1), or equivalently, the Milnor number of the singularity at P of the morphism $\pi_{|H} : H \to C$.

The fact that this multiplicity is equal to $(\delta_P - 1)^N$ is therefore a consequence of the computation of the Milnor number of an isolated semiquasihomogeneous singularity, in the particular semihomogeneous case where the weights are all equal to 1. This computation was done in [MO70] in the quasihomogeneous case where $F_{>\delta_P}$ vanishes; the reduction of the semiquasihomogeneous case to this case appears for instance in [AGZV85, Theorem p. 194]. □

From Proposition 5.1, we may derive the equality (1.4) as in the proof of [Mor22, (6.1.10)], simply replacing the 0-cycle $i_*[\Sigma]$ in *loc. cit.* by $\sum_{P\in\Sigma}(\delta_P - 1)^N[P]$. This proof uses the fact that the cycle $[(s = 0)]$ defined by the vanishing of the regular section s is precisely the top Chern class of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$ (see for instance [Ful98, Example 3.2.16, (ii)]), together with simple computations of Chern classes of vector bundles.

These computations also yield the following equality in $\mathrm{CH}_0(X)$, which shall be useful to us later:

$$\sum_{P\in\Sigma}(\delta_P - 1)^N\,[P] = \sum_{0\leq k\leq N} c_1(L)^{N+1-k} c_k(\Omega^1_{X/C}). \tag{5.2}$$

5.2. **Characteristic classes of the hypersurfaces H and H'.** With the notation introduced in 3.2.1 and 3.2.2, let us denote by η' the generic point of the curve C'. Recall that the complex scheme $\widetilde{H}'$ is smooth and that the divisor $\widetilde{H}'_{\Delta'}$ is reduced with strict normal crossings, so that for every integer n, the local monodromy at every point in Δ' of the variation of Hodge structures:

$$\mathbb{H}^n(H'_{\eta'}/C'_{\eta'}) \simeq \mathbb{H}^n(\widetilde{H}'_{\eta'}/C'_{\eta'})$$

is unipotent.

The following proposition is an analogue of [Mor22, Proposition 6.1.5].

Proposition 5.2. *If the line bundle L' on X' is ample relatively to the morphism π', then for every integer n such that $n < N - 1$, the following equality holds in* $\mathrm{CH}_0(C')$*:*

$$c_1(\mathcal{GK}_{C'}(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'}))) = c_1(\mathcal{GK}_{C'}(\mathbb{H}^n(X'/C'))), \tag{5.3}$$

and for every integer n such that $n > N - 1$, the following equality holds in $\mathrm{CH}_0(C')_{\mathbb{Q}}$*:*

$$c_1(\mathcal{GK}_{C'}(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'}))) = c_1(\mathcal{GK}_{C'}(\mathbb{H}^{n+2}(X'/C'))). \tag{5.4}$$

Proof. The proof uses the arguments used in the proof of [Mor22, Proposition 6.1.5]. Namely, for proving (5.3), we apply Lefschetz's weak theorem to the smooth horizontal hypersurface $H' \setminus H'_{\Delta'}$ in $X' \setminus X'_{\Delta'}$ to obtain an isomorphism, for every $n < N - 1$, between the variations of Hodge structures

$\mathbb{H}^n(H' \setminus H'_{\Delta'}/C' \setminus \Delta')$ and $\mathbb{H}^n(X' \setminus X'_{\Delta'}/C' \setminus \Delta')$ on $C' \setminus \Delta'$. Then we use the unipotence of the local monodromy of both variations of Hodge structures to extend this isomorphism into an isomorphism of Deligne extensions over C', and therefore of Griffiths line bundles.

Moreover equality (5.4) follows from equality (5.3) applied to the integer:

$$n' := 2(N-1) - n < N-1,$$

and from the consequence of Poincaré's duality [Mor22, Proposition 2.5.2], applied to the degeneration $\widetilde{H}'/C'$ whose singular fibers form a reduced divisor with strict normal crossings and to the smooth degeneration X'/C'. Indeed, applied to these data, [Mor22, Proposition 2.5.2] asserts that for every $n > N-1$, the following line bundles on C':

$$\mathcal{GK}_{C'}(\mathbb{H}^{2(N-1)-n}(H'_{\eta'}/C'_{\eta'})) \otimes \mathcal{GK}_{C'}(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'}))^\vee$$

and:

$$\mathcal{GK}_{C'}(\mathbb{H}^{2N-n}(X'/C')) \otimes \mathcal{GK}_{C'}(\mathbb{H}^n(X'/C'))^\vee$$

are of 2-torsion. □

The following proposition is a generalization of [Mor22, (6.1.11)].

Proposition 5.3. *Denoting by i the closed inclusion of H in X, the following equality holds in* $\mathrm{CH}_0(X)$*:*

$$(5.5)\quad \begin{aligned} i_*(c_1c_{N-1})([T_{\pi_{|H}}]) &= (c_1c_N)([T_\pi]) \\ &+ (-1)^N\big([(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})]^{(N+1)} + \sum_{P\in\Sigma}(\delta_P-1)^N[P] - c_1(L)c_N(\Omega^1_{X/C})\big). \end{aligned}$$

Proof. The proof relies on the same computations of Chern classes on H and X as in the proof of [Mor22, (6.1.11)], simply replacing the 0-cycle $i_*[\Sigma]$ in *loc. cit.* by $\sum_{P\in\Sigma}(\delta_P-1)^N[P]$, using the expression of this sum given by equality (5.2), and using that the morphism of smooth schemes $\pi_{|H}: H \to C$ is smooth on a dense open subset of H so that the following equality holds in $K^0(H)$:

$$[T_{\pi_{||H}}] = [\Omega^1_{H/C}]^\vee. \qquad \square$$

5.3. **Completing the proof of Theorem 1.2.** As equality (1.4) was established in Subsection 5.1, to complete the proof of Theorem 1.2, it only remains to show equality (1.5) under the hypothesis that the line bundle L on X is ample relatively to the morphism π.

5.3.1. *The alternating sum* $\sum_{n=0}^{2(N-1)}(-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big)$. We shall use the notation introduced in 3.2.1 and 3.2.2 and denote by η' the generic point of the curve C'. As already observed at the beginning of 5.2, for every integer n, the local monodromy at every point in Δ' of the variation of Hodge structures:

$$\mathbb{H}^n(H'_{\eta'}/C'_{\eta'}) \simeq \mathbb{H}^n(\widetilde{H}'_{\eta'}/C'_{\eta'})$$

is unipotent.

Recall also that by hypothesis the restriction $\pi_{|\Sigma}: \Sigma \to C$ is injective, so that the restriction $\pi'_{|\Sigma'}: \Sigma' \to C'$ also is injective.

Consequently, applying the special case of Theorem 2.1 stated in equality (2.3) to the degeneration $\widetilde{H}'/C'$, which over η' coincides with H'/C' and whose divisor of singular fibers $\widetilde{H}'_{\Delta'}$ is a reduced divisor with strict normal crossings as shown by equality (3.4), the following equality holds:

$$(5.6)\qquad \sum_{n=0}^{2(N-1)}(-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) = 1/12\int_{\widetilde{H}'}(c_1c_{N-1})(\omega^{1\vee}_{\widetilde{H}'/C'}) + 1/12\,\chi_{\mathrm{top}}(\widetilde{H'_{\Delta'}} \cap E')).$$

Using Proposition 3.2, for every point P' in Σ' the Euler characteristic of the connected component $\widetilde{H'_{\Delta'}} \cap E'_{P'}$ of $\widetilde{H'_{\Delta'}} \cap E'$ is given by:

$$\chi_{\mathrm{top}}(\widetilde{H'_{\Delta'}} \cap E'_{P'}) = (-1)^N/\delta_{\tau(P')}\ [(\delta_{\tau(P')} - 1)^N + (-1)^N(N\delta_{\tau(P')} - 1)].$$

Since the morphism of smooth curves $\sigma : C' \to C$ is ramified at every point x' in Δ' with ramification index $\delta_{P_{\sigma(x')}}$ where $P_{\sigma(x')}$ denotes the unique point in $\Sigma \cap \pi^{-1}(\sigma(x'))$, each point x in Δ has precisely $\deg(\sigma)/\delta_{P_x}$ pre-images in Δ', and therefore each point P in Σ has precisely $\deg(\sigma)/\delta_P$ pre-images in Σ'. Consequently the Euler characteristic of $\widetilde{H'_{\Delta'}} \cap E'$ can be written as follows:

$$\begin{aligned}\chi_{\mathrm{top}}(\widetilde{H'_{\Delta'}} \cap E') &= \sum_{P\in\Sigma} \deg(\sigma)/\delta_P\ (-1)^N/\delta_P\ [(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)]\\ &= (-1)^N \deg(\sigma) \sum_{P\in\Sigma} 1/\delta_P^2\ [(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)]. \qquad (5.7)\end{aligned}$$

Replacing (5.7) in (5.6) yields the following equality:

$$\begin{aligned}\sum_{n=0}^{2(N-1)} (-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) &= 1/12 \int_{\widetilde{H'}} (c_1c_{N-1})(\omega^{1\vee}_{\widetilde{H}'/C'})\\ &\quad + (-1)^N \deg(\sigma)/12 \sum_{P\in\Sigma} 1/\delta_P^2\ [(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)]. \qquad (5.8)\end{aligned}$$

According to Proposition 4.1, the vector bundle $\omega^{1\vee}_{\widetilde{H}'/C'}$ is isomorphic to $\upsilon^*\omega^{1\vee}_{\widetilde{H}/C}$. Moreover the restriction of the morphism $\upsilon : \widetilde{H}' \to \widetilde{H}$ to the dense open subset $\widetilde{H}' \setminus E' \simeq H' \setminus \Sigma'$ coincides with the morphism

$$\tau_{|H'\setminus\Sigma'} : H' \setminus \Sigma' \longrightarrow H \setminus \Sigma$$

which is, by base change, a finite morphism of degree $\deg(\sigma)$. Therefore the morphism υ is generically finite, of degree $\deg(\sigma)$. Consequently, using the projection formula applied to υ, we may rewrite equality (5.8) as follows:

$$\begin{aligned}\sum_{n=0}^{2(N-1)} (-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) &= \deg(\sigma)/12 \int_{\widetilde{H}} (c_1c_{N-1})(\omega^{1\vee}_{\widetilde{H}/C})\\ &\quad + (-1)^N \deg(\sigma)/12 \sum_{P\in\Sigma} 1/\delta_P^2\ [(\delta_P - 1)^N + (-1)^N(N\delta_P - 1)]. \qquad (5.9)\end{aligned}$$

Using Proposition 4.2 and the projection formula applied to the birational morphism ν, we may rewrite equality (5.9) as follows:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) = \deg(\sigma)/12 \int_H (c_1c_{N-1})([T_{\pi_{|H}}]) + \deg(\sigma) \sum_{P\in\Sigma} u_{N,\delta_P}, \qquad (5.10)$$

where $u_{N,\delta}$ is the rational number given by:

$$\begin{aligned}u_{N,\delta} &:= (-1)^{N-1}/12\ (1 - N/\delta)[(\delta - 1)^N + (-1)^{N+1}] + (-1)^N/(12\delta^2)\ [(\delta - 1)^N + (-1)^N(N\delta - 1)]\\ &= (-1)^N(\delta - 1)/(12\delta^2)\ [(N\delta - \delta^2 + 1)(\delta - 1)^{N-1} + (-1)^N(\delta + 1)].\end{aligned}$$

Using Proposition 5.3, we may rewrite equality (5.10) as follows:

$$
\begin{aligned}
(5.11)\qquad \sum_{n=0}^{2(N-1)} (-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) &= \deg(\sigma)/12 \int_X (c_1c_N)([T_\pi]) \\
+ (-1)^N \deg(\sigma)/12 \left(\int_X [(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})]^{(N+1)} - \int_X c_1(L)c_N(\Omega^1_{X/C})\right) \\
+ \deg(\sigma)\sum_{P\in\Sigma} v_{N,\delta_P},
\end{aligned}
$$

where $v_{N,\delta}$ is the rational number given by:

$$
\begin{aligned}
v_{N,\delta} &:= u_{N,\delta} + (-1)^N(\delta-1)^N/12 \\
&= (-1)^N w_{N,\delta}. \qquad (5.12)
\end{aligned}
$$

5.3.2. Using again Theorem 2.1, this time applied to the smooth morphism $\pi' : X' \to C'$, yields the following equality:

$$
\sum_{n=0}^{2N}(-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(X'/C')\big) = 1/12 \int_{X'} (c_1c_N)([T_{\pi'}]). \tag{5.13}
$$

Moreover, using that by base change the line bundle L' on X' is ample relatively to the morphism π', since L is ample relatively to π, Proposition 5.2 implies the following equality:

$$
\begin{aligned}
(5.14)\qquad \sum_{n=0}^{2(N-1)} (-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(H'_{\eta'}/C'_{\eta'})\big) - \sum_{n=0}^{2N}(-1)^{n-1}\mathrm{ht}_{GK}\big(\mathbb{H}^n(X'/C')\big) \\
= (-1)^N\Big[\mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(H'_{\eta'}/C'_{\eta'})\big) - \mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(X'/C')\big) + \mathrm{ht}_{GK}\big(\mathbb{H}^{N}(X'/C')\big) \\
- \mathrm{ht}_{GK}\big(\mathbb{H}^{N+1}(X'/C')\big)\Big].
\end{aligned}
$$

Combining equalities (5.11), (5.12), (5.13), and (5.14) yields the following equalities:

$$
\begin{aligned}
\mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(H'_{\eta'}/C'_{\eta'})\big) &= \mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(X'/C')\big) - \mathrm{ht}_{GK}\big(\mathbb{H}^{N}(X'/C')\big) + \mathrm{ht}_{GK}\big(\mathbb{H}^{N+1}(X'/C')\big) \\
&\quad - (-1)^N/12 \int_{X'} (c_1c_{N-1})([T_{\pi'}]) + (-1)^N\deg(\sigma)/12 \int_X (c_1c_N)([T_\pi]) \\
&\quad + \deg(\sigma)/12 \left(\int_X [(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})] - \int_X c_1(L)c_N(\Omega^1_{X/C})\right) \\
&\quad + \deg(\sigma)\sum_{P\in\Sigma} w_{N,\delta_P} \\
&= \deg(\sigma)\Big[\mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(X/C)\big) - \mathrm{ht}_{GK}\big(\mathbb{H}^{N}(X'/C')\big) + \mathrm{ht}_{GK}\big(\mathbb{H}^{N+1}(X/C)\big)\Big] \\
&\quad + \deg(\sigma)/12 \left(\int_X [(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})] - \int_X c_1(L)c_N(\Omega^1_{X/C})\right) \\
&\quad + \deg(\sigma)\sum_{P\in\Sigma} w_{N,\delta_P}, \qquad (5.15)
\end{aligned}
$$

where in (5.15) we used that $\pi' : X' \to C'$ is the base change of $\pi : X \to C$ by $\sigma : C' \to C$.

Since the local monodromy of the variation of Hodge structures $\mathbb{H}^{N-1}(H'_{\eta'}/C'_{\eta'})$ is unipotent, the stable Griffiths height $\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big)$ is by definition given by:

$$
\mathrm{ht}_{GK,stab}\big(\mathbb{H}^{N-1}(H_\eta/C_\eta)\big) = 1/\deg(\sigma)\, \mathrm{ht}_{GK}\big(\mathbb{H}^{N-1}(H'_{\eta'}/C'_{\eta'})\big).
$$

Combining this relation and equality (5.15) concludes the proof of equality (1.5) and of Theorem 1.2.

6. Proof of Proposition 4.1

The existence of υ such that (4.1) is commutative will follow from the fact that $(\tau_{|H'} \circ \nu')^*(\Sigma)$ is a Cartier divisor and from the universal property of blow-ups, and its unicity is clear.

The assertions that $(\tau_{|H'} \circ \nu')^*(\Sigma)$ and E' coincide and that φ extends to an isomorphism $\widetilde{\varphi}$ between $\upsilon^*\omega^1_{\widetilde{H}/C}$ and $\omega^1_{\widetilde{H}'/C'}$ are local on $\widetilde{H}'$, and clearly satisfied outside of the closed subscheme $E' := \nu'^*(\Sigma')$.

Consequently, we may choose a complex point Q' in E', and work locally near Q' to establish these assertions.

We shall denote by P' the image of Q' by ν' and by P the image of P' by τ:

$$P' := \nu'(Q') \in \Sigma' \quad \text{and} \quad P := \tau(P') \in \Sigma.$$

For simplicity's sake, we shall write δ for δ_P.

6.1. Coordinate systems on X and X'.

6.1.1. *Notation.* As in Subsection 3.1 and Proposition 3.1 above, we denote by t a local coordinate of C near $\pi(P)$, and we complete its pullback π^*t into a local coordinate system $(\pi^*t, x_1, \dots, x_N)$ of X near P.

We still denote by $F_\delta(T_1, \dots, T_N)$ an equation in $\mathbb{C}[T_1, \dots, T_N]$ of the projective tangent cone $\mathbb{P}(C_P H_{\pi(P)})$ seen as a projective hypersurface in $\mathbb{P}(T_P H)$ that we identify with $\mathbb{P}^{N-1}_{\mathbb{C}}$ through the (differential at P of the) coordinate system $(x_{1|H}, \dots, x_{N|H})$. It is a homogeneous polynomial of degree δ, with nonzero discriminant. According to Proposition 3.1, we may assume that the germ at P of the divisor H in X is defined by the equation:

$$\pi^*t = F_\delta(x_1, \dots, x_N) + F_{>\delta}, \tag{6.1}$$

for some $F_{>\delta_P}$ in $\mathfrak{m}_{X,P}^{\delta_P+1}$.

Let t' be a local coordinate of C' near $\pi'(P')$. As the finite covering $\sigma : C' \to C$ is ramified with index δ at $\pi'(P')$, there exists a unit u in $\mathcal{O}_{C',\pi'(P')}$ such that the following relation holds:

$$\sigma^*t = u\, t'^\delta.$$

Consequently the smooth complex scheme X' admits $(\pi'^*t', x'_1, \dots, x'_N)$ as local coordinate system near P', where for every integer i such that $1 \leq i \leq N$:

$$x'_i := \tau^* x_i.$$

Moreover the divisor H' in X' is defined by the following equation:

$$\pi'^*u.\pi'^*t'^\delta = F_\delta(x'_1, \dots, x'_N) + \tau^* F_{>\delta}. \tag{6.2}$$

6.1.2. *Description of the blow-ups $\widetilde{X}'$ and $\widetilde{X}$.* As the X'-scheme $\widetilde{X}'$ (resp. the X-scheme $\widetilde{X}$) is locally the blow-up of P' (resp. P) in X' (resp. X), it can locally[7] be identified with the closed subscheme of $X' \times \mathbb{P}^N_{\mathbb{C}}$ (resp. $X \times \mathbb{P}^N_{\mathbb{C}}$) defined by the following equations, where $(v'_0 : v'_1 : \dots : v'_N)$ (resp. $(v_0 : v_1 : \dots : v_N)$) denote homogeneous coordinates on $\mathbb{P}^N_{\mathbb{C}}$:

$$v'_0.\chi'^* x'_i = \chi'^*\pi'^*t'.v'_i \quad (\text{resp. } v_0.\chi^* x_i = \chi^*\pi^*t.v_i)$$

for every integer $i \in \{1, \dots, N\}$, and:

$$v'_i.\chi'^* x'_j = \chi'^* x'_i.v'_j \quad (\text{resp. } v_i.\chi^* x_j = \chi^* x_i.v_j)$$

[7]Namely on some open neighborhood of $Z'_{P'} := \chi'^{-1}(P')$ (resp. of $Z_P := \chi^{-1}(P)$).

for every pair of integers $(i,j) \in \{1,\dots,N\}^2$ such that $i \neq j$.

In particular, an open neighborhood of $Z'_{P'} := \chi'^{-1}(P')$ (resp. of $Z_P := \chi^{-1}(P)))$ in the complex scheme $\widetilde{X}'$ (resp. in $\widetilde{X}$) admits a covering by the open subsets $V'_0,\dots,V'_N$ (resp. $V_0,\dots,V_N$) defined by the non-vanishing of $v'_0,\dots,v'_N$ (resp. of $v_0,\dots,v_N$).

6.2. **Description of $\omega^1_{\widetilde{H}/C|V_1}$.** For future use, let us describe the open subset V_1 of $\widetilde{X}$. This open subset admits as coordinate system[8]:

$$(\widetilde{\pi^* t_1}, \chi^* x_1, \widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1})$$

where $\widetilde{\pi^* t_1}$ is defined by:

$$\widetilde{\pi^* t_1} := u_0/u_1$$

and satisfies the following equality:

$$\chi^*\pi^* t = \widetilde{\pi^* t_1}.\chi^* x_1, \tag{6.3}$$

and where for every integer $i \in \{2,\dots,N\}$, $\widetilde{x}_{i,1}$ is defined by:

$$\widetilde{x}_{i,1} := u_i/u_1$$

and satisfies the following equality:

$$\chi^* x_i = \widetilde{x}_{i,1}.\chi^* x_1. \tag{6.4}$$

Let us describe the divisor $\widetilde{H} \cap V_1$ in V_1. Using firstly the equation (6.1) of the divisor H in X and equalities (6.3) and (6.4), and then the homogeneity of F_δ, the divisor $\chi^* H \cap V_1$ in V_1 is defined by the following equation:

$$\begin{aligned}\widetilde{\pi^* t_1}.\chi^* x_1 &= F_\delta(\chi^* x_1, \widetilde{x}_{2,1}.\chi^* x_1,\dots,\widetilde{x}_{N,1}.\chi^* x_1) + F_{>\delta}\\ &= \chi^* x_1^\delta\left[F_\delta(1,\widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1}) + \chi^* x_1.h_1\right],\end{aligned}$$

where h_1 is defined by:

$$h_1 := \chi^* x_1^{-\delta-1}\cdot\chi^* F_{>\delta},$$

and actually is a (germ of) regular function on V_1 along $Z_P \cap V_1$, since $F_{>\delta}$ belongs to the ideal $\mathfrak{m}^{\delta+1}_{H,P}$ of $\mathcal{O}_{H,P}$, and therefore the pull-back $\chi^* F_{>\delta}$ vanishes at order $\delta+1$ on the exceptional divisor Z_P.

For convenience's sake, we define the following function on V_1:

$$w := F_\delta(1,\widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1}) + \chi^* x_1.h_1.$$

Consequently, the divisor $\widetilde{H} \cap V_1 = (\chi^* H - Z) \cap V_1$ in V_1 is defined by the following equation:

$$\widetilde{\pi^* t_1} = \chi^* x_1^{\delta-1}.w, \tag{6.5}$$

and admits $(\chi^* x_1, \widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1})$ as coordinate system.

Moreover, the divisor $\widetilde{H}_\Delta \cap V_1$ in $\widetilde{H} \cap V_1$ is defined by the following equation:

$$\widetilde{\pi^* t_1}.\chi^* x_1 = 0,$$

or equivalently:

$$\chi^* x_1^\delta.w = 0. \tag{6.6}$$

Since the homogeneous polynomial F_δ has non-zero discriminant, for every point Q of $E_P \cap V_1 = Z_P \cap \widetilde{H} \cap V_1$, there exists an integer $i(Q)$ in $\{2,\dots,N\}$ such that the derivative

$$\partial_{\widetilde{x}_{i(Q),1}} F_\delta(1,\widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1})$$

does not vanish at Q. Moreover, since $\chi^* x_1$ vanishes at Q because Q is in Z_P, the partial derivative $\partial_{\widetilde{x}_{i(Q),1}} w$ also does not vanish.

[8]In other words, the morphism $(\widetilde{\pi^* t_1}, \chi^* x_1, \widetilde{x}_{2,1},\dots,\widetilde{x}_{N,1}) : V_1 \to \mathbb{A}^N_{\mathbb{C}}$ is étale.

Consequently, in some neighborhood of Q, the complex scheme $\widetilde{H} \cap V_1$ admits as coordinate system:

$$(\chi^* x_1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{i(Q)-1,1}, w, \widetilde{x}_{i(Q)+1,1}, \dots, \widetilde{x}_{N,1}).$$

This coordinate system is well-suited to the equation (6.6) of the divisor with normal crossings $\widetilde{H}_\Delta \cap V_1$ in $\widetilde{H} \cap V_1$, and the locally free coherent sheaf $\omega^1_{\widetilde{H}/C}$ of rank $N-1$ on $\widetilde{H} \cap V_1$ admits the following set of N local generators near Q:

$$\big(\chi^*[dx_1/x_1], [dw/w], [d\widetilde{x}_{2,1}], \dots, [d\widetilde{x}_{i(Q)-1,1}], [d\widetilde{x}_{i(Q)+1,1}], \dots, [d\widetilde{x}_{N,1}]\big),$$

satisfying the following relation, which follows from (6.3):

$$0 = \chi^*\pi^*[dt/t] = \delta\chi^*[dx_1/x_1] + [dw/w].$$

In particular, the vector bundle $\omega^1_{\widetilde{H}/C}$ admits the following local frame near Q:

$$\big(\chi^*[dx_1/x_1], [d\widetilde{x}_{2,1}], \dots, [d\widetilde{x}_{i(Q)-1,1}], [d\widetilde{x}_{i(Q)+1,1}], \dots, [d\widetilde{x}_{N,1}]\big). \tag{6.7}$$

The remainder of the proof of Proposition 4.1 shall be divided into two cases, according to the location of the point Q' of $\widetilde{H}'$ relatively to the covering $V'_0, \dots, V'_N$.

In both cases, we shall give a local equation of the divisor $\widetilde{H}'$ in $\widetilde{X}'$ in terms of the coordinate systems introduced in 6.1 above, and establish the existence of the morphism $\upsilon : \widetilde{H}' \to \widetilde{H}$. Then we shall give a local frame near Q' of the locally free $\mathcal{O}_{\widetilde{H}'}$-module $\omega^1_{\widetilde{H}'/C'}$. Comparing this local frame with the pullback by υ of the local frame (6.7) will finally allow us to conclude that $\omega^1_{\widetilde{H}'/C'}$ and $\upsilon^*\omega^1_{\widetilde{H}/C}$ are naturally isomorphic.

6.3. Case 1: The point Q' belongs to the open subset V'_0 of $\widetilde{X}'$.

6.3.1. The open subset V'_0 admits as coordinate system:

$$(\chi'^*\pi'^*t', \widetilde{x}'_{1,0}, \dots, \widetilde{x}'_{N,0}),$$

where for every integer $i \in \{1, \dots, N\}$, $\widetilde{x}'_{i,0}$ is defined by:

$$\widetilde{x}'_{i,0} := v'_i/v'_0,$$

and satisfies the following equality of functions on V'_0:

$$\chi'^* x'_i = \widetilde{x}'_{i,0}.\chi'^*\pi'^*t'. \tag{6.8}$$

Let us describe the divisor $\widetilde{H}' \cap V'_0$ in V'_0. Using firstly the equation (6.2) of the divisor H' in X' and equality (6.8), then the homogeneity of F_δ, the divisor $\chi'^*H' \cap V'_0$ in V'_0 is defined by the following equation:

$$\begin{aligned}\chi'^*\pi'^*(u.t'^\delta) &= F_\delta(\widetilde{x}'_{1,0}.\chi'^*\pi'^*t', \dots, \widetilde{x}'_{N,0}.\chi'^*\pi'^*t') + \chi'^*\tau^*F_{>\delta} \\ &= \chi'^*\pi'^*t'^\delta \big[F_\delta(\widetilde{x}'_{1,0}, \dots, \widetilde{x}'_{N,0}) + \chi'^*\pi'^*t'\, h'_0\big],\end{aligned}$$

where h'_0 is defined by:

$$h'_0 := \chi'^*\pi'^*t'^{-\delta-1}\,.\,\chi'^*\tau^*F_{>\delta},$$

and, similarly to h_1, is actually a (germ of) regular function on V'_0 along $Z'_{P'} \cap V'_0$, since the pull-back $\chi'^*\tau^*F_{>\delta}$ vanishes at order $\delta+1$ on the exceptional divisor $Z'_{P'}$.

Consequently, the divisor $\widetilde{H}' \cap V'_0 = (\chi'^*H' - Z'_\delta) \cap V'_0$ in V'_0 is defined by the following equation:

$$\chi'^*\pi'^*u = F_\delta(\widetilde{x}'_{1,0}, \dots, \widetilde{x}'_{N,0}) + \chi'^*\pi'^*t'\, h'_0. \tag{6.9}$$

By hypothesis, Q' is in $\widetilde{H}'_{\Delta'}$, so that this equation is satisfied at Q' and $\chi'^*\pi'^*t'$ vanishes at Q'. Consequently there exists an integer i such that $1 \le i \le N$ and $\widetilde{x}'_{i,0}$ does not vanish at Q'.

Without loss of generality, we may assume that $i = 1$. Let $W \subset V'_0$ be the open subset $(\widetilde{x'}_{i,0} \neq 0)$, which contains Q'.

The restriction to W of the morphism $\tau \circ \chi' : \widetilde{X}' \to X$ is described by the following equalities of functions on W:

$$(\tau \circ \chi')^* \pi^* t = \chi'^* \pi'^* (u.t'^{\delta}),$$

and, for every integer $i \in \{1, \dots, N\}$:

$$(\tau \circ \chi')^* x_i = \chi'^* x'_i = \widetilde{x}'_{i,0}.\chi'^* \pi'^* t'.$$

These equalities, along with the non-vanishing of $\widetilde{x}_{1,0}$ on W, imply that the subscheme $(\tau \circ \chi')^{-1}(\Sigma) \cap W$ of W is defined by the vanishing of $\chi'^* \pi'^* t'$, and is therefore the exceptional divisor $Z'_{P'} \cap W$.

Along with the universal property of blow-ups, this shows that there exists a unique morphism[9] $\upsilon_{|W} : W \to \widetilde{X}'$ such that the following diagram is commutative:

$$\begin{tikzcd} W \arrow[r, "\upsilon_{|W}"] \arrow[d, "\chi'_{|W}"'] & \widetilde{X} \arrow[d, "\chi"] \\ X' \arrow[r, "\tau"'] & X. \end{tikzcd} \tag{6.10}$$

Moreover $\upsilon_{|W}$ satisfies the following equality of morphisms from W to $\mathbb{P}^N$:

$$\upsilon_{|W}^* [v_0 : \dots : v_N] = [\chi'^* \pi'^* (u.t'^{\delta - 1}) : \widetilde{x}'_{1,0} : \dots : \widetilde{x}'_{N,0}].$$

In particular, the restriction $\upsilon_{|W}$ has values in the open subset $V_1 := (v_1 \neq 0)$ of $\widetilde{X}$, and its description in the coordinate system $(\widetilde{\pi^* t_1}, \chi^* x_1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1})$ of V_1 defined above is given by the following equalities of functions on W:

$$\upsilon_{|W}^* \widetilde{\pi^* t_1} = \chi'^* \pi'^* (u.t'^{\delta - 1})\, \widetilde{x}'^{-1}_{1,0}, \tag{6.11}$$

$$\upsilon_{|W}^* \chi^* x_1 = \widetilde{x}'_{1,0}.\chi'^* \pi'^* t', \tag{6.12}$$

and, for every integer $i \in \{2, \dots, N\}$:

$$\upsilon_{|W}^* \widetilde{x}_{i,1} = \widetilde{x}'_{i,0}\, \widetilde{x}'^{-1}_{1,0}. \tag{6.13}$$

Since the subscheme $(\tau \circ \chi')^{-1}(\Sigma) \cap W$ of W coincides with the exceptional divisor $Z'_{P'} \cap W$, we obtain by restriction that the subscheme $(\tau_{|H'} \circ \nu')^{-1}(\Sigma) \cap W$ of $\widetilde{H}' \cap W$ coincides with the divisor $E'_{P'} \cap W$. Along with the existence of a morphism $\upsilon_{|W}$ such that the diagram (6.10) is commutative, this establishes the first assertion of Proposition 4.1 in Case 1.

Moreover equalities (6.11), (6.12), and (6.13) imply by restriction the following equalities of functions on $\widetilde{H}' \cap W$:

$$\upsilon_{|\widetilde{H}' \cap W}^* \widetilde{\pi^* t_1} = \chi'^* \pi'^* (u.t'^{\delta - 1})\, \widetilde{x}'^{-1}_{1,0}, \tag{6.14}$$

$$\upsilon_{|\widetilde{H}' \cap W}^* \chi^* x_1 = \widetilde{x}'_{1,0}.\chi'^* \pi'^* t', \tag{6.15}$$

and, for every integer $i \in \{2, \dots, N\}$:

$$\upsilon_{|\widetilde{H}' \cap W}^* \widetilde{x}_{i,1} = \widetilde{x}'_{i,0}\, \widetilde{x}'^{-1}_{1,0}. \tag{6.16}$$

[9]The notation $\upsilon_{|W}$ is slightly abusive, since there is no morphism from $\widetilde{X}'$ to $\widetilde{X}$ that would fit in the top line of the diagram (6.10).

6.3.2. Now let us describe locally the vector bundle $\omega^1_{\widetilde{H}'/C'}$ near Q'. Similarly to the function w on V_1, a function w'_0 on W may be defined as follows:

$$\begin{aligned} w'_0 &:= F_\delta(1, \widetilde{x}'_{2,0}.\widetilde{x}'^{-1}_{1,0}, \ldots, \widetilde{x}'_{N,0}.\widetilde{x}'^{-1}_{1,0}) + \chi'^*\pi'^*t'\widetilde{x}'^{-\delta}_{1,0}h'_0 \\ &= \widetilde{x}'^{-\delta}_{1,0}.[F_\delta(\widetilde{x}'_{1,0}, \ldots, \widetilde{x}'_{N,0}) + \chi'^*\pi'^*t'.h'_0], \end{aligned}$$

where the second equality follows from the homogeneity of F_δ.

Since the homogeneous polynomial F_δ has non-zero discriminant, there exists an integer j in $\{2, \ldots, N\}$ such that the derivative

$$\partial_{\widetilde{x}_{j,0}} F_\delta(1, \widetilde{x}'_{2,0}.\widetilde{x}'^{-1}_{1,0}, \ldots, \widetilde{x}'_{N,0}.\widetilde{x}'^{-1}_{1,0})$$

does not vanish at Q'. Without loss of generality, we may assume that $j = 2$. Moreover, since $\chi'^*\pi'^*t'$ vanishes at Q', the partial derivative $\partial_{\widetilde{x}_{2,0}} w'_0$ also does not vanish. Consequently, after possibly shrinking the open neighborhood W of Q' in $\widetilde{X}'$, this neighborhood admits as coordinate system:

$$(\chi'^*\pi'^*t', \widetilde{x}'_{1,0}, w'_0, \widetilde{x}'_{3,0} \ldots, \widetilde{x}_{N,0}).$$

In this coordinate system, the equation (6.9) defining the divisor $\widetilde{H}' \cap W$ in W can be rewritten as follows:

$$\chi'^*\pi'^*u = \widetilde{x}'^{\delta}_{1,0}.w'_0,$$

or equivalently:

$$w'_0 = \widetilde{x}'^{-\delta}_{1,0}\chi^*\pi^*u, \tag{6.17}$$

and therefore admits as coordinate system:

$$(\chi'^*\pi'^*t', \widetilde{x}'_{1,0}, \widetilde{x}'_{3,0} \ldots, \widetilde{x}_{N,0}).$$

Moreover, the divisor $\widetilde{H}'_{\Delta'} \cap W$ in $\widetilde{H'} \cap W$ is defined by the following equation:

$$\chi'^*\pi'^*t' = 0,$$

and is therefore non-singular.

Consequently, the vector bundle

$$\omega^1_{\widetilde{H}'/C'|\widetilde{H}'\cap W} \simeq \Omega^1_{\widetilde{H}'/C'|\widetilde{H}'\cap W}$$

of rank $N - 1$ admits the following local frame:

$$([d\widetilde{x}'_{1,0}], [d\widetilde{x}'_{3,0}], \ldots, [d\widetilde{x}'_{N,0}]). \tag{6.18}$$

6.3.3. Now let us describe the vector bundle $\upsilon^*\omega^1_{\widetilde{H}/C}$ near Q'.

Using successively the definition of the function h_1 on V_1, the commutativity of the diagram (6.10), the equality (6.8) with $i := 1$ and the definition of the function h'_0 on V'_0, one easily obtains the following equalities of functions on W:

$$\begin{aligned} \upsilon^*_{|W}h_1 &= \upsilon^*_{|W}(\chi^*x_1^{-\delta-1}.\chi^*F_{>\delta}) \\ &= \chi'^*(x_1'^{-\delta-1}.\tau^*F_{>\delta}) \\ &= (\widetilde{x}'_{1,0}.\chi'^*\pi'^*t')^{-\delta-1}.(\chi'^*\pi'^*t'^{\delta+1}.h'_0) \\ &= \widetilde{x}'^{-\delta-1}_{1,0}.h'_0. \end{aligned} \tag{6.19}$$

Consequently, using first the definition of the function w on V_1, then equalities (6.12), (6.13) and (6.19), and finally the definition of the function w'_0 on W, one obtains the following equalities of functions on W:

$$\begin{aligned}
\upsilon^*_{|W} w &= \upsilon^*_{|W} F_\delta(1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1}) + \upsilon^*_{|W}(\chi^* x_1 . h_1) \\
&= F_\delta(1, \widetilde{x}'_{2,0}.\widetilde{x}'^{-1}_{1,0}, \dots, \widetilde{x}'_{N,0}.\widetilde{x}'^{-1}_{1,0}) + (\widetilde{x}'_{1,0}.\chi'^*\pi'^* t').(\widetilde{x}'^{-\delta-1}_{1,0} h'_0) \\
&= w'_0. \qquad (6.20)
\end{aligned}$$

Using firstly equality (6.20), then the chain rule applied to the coordinate system on V_1:

$$(\widetilde{\pi^* t_1}, \chi^* x_1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1}),$$

then equalities (6.14), (6.15) and (6.16), we obtain the following expression for the partial derivative $\partial_{\widetilde{x}'_{2,0}} w'_0$ in the local coordinate system $(\chi'^*\pi'^* t', \widetilde{x}'_{1,0}, \dots, \widetilde{x}'_{N,0})$ on W:

$$\begin{aligned}
\partial_{\widetilde{x}'_{2,0}} w'_0 &= \partial_{\widetilde{x}'_{2,0}}(\upsilon^*_{|W} w) \\
&= \partial_{\widetilde{x}'_{2,0}}(\upsilon^*_{|W} \widetilde{\pi^* t_1}).\upsilon^*_{|W} \partial_{\widetilde{\pi^* t_1}} w + \partial_{\widetilde{x}'_{2,0}}(\upsilon^*_{|W} \chi^* x_1).\upsilon^*_{|W} \partial_{\chi^* x_1} w \\
&\quad + \sum_{j=2}^N \partial_{\widetilde{x}'_{2,0}}(\upsilon^*_{|W} \widetilde{x}_{j,1}).\upsilon^*_{|W} \partial_{\widetilde{x}_{j,1}} w \\
&= \partial_{\widetilde{x}'_{2,0}}(\widetilde{x}'_{2,0}\, \widetilde{x}'^{-1}_{1,0}).\upsilon^*_{|W} \partial_{\widetilde{x}_{2,1}} w \\
&= \widetilde{x}'^{-1}_{1,0}.\upsilon^*_{|W} \partial_{\widetilde{x}_{2,1}} w.
\end{aligned}$$

Consequently, since $\partial_{\widetilde{x}'_{2,0}} w'_0$ does not vanish at Q', we obtain that $\partial_{\widetilde{x}_{2,1}} w$ does not vanish at $\upsilon(Q')$.

We can therefore apply the results of Subsection 6.2 near $Q := \upsilon(Q')$ with $i(Q) := 2$. This allows us to describe a local frame of the vector bundle $\upsilon^*_{|\widetilde{H}'\cap W} \omega^1_{\widetilde{H}/C}$, given by pulling back by $\upsilon_{|\widetilde{H}'\cap W}$ the local frame (6.7) of the vector bundle $\omega^1_{\widetilde{H}/C|V_1}$; its components are:

$$\upsilon^*_{|\widetilde{H}'\cap W} \chi^*[dx_1/x_1] = \widetilde{x}'^{-1}_{1,0}[d\widetilde{x}'_{1,0}] + \chi'^*\pi'^*[dt'/t'],$$

and for every j in $\{3, \dots, N\}$:

$$[\upsilon^*_{|\widetilde{H}'\cap W} d\widetilde{x}_{j,1}] = [d(\widetilde{x}'_{j,0}.\widetilde{x}'^{-1}_{1,0})] = -\widetilde{x}'_{j,0}.\widetilde{x}'^{-2}_{1,0}.[d\widetilde{x}'_{1,0}] + \widetilde{x}'^{-1}_{1,0}.[d\widetilde{x}'_{j,0}].$$

The vanishing of the following class in $\upsilon^*_{|\widetilde{H}'\cap W} \omega^1_{\widetilde{H}/C}$:

$$\upsilon^*_{|\widetilde{H}'\cap W} \chi^*\pi^*[dt/t] = \chi'^*\pi'^*([du/u] + \delta[dt'/t']),$$

together with the invertibility of u, implies the vanishing of the class $\chi'^*\pi'^*[dt'/t']$. Using also the invertibility of $\widetilde{x}'_{1,0}$ on W, this shows that:

$$([d\widetilde{x}'_{1,0}], [d\widetilde{x}'_{3,0}], \dots, [d\widetilde{x}'_{N,0}])$$

constitutes a frame of the vector bundle $\upsilon^*_{|\widetilde{H}'\cap W} \omega^1_{\widetilde{H}/C}$.

Comparing this frame with the frame (6.18) for the vector bundle $\omega^1_{\widetilde{H}'/C'|\widetilde{H}'\cap W}$ proves that, over $\widetilde{H}'\cap W$, the isomorphism φ indeed extends to an isomorphism $\widetilde{\varphi}$ between the vector bundles $\upsilon^*\omega^1_{\widetilde{H}/C}$ and $\omega^1_{\widetilde{H}'/C'}$. This concludes the proof of Proposition 4.1 in Case 1.

6.4. Case 2: The point Q' belongs to the open subset V'_i for some $i \in \{1, \dots, N\}$.

6.4.1. Without loss of generality, we may assume that $i=1$. The point Q' is then in the open subset V'_1, which admits a local coordinate system $(\widetilde{\pi'^*t'}_1, \chi'^*x'_1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1})$, where $\widetilde{\pi'^*t'}_1$ is defined by:

$$\widetilde{\pi'^*t'}_1 := v'_0/v'_1,$$

and satisfies the following equality of functions on V'_1:

$$\chi'^*\pi'^*t' = \widetilde{\pi'^*t'}_1.\chi'^*x'_1, \tag{6.21}$$

and where for every integer $i \in \{2, \dots, N\}$, $\widetilde{x}'_{i,1}$ is defined by:

$$\widetilde{x}'_{i,1} := v'_i/v'_1$$

and satisfies the following equality:

$$\chi'^*x'_i = \widetilde{x}'_{i,1}.\chi'^*x'_1. \tag{6.22}$$

One easily checks that the intersection $(\tau \circ \chi')^{-1}(\Sigma) \cap V'_1$ is defined in V'_1 by the equation $(\chi'^*x'_1 = 0)$, so that it coincides with the exceptional divisor $Z'_{P'} \cap V'_1$ as a subscheme of V'_1. Together with the universal property of blow-ups, this shows, as in Case 1, the existence of a unique morphism $\upsilon_{|V'_1} : V'_1 \to \widetilde{X}'$ such that the following diagram is commutative:

$$\begin{tikzcd} V'_1 \arrow[r, "\upsilon_{|V'_1}"] \arrow[d, "\chi'_{|V'_1}"'] & \widetilde{X} \arrow[d, "\chi"] \\ X' \arrow[r, "\tau"'] & X. \end{tikzcd} \tag{6.23}$$

Let us now describe the morphism $\upsilon_{|V'_1}$. Using equalities (6.21) and (6.22), the morphism $\tau \circ \chi'$ from V'_1 to X satisfies the following equalities of functions on V'_1:

$$(\tau \circ \chi')^*\pi^*t = \chi'^*\pi'^*(u.t'^{\delta}) = \chi'^*\pi'^*u \,.\, \widetilde{\pi'^*t'}_1^{\delta} \,.\, \chi'^*x_1'^{\delta},$$
$$(\tau \circ \chi')^*x_1 = \chi'^*x'_1,$$

and for every integer $i \in \{2, \dots, N\}$:

$$(\tau \circ \chi')^*x_i = \widetilde{x}'_{i,1} \,.\, \chi'^*x'_1.$$

Consequently $\upsilon_{|V'_1}$ satisfies the following equality of morphisms from V'_1 to $\mathbb{P}^N$:

$$\upsilon^*_{|V'_1}[v_0 : \dots : v_N] = [\chi'^*\pi'^*u \,.\, \widetilde{\pi'^*t'}_1^{\delta} \,.\, \chi'^*x_1'^{\delta-1} : 1 : \widetilde{x}'_{2,1} : \dots : \widetilde{x}'_{N,1}].$$

In particular, this morphism has values in the open subset $V_1 := (v_1 \neq 0)$ in $\widetilde{X}$.

In the coordinate system $(\widetilde{\pi^*t}_1, \chi^*x_1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1})$ of V_1 defined above, the morphism

$$\upsilon_{|V'_1} : V'_1 \longrightarrow V_1$$

may be described by the following equalities:

$$\upsilon^*_{|V'_1}\widetilde{\pi^*t}_1 = \chi'^*\pi'^*u \,.\, \widetilde{\pi'^*t'}_1^{\delta} \,.\, \chi'^*x_1'^{\delta-1}, \tag{6.24}$$

$$\upsilon^*_{|V'_1}\chi^*x_1 = \chi'^*x'_1, \tag{6.25}$$

and for every integer $i \in \{2, \dots, N\}$:

$$\upsilon^*_{|V'_1}\widetilde{x}_{i,1} = \widetilde{x}'_{i,1}. \tag{6.26}$$

We may now proceed as in Case 1. Since the subscheme $(\tau \circ \chi')^{-1}(\Sigma) \cap V'_1$ of V'_1 coincides with the exceptional divisor $Z'_{P'} \cap V'_1$, we obtain by restriction that the subscheme $(\tau_{|H'} \circ \nu')^{-1}(\Sigma) \cap V'_1$ of $\widetilde{H}' \cap V'_1$ coincides with the divisor $E'_{P'} \cap V'_1$. Along with the existence of a morphism $\upsilon_{|V'_1}$ such that the diagram (6.23) is commutative, this establishes the first assertion of Proposition 4.1 in Case 2.

Moreover equalities (6.24), (6.25), and (6.26) imply by restriction the following equalities of functions on $\widetilde{H}' \cap V_1'$:

$$\upsilon^*_{|\widetilde{H}' \cap V_1'} \widetilde{\pi^* t_1} = \chi'^* \pi'^* u \, . \, \widetilde{\pi'^* t'}_1^{\delta} \, . \, \chi'^* x_1'^{\delta - 1}, \tag{6.27}$$

$$\upsilon^*_{|\widetilde{H}' \cap V_1'} \chi^* x_1 = \chi'^* x_1', \tag{6.28}$$

and for every integer $i \in \{2, \dots, N\}$:

$$\upsilon^*_{|\widetilde{H}' \cap V_1'} \widetilde{x}_{i,1} = \widetilde{x}'_{i,1}. \tag{6.29}$$

6.4.2. Let us now describe the divisor $\widetilde{H}' \cap V_1'$ in V_1'. Using first the equation (6.2) of the divisor H' in X' and equalities (6.21) and (6.22), then the homogeneity of F_δ, one obtains that the divisor $\chi'^* H' \cap V_1'$ in V_1' is defined by the following equation:

$$\begin{aligned}\chi'^* \pi'^* u \, . \, \widetilde{\pi'^* t'}_1^{\delta} \, . \, \chi'^* x_1'^{\delta} &= F_\delta(\chi'^* x_1', \widetilde{x}'_{2,1} . \chi'^* x_1', \dots, \widetilde{x}'_{N,1} . \chi'^* x_1') + \chi'^* \tau^* F_{>\delta} \\ &= \chi'^* x_1'^{\delta} . \big[F_\delta(1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1}) + \chi'^* x_1' \, h_1' \big],\end{aligned}$$

where h_1' is defined by:

$$h_1' := \chi'^* x_1'^{-\delta-1} \, . \, \chi'^* \tau^* F_{>\delta},$$

and, similarly to h_0', is actually a (germ of) regular function on V_1' along $Z'_{P'} \cap V_1'$, since the pull-back $\chi'^* \tau^* F_{>\delta}$ vanishes at order $\delta + 1$ on the exceptional divisor $Z'_{P'}$.

Consequently the divisor $\widetilde{H}' \cap V_1' = (\chi'^* H' - Z'_\delta) \cap V_1'$ in V_1' is defined by the following equation:

$$\chi'^* \pi'^* u \, . \, \widetilde{\pi'^* t'}_1^{\delta} = F_\delta(1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1}) + \chi'^* x_1' \, h_1'. \tag{6.30}$$

Now let us describe locally the vector bundle $\omega^1_{\widetilde{H}'/C'}$ near Q'.

Similarly to the function w on V_1, a function w_1' on V_1' may be defined as follows:

$$w_1' := F_\delta(1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1}) + \chi'^* x_1' h_1'.$$

Since the homogeneous polynomial F_δ has non-zero discriminant, there exists an integer j in $\{2, \dots, N\}$ such that the derivative:

$$\partial_{\widetilde{x}_{j,1}} F_\delta(1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1})$$

does not vanish at Q'. Without loss of generality, we may assume that $j = 2$. Moreover, since $\chi'^* x_1'$ vanishes at Q' because Q' is in $E'_{P'}$, the partial derivative $\partial_{\widetilde{x}_{2,1}} w_1'$ also does not vanish.

Consequently some neighborhood W of Q' in V_1' admits as coordinate system:

$$(\widetilde{\pi'^* t'}_1, \chi'^* x_1', w_1', \widetilde{x}'_{3,1}, \dots, \widetilde{x}'_{N,1}).$$

In this coordinate system, the restriction to W of the equation (6.30) defining the divisor $\widetilde{H}' \cap V_1'$ in V_1' can be rewritten as follows:

$$\chi'^* \pi'^* u \, . \, \widetilde{\pi'^* t'}_1^{\delta} = w_1', \tag{6.31}$$

and therefore the divisor $\widetilde{H}' \cap W$ admits as coordinate system:

$$(\widetilde{\pi'^* t'}_1, \chi'^* x_1', \widetilde{x}'_{3,1}, \dots, \widetilde{x}'_{N,1}).$$

Moreover, it follows from equality (6.21) that the divisor with normal crossings $\widetilde{H}'_{\Delta'} \cap W$ in $\widetilde{H}' \cap W$ is defined by the following equation:

$$\chi'^* \pi'^* t' = \widetilde{\pi'^* t'}_1 . \chi'^* x_1' = 0.$$

Consequently the vector bundle $\omega^1_{\widetilde{H}'/C'|\widetilde{H}'\cap W}$ of rank $N-1$ admits the following local set of generators:

$$([d\widetilde{\pi'^*t'}_1/\widetilde{\pi'^*t'}_1], \chi'^*[dx'_1/x'_1], [d\widetilde{x}'_{3,1}], \dots, [d\widetilde{x}'_{N,1}]),$$

which satisfy the following relation:

$$[d\widetilde{\pi'^*t'}_1/\widetilde{\pi'^*t'}_1] + \chi'^*[dx'_1/x'_1] = 0.$$

In particular, it admits the following local frame:

$$(\chi'^*[dx'_1/x'_1], [d\widetilde{x}'_{3,1}], \dots, [d\widetilde{x}'_{N,1}]). \tag{6.32}$$

6.4.3. Now let us describe the vector bundle $\upsilon^*\omega^1_{\widetilde{H}/C}$ near Q'. Using successively the definition of the function h_1 on V_1, the commutativity of the diagram (6.23), and the definition of the function h'_1 on V'_1, one easily obtains the following equalities of functions on V'_1:

$$\begin{aligned} \upsilon^*_{|V'_1} h_1 &= \upsilon^*_{|V'_1}(\chi^* x_1^{-\delta-1}.\chi^* F_{>\delta}) \\ &= \chi'^*(x_1'^{-\delta-1}.\tau^* F_{>\delta}) \\ &= h'_1. \end{aligned} \tag{6.33}$$

Consequently, using first the definition of the function w on V_1, then equalities (6.25), (6.26) and (6.33), and finally the definition of the function w'_1 on V'_1, one obtains the following equalities of functions on V'_1:

$$\begin{aligned} \upsilon^*_{|V'_1} w &= \upsilon^*_{|V'_1} F_\delta(1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1}) + \upsilon^*_{|W}(\chi^* x_1.h_1) \\ &= F_\delta(1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1}) + \chi'^* x'_1.h'_1 \\ &= w'_1. \end{aligned} \tag{6.34}$$

Using first equality (6.34), then the chain rule applied to the coordinate system

$$(\widetilde{\pi^* t}_1, \chi^* x_1, \widetilde{x}_{2,1}, \dots, \widetilde{x}_{N,1})$$

of V'_1, and then equalities (6.24), (6.25), and (6.26), we obtain the following expression for the partial derivative $\partial_{\widetilde{x}'_{2,1}} w'_1$ in the local coordinate system $(\widetilde{\pi'^*t'}_1, \chi'^* x'_1, \widetilde{x}'_{2,1}, \dots, \widetilde{x}'_{N,1})$ of V'_1:

$$\begin{aligned} \partial_{\widetilde{x}'_{2,1}} w'_1 &= \partial_{\widetilde{x}'_{2,1}}(\upsilon^*_{|V'_1} w) \\ &= \partial_{\widetilde{x}'_{2,1}}(\upsilon^*_{|V'_1}\widetilde{\pi^* t}_1).\upsilon^*_{|V'_1}\partial_{\widetilde{\pi^* t}_1} w + \partial_{\widetilde{x}'_{2,1}}(\upsilon^*_{|V'_1}\chi^* x_1).\upsilon^*_{|V'_1}\partial_{\chi^* x_1} w \\ &\quad + \sum_{j=2}^{N} \partial_{\widetilde{x}'_{2,1}}(\upsilon^*_{|V'_1}\widetilde{x}_{j,1}).\upsilon^*_{|V'_1}\partial_{\widetilde{x}_{j,1}} \\ &= \partial_{\widetilde{x}'_{2,1}}(\widetilde{x}'_{2,1}).\upsilon^*_{|V'_1}\partial_{\widetilde{x}_{2,1}} w \\ &= \upsilon^*_{|V'_1}\partial_{\widetilde{x}_{2,1}} w. \end{aligned}$$

Consequently, since $\partial_{\widetilde{x}'_{2,1}} w'_1$ does not vanish at Q', we obtain that $\partial_{\widetilde{x}_{2,1}} w$ does not vanish at $\upsilon(Q')$.

We can therefore apply the results of Subsection 6.2 near $Q := \upsilon(Q')$ with $i_Q := 2$. This allows us to describe a local frame of the vector bundle $\upsilon^*_{|\widetilde{H}'\cap V'_1}\omega^1_{\widetilde{H}/C}$, given by pulling back by $\upsilon_{|\widetilde{H}'\cap V'_1}$ the local frame (6.7) of the vector bundle $\omega^1_{\widetilde{H}/C|V_1}$; its components are:

$$\upsilon^*_{|\widetilde{H}'\cap V'_1}\chi^*[dx_1/x_1] = \chi'^*[dx'_1/x'_1],$$

and for every j in $\{3, \dots, N\}$:

$$[\upsilon^*_{|\widetilde{H}'\cap V'_1} d\widetilde{x}_{j,1}] = [d\widetilde{x}'_{j,1}].$$

As in Case 1 above, comparing this frame with the frame (6.32) for the vector bundle $\omega^1_{\widetilde{H}'/C'|\widetilde{H}'\cap W}$ proves that, over $\widetilde{H}'\cap W$, the isomorphism φ indeed extends to an isomorphism $\widetilde{\varphi}$ between the vector bundles $\upsilon^*\omega^1_{\widetilde{H}/C}$ and $\omega^1_{\widetilde{H}'/C'}$. This concludes the proof of Proposition 4.1 in Case 2.

References

[AGZV85] V. I. Arnold, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds.

[Del70] P. Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[Ful98] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, second edition, 1998.

[MO70] J. Milnor and P. Orlik. Isolated singularities defined by weighted homogeneous polynomials. *Topology*, 9:385–393, 1970.

[Mor22] T. Mordant. Griffiths heights and pencils of hypersurfaces. `https://arxiv.org/abs/2212.11019v3`, to appear in *Mém. Soc. Math. Fr.*, 2022.

[Mor25] T. Mordant. Pencils of projective hypersurfaces, Griffiths heights and geometric invariant theory I. `http://arxiv.org/abs/2506.15334`, 2025.