Workshop: TITLE-OF-WORKSHOP

Table of Contents

Abstracts

Lectures C.7 and C.8.
One-variable proofs of the holonomy bounds via the method of slopes

Luigi Martinelli, Thomas Mordant

The aim of these talks was to present one-variable proofs of the holonomy bounds (notably of [4, Theorem 7.0.1]) based on the content of [4, §7] and on the relevant material in [2] and [3], while putting emphasis on the geometry underlying these bounds.

We like to think of these proofs as instances of "arithmetic Italian geometry:" they involve inequalities relating intersection numbers attached to projective surfaces, now in the arithmetic context of Arakelov geometry, where classical intersection numbers are replaced by real numbers defined in terms of heights and integrals of Green functions.

The use of auxiliary functions and Siegel's lemma in classical Diophantine approximation proofs is incorporated into the slopes method: the main objects of interest become the evaluation morphisms from spaces of sections of ample line bundles — a.k.a. auxiliary polynomials — to spaces of formal sections of those. The holonomy bounds appear as (limit cases of) estimates relating the heights of these evaluation morphisms, and the slopes of their source and target spaces.

In this abstract, we focus on geometric versions of the holonomy bounds in [4], which notably cover [4, Theorem 7.0.1] in the special case where $\mathbf{e} = 0$. The level of generality of these versions has been chosen to clarify the geometric meaning of the data and of the hypotheses in these holonomy bounds, and to indicate how a geometric approach leads to holonomy bounds concerning formal functions on arbitrary projective curves.

These geometric holonomy bounds were presented in the second half of Lecture C.8, by Luigi Martinelli, after the necessary material from Arakelov geometry had been introduced in Lecture C.7 by Thomas Mordant, and used in the first half of Lecture C.8 to present the slopes method. Further details will appear in a forthcoming joint paper with Jean-Benoît Bost.

1. A general geometric framework

Let us describe the framework of our versions of the holonomy bounds.

We consider an arbitrary smooth projective curve C over $\mathbb{Q}$, equipped with some $\mathbb{Q}$-rational point P, and the holonomy bounds will provide upper bounds on the size m of some m-tuple $(f_1, \ldots, f_m)$ of elements of the completion of the local ring $\mathcal{O}_{C,P}$, when these elements are $\mathbb{Q}(C)$-linearly independent and holonomic, and satisfy suitable arithmetic and analytic conditions.

We might in fact consider C to be a smooth projective curve over an arbitrary number field K, and P a closed point of C. However, for simplicity's sake, we discuss only the situation where K is $\mathbb{Q}$ and P belongs to $C(\mathbb{Q})$.

1.1. As above, let C be an irreducible smooth projective curve over $\mathbb{Q}$, equipped with a point $P \in C(\mathbb{Q})$. As is customary, we denote by $\widehat{C}_P$ the formal completion of C at P. The choice of a uniformizer of C at P provides an isomorphism from $\widehat{C}_P$ to $\operatorname{Spf} \mathbb{Q}[[T]]$.

Moreover we choose a regular projective arithmetic surface $\pi : \mathcal{C} \to \operatorname{Spec} \mathbb{Z}$ which is a model of the curve C. In other words, $\mathcal{C}$ is an integral regular scheme of dimension 2, projective and flat over $\operatorname{Spec} \mathbb{Z}$, whose generic fiber $\mathcal{C}_{\mathbb{Q}}$ is identified to C.

The $\mathbb{Q}$-point P of C uniquely extends to a section $\mathcal{P}$ of the morphism π. Its image lies in the smooth locus of this morphism, and we may consider the normal bundle $N_{\mathcal{P}}\mathcal{C} \simeq \mathcal{P}^* T_\pi$. It is a line bundle of rank one over $\operatorname{Spec} \mathbb{Z}$, and its generic fiber $(N_{\mathcal{P}}\mathcal{C})_{\mathbb{Q}}$ may be identified with the $\mathbb{Q}$-line $T_P C$.

For every non-negative integer i, we shall denote by $\mathcal{P}_i$ the i-th infinitesimal neighborhood of $\mathcal{P}$ in $\mathcal{C}$. It is an affine subscheme of $\mathcal{C}$, finite and flat over $\mathbb{Z}$. The algebra $\mathcal{O}_{\mathcal{P}_i}$ of regular functions over $\mathcal{P}_i$ is a free $\mathbb{Z}$-module of rank $i+1$, and its generic fiber $\mathcal{O}_{\mathcal{P}_i,\mathbb{Q}}$ is the $\mathbb{Q}$-algebra $\mathcal{O}_{P_i} \simeq \mathbb{Q}[[T]]/T^{i+1}\mathbb{Q}[[T]]$ of functions on the i-th infinitesimal neighborhood P_i of P in C.

1.2. By a *denominator condition*, we mean a map:

$$\mathbf{d} : \mathbb{N} \longrightarrow \mathbb{Z}_{>0}$$

such that, for every $i \in \mathbb{N}$, $\mathbf{d}(i) \mid \mathbf{d}(i+1)$. We shall say that a formal function $f \in \mathcal{O}_{\widehat{C}_P}$ satisfies the denominator condition $\mathbf{d}$ when, for every $i \in \mathbb{N}$, its truncation $f_{|P_i}$ in $\mathcal{O}_{P_i}$ satisfies:

$$\mathbf{d}(i)\, f_{|P_i} \in \mathcal{O}_{\mathcal{P}_i}.$$

1.3. Besides suitable denominator conditions, the formal functions in holonomy bounds are assumed to satisfy analytic conditions of overconvergence. These are formulated in terms of a marked compact Riemann surface with boundary (M, O) and of a complex analytic map:

$$\varphi : (M, O) \longrightarrow (C(\mathbb{C}), P_{\mathbb{C}}), \tag{1}$$

which we assume to be compatible with the markings — namely:

$$\varphi(O) = P_{\mathbb{C}}$$

— and to be étale at O.

Here M is a connected compact Riemann surface with non-empty boundary ∂M, defined as a domain with C^∞ boundary in some (germ of open) Riemann surface M^+. By O we denote a point in the interior $\mathring{M} := M \setminus \partial M$ of M, and by φ a map which is $\mathbb{C}$-analytic up to the boundary.[1]

We shall say that a formal function f in $\mathcal{O}_{\widehat{C_{\mathbb{C}}}_{,P_{\mathbb{C}}}} \simeq \mathcal{O}_{\widehat{C}_P} \widehat{\otimes}_{\mathbb{Q}} \mathbb{C}$ is *φ-overconvergent* when f defines a germ of $\mathbb{C}$-analytic function at the point $P_{\mathbb{C}}$ on the Riemann surface $C(\mathbb{C})$ and if its pull-back $\varphi^*(f)$ — which is a germ of $\mathbb{C}$-analytic function

[1]Or equivalently, a germ of analytic map from M^+ to $C(\mathbb{C})$ along M.

at the point O of M — analytically extends to some germ of meromorphic map on M^+ along M.

When f is a formal function in $\mathcal{O}_{\widehat{C}_P}$, we shall say that f is φ-overconvergent when its image in $\mathcal{O}_{\widehat{C_{\mathbb{C}}}, P_{\mathbb{C}}}$ is.

1.4. Recall that to a marked compact connected Riemann surface with non-empty boundary (M, O) as above is attached the *classical*, or Dirichlet, *Green function* $g_{M,O}$. It is defined as the unique function:

$$g_{M,O} : M \setminus \{O\} \longrightarrow \mathbb{R}$$

which satisfies the following conditions:

(1) $g_{M,O}$ is harmonic on $\mathring{M} \setminus \{O\}$;
(2) $g_{M,O}$ admits a logarithmic singularity at O; namely, if (U, z) denotes a $\mathbb{C}$-analytic chart on $\mathring{M}$ whose domain U contains O, the function $g_{M,O} - \log |z - z(O)|^{-1}$ is bounded on some punctured neighborhood of O in U;
(3) $g_{M,O}$ is continuous on $M \setminus \{O\}$ and vanishes on ∂M.

With the above notation, the difference $g_{M,O} - \log |z - z(O)|^{-1}$, initially defined on $U \setminus \{O\}$, actually extends to some harmonic function $h : U \to \mathbb{R}$.

According to its very definition, $g_{M,O}$ admits a continuous extension on the open Riemann surface $M^+ \setminus \{O\}$ defined by the equality:

$$g_{M,O}(x) = 0 \quad \text{for every } x \in M^+ \setminus M,$$

and then $g_{M,O}$ is a locally L^1 function on M^+. As such, it satisfies the following equation of currents:

$$\frac{i}{\pi} \partial\overline{\partial} g_{M,O} + \delta_O = \mu_{M,O},$$

where δ_O is the Dirac measure at O, and where $\mu_{M,O}$ denotes the *harmonic measure* attached to (M, O): it is a probability measure supported by the boundary ∂M, defined by a C^∞ everywhere positive one-form on ∂M.

To the Green function $g_{M,O}$ is attached a C^∞ Hermitian metric $\|.\|_{M,O}$ on the line bundle $\mathcal{O}(O)$ on M, defined by the following equality on $M \setminus \{O\}$:

$$\|\mathbf{1}_{\mathcal{O}(O)}\|_{M,O} := \exp(-g_{M,O}).$$

Transported by the adjunction isomorphism $\mathcal{O}(O)_O \simeq T_O M$, the restriction to $\mathcal{O}(O)_O$ of the Hermitian metric $\|.\|_{M,O}$ becomes the *capacitary norm* $\|.\|^{\mathrm{cap}}_{M,O}$ on $T_O M$. In terms of the above coordinate chart (U, z) and of the harmonic function h, it is also defined by the following equality:

$$\left\| \frac{\partial}{\partial z} \right\|^{\mathrm{cap}}_{M,O} = \exp\bigl(-h(O)\bigr).$$

1.5. Using the data (M, O) and φ introduced in **1.3**, as well as the classical Green function $g_{M,O}$ and the capacitary norm $\|.\|^{\mathrm{cap}}_{M,O}$ on $T_O M$ defined in **1.4**, we may construct some natural Hermitian line bundles on the arithmetic curve $\mathcal{P}$ ($\simeq \operatorname{Spec} \mathbb{Z}$) and on the arithmetic surface $\mathcal{C}$.

Firstly, since φ is étale at O, the isomorphism

$$D\varphi(O) : T_O M \xrightarrow{\sim} T_{P_{\mathbb{C}}} C(\mathbb{C}) \simeq (T_P C)_{\mathbb{C}}.$$

allows us to transport the capacitary norm $\|.\|^{\mathrm{cap}}_{M,O}$ on $T_O M$ to a norm on $(T_P C)_{\mathbb{C}} \simeq (N_{\mathcal{P}}\mathcal{C})_{\mathbb{C}}$, which we shall denote by $\|.\|^{\mathrm{cap}}_{M,O,\varphi}$. Endowed with this norm, $N_{\mathcal{P}}\mathcal{C}$ becomes a Hermitian line bundle over $\mathcal{P} \simeq \operatorname{Spec}\mathbb{Z}$:

$$\overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} := (N_{\mathcal{P}}\mathcal{C}, \|.\|^{\mathrm{cap}}_{M,O,\varphi}).$$

Secondly, the direct image $\varphi_* g_{M,O}$ of the classical Green function $g_{M,O}$ defines a Green function for the divisor $\mathcal{P}$ on $\mathcal{C}$, and we shall denote by $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ the corresponding Hermitian line bundle on $\mathcal{C}$. In other words, $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ is the pair $(\mathcal{O}_{\mathcal{C}}(\mathcal{P}), \|.\|_{\varphi_* g_{M,O}})$, where $\|.\|_{\varphi_* g_{M,O}}$ is the Hermitian metric on the line bundle $\mathcal{O}_{\mathcal{C}}(\mathcal{P})$ defined by the following equality, for every $x \in C(\mathbb{C})$:

$$\log \|\mathbf{1}_{\mathcal{O}_{C(\mathbb{C})}(P_{\mathbb{C}})}(x)\|^{-1}_{\varphi_* g_{M,O}} = \varphi_* g_{M,O}(x) := \sum_{y \in \varphi^{-1}(x)} g_{M,O}(y).$$

where, in the last sum, the multiplicities of y in the divisor $\varphi^{-1}(x)$ have to be taken into account.

The Arakelov self-intersection of the Hermitian line bundle $\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}$ over $\mathcal{C}$, the height $\mathrm{ht}_{\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}}(P)$ of P with respect to this Hermitian line bundle, and the Arakelov degree $\widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi}$ of the Hermitian line bundle $\overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi}$ over $\mathcal{P}$ are related by the following equalities:

$$\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi} \cdot \overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi} = \mathrm{ht}_{\overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}}(P) + \int_M g_{M,O}\, \varphi^* \varphi_* \mu_{M,O} \tag{2}$$

$$= \widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} + \mathrm{Ex}\big(\varphi : (M,O) \to C(\mathbb{C})\big), \tag{3}$$

where $\mathrm{Ex}(\varphi : (M,O) \to C(\mathbb{C}))$ denotes the *overflow* associated to the morphism of marked Riemann surfaces φ, introduced in [3, Chapter 5], and defined as follows:

$$\mathrm{Ex}\big(\varphi : (M,O) \to C(\mathbb{C})\big) := \int_M g_{M,O}\, (\delta_{\varphi^*(P_{\mathbb{C}}) - O} + \varphi^* \varphi_* \mu_{M,O}).$$

Equalities (2) and (3) are indeed straightforward consequences of the basic properties of the Arakelov intersection pairing, in the setting of [3, Chapter 4].

2. Eleutherian families

In this section, we denote by C a smooth projective integral curve over an arbitrary field k, endowed with an ample line bundle L and a point $P \in C(k)$.

Consider a tuple of formal functions $(f_\alpha)_{1 \leq \alpha \leq m}$ in $\mathcal{O}^{\oplus m}_{\widehat{C}_P}$. For any $D \in \mathbb{N}$, we have an evaluation morphism:

$$\eta_D \colon \Gamma(C, L^{\otimes D})^{\oplus m} \longrightarrow L^{\otimes D}|_{\widehat{C}_P}, \quad (s_\alpha)_{1 \leq \alpha \leq m} \longmapsto \sum_{\alpha=1}^{m} s_\alpha|_{\widehat{C}_P} \cdot f_\alpha.$$

The composition of η_D with the restriction morphism $L^{\otimes D}|_{\widehat{C}_P} \to L^{\otimes D}|_{P_{i-1}}$ for any $i > 0$ yields a collection of evaluation morphisms:

$$\eta_{D,i}\colon \Gamma(C, L^{\otimes D})^{\oplus m} \longrightarrow L^{\otimes D}|_{P_{i-1}}. \tag{4}$$

Some of the properties of the family $(f_\alpha)_{1\leq\alpha\leq m}$ can be expressed in terms of these evaluation morphisms. For instance, the following conditions are equivalent:

(1) $f_1, \ldots, f_m \in \mathcal{O}_{\widehat{C}_P} \subset \mathrm{Frac}(\mathcal{O}_{\widehat{C}_P})$ are linearly independent over $k(C)$;
(2) for any $D \geq 0$, the morphism η_D is injective;
(3) for any $D \geq 0$, and for any $i \gg_D 0$, the morphism $\eta_{D,i}$ is injective.

If one these equivalent conditions is satisfied, we set:

$$\iota_L(D) = \min\{i \in \mathbb{Z}_{>0} \mid \eta_{D,i} \text{ is injective}\}.$$

In particular, we have the inequality:

$$\iota_L(D) \geq \dim_k \Gamma(C, L^{\otimes D})^{\oplus m} \geq m \cdot (D \cdot \deg_C L + 1 - g).$$

We shall say that a family $(f_\alpha)_{1\leq\alpha\leq m} \in \mathcal{O}_{\widehat{C}_P}^{\oplus m}$ is *eleutherian* if the following asymptotic holds:

$$\iota_L(D) \sim m \cdot D \cdot \deg_C L \quad \text{when } D \to +\infty.$$

Standard arguments indeed show that this condition does not depend on the choice of the ample line bundle L over C. It is also invariant under extension of the base field k.

When $C = \mathbb{P}^1_{\mathbb{C}}$ is the projective complex line and $P = [1 : 0]$, examples of eleutherian families already appear in the works of Hermite discussed in talk **B.4**. Consider the exponential functions $f_1 = e^{\gamma_1 T}, \ldots, f_m = e^{\gamma_m T} \in \mathbb{C}[[T]]$, for pairwise distinct complex numbers $\gamma_1, \ldots, \gamma_m$. In [6], Hermite proved that the family $(f_1, \ldots, f_m)$ is eleutherian. Actually Hermite showed that, for this family, $\iota_{\mathcal{O}(1)}(D) = m \cdot D$ for every $D \in \mathbb{N}$.

When k is a field of characteristic 0 and C is $\mathbb{P}^1_k$, the Chudnovsky–Osgood theorem (see [5], [7]) asserts that a family $(f_1, \ldots, f_m)$ of holonomic formal functions is eleutherian as soon as it is linearly independent over $k(C) \simeq k(T)$.

3. A basic bound on eleutherian families

Besides the data introduced in **1.1-3** above — which will be used to formulate the denominator and the overconvergence conditions required on the formal functions considered in holonomy bounds — the basic version of those involve some auxiliary data. By specific choices of these auxiliary data, our basic holonomy bounds will take more convenient forms, which specialized to $C = \mathbb{P}^1_{\mathbb{Q}}$ will cover various versions of the holonomy bounds in [4].

3.1. The first of these auxiliary data is a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|.\|)$ over the arithmetic surface $\mathcal{C}$, defined by a line bundle $\mathcal{L}$ over $\mathcal{C}$ — whose restriction $L := \mathcal{L}_{\mathcal{C}_{\mathbb{Q}}}$ to the curve C is ample — and a Hermitian metric $\|.\|$ (invariant under complex conjugation) on the line bundle $L_{\mathbb{C}}$ over the Riemann surface $C(\mathbb{C})$ locally of the form $\|.\| = e^{-\psi}|.|$ where ψ is a continuous subharmonic function. In particular the "first Chern form" $c_1(\overline{L}_{\mathbb{C}})$ of $(L_{\mathbb{C}}, \|.\|)$ is a well-defined positive measure on $C(\mathbb{C})$, locally given by $i\partial\overline{\partial}\psi/\pi$, of total mass $\deg_C L$.

For any integer D, we shall denote by $\overline{\mathcal{L}}^{\otimes D} := (\mathcal{L}^{\otimes D}, \|.\|_D)$ the D-th tensor power of $\overline{\mathcal{L}}$, and by $\pi_*\overline{\mathcal{L}}^{\otimes D}$ the Euclidean lattice defined by the $\mathbb{Z}$-module $\Gamma(\mathcal{C}, \mathcal{L}^{\otimes D})$ and the John norm $\|.\|^J_D$ on $\Gamma(\mathcal{C}, \mathcal{L}^{\otimes D})_{\mathbb{C}} \simeq \Gamma(C(\mathbb{C}), L^{\otimes D}_{\mathbb{C}})$ attached to the L^∞-norm $\|.\|_{D,L^\infty}$ defined by $\|s\|_{D,L^\infty} := \max_{x\in C(\mathbb{C})} \|s(x)\|_D$. The slope

$$\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D}) := \frac{\widehat{\deg}\, \pi_*\overline{\mathcal{L}}^{\otimes D}}{\operatorname{rk} \pi_*\overline{\mathcal{L}}^{\otimes D}}$$

of this Euclidean lattice is easily seen to satisfy:

$$\left|\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D})\right| = O(D) \quad \text{when } D \to +\infty.$$

3.2. The second of these auxiliary data is associated to a tuple $(\mathbf{d}_\alpha)_{1\leq\alpha\leq m}$ of denominator conditions. We shall say that a m-tuple $(\widetilde{\mathbf{d}}_\alpha)_{1\leq\alpha\leq m}$ of functions $\widetilde{\mathbf{d}}_\alpha : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and a non-decreasing function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ are *suited to* $(\mathbf{d}_\alpha)_{1\leq\alpha\leq m}$ when, for every α in $\{1, \ldots, m\}$, we have:

$$\widetilde{\delta}_\alpha := \limsup_{D\to+\infty} D^{-1} \log \widetilde{\mathbf{d}}_\alpha(D) < +\infty, \tag{5}$$

and when, for every $(i, D) \in \mathbb{N} \times \mathbb{Z}_{>0}$, the following inequality holds:

$$\lambda(i, D) := \log \operatorname{lcm}_{1\leq\alpha\leq m}\big(\mathbf{d}_\alpha(i) \cdot \widetilde{\mathbf{d}}_\alpha(D)\big) \leq D\, \chi(i/D). \tag{6}$$

3.3. We are now in position to state a basic version of the holonomy bounds.

Theorem 1. *Let us consider a marked smooth projective curve (C, P) over $\mathbb{Q}$, and a model $\pi : \mathcal{C} \to \operatorname{Spec}\mathbb{Z}$ of C and the section $\mathcal{P}$ of π as in* **1.1***. Let moreover $\varphi : (M, O) \to (C(\mathbb{C}), P_{\mathbb{C}})$ be as in* **1.3***.*

Consider a tuple $(f_\alpha)_{1\leq\alpha\leq m}$ of formal functions in $\mathcal{O}_{\widehat{C}_P}$ such that, for every α in $\{1, \ldots, m\}$, f_α satisfies some denominator condition $\mathbf{d}_\alpha$ and is φ-overconvergent. Moreover let $(\widetilde{\mathbf{d}}_\alpha)_{1\leq\alpha\leq m}$ and χ be functions suited to $(\mathbf{d}_\alpha)_{1\leq\alpha\leq m}$.

If the family $(f_\alpha)_{1\leq\alpha\leq m}$ is eleutherian, then the following inequality holds:

$$\begin{aligned} m \deg_C L \left(\frac{1}{2}\widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_\varphi - \frac{1}{(m \deg_C L)^2} \int_0^{m \deg_C L} \chi(t)dt\right) \\ \leq \operatorname{ht}_{\overline{\mathcal{L}}}(P) + \int_M g_{M,O}\, \varphi^* c_1(\overline{L}_{\mathbb{C}}) - \frac{1}{m}\sum_{\alpha=1}^m \widetilde{\delta}_\alpha - \limsup_{D\to+\infty} \frac{1}{D}\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D}). \end{aligned} \tag{7}$$

The inequality (7) follows from the slope estimates applied to the evaluation morphism $\eta_{D,i}$ introduced in (4) when $i = \iota_L(D)$ and by letting D go to infinity.

The derivation of bounds involving formal functions satisfying possibly different denominator conditions $(\mathbf{d}_\alpha)_{1\leq\alpha\leq m}$ crucially relies on the choice of integral structures on the spaces of auxiliary polynomials $\Gamma(C, L^{\otimes D})^{\oplus m}$ introduced in [4, (7.3.6)] in terms of the $(\widetilde{\mathbf{d}}_\alpha)_{1\leq\alpha\leq m}$. Besides this construction, the proof of (7) relies on the basic formalism of slopes estimates, as presented in [1, 4.1-2], and the Schwarz Lemma in terms of Green functions and capacitary metrics in [2, Section 10.5].

4. From basic bounds to holonomy bounds

4.1. The denominator conditions that appear in [4, Theorems 6.0.2 and 7.0.1], when $\mathbf{e} = 0$, are covered by the following choices of $\mathbf{d}_\alpha$, $\widetilde{\mathbf{d}}_\alpha$ and χ in **3.2**, when $\deg_C L = 1$.

We consider a matrix $\mathbf{b} = (b_{\alpha\beta})_{1\leq\alpha\leq m, 1\leq\beta\leq r}$ of non-negative real numbers as in [4, Theorem 6.0.2], and we let, with the notation of [4, Section 7.3.3]:

$$\mathbf{d}_\alpha(i) := \prod_{\beta=1}^{h_\alpha} [1, \ldots, b_\beta i] \quad \text{and} \quad \widetilde{\mathbf{d}}_\alpha(D) := \prod_{\beta=h_\alpha+1}^{r} [1, \ldots, y_\beta D].$$

Then conditions (5) and (6) are satisfied with:

$$\widetilde{\delta}_\alpha = \sum_{\beta=h_\alpha+1}^{r} y_\beta \quad \text{and} \quad \chi(t) = \sum_{\beta=1}^{r} \max(y_\beta, b_\beta t).$$

When the y_β are chosen as in [4, Section 7.3.3, Proof of Theorem 7.0.1], namely when $y_\beta = u_\beta b_\beta$, we obtain:

$$\frac{1}{(m \deg_C L)^2} \int_0^{m \deg_C L} \chi(t)dt = \frac{1}{2m^2} \sum_{\beta=1}^{r} b_\beta (u_\beta^2 + m^2)$$

and

$$\frac{1}{m} \sum_{\alpha=1}^{m} \widetilde{\delta_\alpha} = \frac{1}{m} \sum_{\beta=1}^{r} b_\beta u_\beta^2.$$

Finally if, as in [4, Theorem 6.0.2], we let:

$$\tau(\mathbf{b}) := \sum_{\beta=1}^{r} b_\beta - \frac{1}{m^2} \sum_{\beta=1}^{r} b_\beta u_\beta^2,$$

the estimate (7) becomes:

$$(8) \quad m\big(\widehat{\deg}\, \overline{N_{\mathcal{P}} \mathcal{C}}_\varphi - \tau(\mathbf{b})\big) \leq 2\mathrm{ht}_{\overline{\mathcal{L}}}(P) + 2\int_M g_{M,O}\, \varphi^* c_1(\overline{L}_{\mathbb{C}}) - 2 \limsup_{D\to+\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}).$$

4.2. When the auxiliary Hermitian line bundle $\overline{\mathcal{L}}$, besides fulfilling the conditions introduced in **3.1**, has an underlying line bundle $\mathcal{L}$ which is nef on the closed fibers $\mathcal{C}_{\mathbb{F}_p}$ of π, the arithmetic Hilbert–Samuel formula applies, as shown in [8], and accordingly we have:

$$(9) \qquad \lim_{D\to+\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{\overline{\mathcal{L}} \cdot \overline{\mathcal{L}}}{2 \deg_C L}.$$

4.3. When $\mathcal{C} = \mathbb{P}^1_{\mathbb{Z}}$ and $\overline{\mathcal{L}}$ is the Hermitian line bundle $\overline{\mathcal{O}}_{\mathbb{P}^1}(1) := (\mathcal{O}_{\mathbb{P}^1}(1), \|.\|)$, defined by the standard Hermitian metric $\|.\|$ on $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ whose first Chern form is the Fubini–Study 2-form:

$$\omega_{\mathrm{FS}} := c_1(\overline{\mathcal{O}}_{\mathbb{P}^1}(1)),$$

then $\deg_C L = 1$, and, as a special case[2] of (9), we have:

$$\lim_{D \to +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{O}}_{\mathbb{P}^1}(D)) = 1/4. \tag{10}$$

When moreover $P = [1:0]$ and $(M, O) = (\overline{D}(0,1), 0)$, then we have:

$$\widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} = \log |\varphi'(0)|,$$

the height $\mathrm{ht}_{\overline{\mathcal{L}}}(P)$ vanishes, and the integral in the right-hand side of (7) is the Nevanlinna–Ahlfors–Shimizu characteristic function of φ:

$$\int_M g_{M,O}\, \varphi^* c_1(\overline{L}_{\mathbb{C}}) = \int_{\overline{D}(0,1)} \log |z|^{-1}\, \varphi^* \omega_{\mathrm{FS}} =: T_{\varphi}(1).$$

Finally, when moreover the $\mathbf{d}_\alpha(i)$ and the $\widetilde{\mathbf{d}}_\alpha(D)$ are chosen as in **4.1**, the estimate (8) becomes:

$$m\big(\log |\varphi'(0)| - \tau(\mathbf{b})\big) \leq 2\, T_{\varphi}(1) - 1/2.$$

This is an improved version of [4, (2.2.3)], with e replaced by 2, as mentioned at the end of [4, Section 2.2].

4.4. The conditions in **3.1** are also satisfied when $\overline{\mathcal{L}} = \overline{\mathcal{O}(\mathcal{P})}_{\varphi}$. Since $\mathcal{O}(\mathcal{P})$ is nef on the closed fiber of π, we may also apply the arithmetic Hilbert–Samuel formula (9), and accordingly:

$$\lim_{D \to +\infty} \frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{1}{2}\, \overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi} \cdot \overline{\mathcal{O}_{\mathcal{C}}(\mathcal{P})}_{\varphi}.$$

Together with the relations (2) and (3), this shows that, with this choice of $\overline{\mathcal{L}}$, the estimate (8) becomes:

$$m\big(\widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} - \tau(\mathbf{b})\big) \leq \widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} + \mathrm{Ex}\big(\varphi : (M,O) \to C(\mathbb{C})\big). \tag{11}$$

In turn, when $\mathcal{C} = \mathbb{P}^1_{\mathbb{Z}}$, $P = [1:0]$, $(M, O) = (\overline{D}(0,1), 0)$, and the image of φ lies in $\mathbb{C}$, the overflow of φ may be written:

$$\mathrm{Ex}\big(\varphi : (M,O) \to C(\mathbb{C})\big) = \iint_{\mathbb{T}^2} \log |\varphi(z) - \varphi(w)|\, d\mu(z) d\mu(w) - \log |\varphi'(0)|$$

(see [3, Proposition 5.4.3]), and therefore (11) becomes:

$$m\big(\log |\varphi'(0)| - \tau(\mathbf{b})\big) \leq \iint_{\mathbb{T}^2} \log |\varphi(z) - \varphi(w)|\, d\mu(z) d\mu(w),$$

which is precisely the estimate [4, (7.0.3)] with $\mathbf{e} = 0$.

The correspondence between our geometric framework and the original setting in [4] is summarized in the following table.

---

[2] It is also easily derived by a direct computation.

Geometric framework	**Calegari–Dimitrov–Tang**
Data	
C smooth projective curve/ $\mathbb{Q}$ $P \in C(\mathbb{Q})$	$\mathbb{P}^1_{\mathbb{Q}}$ $[1:0] \in \mathbb{P}^1(\mathbb{Q})$
$\mathcal{C}$ regular projective model of $C/\ \mathbb{Z}$ $\mathcal{P} \in \mathcal{C}(\mathbb{Z})$ extending $P \in C(\mathbb{Q})$	$\mathbb{P}^1_{\mathbb{Z}}$ $[1:0] \in \mathbb{P}^1(\mathbb{Z})$
M connected compact Riemann surface with boundary $\partial M \neq \emptyset$ $O \in M \setminus \partial M$	$\overline{D}(0,1) = \{z \in \mathbb{C} \mid \lvert z\rvert \leq 1\}$ $\mathbb{T} = \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$ $0 \in \overline{D}(0,1) \setminus \mathbb{T}$
$\varphi \colon M \to C(\mathbb{C})$ holomorphic $\varphi(O) = P_{\mathbb{C}}$ $D\varphi(O) \colon T_O M \xrightarrow{\sim} (T_P C)_{\mathbb{C}}$ $\Vert.\Vert^{\mathrm{cap}}_{M,O} \rightsquigarrow \Vert.\Vert^{\mathrm{cap}}_{M,O,\varphi}$	$\varphi \colon \overline{D}(0,1) \to \mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ holomorphic $\varphi(0) = [1:0]$ $\varphi'(0) \neq 0$
$\overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi} = (N_{\mathcal{P}}\mathcal{C}, \Vert.\Vert^{\mathrm{cap}}_{M,O,\varphi})$ $\widehat{\deg}(\overline{N_{\mathcal{P}}\mathcal{C}}_{\varphi})$	$\log \lvert\varphi'(0)\rvert$
$(f_\alpha)_{1 \leq \alpha \leq m} \in \mathcal{O}^{\oplus m}_{\widehat{C}_P}$	$(f_\alpha)_{1 \leq \alpha \leq m} \in \mathbb{Q}[[T]]^{\oplus m}$
$\mathbf{d}_\alpha \colon \mathbb{N} \to \mathbb{Z}_{>0}, \quad \mathbf{d}_\alpha(i) \mid \mathbf{d}_\alpha(i+1)$ $\mathbf{d}_\alpha(i) \cdot f_{\alpha\lvert P_i} \in \mathcal{O}_{\mathcal{P}_i}$	$\mathbf{d}_\alpha(i) = \prod_{\beta=1}^{h_\alpha} [1, \ldots, b_\beta i]$ $f_\alpha(T) = \sum_{i \geq 0} \frac{a_{\alpha,i}}{\mathbf{d}_\alpha(i)} T^i, \quad a_{\alpha,i} \in \mathbb{Z}$
$\varphi^*(f_{\alpha,\mathbb{C}})$ extends to a meromorphic function on M	$\varphi^*(f_{\alpha,\mathbb{C}})$ extends to a meromorphic function on $\overline{D}(0,1)$
$(f_\alpha)_{1 \leq \alpha \leq m} \in \mathcal{O}^{\oplus m}_{\widehat{C}_P}$ is an eleutherian family	$(f_\alpha)_{1 \leq \alpha \leq m} \in \mathbb{Q}[[T]]$ is free / $\mathbb{Q}(T)$, and every f_α is holonomic

Geometric framework	**Calegari–Dimitrov–Tang**
Auxiliary data	
$\overline{\mathcal{L}}$ Hermitian line bundle on $\mathcal{C}$	$\overline{\mathcal{O}(1)}_{\varphi} = (\mathcal{O}([1:0]), \varphi_* \log^+ \lvert z\rvert^{-1})$
$L = \mathcal{L}\vert_C$	$\mathcal{O}(1)$
$\limsup_{D\to+\infty} \frac{1}{D}\widehat{\mu}(\pi_*\overline{\mathcal{L}}^{\otimes D})$	$\frac{1}{2}(\overline{\mathcal{O}(1)}_{\varphi} \cdot \overline{\mathcal{O}(1)}_{\varphi})$
$\mathrm{ht}_{\overline{\mathcal{L}}}(P) + \int_M g_{M,O}\, \varphi^* c_1(\overline{L}_{\mathbb{C}})$	$\iint_{\mathbb{T}^2} \log \lvert\varphi(z) - \varphi(w)\rvert \, d\mu(z) d\mu(w)$
$\widetilde{\mathbf{d}}_\alpha \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$	$\widetilde{\mathbf{d}}_\alpha(D) = \prod_{\beta=h_\alpha+1}^{r} [1, \ldots, y_\beta D]$
$\widetilde{\delta}_\alpha = \limsup_{D\to+\infty} \frac{\log \widetilde{\mathbf{d}}_\alpha(D)}{D} < +\infty$	$\widetilde{\delta}_\alpha = \sum_{\beta=h_\alpha+1}^{r} y_\beta$
$\lambda(i, D) = \log \mathrm{lcm}_\alpha(\mathbf{d}_\alpha(i) \cdot \widetilde{\mathbf{d}}_\alpha(D))$	
$\lambda(i, D) \leq D\chi(\frac{i}{D})$,	
$\chi \colon \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing	$\chi(t) = \sum_{\beta=1}^{r} \max(y_\beta, b_\beta t)$
$\int_0^{m \deg_C L} \chi(t) dt$	$\frac{1}{2}\sum_{\beta=1}^{r}\big(\frac{y_\beta^2}{b_\beta} + b_\beta m^2\big)$

References

[1] J.-B. Bost, *Algebraic leaves of algebraic foliations over number fields*, Publications Mathématiques. Institut de Hautes Études Scientifiques (2001) no. 93, 161–221.

[2] J.-B. Bost, *Theta invariants of Euclidean lattices and infinite-dimensional Hermitian vector bundles over arithmetic curves*, Progress in Mathematics, Vol. 334 (2020), Birkhäuser.

[3] J.-B. Bost, F. Charles, *Quasi-projective and formal-analytic arithmetic surfaces*, Annals of Mathematics Studies, Vol. 224 (2026) Princeton University Press.

[4] F. Calegari, V. Dimitrov, Y. Tang, *The linear independence of* $1, \zeta(2)$ *and* $L(2, \chi_{-3})$ (2024) `https://arxiv.org/abs/2408.15403v2`.

[5] D. V. Chudnovsky, G. V. Chudnovsky, *Rational approximations to solutions of linear differential equations*, Proceedings of the National Academy of Sciences of the United States of America **80** (1983), no. 16, 5158–5162.

[6] C. Hermite, *Sur la fonction exponentielle*, Comptes rendus de l'Académie des Sciences, **77** (1873), 18–24, 74–79, 226–233, 285–293.

[7] C. F. Osgood, *Sometimes effective Thue–Siegel–Roth–Schmidt–Nevanlinna bounds, or better*, Journal of Number Theory **21** (1985), no. 3, 347–389.

[8] S. Zhang, *Positive line bundles on arithmetic varieties*, Journal of the American Mathematical Society **8** (1995), 187–221.

Reporter: FIRST-NAME-OF-REPORTER LAST-NAME-OF-REPORTER