Optimal Permutation Estimation in Crowd-Sourcing Problems

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Based on joint works with **Emmanuel Pilliat** (Uni Montpellier and INRAE) and **Nicolas Verzelen** (INRAE)

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Typical Dataset in Crowd-Sourcing

Cifar10H dataset: 10000 images, 10 labels.

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Typical Dataset in Crowd-Sourcing

Frog (??)

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This Talk

We consider a **ranking** problem:

- Given the observation of the correctness of answers of n experts on d questions,
- We want to rank the experts according to their ability.

Question: how well can we recover their ranking in a minimax sense?

Example of Possible Data

10 questions

$$
\text{4 experts} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}
$$

0: Wrong answer 1: Correct answer

Example of Pos

4 experts $\begin{pmatrix} \\ \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

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Bad Experts   **Good Experts**

Example of Pos

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Hard Questions   **Easy Questions**

Example of Pos

4 experts

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This talk: **Ranking of Experts**

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This talk: **Ranking of Experts**

Under **Known Difficulty** of the questions

Experts/Questions Setting

Experts $i \in \{1, \ldots, n\}$ and **questions** $k \in \{1, \ldots, d\}$. We observe for all i, k:

$$Y_{ik} \sim \text{Bern}\left(M_{ik}\right) \quad .$$

1: Correct 0: Wrong

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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- $M_{ik} = 1/2$: random choice of expert i at question k
- $M_{ik} = 1$: Expert i knows perfectly the answer of question k

Statistical Models

Observation Model

$Y = M + \varepsilon \in \mathbb{R}^{n \times d}$

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Parametric Models for M:

- Questions Equaly Difficult $\rightsquigarrow M_{ik} = a_i$ ≈[Dawid and Skene, 1979]
- Ability/Difficulty $\rightsquigarrow M_{ik} = \phi(\alpha_i - \beta_k)$ ≈[Bradley and Terry, 1952]

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Non-Parametric Models for M ≈[Mao et al., 2018]

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White $=0$; Black $=1$

Matrix M_{π^*}. (isotonic).

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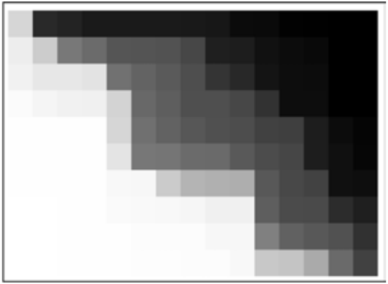

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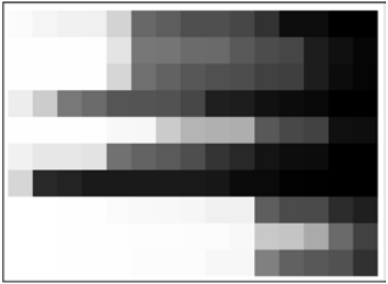

Matrix M (isotonic up to a permutation of experts).

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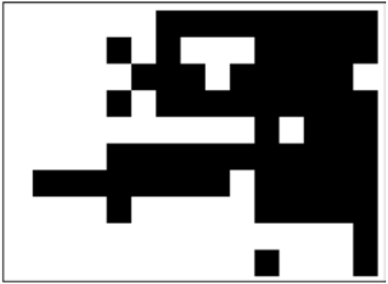

Matrix Y (M in noise).

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Aim

Estimation of π^*.

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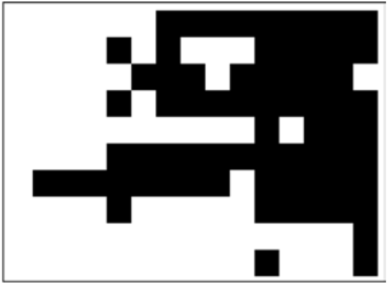
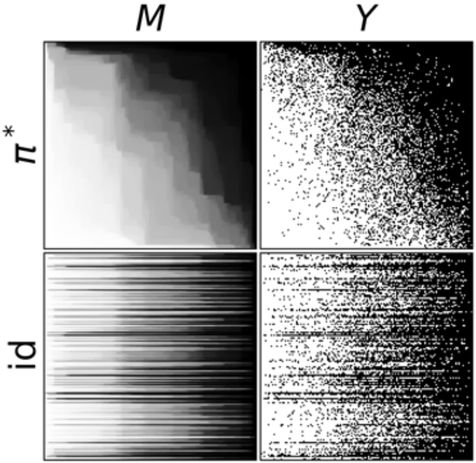
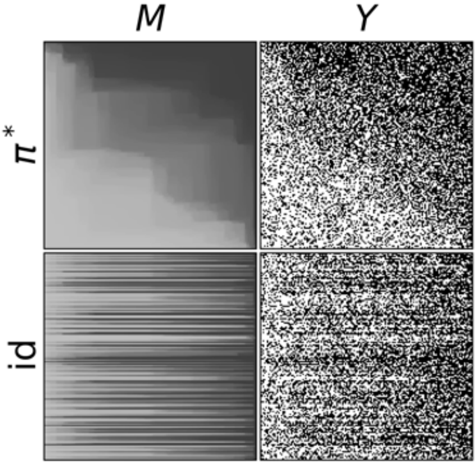


Matrix Y (M in noise).

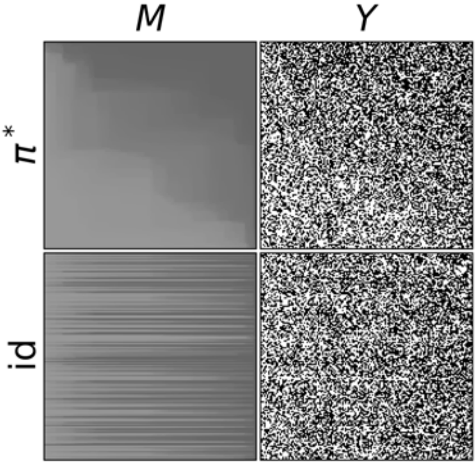
Example with $n, d = 150$, $M \in [0, 1]$



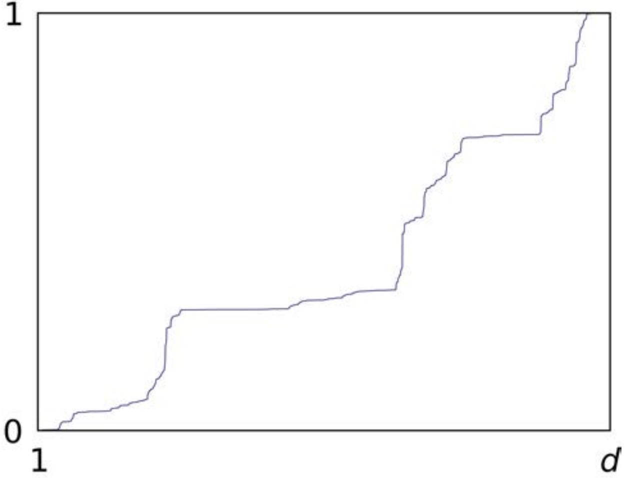
Example with $n, d = 150$, $M \in [0.25, 0.75]$



Example with $n, d = 150$, $M \in [0.4, 0.6]$



Bi-isotonic M - Other representation



Each line $M_{i,\cdot}$ represents an expert i

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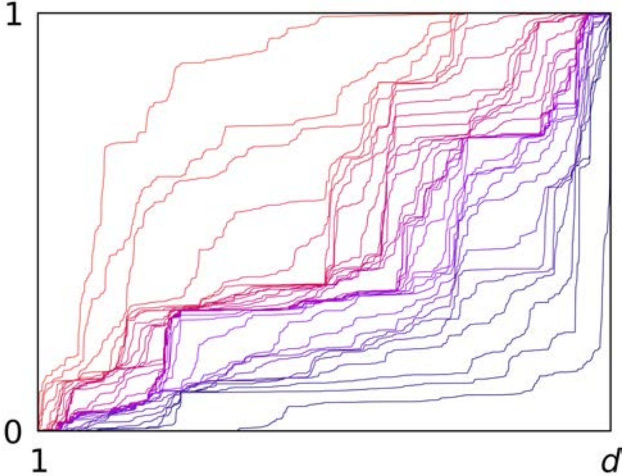

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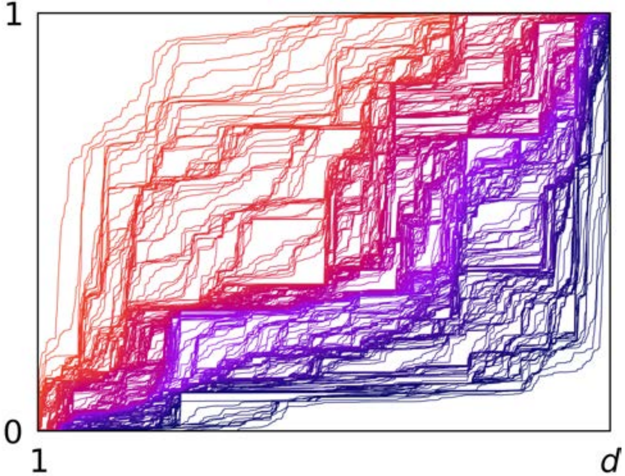

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Permutation loss

For an estimator $\hat{\pi}$ of π^*

$$\text{Perm-Loss} := \|M_{\hat{\pi}} - M_{\pi^*}\|_F^2$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{d}(M_{\pi(i),k} - M_{\pi^*(i),k})^2$$

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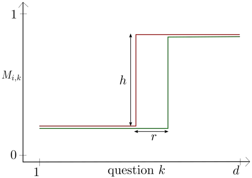

If the two lines are misclassified:
$\text{Perm-Loss} = 2rh^2$

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Estimation loss

For an estimator $\hat{M}$ of M

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Max-Risk and MiniMax-Risk

If $\hat{\pi}$ is an estimator of π^*, we define

$$\begin{aligned}&\text{Max-Perm}(\hat{\pi})\\ &= \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2],\end{aligned}$$

$$\text{MiniMax-Perm} = \inf_{\hat{\pi}}(\text{Max-Perm}(\hat{\pi}))$$

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Define similarly Max-Estim and MiniMax-Estim for estimation of M with $\hat{M}$.

Other Ranking and Permutation Problems

Related rectangular problems:

- **Two permutations** [Mao et al., 2018, Shah et al., 2019]
  M is bi-isotonic up to permutations π^* and σ^* of rows and columns.
  Objective: ranking the experts and the questions.

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Short story:

- No computational gap for *parametric models* (BLT, noisy sorting)
- Mostly unknown for *non-parametric* models: computational gaps were conjectured

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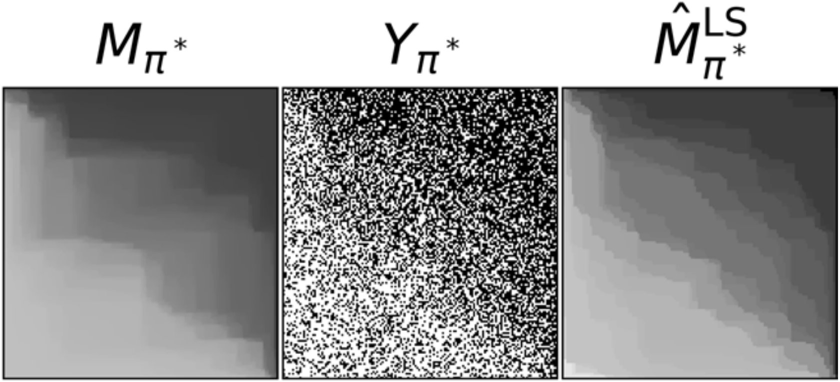
Our Contributions

- Control of MiniMax-Perm(n, d)
- A polynomial-time procedure achieves MiniMax-Perm(n, d)

Existing Methods

- Non-Polynomial Time Methods with Least Square
- Simple Global Average Comparison
- [Liu and Moitra, 2020] based on Hierarchical Clustering

Least Square on Bi-isotinic Matrix



Non-Polynomial Time Method [Mao et al., 2018]

- Perm the set of all permutation of $\{1, \ldots, n\}$
- Mon be the set of all bi-isotonic matrix in $[0, 1]$

Least-square estimator

$$(\hat{M}^{\mathrm{LS}}, \hat{\pi}^{\mathrm{LS}}) = \underset{\widetilde{M} \in \mathrm{Mon}, \widetilde{\pi} \in \mathrm{Perm}}{\arg\min} \|\widetilde{M}_{\widetilde{\pi}} - Y\|_F^2$$

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Matrix $M_{\pi^*,.}$ (bi-isotonic).

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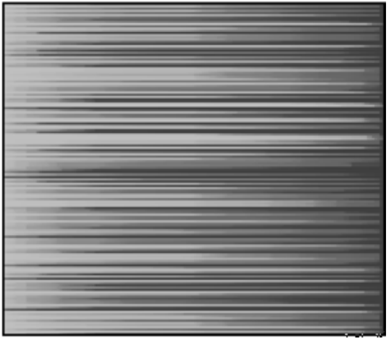

Matrix M.

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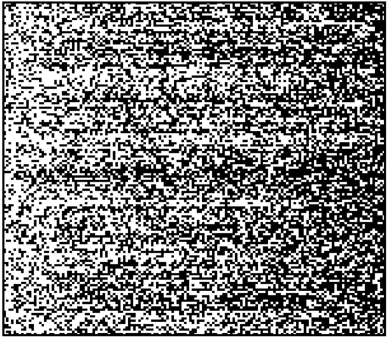

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No know polynomial-time method to compute $Y_{\hat{\pi}^{\mathrm{LS}},.}$

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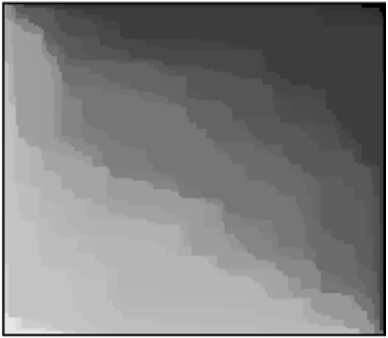
- Mon be the set of all bi-isotonic matrix in $[0, 1]$

Least-square estimator

$$(\hat{M}^{\mathrm{LS}}, \hat{\pi}^{\mathrm{LS}}) = \underset{\widetilde{M} \in \mathrm{Mon}, \widetilde{\pi} \in \mathrm{Perm}}{\arg\min} \|\widetilde{M}_{\widetilde{\pi}} - Y\|_F^2$$



Matrix $\hat{M}^{\mathrm{LS}}_{\pi^*,.}$.

Non-Polynomial Time Method [Mao et al., 2018]

Least-square guarantees

$(\hat{\pi}^{\mathrm{LS}}, \hat{M}^{\mathrm{LS}})$ satisfy -up to polylogs:

$$\text{Max-Estim}(\hat{M}^{\mathrm{LS}}) \lesssim n \vee (\sqrt{nd} \wedge nd^{1/3})$$
$$\text{Max-Perm}(\hat{\pi}^{\mathrm{LS}}) \lesssim n \vee (\sqrt{nd} \wedge nd^{1/3})$$

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- $n!$ permutations: $n \asymp \log(n!)$

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Remarks:

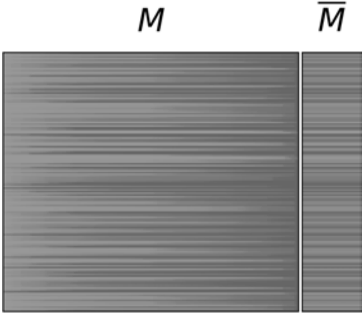
- MiniMax-Estim Optimal [Mao et al., 2018]
- not proven to be MiniMax-Perm Optimal

Summary

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
MiniMax-Perm	??	??	n
MiniMax-Estim	$nd^{1/3}$	$\sqrt{nd}$	n

But algo. not polynomial time.

Global Average Comparison [Pananjady and Samworth, 2020, Shah et al., 2019]



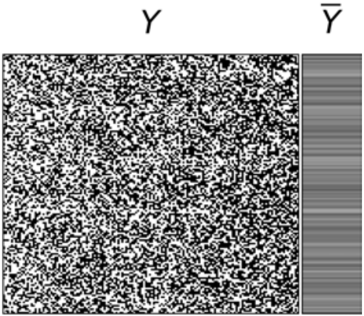
Matrix M.

Global Average Comparison [Pananjady and Samworth, 2020, Shah et al., 2019]

Method:

- Compute expert i average performances on all questions:

$$\overline{Y}_i = \frac{1}{d}\sum_{k=1}^{d} Y_{ik}$$



Matrix Y (M in noise).

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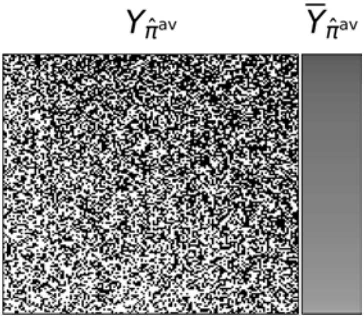

Matrix $Y_{\hat{\pi}^{\mathrm{av}}}$ (M in noise).

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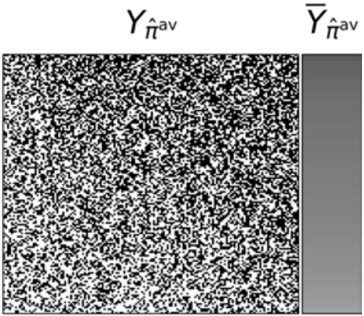

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Idea of Proof

Perfect expert on $\sqrt{d}$ questions VS random:

$$M_{1,\cdot} = (.5.5.5.5.5\dots.5.5\,\underbrace{1111111111})$$

$$M_{2,\cdot} = (.5.5.5.5.5\dots.5.5\,\underbrace{.5.5.5.5.5.5.5}_{\sim\sqrt{d}})$$

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(Example of Observations)

Global Average Comparison [Pananjady and Samworth, 2020, Shah et al., 2019]

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- **Lower Bound** for $\hat{\pi}^{\text{av}}$: There exists M s.t. Max-Perm($\hat{\pi}^{\text{av}}$) $\gtrsim n\sqrt{d}$

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- **Upper Bound**: For any M, π^*, Max-Perm$(\hat{\pi}^{\mathrm{av}}) \lesssim n\sqrt{d}$

Summary

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
MiniMax-Perm	??	??	n
MiniMax-Estim	$nd^{1/3}$	$\sqrt{nd}$	n
Global average (UB)	$n\sqrt{d}$	$n\sqrt{d}$	$n\sqrt{d}$

Remarks:

- Algo. for rates in MiniMax-Estim and MiniMax-Perm not in polynomial time.
- One to one comparisons give UB but sub-optimal whenever $d \gtrsim 1$.

CP and Hierarchical Clustering Based Algo. [Liu and Moitra, 2020]

[Liu and Moitra, 2020] **consider only the case** $d = n$, and provide a **poly. time** algo. returning $\hat{\pi}^{(LM)}$ such that

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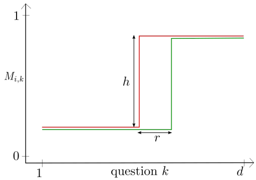

Localisation through CP detection.

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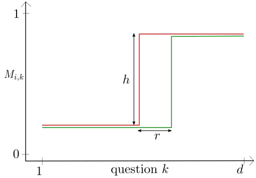

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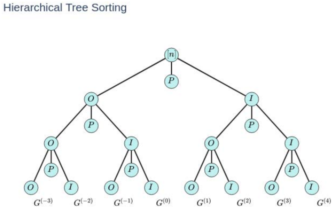

Hierarchical clustering.

Summary

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
MiniMax-Perm	??	??	n
MiniMax-Estim	$nd^{1/3}$	$\sqrt{nd}$	n
Global average (UB)	$n\sqrt{d}$	$n\sqrt{d}$	$n\sqrt{d}$
Ext. of LM (UB)	d	d	n

Remarks:

- **Poly. time algo of LM achieves MiniMax-Perm and MiniMax-Estim for** $d = n$
- This algorithm can be analysed in a more refined way for $d \neq n$ - but not done in [Liu and Moitra, 2020].

Minimax and Poly. Time

Theorem [P., Carpentier, Verzelen, 2022]

Assume we have polylog samples.
There exists a estimator $\hat{\pi}$ of π^* **which is poly. time** and **minimax optimal**

$$\mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2] \lesssim n \vee (n^{3/4}d^{1/4} \wedge nd^{1/6}) \asymp \text{MiniMax-Perm} \ .$$

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Can be combined with bi-isotonic regression to have a **poly. time MiniMax-Estim algo**!

Summary

Poly. time algo achieving the minimax rates:

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
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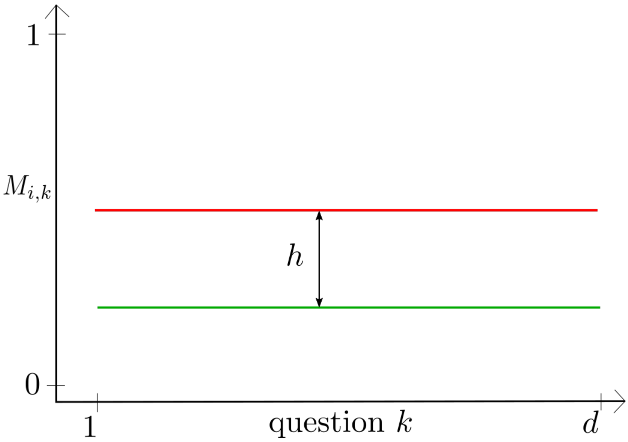
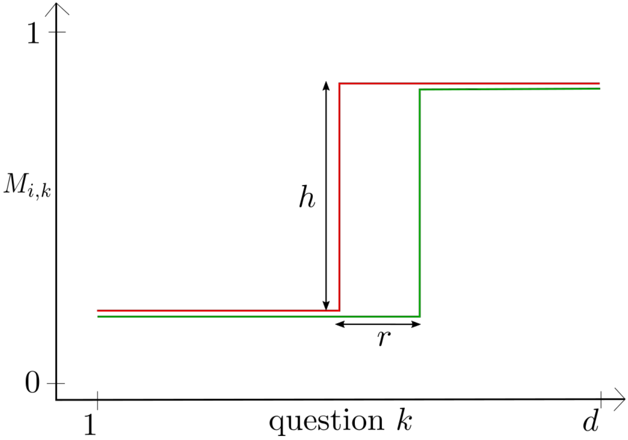
Uniform distance between two experts



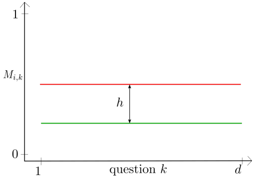
Global average comparison is optimal:
Constant Perm-Risk - Confusion only if $h \lesssim 1/\sqrt{d}$.

Localised distance between two experts

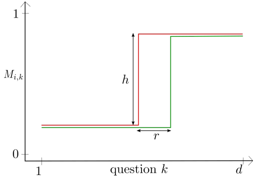


$\psi([d]) = \frac{1}{d}\sum_{i=1}^{d} Y_i$ achieves
Perm-Risk $\asymp \sqrt{d} \gg d^{1/6}$

From Global to Local Averages



Global average good.



Global average bad → need to **localise**.

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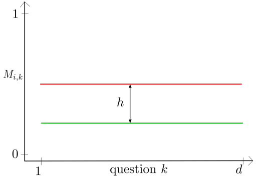

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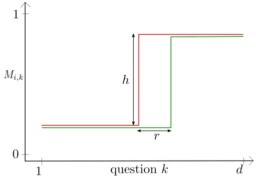

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Idea:

- Estimate by a **change point** (CP) method **windows** where any of the two experts changes by more than h.

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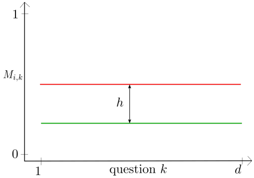

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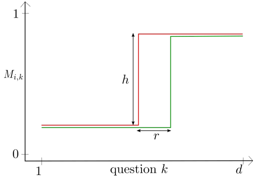

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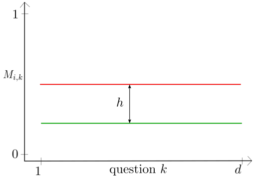

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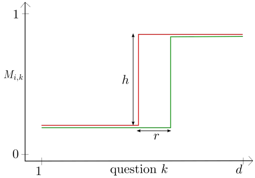

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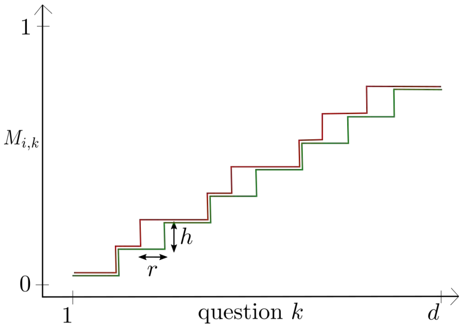
- Compute **local average** on these **windows**.

[Liu and Moitra, 2020] introduced this idea of localisation with CP - in a different context and regime.

Toward a Worst Case Scenario



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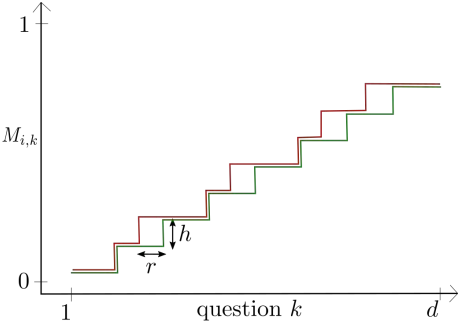

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- A CP of size h can be detected on a window of $1/h^2$ questions.

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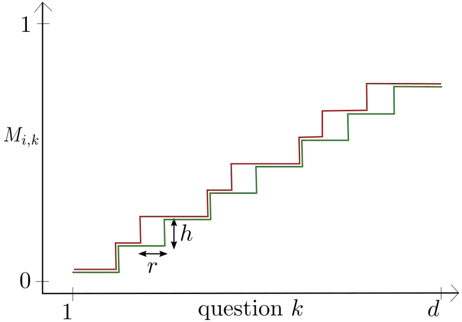

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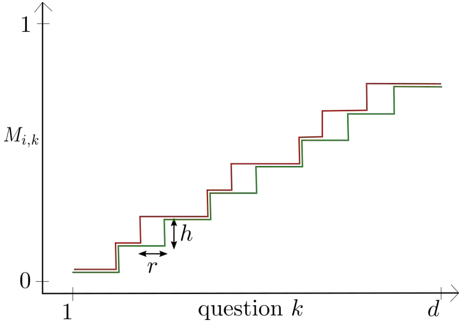

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$$\begin{aligned}\|M_{1\cdot} - M_{2\cdot}\|_2^2 &\le h\|M_{1\cdot} - M_{2\cdot}\|_1 \\ &\le h\sqrt{\frac{1}{h^2}\frac{1}{h} \wedge d} \\ &\le d^{1/6} \quad .\end{aligned}$$

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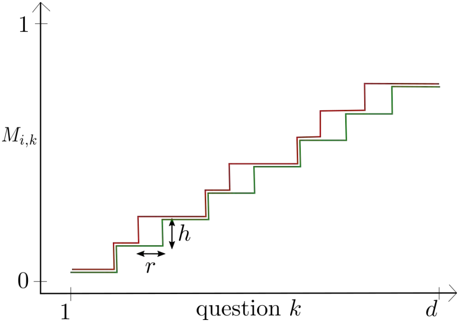

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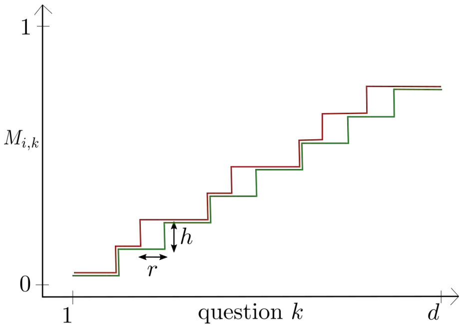

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- $d^{1/6}$ is optimal for two experts: MiniMax-Perm $\asymp d^{1/6}$.
- For any n (UB): MiniMax-Perm $\lesssim nd^{1/6}$.

Summary

Poly. time algo achieving the minimax rates:

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Ext. of LM (UB) extends [Liu and Moitra, 2020] to $d \neq n$

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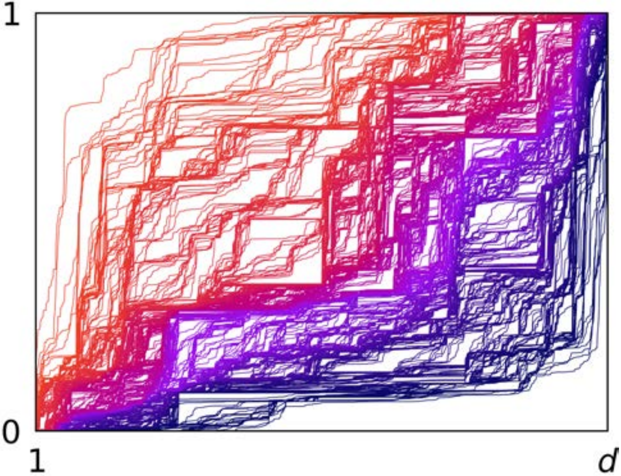
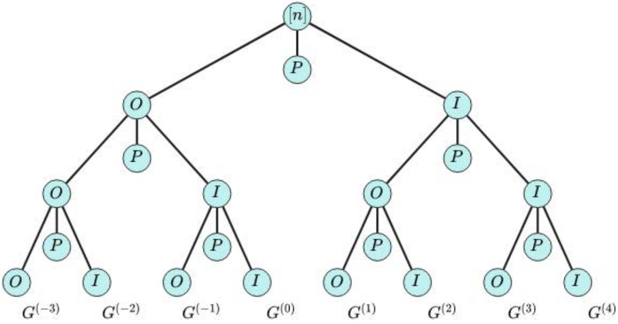

1
0
1
d

Hierarchical Clustering

Beyond [Liu and Moitra, 2020] for $d \neq n$

Hierarchical Tree Sorting



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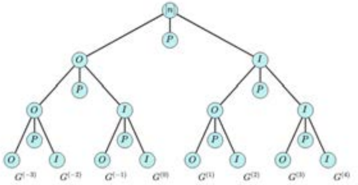
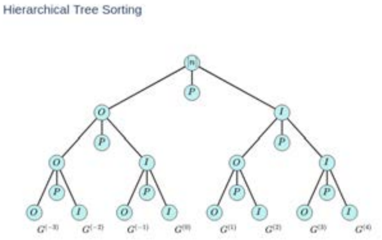
Beyond [Liu and Moitra, 2020] for $d \neq n$

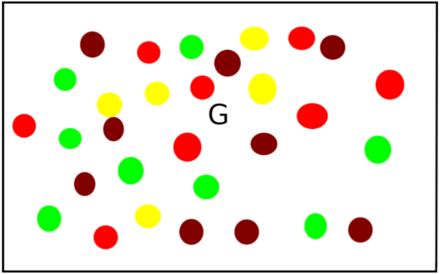
Hierarchical Tree Sorting



Hierarchical Clustering

Beyond [Liu and Moitra, 2020] for $d \neq n$





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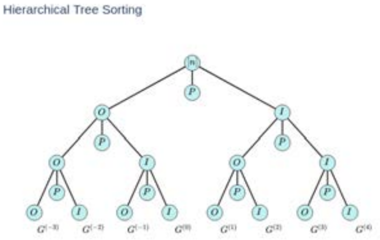

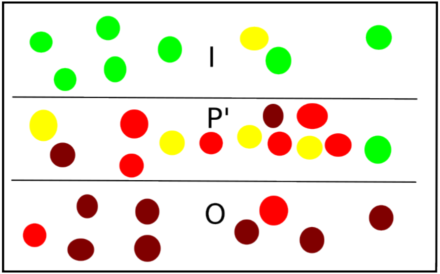

Hierarchical Clustering

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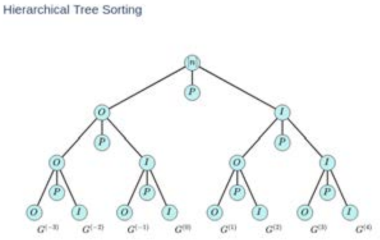

$G^{(4)}$
$G^{(3)}$
$G^{(2)}$
$G^{(1)}$
$G^{(0)}$
$G^{(-1)}$
$G^{(-2)}$
$G^{(-3)}$

Hierarchical Clustering

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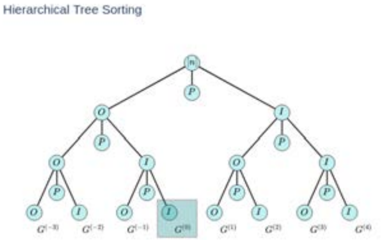
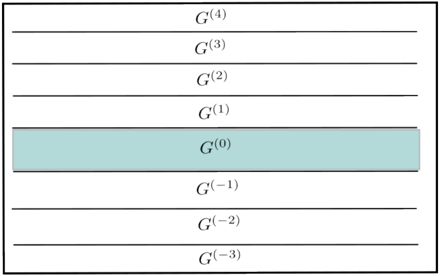
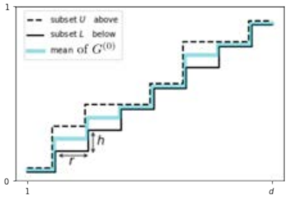
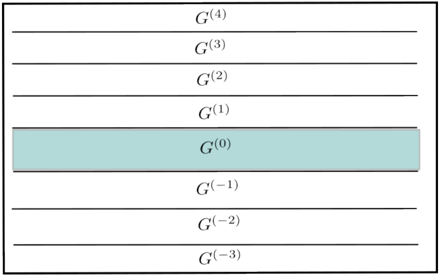




Worst Case for a Group $G^{(0)}$ $(n \gg d^{1/3})$



In $G^{(0)}$, an expert is either in L or in U.



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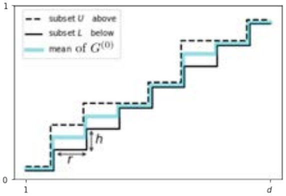


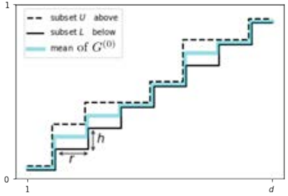
In $G^{(0)}$, an expert is either in L or in U.

After Aggregation

$$\frac{\sqrt{r}h}{2}\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

1: above the mean (U)
-1: below the mean (L)

Worst Case for a Group $G^{(0)}$ $(n \gg d^{1/3})$



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1: above the mean (U)
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Rank one matrix $\rightsquigarrow$ (PCA):
1^{st} left singular vector: better clustering than local averages in some regimes

Beyond [Liu and Moitra, 2020] for $d \neq n$

The corresponding Max-Perm is upper bounded by

$$n \vee \left(n^{2/3} d^{1/3}\right) .$$

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- But not Optimal !

$$n \vee (n^{2/3} d^{1/3}) \gg n \vee (n^{3/4} d^{1/4}) \ .$$

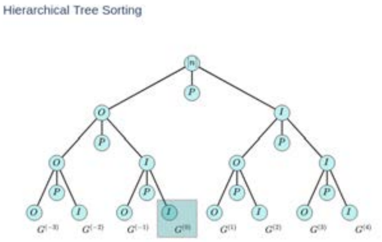
Summary

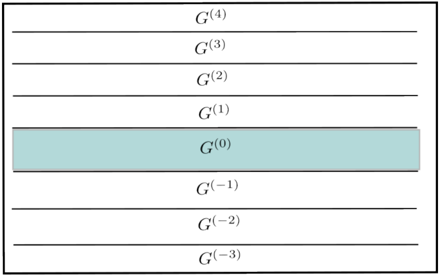
Poly. time algo achieving the minimax rates:

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
MiniMax-Perm	$nd^{1/6}$	$n^{3/4}d^{1/4}$	n
MiniMax-Estim	$nd^{1/3}$	$\sqrt{nd}$	n
Global average (UB)	$n\sqrt{d}$	$n\sqrt{d}$	$n\sqrt{d}$
Ext. of LM (UB)	d	d	n
Super ext. of LM	$nd^{1/6}$	$n^{2/3}d^{1/3}$	n

Remark: Super ext. of LM requires a lot of additional work w.r.t. [Liu and Moitra, 2020]

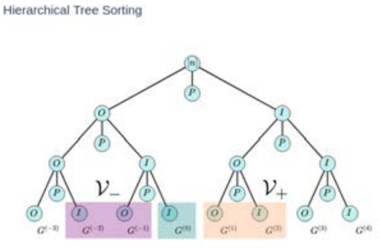
Ideas to achieve $n^{3/4}d^{1/4}$





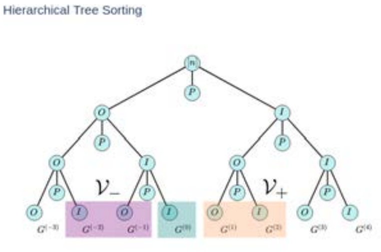
From an oblivious Hierarchical Clustering

Ideas to achieve $n^{3/4}d^{1/4}$



To using the Memory of the Tree

Ideas to achieve $n^{3/4}d^{1/4}$



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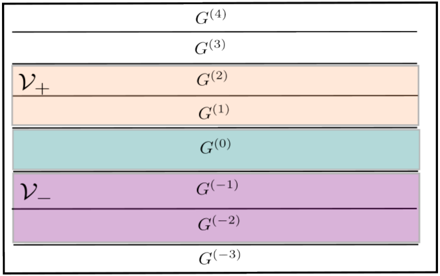
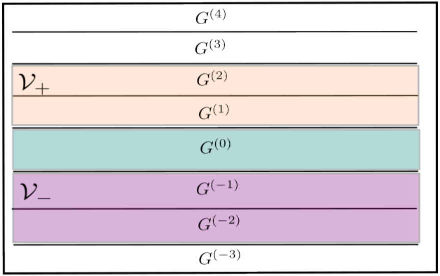


$G^{(0)}$ is sandwiched between $\mathcal{V}_-$ and $\mathcal{V}_+$

Two Types of Information



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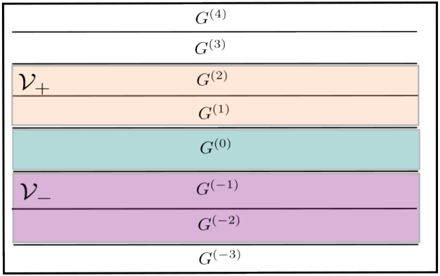

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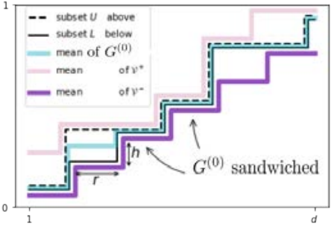
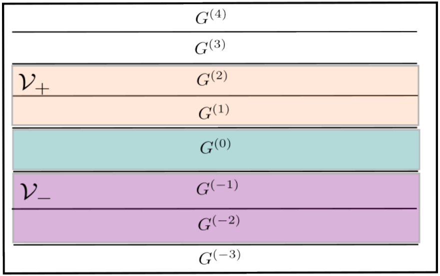
First Type



Removing regions where $G^{(0)}$ is sandwiched

Two Types of Information



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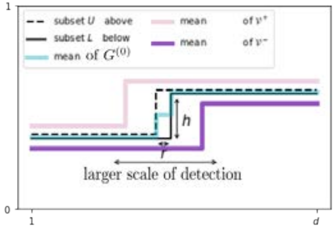
Second Type



Better Change-Point Detection

Conclusion of the Method with Memory

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Conclusion

For all n, d:

- The rate MiniMax-Perm which is of order $n \vee (n^{3/4} d^{1/4} \wedge n d^{1/6})$ (UB and LB).
- An associated poly.-time ranking method.
- Together with bi-isotonic regression, this provides a poly.-time method for Minimax-Estim.
- Related to [Liu and Moitra, 2020] but **new concepts** necessary for minimax rate (memory of the tree).
- Setting can be relaxed without problems to partial observations.

Reference: [arXiv:2211.04092], accepted in AOS (2022)

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Research Directions:

- Unknown order on questions.
- Removing the isotonicity constraint on questions.
- Unknown answers: -observing labels instead of correctness.

(Isotonic)-π^*

- Isotonicity in experts **for an unknown permutation** π^*

- $M_{ik} \in [0, 1]$
- (ε_{ik}) independent and Subgaussian

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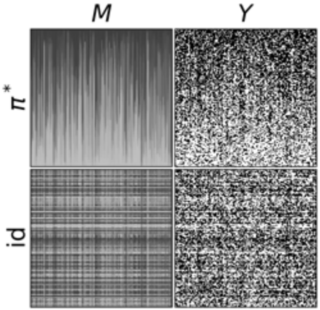
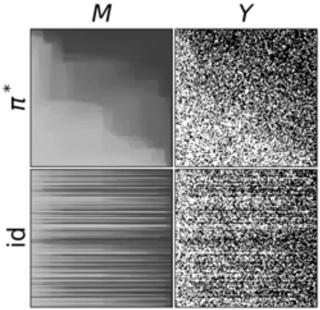


(Bi-isotonic)-π^*



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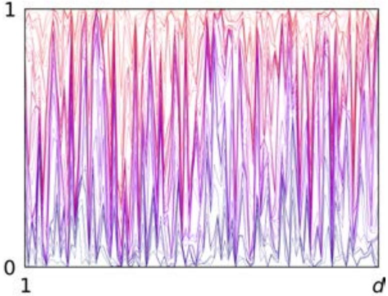

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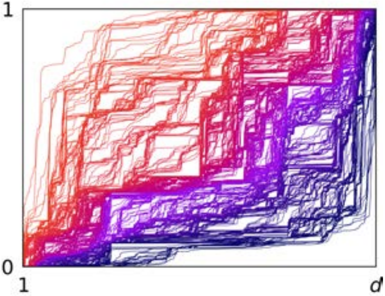

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Statistical difficulty:

$$\textbf{(Isotonic)-}\pi^* \succ \textbf{(Bi-isotonic)-}(\pi^*, \sigma^*) \succ \textbf{(Bi-isotonic)-}\pi^*$$

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