A typical afternoon...

... covering circles with random arcs

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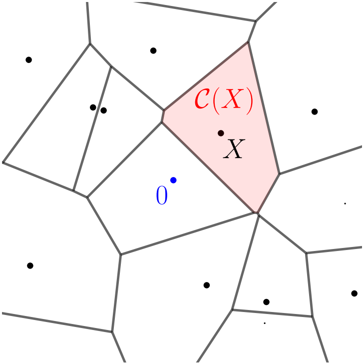
Poisson-Voronoi tessellation

Spatial Poisson point process, i.e. a uniform rain of points • on $\mathbb{R}^2$: $\Phi \cup \{0\}$.

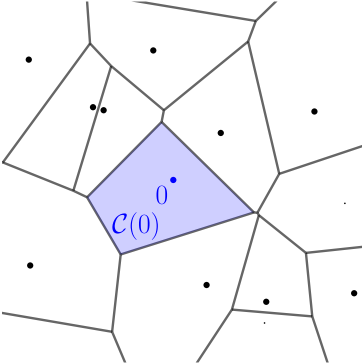
To each (random) raindrop X, associate the set :

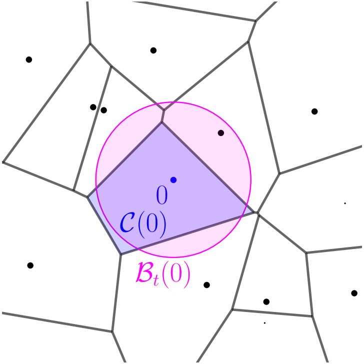
$$\mathcal{C}(X) = \{y \in \mathbb{R}^2 : \|y - X\| \leq \|y - X'\| \,\forall X' \in \Phi\}.$$



Our question:

$$\text{for } t >> 0 \text{ fixed, } \mathbb{P}\left(\mathcal{C}(0) \not\subseteq \mathcal{B}_t(0)\right) = ?$$





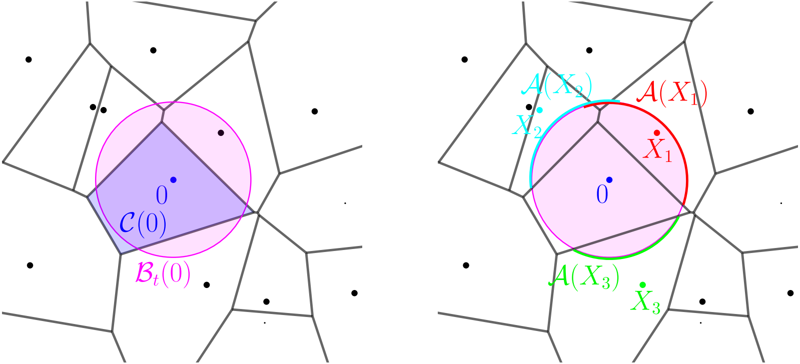
Original question:

$$\text{for } t >> 0,\ \mathbb{P}\left(\mathcal{C}(0) \not\subseteq \mathcal{B}_t(0)\right) = ?$$

Idea of Calka (2002): the raindrops $X \in \Phi \cup \mathcal{B}_{2t}(0)$ close to 0 define a random arc $\mathcal{A}(X)$ on $\partial\mathcal{B}_t(0)$... The original question is equivalent to:

$$\mathbb{P}\left(\partial\mathcal{B}_t(0) \text{ is not covered by } \{\mathcal{A}(X), X \in \Phi \cup \mathcal{B}_{2t}(0)\}\right) = ?$$

There are $N \sim \text{Poisson}(\pi(2t)^2)$ i.i.d. arcs positioned unif at random on the circle and whose lengths have density $l \in [0, \pi t] \mapsto \frac{1}{2t} \sin\left(\frac{l}{t}\right)$.



- TO DO: Bibliography! $\rightarrow$ Stevens (1939)

Stevens (1939) TABLEAU

On fixe $n \in \mathbb{N}$, $a \in [0,1[$.

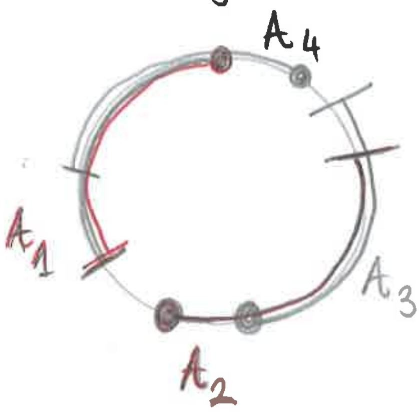
On considère un **cercle** de circumference $= 1$.

On lui fait tomber dessus n **points** $\bullet$ indép. et de manière uniforme :

$$\mathbb{P}(\bullet \in \text{arc } \ell) = \frac{\text{long}(\text{arc})}{\text{long}(\bigcirc)} = \ell$$

Depuis chaque point $\bullet$ on fait pousser en sens antihoraire un arc de cercle de longeur a.

<u>ex</u> : $a = \frac{1}{3}$, $n = 4$



on les numerotte en sens antihoraire.

$$\mathbb{P}\left(\begin{array}{l}\text{cercle pas couvert}\\ \text{par } A_1, \dots, A_n\end{array}\right) = \ ?$$

<u>$n = 2$</u> (avec $a \geq \frac{1}{2}$)

$$\left\{\begin{array}{c}\text{cercle pas}\\ \text{couvert}\end{array}\right\} = \underbrace{\left\{\begin{array}{c}\text{il y a un}\\ \text{trou après}\\ A_1\end{array}\right\}}_{=:\, T_1} \cup \underbrace{\left\{\begin{array}{c}\text{il y a un}\\ \text{trou après}\\ A_2\end{array}\right\}}_{=:\, T_2}$$

T_1 est realisé

T_2 pas realisé

$$\mathbb{P}\left(\begin{smallmatrix}\text{cercle pas}\\ \text{couvert}\end{smallmatrix}\right) = \mathbb{P}(T_1 \cup T_2) = \mathbb{P}(T_1) + \mathbb{P}(T_2) - \mathbb{P}(T_1 \cap T_2)$$

$$\mathbb{P}(T_1) = \mathbb{P}\Big(\bullet(A_2) \in$$ 

$$\Big) = 1-a \underset{\text{symmétrie}}{=} \mathbb{P}(T_2)$$

$$\mathbb{P}(T_1 \cap T_2) = \mathbb{P}\Big(\bullet(A_2) \in$$ 

$$\Big) = (1-2a)_+$$

$$f(x)_+ = \begin{cases} f(x) & \text{si } f(x) \geq 0 \\ 0 & \text{si } f(x) < 0 \end{cases}$$

$$\Rightarrow \mathbb{P}\left(\begin{smallmatrix}\text{cercle pas}\\ \text{couvert}\end{smallmatrix}\right) = 2\cdot(1-a)_+ - (1-2a)_+$$

<u>n générique</u> :

inclusion-exclusion

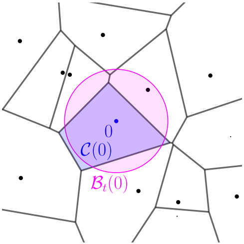
$$\mathbb{P}\left(\begin{smallmatrix}\text{cercle pas}\\ \text{couvert}\end{smallmatrix}\right) = \mathbb{P}(T_1 \cup T_2 \cup \dots \cup T_n) =$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} \underbrace{\mathbb{P}(T_{i_1} \cap T_{i_2} \cap \dots \cap T_{i_k})}_{=(1-ka)_+}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (1-k.a)_+$$

Calka (2002) proves that Stevens (1939) generalizes, but only gives a double bound

$$2\pi t^2 e^{-\pi t^2} \leq \mathbb{P}\left(\mathcal{C}(0) \not\subseteq \mathcal{B}_t(0)\right) \leq 4\pi t^2 e^{-\pi t^2}.$$



C. D'Errico, P. Calka, N. Enriquez: we took a completely different approach to find

$$\mathbb{P}\left(\mathcal{C}(0) \not\subseteq \mathcal{B}_t(0)\right) \underset{t\to\infty}{\sim} 4\pi t^2 e^{-\pi t^2}$$

and for the same construction in $\mathbb{R}^d$:

$$\mathbb{P}\left(\mathcal{C}(0) \not\subseteq \mathcal{B}_t(0)\right) \underset{t\to\infty}{\sim} \frac{2\pi^{\frac{d^2-1}{2}}}{(d-1)!\Gamma(\frac{d+1}{2})^{d-1}} t^{d(d-1)} e^{-\kappa_d t^d}$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$.

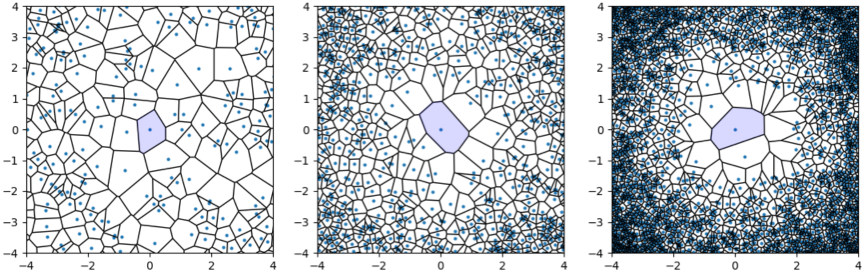
Further results and questions...

- Siegel & Holst (1980) found explicit formulas for the probability of covering the circle of circumference 1 with n arcs of i.i.d. random lengths $L_1, .., L_n \sim L$ where L is any distribution on the interval $[0,1]$:

  $\mathbb{P}($ circle of circumference 1 is covered $) =$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_{\{\sum_{i=1}^{k} u_i = 1\}} \left(\prod_{i=1}^{k} \mathbb{P}(L \leq u_i) \right) \left(\prod_{j=1}^{k} \int_0^{u_j} \mathbb{P}(L \leq v) dv \right)^{n-k} du_1 \cdots du_k$$

- TO DO: Asymptotic of $\uparrow$ for $n \to \infty$ for different CDF $x \mapsto \mathbb{P}(L \leq x)$?
- TO DO: Generalize S. & H. for specific distribution of N, a random number of arcs?
- TO DO: What if the density of raindrops is not uniform (but still isotropic): can we generalize Stevens/ Siegel & Holst, or can we apply a new method?



- ... ?