WELL-BEHAVED CONVOLUTION AVERAGES AND THE NON-ACCUMULATION THEOREM FOR LIMIT-CYCLES.

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Abstract : This expository paper introduces several uniformizing averages, which are serviceable in resummation theory because they manage to reconcile three essential, but at first sight quite conflicting demands : respectin g convolution ; preserving realness ; reproducing lateral growth. Their potential range of application covers most situations characterized by a combination of (1) non-linearity, (2) divergence, (3) realness. We sketch three typical applications, the last of which leads to a marginal, yet significant simplification of the constructive (i.e. resummation-theoretical) proof of the non-accumulation theorem for the limit-cycles of a real-analytic vector field on $\mathbf{R}^2$.
Résumé : Nous introduisons, comme auxiliaire pour l'accéléro-sommation des séries divergentes, diverses *moyennes uniformisantes* qui concilient trois propriétés essentielles, mais à première vue antagonistes : respecter le produit de convolution ; préserver la réalité des séries ; reproduire la croissance latérale de leurs transformées de Borel. Ces moyennes uniformisantes ont un domaine naturel d'application qui couvre peu ou prou toutes les situations mêlant (1) non-linéarité, (2) divergence, (3) réalité. Nous esquissons trois applications typiques, dont la dernière apporte des simplifications appréciables à la preuve constructive (fondée sur la théorie de la resommation) du théorème de non-accumulation des cycles-limite pour un champ de vecteurs analytique-réel sur $\mathbf{R}^2$.
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Contents

1 Some heuristics. The need for well-behaved, convolution-preserving averages.

The general resummation scheme .

The general scheme for resumming divergent series $\tilde{\varphi}(t)$ of "natural origin" goes like this :

$$\begin{array}{ccccccc}
\tilde{\varphi}_1(z_1) \equiv \tilde{\varphi}(t) & \dashrightarrow & & & & \longrightarrow & \varphi(t) \equiv \varphi_r(z_r) \\
\big\downarrow & & & & & & \big\uparrow \\
\hat{\varphi}_1(\zeta_1) & \longrightarrow & \hat{\varphi}_2(\zeta_2) & \longrightarrow & \cdots \ \hat{\varphi}_{r-1}(\zeta_{r-1}) & \longrightarrow & \hat{\varphi}_r(\zeta_r)
\end{array}$$

figure 1

(a) Here $\tilde{\varphi}(t)$ denotes a divergent power series (or transseries) of *natural origin* ; for instance the formal solution of a local analytic equation or system (differential ; partial differential ; functional ; etc...).

(b) t is the natural variable (usually $t \sim 0$) and $z_1 \ll z_2 \ll \ldots \ll z_r$ are the so-called *critical times* (or *critical variables*), defined up to equivalence and ordered from *slower* to *faster*. Each of them is large ($z_i \sim \infty$) and a function of t. Most often, they are just plain negative powers of t :

(1) $$z_1 \equiv t^{-p_1} \,;\, z_2 \equiv t^{-p_2} \,;\, \ldots ;\, z_r \equiv t^{-p_r} \;\; (\, 0 < p_1 < p_2 < \ldots < p_r \,)$$

(c) We begin (step $t \to z_1$) by expressing $\tilde{\varphi}(t)$ in terms of its slowest time z_1 and then (step $z_1 \to \zeta_1$) we subject $\tilde{\varphi}_1(z_1)$ to the *formal Borel transform*, which for instance turns each monomial $(z_1)^{-\sigma}$ into $(\zeta_1)^{\sigma-1}/\Gamma(\sigma)$.

(d) then we go successively through the steps $\zeta_i \to \zeta_{i+1}$. These stand for the so-called *acceleration transforms* :

$$\hat{\varphi}_i(\zeta_i) \to \hat{\varphi}_{i+1}(\zeta_{i+1}) \equiv \int_0^{+\infty} C_{F_i}(\zeta_{i+1}, \zeta_i)\, \hat{\varphi}_i(\zeta_i)\, d\zeta_i \tag{2}$$

which are the pull-back, under Borel-Laplace, of the mere changes of "time" $\varphi_i(z_i) \equiv \varphi_{i+1}(z_{i+1})$ with $z_i \equiv F_i(z_{i+1})$.

(e) then (step $\zeta_r \to z_r$) we carry out a *Laplace transform* :

$$\hat{\varphi}_r(\zeta_r) \to \varphi_r(z_r) \equiv \int_0^{+\infty} e^{-z_r \zeta_r}\, \hat{\varphi}_r(\zeta_r)\, d\zeta_r \tag{3}$$

and lastly we revert to the original variable (step $z_r \to t$).

This seemingly round-about procedure for "dropping the twiddle", i.e. for turning the formal object $\tilde{\varphi}(t)$ into a geometric one $\varphi(t)$, is known as *accelero-summation.* It has nothing arbitrary about it, and the various steps must be enacted in precisely the specified order, because the growth rate in each $\zeta_i - plane$ allows acceleration to ζ_{i+1}, but (usually) not to ζ_{i+2} , ζ_{i+3} , ...

Three steps in one.

Although each move from ζ_i to ζ_{i+1} (or from ζ_r to z_r) looks like being one single step, it actually involves three distinct substeps.

(i) *First substep : calculating a germ.* We first obtain $\hat{\varphi}_i(\zeta_i)$ as a germ near $\zeta_i = +0$ either (if $i = 1$) by the formal Borel transform or (if $i \geq 2$) by an acceleration integral which, generally speaking, converges only for small enough values of ζ_i.

(ii) *Second substep : getting a global function.* We must continue this germ $\hat{\varphi}_i(\zeta_i)$ from $+0$ to $+\infty$, so as to get hold of a *global function.* This turns out to be possible because $\hat{\varphi}_i(\zeta_i)$ is always *cohesive* (either *analytic* or *regular quasianalytic*) and because, owing to the "natural origin" of $\tilde{\varphi}(t)$, there are no obstacles to cohesive continuation from $+0$ to $+\infty$.

(iii) *Third substep : uniformizing the global function.* Although there are no obstacles to *cohesive* (analytic or quasianalytic) *continuation*, there may well be *cohesive* (analytic or quasianalytic) *singularities.* Indeed, we must recall that the existence of various singularities in the various $\zeta_i - planes$ is precisely what causes the divergence of the initial series $\tilde{\varphi}(t)$, and that there is nothing to prevent those singularities from lying over $\mathbf{R}^+$. On the contrary, there are often compelling reasons for them to be located there. Whenever this is the case, the global function $\hat{\varphi}_i(\zeta_i)$ is *multivalued* (i.e. *many-branched*) over $\mathbf{R}^+$, and we must turn it, *in some suitable way* (here lies the hitch !) into a *univalued function* $(\mathbf{m}\hat{\varphi}_i)(\zeta_i)$, so as to be in a position to perform the next acceleration transform $\zeta_i \to \zeta_{i+1}$ or (if $i = r$) the concluding Laplace transform $\zeta_r \to z_r$.

The space $RAMIF$ and its natural convolution product.

So let us fix some i and put $\zeta_i = \zeta$ for simplicity. What we require, in order to carry out the third substep in the above scheme, is a proper *uniformizing average* from the space $RAMIF(\mathbf{R}^+)$ of *forward-ramified functions over* $\mathbf{R}^+$ into the space $UNIF(\mathbf{R}^+)$ of *uniform functions* on $\mathbf{R}^+$. But first, we must get a few definitions out of the way.

To each (finite or infinite) sequence :

$$\Omega = \{0 = \eta_0 < \eta_1 < \eta_2 < \eta_3 < \cdots\} \subset \mathbf{R}^+ \tag{4}$$

we associate the space $RAMIF(\mathbf{R}^+//\Omega)$ of all complex-valued functions $\hat{\varphi}(\zeta)$ defined on the set $\mathbf{R}^+//\Omega$ consisting of one branch over the interval $]0, \eta_1[$, two branches over the interval $]\eta_1, \eta_2[$, ..., and 2^r branches over the interval $]\eta_r, \eta_{r+1}[$. Each branch over $]\eta_r, \eta_{r+1}[$ is characterized by its "address" $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ consisting of r signs $\varepsilon_i = \pm$. If $\varepsilon_r = +$ (resp. $-$), the branch over $]\eta_r, \eta_{r+1}[$ with address $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ is regarded as being the right (resp. the left) continuation of the branch over $]\eta_{r-1}, \eta_r[$ with address $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{r-1})$. See figure 2 below.

If $\Omega_1 \subset \Omega_2$, there is a trivial projection of $\mathbf{R}^+//\Omega_2$ onto $\mathbf{R}^+//\Omega_1$ and so too (with provision for the usual reversal) a trivial injection of $RAMIF(\mathbf{R}^+//\Omega_1)$ into $RAMIF(\mathbf{R}^+//\Omega_2)$. This enables us to define $RAMIF(\mathbf{R}^+)$ as the inductive limit of all spaces $RAMIF(\mathbf{R}^+//\Omega)$ relatively to the inclusion of the indexing sets Ω :

$$RAMIF(\mathbf{R}^+) \overset{def}{=} lim\ ind_{\Omega}\ \ RAMIF(\mathbf{R}^+//\Omega) \tag{5}$$

But we must also endow $RAMIF(\mathbf{R}^+)$, or rather the subspace $RAMIF(\mathbf{R}^+\,;\,int.)$ of *locally integrable functions*, with a convolution product $*$ that extends the natural convolution $*$ defined on the space of locally integrable germs at $+0$:

$$\hat{\varphi}_3(\zeta) = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) = \int_0^{\zeta} \hat{\varphi}_1(\zeta_1)\,\hat{\varphi}_2(\zeta - \zeta_1)\,d\zeta_1 \ \ (0 < \zeta \ll 1) \tag{6}$$

This is done by piecing together the three following lemmas :

Lemma 1 *Let Ω be any discrete additive semigroup of $\mathbf{R}^+$. If the analytic germs $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are defined on $]0, ...[$, integrable at 0, and possess analytic continuations uniform on $RAMIF(\mathbf{R}^+//\Omega)$, so too does the germ $\hat{\varphi}_3$ defined by the local convolution above. Such germs span a space $RAMIF(\mathbf{R}^+//\Omega\,;\,ana.)$ embedded in $RAMIF(\mathbf{R}^+//\Omega\,;\,int.)$ and endowed with a global convolution.*

Lemma 2 *Relative to a suitable system of L_1-norms on increasing compact subsets of $\mathbf{R}^+//\Omega$, the space $RAMIF(\mathbf{R}^+//\Omega\,;\,ana.)$ is actually dense in $RAMIF(\mathbf{R}^+//\Omega\,;\,int.)$. This induces on $RAMIF(\mathbf{R}^+\,;\,int.)$ a global convolution which owes nothing to analytic continuability (there is no analyticity or cohesiveness any more), but is directly calculable by :*

$$\hat{\varphi}_3(\zeta) = \int \mathbf{T}_{\zeta_3}^{\zeta_1,\zeta_2}(\Omega)\, \hat{\varphi}_1(\zeta_1)\, \hat{\varphi}_2(\zeta_2)\, d\zeta_1 \ (or\ \ d\zeta_2) \tag{7}$$

$$\zeta_1, \zeta_2, \zeta \in \mathbf{R}^+//\Omega \ ;\ \dot{\zeta}_1 + \dot{\zeta}_2 = \dot{\zeta} \ ;\ 0 < \dot{\zeta}_1 < \dot{\zeta} \ ;\ 0 < \dot{\zeta}_2 < \dot{\zeta} \tag{8}$$

Here, ζ_1, ζ_2, ζ are points of $\mathbf{R}^+//\Omega$ with projections $\dot{\zeta}_1$, $\dot{\zeta}_2$, $\dot{\zeta}$ on $\mathbf{R}^+$. The well-defined "structure tensor" $\mathbf{T}_{\zeta_3}^{\zeta_1,\zeta_2}(\Omega)$ is locally constant in its three variables ; assumes whole values (positive or negative) only ; and vanishes unless $\dot{\zeta}_1+\dot{\zeta}_2 = \dot{\zeta}$.

Lemma 3 *The convolutions defined on the various spaces $RAMIF(\mathbf{R}^+//\Omega\,;\,int.)$ are compatible with the natural embeddings :*

$$RAMIF(\mathbf{R}^+//\Omega_1\,;\,int.) \rightarrow RAMIF(\mathbf{R}^+//\Omega_2\,;\,int.) \tag{9}$$

$$\Omega_1 \subset \Omega_2 \subset \mathbf{R}^+\,;\ \Omega_1 \text{ and } \Omega_2 \text{ discrete additive semigroups} \tag{10}$$

and this induces a global convolution on the limit space :

$$RAMIF(\mathbf{R}^+\,;\,int.) \stackrel{def}{=} lim\ ind_{\Omega}\ \ RAMIF(\mathbf{R}^+//\Omega\,;\,int.) \tag{11}$$

For details, see [8].

There is a very rich array of internal operators acting on $\hat{\varphi}$. Apart from the *natural derivation* $\hat{\partial}$:

$$\hat{\partial}\hat{\varphi}(\zeta) \stackrel{def}{=} -\dot{\zeta}\hat{\varphi}(\zeta) \tag{12}$$

(which is simply the image under Borel of the plain derivation $\partial = \partial/\partial z$)and the *complex conjugation* (which exchanges conjugate branches), there are the far more interesting *alien operators*, which are characterized by commuting with the natural derivation $\hat{\partial}$. They span an associative algebra $ALIEN$ which is essentially generated by the *alien derivations* (under allowance of infinite sums ; see [8]). In this paper, however, the internal operators on $RAMIF$ shall play an ancillary part at best, and we shall focus instead on the "projections" or "averages" from $RAMIF$ onto $UNIF$.

Uniformizing averages. Six desirable properties

Uniformizing averages are linear maps from $RAMIF$ to $UNIF$ which reverse the natural embedding of $UNIF$ into $RAMIF$. In concrete term, each uniformizing average :

$$\mathbf{m}\ :\ \hat{\varphi} \mapsto \mathbf{m}\hat{\varphi}\ ; \qquad (\mathbf{m}) \circ (embed.) = id_{UNIF} \tag{13}$$

$$RAMIF(\mathbf{R}^+) \rightarrow UNIF(\mathbf{R}^+)\ ; \quad RAMIF(\mathbf{R}^+\,;\,int.) \rightarrow UNIF(\mathbf{R}^+\,;\,int.) \tag{14}$$

is defined by an infinite set of *averaging weights* $\mathbf{m}^{\boldsymbol{\varpi}}$:

$$(15)\{\mathbf{m}^{\boldsymbol{\varpi}} = \mathbf{m}^{\varpi_1,\ldots,\varpi_r}\,; with\ r \in \mathbf{N}; \varpi_i = \begin{pmatrix} \varepsilon_i \\ \omega_i \end{pmatrix}; \varepsilon_i = \pm; \omega_i \in \mathbf{R}^+\,;\ \mathbf{m}^{\boldsymbol{\varpi}} \in \mathbf{C}\}$$

and it acts on a given ramified function :

$$(16)\quad \begin{array}{c} \hat{\varphi} \in RAMIF(\mathbf{R}^+//\Omega) \\ with\ \Omega = \{\eta_0 = 0\,;\ \eta_1 = \omega_1\,;\ \eta_2 = \omega_1 + \omega_2\,;\ \eta_3 = \omega_1 + \omega_2 + \omega_3\,; \ldots\} \end{array}$$

according to the rule :

$$(17)\qquad (\mathbf{m}\hat{\varphi})(\zeta) \overset{def}{=} \sum_{\varepsilon_1=\pm;\ldots;\varepsilon_r=\pm} \mathbf{m}^{\varpi_1,\ldots,\varpi_r}\hat{\varphi}(\zeta^{\varpi_1,\ldots,\varpi_r}) \ \ (if\ \eta_r < \zeta < \eta_{r+1})$$

Here, of course, $\varpi_i = \begin{pmatrix} \varepsilon_i \\ \omega_i \end{pmatrix}$ and $\zeta^{\varpi_1,\ldots,\varpi_r}$ denotes the point of $\mathbf{R}^+//\Omega$ that lies *over* ζ on the branch $]\eta_r, \eta_{r+1}[$ of address $(\varepsilon_1, \ldots, \varepsilon_r)$.

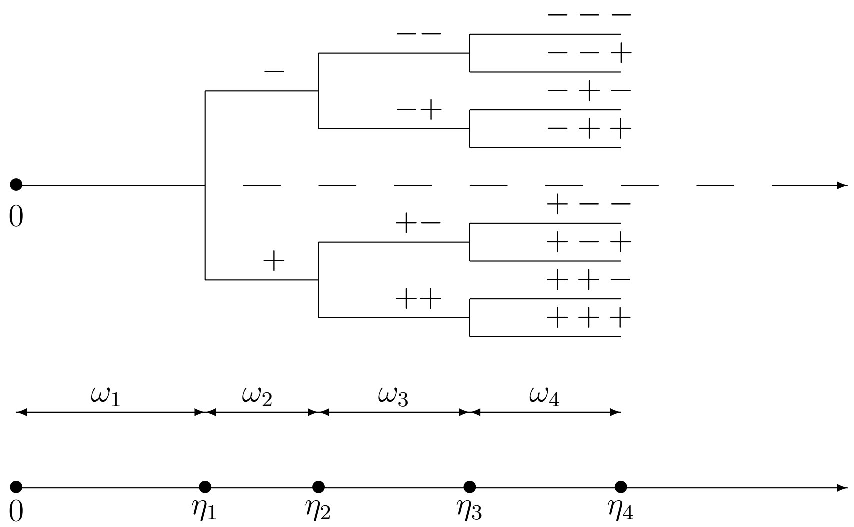

figure 2

Obviously, in order for the averaging to be independent of Ω (i.e. compatible with the canonical embeddings of $RAMIF(\mathbf{R}^+//\Omega_1)$ into $RAMIF(\mathbf{R}^+//\Omega_2)$ for $\Omega_1 \subset \Omega_2$) and therefore to induce a map from the limit space $RAMIF(\mathbf{R}^+)$ into $UNIF(\mathbf{R}^+)$, the weights must satisfy the following condition :

Property 0 : self-consistency

$$\sum_{\varepsilon_1=\pm} \mathbf{m}^{\begin{pmatrix} \varepsilon_1 \\ \omega_1 \end{pmatrix}} = 1\ \ and$$

$$\sum_{\varepsilon_j=\pm} \mathbf{m}\begin{pmatrix} \varepsilon_1,\dots,\varepsilon_j,\dots,\varepsilon_r \\ \omega_1,\dots,\omega_j,\dots,\omega_r \end{pmatrix} = \mathbf{m}\begin{pmatrix} \varepsilon_1,\dots,\varepsilon_{j-1},\dots & ,\varepsilon_r \\ \omega_1,\dots,\omega_{j-1}+\omega_j,\dots & ,\omega_r \end{pmatrix} \quad (\forall j) \tag{18}$$

for all ω_i *in* $\mathbf{R}^+$ *and all* ε_i *in* $\{+,-\}$.

This is the very least we must ask, and Property 0 will always be tacitly assumed. But we may, and often must, make additional demands on our uniformizing averages, such as :

P.1 : respecting convolution.
P.2 : respecting realness.
P.3 : respecting lateral growth.
P.4 : being positive.
P.5 : being secable.
P.6 : being scale-invariant.

Let us now spell out the exact import of each property, and examine its algebraic translation in terms of averaging weights.

Property 1 : respecting convolution.

This is arguably the main demand. It means that the map (13) should be an algebra homomorphism :

$$\mathbf{m}(\hat\varphi_1 * \hat\varphi_2) = (\mathbf{m}\hat\varphi_1) * (\mathbf{m}\hat\varphi_2) \quad \forall \hat\varphi_i \in RAMIF(\mathbf{R}^+\,;\, int.) \tag{19}$$

where the first (resp. second) star $*$ denotes the *global convolution* on $RAMIF(\mathbf{R}^+\,;\, int.)$ (resp. on $UNIF(\mathbf{R}^+\,;\, int.)$). Actually, if we don't want to bother with the rather complicated convolution on $RAMIF(\mathbf{R}^+\,;\, int.)$, we may simply impose (19) for all pairs $\hat\varphi_1$, $\hat\varphi_2$ in the space $RAMIF(\mathbf{R}^+\,;\, ana.)$ of ramified analytic functions, where the *global convolution* may be decomposed into the elementary *local convolution* (6) at $+0$, followed by *analytic continuation.* This apparently weaker demand is in fact equivalent to (19), and implies the same constraints [1] on the weights, namely :

Lemma 4 *An averaging map* $\mathbf{m}$ *respects convolution if and only if the following equivalent conditions are fulfilled :*
(C.1) The right-associated mould $Rm^\bullet$ *is symmetrel*
(C.2) The left-associated mould $Lm^\bullet$ *is symmetrel*

[1]There are various and quite cogent technical reasons, though, for preferring to work with the pair $\{RAMIF(\mathbf{R}^+\,;\, int.)\,;\, UNIF(\mathbf{R}^+\,;\, int.)\}$ rather than with the pair $\{RAMIF(\mathbf{R}^+\,;\, ana.)\,;\, UNIF(\mathbf{R}^+\,;\, int.)\}$. With the first pair, all averaging maps are *onto*, instead of *into* with the second pair. Also, the first pair, being of type {non cohesive ; non cohesive }, is more homogeneous than the second pair, which is of type {cohesive ; non cohesive} and cannot, moreover, be replaced by a pair of type {cohesive ; cohesive}.

(C.3) For each discrete additive semigroup $\Omega \subset \mathbf{R}^+$, the discretized weights $\mathbf{m}^{\boldsymbol{\varepsilon}}(\Omega)$ verify the canonical multiplication table :

$$\mathbf{m}^{\boldsymbol{\varepsilon}}(\Omega) = \sum_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''} \mathbf{T}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}_{\boldsymbol{\varepsilon}}(\Omega)\, \mathbf{m}^{\boldsymbol{\varepsilon}'}(\Omega)\, \mathbf{m}^{\boldsymbol{\varepsilon}''}(\Omega) \tag{20}$$

This calls for some explanations. A *mould* $M^\bullet$ is simply a collection of elements $M^{\boldsymbol{\omega}} \equiv M^{\omega_1,\ldots,\omega_r}$ of some commutative algebra. The indexing sequences have arbitrary length r and (usually) real or complex components ω_i. In the above criteria (C.1) and (C.2), the moulds $Rm^\bullet$ and $Lm^\bullet$are defined as follows :

$$Rm^{\omega_1,\ldots,\omega_r} \overset{def}{=} (-1)^r \mathbf{m}\begin{pmatrix} +,+,\ldots,+ \\ \omega_1,\omega_2,\ldots,\omega_r \end{pmatrix} \quad (\forall r \in \mathbf{N}\,;\, \forall \omega_i \in \mathbf{R}^+) \tag{21}$$

$$Lm^{\omega_1,\ldots,\omega_r} \overset{def}{=} (-1)^r \mathbf{m}\begin{pmatrix} -,-,\ldots,- \\ \omega_1,\omega_2,\ldots,\omega_r \end{pmatrix} \quad (\forall r \in \mathbf{N}\,;\, \forall \omega_i \in \mathbf{R}^+) \tag{22}$$

There is a rich structure attached to moulds, but here we only require the notion of *symmetrelness*. *Being symmetrel* for a mould $M^\bullet$ means that for any two sequences $\boldsymbol{\omega}$' and $\boldsymbol{\omega}$" we should have :

$$M^{\boldsymbol{\omega}'}\, M^{\boldsymbol{\omega}''} = \sum M^{\boldsymbol{\omega}} \quad (\boldsymbol{\omega} \in ctsh(\boldsymbol{\omega}'\,;\,\boldsymbol{\omega}'')) \tag{23}$$

with a sum $\sum$ extending to all sequences $\boldsymbol{\omega}$ obtained by *shuffling* the sequence $\boldsymbol{\omega}$' with $\boldsymbol{\omega}$" (i.e. by interdigitating their elements under preservation of the internal order of each parent sequence) and, possibly, *contracting* adjacent components from $\boldsymbol{\omega}$' and $\boldsymbol{\omega}$". Thus, if $M^\bullet$ is symmetrel, we must have,for example :

$$M^{\omega_1}\, M^{\omega_2} = M^{\omega_1,\omega_2} + M^{\omega_2,\omega_1} + M^{\omega_1+\omega_2} \tag{24}$$

$$\begin{aligned} M^{\omega_1}\, M^{\omega_2,\omega_3} = {} & M^{\omega_1,\omega_2,\omega_3} + M^{\omega_2,\omega_1,\omega_3} + M^{\omega_2,\omega_3,\omega_1} \\ & + M^{\omega_1+\omega_2,\omega_3} + M^{\omega_2,\omega_1+\omega_3} \end{aligned} \tag{25}$$

In the criterion (C.3), Ω denotes any fixed additive subgroup of $\mathbf{R}^+$ with *elements* η_i and *increments* ω_i :

$$\Omega = \{\eta_0 = 0\,;\, \eta_1 = \omega_1\,;\, \eta_2 = \omega_1 + \omega_2\,;\, \eta_3 = \omega_1 + \omega_2 + \omega_3\,;\ldots\} \tag{26}$$

The *discretized weights* $\mathbf{m}^{\boldsymbol{\varepsilon}}(\Omega)$ are indexed by sequences $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_r)$ of plus or minus signs, and defined by :

$$\mathbf{m}^{\varepsilon_1,\ldots,\varepsilon_r}(\Omega) \overset{def}{=} \mathbf{m}\begin{pmatrix} \varepsilon_1,\ldots,\varepsilon_r \\ \omega_1,\ldots,\omega_r \end{pmatrix} \quad (\forall r \in \mathbf{N}\,;\, \forall \varepsilon_i \in \{+,-\}) \tag{27}$$

The sum in (20) extends to all sign sequences $\boldsymbol{\varepsilon}' = (\varepsilon'_1,\ldots,\varepsilon'_{r'})$ and $\boldsymbol{\varepsilon}'' = (\varepsilon''_1,\ldots,\varepsilon''_{r''})$ such that :

$$\eta_r = \eta_{r'} + \eta_{r''} \tag{28}$$

and the structure tensor in (2) is elementarily related to the structure tensor in (7) by :

$$\mathbf{T}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}_{\boldsymbol{\varepsilon}}(\Omega) = \mathbf{T}^{\zeta',\zeta''}_{\zeta}(\Omega) \tag{29}$$

where ζ, ζ', ζ'' denote the points of $\mathbf{R}^+//\Omega$ that lie over $\eta_r(1+t)$, $\eta_{r'}(1+t)$, $\eta_{r''}(1+t)$ with addresses $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}$', $\boldsymbol{\varepsilon}$" (and t is any small enough positive real number ; recall that the structure tensor in (7) is *locally constant*). The criteria (C.1) and (C.2) are handier than (C.3), but (C.3) has its usefulness, too, especially when $\Omega = \mathbf{N}$. In this case, (28) reduces to :

$$r = r' + r'' \tag{30}$$

and the "basic" structure tensor $\mathbf{T}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}_{\boldsymbol{\varepsilon}} \stackrel{def}{=} \mathbf{T}^{\zeta',\zeta''}_{\zeta}(\mathbf{N})$ can be calculated by either of the following inductions :

$$|\,\mathbf{T}^{\mathbf{a},\mathbf{b}}_{\mathbf{c}}\,| = \delta^{a_1,c_1}\,|\,\mathbf{T}^{\text{‘}\mathbf{a},\mathbf{b}}_{\text{‘}\mathbf{c}}\,| + \delta^{b_1,c_1}\,|\,\mathbf{T}^{\mathbf{a},\text{‘}\mathbf{b}}_{\text{‘}\mathbf{c}}\,| - \delta^{a_1,c_2}\delta^{b_1,c_2}\,|\,\mathbf{T}^{\text{‘}\mathbf{a},\text{‘}\mathbf{b}}_{\text{“}\mathbf{c}}\,| \tag{31}$$

$$|\,\mathbf{T}^{\mathbf{a},\mathbf{b}}_{\mathbf{c}}\,| = \delta^{a_{r_1},c_{r_3}}\,|\,\mathbf{T}^{\mathbf{a}\text{’},\mathbf{b}}_{\mathbf{c}\text{’}}\,| + \delta^{b_{r_2},c_{r_3}}\,|\,\mathbf{T}^{\mathbf{a},\mathbf{b}\text{’}}_{\mathbf{c}\text{’}}\,| - \delta^{a_{r_1},c_{r_3}}\delta^{b_{r_2},c_{r_3}}\,|\,\mathbf{T}^{\mathbf{a}\text{’},\mathbf{b}\text{’}}_{\mathbf{c}\text{”}}\,| \tag{32}$$

supplemented by the "initial conditions" :

$$\mathbf{T}^{\emptyset,\emptyset}_{\emptyset} \equiv \mathbf{T}^{\mathbf{a},\emptyset}_{\mathbf{a}} \equiv \mathbf{T}^{\emptyset,\mathbf{b}}_{\mathbf{b}} \equiv 1 \quad (\forall \mathbf{a}\,,\, \forall \mathbf{b}) \tag{33}$$

and the elementary sign rule :

$$\textit{sign of } \mathbf{T}^{\mathbf{a},\mathbf{b}}_{\mathbf{c}} = (\prod a_i)(\prod b_i)(\prod c_i) \tag{34}$$

Needless to say, **a**, **b**, **c** denote three sign sequences :

$$\mathbf{a} = (a_1, \ldots, a_{r_1})\ ,\ \mathbf{b} = (b_1, \ldots, b_{r_2})\ ,\ \mathbf{c} = (c_1, \ldots, c_{r_3}) \tag{35}$$

and **‘a**, **“a**, etc...(resp. **a’**, **a”**, etc...) denote the sequence **a** deprived of its first term, first and second terms, etc...(resp. last term, two last terms, etc...). Lastly, $\delta^{\varepsilon_1,\varepsilon_2}$ denotes the classical Kronecker symbol :

$$\delta^{\varepsilon_1,\varepsilon_2} = 1\ (\textit{resp. } 0)\ \textit{if } \varepsilon_1 = \varepsilon_2\ (\textit{resp. } \varepsilon_1 \neq \varepsilon_2) \tag{36}$$

Thus the universal multiplication table for the discretized weights $\mathbf{m}^{\boldsymbol{\varepsilon}} = \mathbf{m}^{\boldsymbol{\varepsilon}}(\mathbf{N})$ reads :

$$\mathbf{m}^+\mathbf{m}^+ = \mathbf{m}^{++} - \mathbf{m}^{-+}\ ;\ \mathbf{m}^+\mathbf{m}^- = \mathbf{m}^{+-} + \mathbf{m}^{-+}\ ;\ \mathbf{m}^-\mathbf{m}^- = \mathbf{m}^{--} - \mathbf{m}^{+-} \tag{37}$$

$$\mathbf{m}^+\mathbf{m}^{++} = \mathbf{m}^{+++} - \mathbf{m}^{+-+} - \mathbf{m}^{-++}\ \ \ldots \tag{38}$$

Of course, each relation remains valid when we simultaneously *change* all signs ε_i ; but it *does not* when we simultaneously *reverse* all three sign sequences. For proofs, complements, and tables, see [8]. We now proceed with our list of "desirable" properties.

Property 2 : respecting realness.

This means that $\mathbf{m}\hat{\varphi}$ must be real whenever $\hat{\varphi}$ is real. For an analytic $\hat{\varphi}$, being real, of course, means assuming real values right of 0 up to the first singularity η_1. For a merely integrable $\hat{\varphi}$, being real means assuming complex conjugate values on *conjugate branches*, i.e. on branches with conjugate addresses $(\varepsilon_1, \ldots, \varepsilon_r)$ and $(\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r)$ (ε_i is any sign and $\bar{\varepsilon}_i$ is the opposite sign). Therefore, an averaging map respects realness if and only if its weights verify :

$$(39) \qquad \mathbf{m}^{\begin{pmatrix} \varepsilon_1, \ldots, \varepsilon_r \\ \omega_1, \ldots, \omega_r \end{pmatrix}} = \overline{\mathbf{m}}^{\begin{pmatrix} \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r \\ \omega_1, \ldots, \omega_r \end{pmatrix}} (\forall \omega_i \in \mathbf{R}^+ \,;\, \forall \varepsilon_i \in \{+,-\})$$

Property 3 : respecting lateral growth.

Here, three basic facts must be borne in mind :

Fact 1 : A function $\hat{\varphi}(\zeta)$ can be accelerated or Laplaced only if it has a well-defined critical growth at infinity. For instance, it can be Laplaced if it grows no faster than exponentially, and it can be accelerated if it grows no faster than $C_F(\zeta_2, \zeta)$ for some fixed $\zeta_2 > 0$ ($\zeta \to +\infty$). Now, in all natural instances, some "preestablished harmony" underlying resummation theory automatically ensures this required growth condition *on singularity-free axes* and also (with certain innocuous provisos) on *both sides*, right and left, *of singularity-carrying axes.*

Fact 2 : However, if we continue $\hat{\varphi}(\zeta)$ along a singularity-carrying direction and follow an "oft-crossing" path (for instance, if we cross the axis between any two consecutive singularities - assuming of course that there are infinitely many), we will quite generically, at least in non-linear problems, encounter faster-than-critical growth. Thus, if $\hat{\varphi}(\zeta)$ has exponential *lateral* growth, on *oft-crossing paths* the bounds are, generically, no better than :

$$(40) \qquad \mid \hat{\varphi}(\zeta) \mid \leq K_1 \exp(K_2 \mid \zeta \mid \mid \log(\zeta) \mid) \quad (K_1, K_2 \; constant)$$

Fact 3 : This remarkable phenomenon, in turn, is due to the special nature of the *acting alien algebra* $ACT(\varphi)$ associated with resurgent functions of natural origin, and defined as being the quotient :

$$(41) \qquad ACT(\varphi) \equiv ALIEN/IDE(\varphi)$$

of the algebra $ALIEN$ of all *alien derivations* by the *ideal* $IDE(\varphi)$ generated by those alien derivations that annihilate φ. Indeed, despite the amazing diversity of resurgence patterns and resurgence equations met with in real life, the *acting alien algebra* $ACT(\varphi)$ turns out, in factually all natural instances, to be isomorphic to some subalgebra of $Endo(\mathbf{C}[[x_1, \ldots, x_\nu]])$. These three facts, taken together, motivate the following definition, which we first state and then explain :

Definition 1 *A uniformizing average* $\mathbf{m}$ *is said to "preserve lateral growth" if the following alien operators :*

$$\begin{pmatrix} \mathbf{mul} \\ \mathbf{m} \end{pmatrix} \quad and \quad \begin{pmatrix} \mathbf{mul} \\ \mathbf{mur} \end{pmatrix} \tag{42}$$

or equivalently :

$$\begin{pmatrix} \mathbf{mur} \\ \mathbf{m} \end{pmatrix} \quad and \quad \begin{pmatrix} \mathbf{mur} \\ \mathbf{mul} \end{pmatrix} \tag{43}$$

are equianalytic.

Explanations : Any two averaging maps $\mathbf{m}_1$ and $\mathbf{m}_2$ are connected by a well-defined alien automorphism $\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}$ of $RAMIF$:

$$\mathbf{m}_2 = \mathbf{m}_1 \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} \tag{44}$$

$$\begin{tikzcd} RAMIF \arrow[rr, "\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}"] \arrow[dr, "\mathbf{m}_2"'] & & RAMIF \arrow[dl, "\mathbf{m}_1"] \\ & UNIF & \end{tikzcd}$$

and, if both $\mathbf{m}_1$ and $\mathbf{m}_2$ respect convolution, so too does $\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}$. Consequently, defining a plain (resp. convolution-respecting) uniformizing average $\mathbf{m}$ is equivalent to defining the plain (resp. convolution-respecting) alien automorphisms $\begin{pmatrix} \mathbf{mur} \\ \mathbf{m} \end{pmatrix}$ or $\begin{pmatrix} \mathbf{mul} \\ \mathbf{m} \end{pmatrix}$ which connect $\mathbf{m}$ with the *trivial lateral "averages"*, namely **mur** (right determination) and **mul** (left determination) (see section 2 below, Example 1).

Next, a *reduction* of the graded algebra $ALIEN$ of alien operators is, by definition, any (continuous) graded algebra homomorphism of $ALIEN$ into *some* algebra $Endo(\mathbf{C}[[x_1, \dots, x_\nu]])$. Obviously, a "reduction" is known the moment we know the images of each alien derivation Δ_ω :

$$\Delta_\omega \mapsto red(\Delta_\omega) \equiv x_1^{n_1(\omega)} \dots x_\nu^{n_\nu(\omega)} \sum_{1 \le i \le \nu} A_\omega^i x_i \frac{\partial}{\partial x_i} \quad (A_\omega^i \in \mathbf{C}) \tag{45}$$

The constraint of having to respect the gradation forces all but enumerably many Δ_ω to have vanishing images $red(\Delta_\omega)$. For instance, the map :

$$\Delta_\omega \mapsto red(\Delta_\omega) = 0 \quad if\ \omega \notin \mathbf{N} \tag{46}$$

$$\Delta_\omega \mapsto red(\Delta_\omega) = A_\omega x^{\omega+1} \frac{\partial}{\partial x} \quad if\ \omega \in \mathbf{N}\ \ with\ A_\omega \in \mathbf{C} \tag{47}$$

is one of the simplest examples of non-trivial "reduction".

Further, any two alien operators Op and Op' are said to be *equianalytic* if, *under any given reduction red*, their images $red(Op)$ and $red(Op')$ are *simultaneously analytic* or *non-analytic.* Of course, since $red(Op)$ and $red(Op')$ are endomorphisms of the space $\mathbf{C}[[x_1, \ldots, x_\nu]]$ of formal power series in $x_1, \ldots, x_\nu$, *being analytic* for them means that they should leave invariant the subspace $\mathbf{C}\{x_1, \ldots, x_\nu\}$ consisting of convergent power series.

Lastly, our definition 1 is self-consistent because, under any "reduction", the images $red\begin{pmatrix}\mathbf{mul}\\ \mathbf{mur}\end{pmatrix}$ and $red\begin{pmatrix}\mathbf{mur}\\ \mathbf{mul}\end{pmatrix}$, *being mutually reciprocal substitution automorphisms of* $\mathbf{C}[[x_1, \ldots, x_\nu]]$, are *automatically equianalytic* (either both analytic or both non-analytic).

Thus, the vague-sounding requirement of "respecting lateral growth" admits of a rigorous and fully algebraized translation (namely the one in definition 1), which is neither too strong nor too weak, but *exactly what the applications require.*

Property 4 : being positive.

Weak positivity for an average $\mathbf{m}$ simply means that all the weights $\mathbf{m}^{\boldsymbol{\varpi}}$ are non-negative. *Strong positivity* is a more recondite property ; it means that, for each $p \geq 1$, there should exist a general co-product :

$$(48) \qquad \mathbf{m}(\hat{\varphi}_1 * \cdots * \hat{\varphi}_p) \equiv \sum P^{i_1,\ldots,i_p}(\mathbf{m}_{i_1}\hat{\varphi}_1) * \cdots * (\mathbf{m}_{i_p}\hat{\varphi}_p) \quad (\forall \hat{\varphi}_i)$$

involving only *weakly positive* averages $\mathbf{m}_{i_q}$. The $P^{\cdots}$ denote universal constants, and the sum $\sum$ in (48) may involve an infinite number of terms. For a convolution-preserving average, weak positivity clearly implies strong positivity (in view of (19)), but for general uniformizing averages, there is a great gap between the two conditions.

Property 5 : being secable.

"Secability" is specially meaningful for convolution-preserving averages. This notion is relative to a fixed semigroup Ω, usually $\Omega = \mathbf{N}$, and means that the averages $\mathbf{m}_{\boldsymbol{\varepsilon}} = \mathbf{m}_{\varepsilon_1,\ldots,\varepsilon_r}$ (mark the *lower position* of the indices, as opposed to their upper position in (27)) obtained by "cutting" $\mathbf{m}$ (i.e. by retaining only that part of $\mathbf{m}$ which is supported by the branches of $\mathbf{R}^+//\mathbf{N}$ whose address starts with $\boldsymbol{\varepsilon}$) should, like $\mathbf{m}$ itself, verify some *finite coproduct* :

$$(49) \qquad \mathbf{m}_{\boldsymbol{\varepsilon}}(\hat{\varphi} * \hat{\psi}) \equiv \sum K_{\boldsymbol{\varepsilon}}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}\, \mathbf{m}_{\boldsymbol{\varepsilon}'}(\hat{\varphi}) * \mathbf{m}_{\boldsymbol{\varepsilon}''}(\hat{\psi}) \quad (K_{\boldsymbol{\varepsilon}}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''} \in \mathbf{C})$$

that is to say, a coproduct with finitely many terms in $\sum$.

Property 6 : being scale-invariant.

Scale invariance means invariance under homothetic rescalings $\zeta \mapsto t\zeta$ of $\mathbf{R}^+$ or, in terms of weights :

$$(50)\quad \mathbf{m}\begin{pmatrix} \varepsilon_1, \dots, \varepsilon_r \\ \omega_1, \dots, \omega_r \end{pmatrix} = \mathbf{m}\begin{pmatrix} \varepsilon_1, \dots, \varepsilon_r \\ t\omega_1, \dots, t\omega_r \end{pmatrix} \quad (\forall \varepsilon_i = \pm \ ; \ \forall \omega_i \in \mathbf{R}^+ \ ; \ \forall t \in \mathbf{R}^+)$$

Let us recapitulate :

Property 0 (being an average) is a logical necessity.

Property 1 (respecting convolution) is indispensible in all non-linear situations (i.e. whenever the divergent series $\tilde{\varphi}(t)$ we wish to resum happens to be the formal solution of a non-linear equation or system), because in that case all the steps in the resummation diagram (figure 1) must involve algebra homomorphisms.

Property 2 (respecting realness) is a must whenever the series $\tilde{\varphi}(t)$ has real coefficients and must be assigned a real sum $\varphi(t)$ for some compelling reason, e.g. because it represents a physical or real-geometric object.

Property 3 (respecting lateral growth) is highly desirable (see the application in section 6) and sometimes indispensible (see the applications in sections 4 and 5), namely when the half-axis $\mathbf{R}^+$ in some $\zeta-$*plane* carries infinitely many singularities and when the added necessity of summing the real series $\tilde{\varphi}(t)$ to a real germ $\varphi(t)$ forces one to take into account the continuation of $\hat{\varphi}(\zeta)$ along all paths, including oft-crossing ones (see the Lemma 5 below).

Properties 4,5,6 are not indispensible, but merely desirable. They are helpful in narrowing down the field of investigation, and in characterizing some of the main uniformizing averages. Thus, the important *Catalan average* (see section 3) was discovered while searching for "secable" averages.

Let us now conclude this extensive introductory section with a lemma that brings out the conflicting nature of Properties 1,2,3.

Lemma 5 *Any uniformizing average* $\mathbf{m}$ *which respects both convolution and realness, necessarily involves the determinations of* $\hat{\varphi}$ *on at least some paths that cross the real axis infinitely many times. More precisely, for each finite* q*, there always exists at least one non-vanishing weight* $\mathbf{m}\begin{pmatrix} \varepsilon_1, \dots, \varepsilon_r \\ \omega_1, \dots, \omega_r \end{pmatrix} \neq 0$ *with an address* $(\varepsilon_1, \dots, \varepsilon_r)$ *displaying more than* q *sign changes.*

It is enough to consider the case of unit increments ($1 = \omega_1 = \omega_2 = \ldots$). Using the induction (31) or (32), it can be checked that, if $\boldsymbol{\varepsilon'}$ and $\boldsymbol{\varepsilon''}$ are two sign sequences starting with different signs but displaying exactly q sign changes each, then the structure tensor $\mathbf{T}_{\boldsymbol{\varepsilon}}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}$ vanishes unless the third sequence $\boldsymbol{\varepsilon}$ displays at least $q+1$ sign changes.

Now, let us assume the above Lemma 5 to be false. Then there exists some critical integer q such that all discretized weights $\mathbf{m}^{\boldsymbol{\varepsilon}}$ with more than q sign changes vanish, while at least one weight $\mathbf{m}^{\boldsymbol{\varepsilon'}}$ with exactly q sign changes is different from 0. But since $\mathbf{m}$ preserves realness, the weight $\mathbf{m}^{\boldsymbol{\varepsilon''}}$ with address $\boldsymbol{\varepsilon''}$ conjugate to $\boldsymbol{\varepsilon'}$ (i.e. with all signs changed) is complex-conjugate to $\mathbf{m}^{\boldsymbol{\varepsilon'}}$, and so different from 0. Therefore :

$$0 \neq \mathbf{m}^{\boldsymbol{\varepsilon'}}\ \mathbf{m}^{\boldsymbol{\varepsilon''}} = \sum \mathbf{T}_{\boldsymbol{\varepsilon}}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}\ \mathbf{m}^{\boldsymbol{\varepsilon}} \tag{51}$$

This, however, is impossible because, due to the above remark, $\mathbf{T}_{\boldsymbol{\varepsilon}}^{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''}$ vanishes unless $\boldsymbol{\varepsilon}$ has more than q sign changes, in which case $\mathbf{m}^{\boldsymbol{\varepsilon}} = 0$ according to our assumption. Contradiction.

The import is this : if there existed realness-preserving convolution averages which involved only paths that cross $\mathbf{R}^+$ finitely often, such averages would automatically "respect lateral growth" since, due to the universal resurgence structure, lateral growth always obtains on such paths. But our Lemma 5 rules out the existence of averages so conveniently simple. It shows that Properties 1,2,3 cannot be reconciled in any "cheap" manner but only, if at all, through a very careful selection of weights (so as to load oft-crossing paths as lightly as possible), or else through some subtle compensation between the various branches. As we shall see, there actually exist averages illustrating either possibility.

2 Examples of convolution-respecting averages.

The abbreviations for averages all begin with the letter **m** (for *mean-value*) followed by some vowel reminiscent of their nature (**u** for *uniform* ; **o** for *organic* ; **a** for *Catalan* ; **y** for *Cauchy* ; **ow** for *Brownian* ; etc...), and they end either with **r** (for *right*) or **l** (for *left*) or with **n** (for *neutral*) in the case of realness-preserving averages ; or again with the lower indices (α, β) in the case of parameter-dependent families.

Example 1 : The right-lateral average mur and the left-lateral average mul.

They may be characterized as the only convolution-preserving "averages" that involve only one determination over each interval. With the criteria of Lemma 4, it may be shown that the determination in question must be either right-lateral or left-lateral, to the exclusion of any other. In terms of weights :

$$\mathbf{mur}^{\varpi_1,\ldots,\varpi_r} = 1 \ (resp.\ 0) \ \ if\ \varepsilon_1 = \varepsilon_2 = \cdots = + \ \ (resp.\ otherwise) \tag{52}$$

$$\mathbf{mul}^{\varpi_1,\ldots,\varpi_r} = 1 \ (resp.\ 0) \ \ if\ \varepsilon_1 = \varepsilon_2 = \cdots = - \ \ (resp.\ otherwise) \tag{53}$$

mur and **mul** clearly fail to preserve realness, but possess all five other properties P_i. Despite being utterly trivial, **mur** and **mul** are quite basic since, owing to formula (44), they generate all other convolution averages by postcomposition with alien automorphisms of $RAMIF$.

Example 2 : The median average mun.

Like **mur** and **mul**, **mun** is "uniform' in the sense that its weights do not depend on the increments $(\omega_1, \ldots, \omega_r)$, but only on the addresses $(\varepsilon_1, \ldots, \varepsilon_r)$. In fact they depend only on the number p (resp. q) of $+$ signs (resp. $-$ signs) in the address $(\varepsilon_1, \ldots, \varepsilon_r)$:

$$\mathbf{mun}^{\varpi_1,\ldots,\varpi_r} \equiv \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{\Gamma(p+q+1)} \equiv \frac{(2p)!\,(2q)!}{4^{p+q}\,(p+q)!\,p!\,q!} \tag{54}$$

All these averages **mur**, **mul**, **mun** can actually be embedded in an interval of "uniform" averages $\mathbf{mu}_{\alpha,\beta}$ with weights of the form :

$$\mathbf{mu}_{\alpha,\beta}^{\varpi_1,\ldots,\varpi_r} \equiv \frac{\Gamma(p+\alpha)\,\Gamma(q+\beta)}{\Gamma(p+q+1)} \quad (\alpha, \beta \in \mathbf{R}\ ;\ \alpha+\beta = 1) \tag{55}$$

$$\mathbf{mu}_{1,0} = \mathbf{mur}\ ;\ \mathbf{mu}_{1/2,1/2} = \mathbf{mun}\ ;\ \mathbf{mu}_{0,1} = \mathbf{mul} \tag{56}$$

Using criteria (C.1) or (C.2) of Lemma 4, one checks that all averages $\mathbf{mu}_{\alpha,\beta}$ respect convolution. It is trivial to see that only **mun** respects realness, but non-trivial that only **mur** and **mul** respect lateral growth. All $\mathbf{mu}_{\alpha,\beta}$ are clearly positive if (and only if) α and β are positive, but it may be shown that none is secable (except for **mur** and **mul**). The "uniform" averages are quite important, due mainly to their independence on the increments ω_i, but their failure to combine P_1, P_2, P_3 is a severe drawback.

Example 3 : the organic average mon.

Its weights depend both on the signs ε_i and the increments ω_i, but the latter dependence is optimally simple, i.e. rational :

$$\mathbf{mon}^{\varpi_1} = 1/2 \quad (\forall \varpi_1 = \begin{pmatrix} \varepsilon_1 \\ \omega_1 \end{pmatrix}) \tag{57}$$

$$\mathbf{mon}^{\varpi_1,\dots,\varpi_r} = \mathbf{mon}^{\varpi_1,\dots,\varpi_{r-1}}\, P_r \tag{58}$$

with a factor P_r defined by :

$$\begin{aligned} P_r &= 1 - \frac{1}{2}\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad \varepsilon_{r-1} = \varepsilon_r \\ P_r &= \frac{1}{2}\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad \varepsilon_{r-1} \neq \varepsilon_r \end{aligned} \tag{59}$$

Like **mun**, **mon** may be imbedded in an interval of averages $\mathbf{mo}_{\alpha,\beta}$ $(\alpha + \beta = 1)$, with extremities $\mathbf{mor} = \mathbf{mo}_{1,0}$ and $\mathbf{mol} = \mathbf{mo}_{0,1}$ that are "tilted" to the right or to the left. In terms of weights :

$$\mathbf{mo}_{\alpha,\beta}^{\varpi_1} = \alpha \ (resp.\ \beta) \ if \ \varepsilon_1 = + \ (resp.\ -) \tag{60}$$

$$\mathbf{mo}_{\alpha,\beta}^{\varpi_1,\dots,\varpi_r} = \mathbf{mo}_{\alpha,\beta}^{\varpi_1,\dots,\varpi_{r-1}}\, P_r \tag{61}$$

with a factor P_r defined by :

$$\begin{aligned} P_r &= 1 - \beta\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad (\varepsilon_{r-1}, \varepsilon_r) = (+,+) \\ P_r &= \beta\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad (\varepsilon_{r-1}, \varepsilon_r) = (+,-) \\ P_r &= \alpha\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad (\varepsilon_{r-1}, \varepsilon_r) = (-,+) \\ P_r &= 1 - \alpha\frac{\omega_r}{\omega_1 + \cdots + \omega_r} \qquad if \quad (\varepsilon_{r-1}, \varepsilon_r) = (-,-) \end{aligned} \tag{62}$$

Clearly, **mon** respects realness, and it can be shown that all $\mathbf{mo}_{\alpha,\beta}$ respect convolution (use the criteria of Lemma 4) and lateral growth (use the fact that each sign change contributes a small factor P_r). These averages are positive if α and β are > 0 (recall that $\alpha + \beta = 1$) and they are clearly scale-invariant. They are not secable, though.

The "organic" average **mon** is thus a marked improvement on the "uniform" average **mun**, as it reconciles the three main demands P_1, P_2, P_3.

Example 4 : averages induced by a diffusion process.

We fix any continuous convolution semigroup on $\mathbf{R}$, i.e any family of integrable functions $f_\omega(x)$ (with $x \in \mathbf{R}$ and $\omega \in \mathbf{R}^+$) such that :

$$\int_{-\infty}^{+\infty} f_\omega(x)\, dx = 1 \quad (\forall \omega \in \mathbf{R}^+) \tag{63}$$

$$\int_{-\infty}^{+\infty} f_{\omega_1}(x_1)\, f_{\omega_2}(x - x_1)\, dx_1 = f_{\omega_1+\omega_2}(x) \quad (\forall \omega_1\, ,\, \omega_2 \in \mathbf{R}^+) \tag{64}$$

(Observe that integration here is over $\mathbf{R}$, not over $\mathbf{R}^+$ as in (6)). Each function f_ω may be viewed as representing the probability distribution at the time $t = \omega$, on the vertical axis $\omega + i\mathbf{R}$, of a particle starting from the origin at $t = 0$, moving along $\mathbf{R}^+$ with unit speed, and diffusing randomly in the vertical direction. To any such "diffusion" (the term is used somewhat loosely here), we may associate a uniformizing average $\mathbf{m}$ with weights defined as follows :

Definition 2 . $\mathbf{m}^{\varpi_1,\ldots,\varpi_r}$ *is the probability for the particle to hit the half-axis* $\eta_r + i\varepsilon_r\mathbf{R}^+$ *at the time* $\eta_r = \omega_1 + \cdots + \omega_r$ *after successively crossing each half-axis* $\eta_j + i\varepsilon_j\mathbf{R}^+$ *at the time* $\eta_j = \omega_1 + \cdots + \omega_j$.

Analytically, this translates into the following formula :

$$\mathbf{m}^{\varpi_1,\ldots,\varpi_r} = \int f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\ldots\sigma_{\varepsilon_r}(x_1+\cdots+x_r)\,dx_1\ldots dx_r \tag{65}$$

with integration over $\mathbf{R}^r$ and with the classical step functions σ_+ and σ_- :

$$\sigma_\pm(x) \equiv 1 \ (resp.\ 0) \quad if \quad \pm x > 0 \quad (resp.\ \pm x \le 0) \tag{66}$$

Due to (63), $\mathbf{m}$ satisfies Property 0, and is indeed a uniformizing average. It also respects convolution, due to the symmetrel nature of the moulds $R^\bullet$ and $L^\bullet$ defined by :

$$R^{\boldsymbol{\omega}} \overset{def}{=} (-1)^r\sigma_+(\omega_1)\sigma_+(\omega_1+\omega_2)\ldots\sigma_+(\omega_1+\ldots+\omega_r) \tag{67}$$

$$L^{\boldsymbol{\omega}} \overset{def}{=} (-1)^r\sigma_-(\omega_1)\sigma_-(\omega_1+\omega_2)\ldots\sigma_-(\omega_1+\ldots+\omega_r) \tag{68}$$

for any sequence of real numbers ω_i. Lastly, if each function f_ω is even, the average $\mathbf{m}$ clearly respects realness. As for settling the other properties P_i, we must know more about the particular "diffusion" which induces $\mathbf{m}$.

We may note that, although quite diverse, the diffusion-induced averages are very thinly spread out in the much larger set of convolution-respecting averages. Most such averages, like **mun** and **mon**, are *not* diffusion-induced.

Example 5 : averages induced by pseudo-diffusions.

We introduce on $\mathbf{R}$ a (commutative and associative) *pseudoaddition* defined, for almost every pair of real numbers, by :

$$x_1 \wedge x_2 = x_1 \quad if \quad |\,x_1\,| > |\,x_2\,| \quad (resp.\ x_2 \quad if \quad |\,x_2\,| > |\,x_1\,|) \tag{69}$$

We then replace the convolution semigroups of Example 4 by pseudoconvolution semigroups. To do so, we leave (63) in force, but change (64) into :

$$\int_{x_1\wedge x_2 = x} f_{\omega_1}(x_1)\,f_{\omega_2}(x_2)\,(dx_1+dx_2) = f_{\omega_1+\omega_2}(x) \quad (\forall\omega_1\,,\,\omega_2 \in \mathbf{R}^+) \tag{70}$$

or less abstrusely :

$$f_{\omega_2}(x)\int_{|x_1|<|x|} f_{\omega_1}(x_1)dx_1 + f_{\omega_1}(x)\int_{|x_2|<|x|} f_{\omega_2}(x_2)dx_2 \; = f_{\omega_1+\omega_2}(x) \;\; (\forall\omega_1\,,\,\omega_2\in\mathbf{R}^+) \tag{71}$$

Lastly, we change (65) into :

$$\mathbf{m}^{\varpi_1,\dots,\varpi_r} = \int f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1\wedge x_2)\dots\sigma_{\varepsilon_r}(x_1\wedge\dots\wedge x_r)\,dx_1\dots dx_r \tag{72}$$

It is not difficult to check that pseudodiffusions, like diffusions, induce convolution-preserving averages. Here, however, an interesting universality phenomenon enters the picture : whenever the pseudodiffusion is *symmetrical* (meaning that each function f_ω is even), it always induces *one and the same* uniformizing average, namely the "organic" average **mon** of Example 3.

Example 6 : the homogeneous averages ${}^\tau$moun.

Scale-invariant convolution averages depend on an infinity of continuous parameters ; and so do the diffusion-induced averages. If we combine both requirements, however, the situation becomes more rigid : there is only one degree of freedom left, and we get the family ${}^\tau$**moun** ($\tau>0$) of "homogeneous" averages. For each given τ, the average ${}^\tau$**moun** is induced by a *convolution semigroup* ${}^\tau f_\omega$, which is the Fourier transform of a quite elementary *multiplication semigroup* ${}^\tau g_\omega$. Indeed :

$$ {}^\tau g_\omega(y)\equiv\exp(-\omega\mid y\mid^\tau)\quad(y\in\mathbf{R}\;;\;\omega,\tau\in\mathbf{R}^+) \tag{73}$$

$$ {}^\tau f_\omega(x)\equiv(2\pi)^{-1}\int_{-\infty}^{+\infty}{}^\tau g_\omega(y)e^{ixy}\,dy\quad(x,y\in\mathbf{R}\;;\;\omega,\tau\in\mathbf{R}^+) \tag{74}$$

For each fixed τ, the variables x and ω essentially coalesce into one :

$$ {}^\tau f_\omega(x)\equiv\omega^{-1/\tau}\;{}^\tau f_1(x/\omega^{1/\tau}) \tag{75}$$

which fact is of course responsible for the scale-invariance of ${}^\tau$**moun**.

When $\tau\to\infty$ or $\tau\to 0$, the above "diffusions" [2] possess no limit, but the weights of the "homogeneous" averages seem to tend to limits, namely to the weights of **mun** and **mon** :

$$\lim_{\tau\to\infty}{}^\tau\mathbf{moun}=\mathbf{mun}\;\;;\;\;\lim_{\tau\to 0}{}^\tau\mathbf{moun}=\mathbf{mon}\quad(?) \tag{76}$$

but this point hasn't been settled yet.

[2]according to standard terminology, we have proper diffusion processes only for $\tau\le 2$, but we need not bother about this distinction

Example 7 : the Cauchy average myn and the Brownian average mown.

In the important special cases $\tau = 1$ and $\tau = 2$, we get quite explicit distribution functions :

$$ {}^1f_\omega(x) \equiv (\omega/\pi)\,(x^2+\omega^2)^{-1} \tag{77}$$

$$ {}^2f_\omega(x) \equiv (1/2)(\omega\pi)^{-1/2}\,\exp(-x^2/4\omega) \tag{78}$$

We recognize the Cauchy kernel, resp. the Gaussian kernel of the Brownian motion on $\mathbf{R}$. Accordingly, the induced averages :

$$ \mathbf{myn} \overset{def}{=} {}^1\mathbf{moun} \;\; ; \;\; \mathbf{mown} \overset{def}{=} {}^2\mathbf{moun} \tag{79}$$

will be referred to as the Cauchy and Brownian averages. The latter will be studied in some detail in the coming section.

3 More examples. The Catalan and Brownian averages.

Example 8 : the Catalan average man.

The Catalan average **man** is induced by the diffusion process corresponding to the convolution semigroup f_ω obtained by Fourier transforming the following multiplicative semigroup g_ω :

$$ g_\omega(y) \overset{def}{=} (y^2+1)^{-\omega} \quad (y \in \mathbf{R}\ ;\ \omega \in \mathbf{R}^+) \tag{80}$$

$$ f_\omega(x) \overset{def}{=} (2\pi)^{-1}\int g_\omega(y)e^{ixy}\,dy \quad (x,y \in \mathbf{R}\ ;\ \omega \in \mathbf{R}^+) \tag{81}$$

(81) yields explicit formulae for integral values of ω :

$$ f_1(x) \equiv \frac{1}{2}\exp(-\mid x\mid) \;\; ; \;\; f_n(x) \equiv P_n(\mid x\mid)\exp(-\mid x\mid) \tag{82}$$

where P_n is a polynomial of degree $(n-1)$ simply connected to the Catalan polynomial Cat_n (see below). By plugging (81) into the general formula (65), we get the weights $\mathbf{man}^{\boldsymbol{\varpi}}$ for arbitrary increments ω_i. For whole increments, however, the weights assume rational values, and may be obtained much more directly by the following formula :

$$ \mathbf{man}^{\varepsilon_1,\dots,\varepsilon_r} \equiv 4^{-r} cat_{n_1} cat_{n_2} \dots cat_{n_s}\,(1+n_s) \tag{83}$$

with the discretized weights relative to $\Omega = \mathbf{N}$:

$$ \mathbf{man}^{\varepsilon_1,\dots,\varepsilon_r} = \mathbf{man}^{\varepsilon_1,\dots,\varepsilon_r}(\mathbf{N}) = \mathbf{man}^{\begin{pmatrix} \varepsilon_1,\dots,\varepsilon_r \\ 1,\dots,1 \end{pmatrix}} \tag{84}$$

and with the classical Catalan numbers :

$$cat_n \overset{def}{=} \frac{(2n)!}{n!\,(n+1)!} \quad (cat_n \in \mathbf{N}) \tag{85}$$

which in this case are indexed by the integers $n_1, n_2, \dots, n_s$ which denote the numbers of identical consecutive signs within the address $(\varepsilon_1, \dots, \varepsilon_r)$:

$$(\varepsilon_1, \dots, \varepsilon_r) = (\pm)^{n_1}(\mp)^{n_2} \dots (\varepsilon_r)^{n_s} \quad (of\ course\ n_1 + \cdots + n_s = r) \tag{86}$$

Like **mun** and **mon**, the Catalan average **man** may be imbedded in an interval of averages $\mathbf{ma}_{\alpha,\beta}$ (as usual $\alpha + \beta = 1$) with weights :

$$\mathbf{ma}_{\alpha,\beta}^{\varepsilon_1,\dots,\varepsilon_r} \overset{def}{=} (\alpha\beta)^r (cat_{n_1} cat_{n_2} \dots cat_{n_{s-1}}) Cat_{n_s}((\alpha/\beta)^{\varepsilon_r}) \tag{87}$$

Here, as in (83), n_i denotes the cardinality of the i^{th} cluster of identical signs. The new formula, however, alongside with the Catalan numbers cat_n, also involves the Catalan polynomials Cat_n, which are distinguished by a capital C and inductively definable by :

$$Cat_0(x) = 1 \tag{88}$$

$$Cat_{1+n}(x) = -(1 + x^{-1})cat_n + (x + 2 + x^{-1})Cat_n(x) \tag{89}$$

All negative powers of x cancel out, and it may be noted that :

$$Cat_n(0) = cat_n \quad ; \quad Cat_n(1) = (1+n)cat_n \tag{90}$$

$$\lim_{x \to -1} (x+1)^{-1} Cat_n(x) = cat_{n-1} \tag{91}$$

The Catalan average has quite a few remarkable properties, about which more in a moment. But it has one blemish : it is not scale-invariant. There is no reason why it should be, and from (83) one can easily infer that it is not. However, under a rescaling and a passage to the limit, the Catalan average gives rise to the so-called Brownian average, which inherits most of its nicer properties, and is scale-invariant into the bargain.

Example 9 : the Brownian average mown.

We already defined **mown**, in Example 7, as being induced by the Brownian diffusion (with the Gaussian distribution (78)). But **mown** is also the limit of the Catalan average (and of many others, besides) under an infinite shrinking (not dilatation !) of the scale. In terms of weights :

$$\mathbf{mown}\begin{pmatrix} \varepsilon_1, \dots, \varepsilon_r \\ \omega_1, \dots, \omega_r \end{pmatrix} = \lim_{t \to +\infty} \mathbf{man}\begin{pmatrix} \varepsilon_1, \dots, \varepsilon_r \\ t\omega_1, \dots, t\omega_r \end{pmatrix} \quad (\varepsilon_i = \pm\ ,\ \omega_i > 0) \tag{92}$$

Or again, since **man** has the merit of possessing simple and rational-valued weights for integer-valued increments ω_i, we may take :

$$\text{(93)} \qquad \mathbf{mown}\begin{pmatrix} \varepsilon_1, \ldots, \varepsilon_r \\ \omega_1, \ldots, \omega_r \end{pmatrix} = \lim \mathbf{man}\begin{pmatrix} \varepsilon_1, \ldots, \varepsilon_r \\ n_1, \ldots, n_r \end{pmatrix}$$

with integers n_i growing in such a way that $n_i/n_j \to \omega_i/\omega_j \ \forall i, j$.

Both **man** and **mown** clearly respect realness, but also convolution. The latter point follows from their being induced by diffusion processes, but in the case of **man**, two alternative proofs, directly based on (83), may also be found in [9]. Then **man** has the outstanding (and nearly characteristic) property of being secable : see [8]. The average **mown** is not secable, but makes up for it by being scale-invariant, unlike **man**. The crucial point, however, is Property 3 :

Proposition 1 *: (F. Menous) Both the Catalan and Brownian averages respect lateral growth.*

Unlike in the case of the "organic" average **mon**, this doesn't directly follow from the smallness of the weights on *bad*, i.e. *oft-crossing*, *paths*. Indeed, if we assume unit increments ($\omega_i = 1$) and look at the worst possible situation, namely fully alternating sign sequences of length $2r$, we find for the discretized weights the following expressions :

$$\text{(94)} \qquad \mathbf{mun}^{(+,-,+,-,\ldots,\varepsilon_{2r})} = 4^{-2r}(2r)!(r!)^{-2} \ \# \ 2^{-2r} \quad (\mathit{from}\ (54))$$

$$\text{(95)} \qquad \mathbf{mon}^{(+,-,+,-,\ldots,\varepsilon_{2r})} = 2^{-2r}((2r)!)^{-1} \quad (\mathit{from}\ (58))$$

$$\text{(96)} \qquad \mathbf{man}^{(+,-,+,-,\ldots,\varepsilon_{2r})} = 2\ 4^{-2r} \quad (\mathit{from}\ (83))$$

$$\text{(97)} \qquad \mathbf{mown}^{(+,-,+,-,\ldots,\varepsilon_{2r})} \ \# \ (\mathit{Const})^{-2r} \ \mathit{with}\ 2 < \mathit{Const} < 4 \quad (\mathit{from}\ (78))$$

Thus, of these four realness and convolution-preserving averages, only the "organic" average displays a factor $1/(2r)!$ which exactly offsets the characteristic " median growth" described in (40) and, by so doing, rather easily ensures the Property 3. But neither **mun** nor **man** nor **mown** possess the required rate of decrease for alternating sequences ; indeed, the weights of **man** and **mown** are only marginally smaller than those of **mun** (compare (96) and (97) with (94)). Now, as we already pointed out, the "uniform" average **mun** *does not respect lateral growth.* So it comes as a pleasant surprise to learn that **man** *and* **mown** *do*, owing to a subtle compensation mechanism from path to path. The proof (see [9]) resolves into five main steps :

Step one : To the family of Catalan averages $\mathbf{ma}_{\alpha,\beta}$ and **man**, one associates a parallel family of *alien derivations* $\mathbf{da}_{\alpha,\beta}$ and **dan**, which are characterized by :

$$\text{(98)} \qquad (\partial/\partial\alpha)\mathbf{ma}_{\alpha,\beta} \equiv -(\partial/\partial\beta)\mathbf{ma}_{\alpha,\beta} \equiv \mathbf{ma}_{\alpha,\beta}\,\mathbf{da}_{\alpha,\beta} \quad (\alpha+\beta=1)$$

$$\mathbf{dan} \overset{def}{=} \mathbf{da}_{1/2,1/2} \tag{99}$$

Step two : One shows that both $\begin{pmatrix} \mathbf{mur} \\ \mathbf{man} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{mul} \\ \mathbf{man} \end{pmatrix}$ are equianalytic to **dan**.

Step three : One shows that both $\begin{pmatrix} \mathbf{mur} \\ \mathbf{mul} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{mul} \\ \mathbf{mur} \end{pmatrix}$ are "at least as analytic" as **dan**.

Step four : One shows that **dan** is "at least as analytic" as $\begin{pmatrix} \mathbf{mur} \\ \mathbf{mul} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{mul} \\ \mathbf{mur} \end{pmatrix}$.

Step five : By rescaling **man** and letting it tend to **mown** as in (93), one checks that each of the four first steps carries over from **man** to **mown**.

The arguments at each step are largely "algebraic", but they also rest on remarkable integral (or combinatorial) identities. Step four is particularly tricky : the analyticity-preserving nature of the transformation from $\begin{pmatrix} \mathbf{mur} \\ \mathbf{mul} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{mul} \\ \mathbf{mur} \end{pmatrix}$ to **dan** is not directly apparent on that transformation itself, but on its *first derivative*. For details, see [9].

4 First application : unitary iteration of unitary diffeomorphisms.

A (local, analytic) diffeomorphism U of $\mathbf{C}^\nu$ is said to be *unitary* if it is reciprocal to its own complex conjugate :

$$\{U \textit{ unitary}\} \Longleftrightarrow \left\{U \circ \bar{U} = id\right\} \quad (U, \bar{U} \ : \ \mathbf{C}^\nu_{,0} \mapsto \mathbf{C}^\nu_{,0}) \tag{100}$$

We shall focus for simplicity on *unitary* and *identity-tangent diffeomorphisms* of $\mathbf{C}$. It is actually more convenient to locate the fixed point at infinity and, as far as analytic difficulties are concerned, quite sufficient to study diffeomorphisms U that are formally conjugate to a pure imaginary shift, say $T_{2\pi i}$ for convenience. Thus we may consider the following data, with twiddles standing for formalness :

$$U \ : \ z \mapsto U(z) = z + 2\pi i + \sum_{n\geq 2} a_n z^{-n} \quad (z \sim \infty \ ; \ a_n \in \mathbf{C}) \tag{101}$$

$$U = ({}^*\widetilde{U}) \circ (T_{2\pi i}) \circ (\widetilde{U}^*) \ \ \textit{and} \ \ ({}^*\widetilde{U}) \circ (\widetilde{U}^*) = id \tag{102}$$

with $T_{2\pi i} \overset{def}{=} z + 2\pi i$ and :

$$U(z) \in z + \mathbf{C}\{z^{-1}\} \ \ ; \ \ \widetilde{U}^*(z) \textit{ and } {}^*\widetilde{U}(z) \textit{ both in } z + \mathbf{R}[[z^{-1}]] \tag{103}$$

Clearly, the mapping U is *unitary* if and only if the formal power series $\widetilde{U}^*$ and ${}^*\widetilde{U}$ are real, in which case the formal iterates of *real* order w :

$$\widetilde{U}^{\circ w} \stackrel{def}{=} ({}^*\widetilde{U}) \circ (T_{2\pi iw}) \circ (\widetilde{U}^*) \quad (with\ T_{2\pi iw} \stackrel{def}{=} z + 2\pi iw) \tag{104}$$

are themselves *formally unitary.* It is well-known, however, that the formal power series $\widetilde{U}^*$, ${}^*\widetilde{U}$, $\widetilde{U}^{\circ w}$ are *generically divergent* and *always resurgent* (see [4] or [6]) and that their Borel transform display, again generically, ramified singularities all over $\mathbf{Z}$ in the $\zeta - plane$. So the challenge here is to resum the series $\widetilde{U}^*$ and ${}^*\widetilde{U}$ (resp. $\widetilde{U}^{\circ w}$) to *real* germs U^* and ${}^*U$ (resp. to a unitary germ $U^{\circ w}$). The difficulty, of course, stems from the singularities which lie over $\mathbf{R}^+$ in the $\zeta - plane$ and obstruct straightforward Laplace integration, and from the faster-than-exponential growth of type (40) which generically occurs on oft-crossing paths. We can apply Laplace neither to the lateral determinations **mur** or **mul** (for they would yield imaginary parts), nor to their half-sum (because (**mur** + **mul**)/2 does not respect convolution), nor to the median average **mun** (because of the faster-than-exponential growth), but only to a suitable average, like **mon**, **man**, or **mown**, which simultaneously respects *convolution*, *realness*, and *lateral growth.*

Similar results hold in all dimensions, not only for unitary diffeomorphisms that are identity-tangent, but also for merely resonant ones.

5 Second application : real normalization of real vector fields.

The remarks in the preceding section also apply, for much the same reasons, to local, real-analytic vector fields on $\mathbf{R}^\nu$, especially when they are *resonant* and some of their *multipliers* λ_i (i.e. the eigenvalues of the field's linear part) *are real*, rather than complex and pairwise conjugate. Indeed, the *formal normalizing maps* attached to such fields will not only be generically divergent, but also resurgent with respect to a well-chosen, infinitely large variable z ; and the Borel transform $z \to \zeta$ will generically produce singularities over the set $\sum(\lambda_i \mathbf{Z})$ and thus over $\mathbf{R}$, with all the attending complications of non-linearity and faster-than-exponential growth on oft-crossing paths. All of which calls for an averaging of type $P_1+P_2+P_3$ if we want resummation to convert the *real formal normalization* into a *real sectorial normalization.*

There exist, however, subtle differences between the various sectors in the $z - plane$, which reflect the unequal "badness" of the singularities over $\lambda_i\mathbf{N}$ and $-\lambda_i\mathbf{N}$ in the $\zeta - plane$, for an *inert multiplier* λ_i (i.e. one that is *not* involved in the resonance relations). On the "good side", namely over $-\lambda_i\mathbf{N}$, the resurgent pattern is so utterly simple (since here only *one* alien derivation, i.e. $\Delta_{-\lambda_i}$, may act effectively) that *all* realness-preserving averages not only yield the *same result*, but also respect lateral growth. On the "bad side", however, that is to say over $\lambda_i\mathbf{N}$, the resurgence structure is far more tangled (because there an *infinity* of alien derivations, i.e. Δ_{λ_i}, $\Delta_{2\lambda_i}$, $\Delta_{3\lambda_i}$, ..., are liable to act), so that the various realness-preserving averages generally produce distinct results, and only the well-behaved ones amongst them (like **mon**, **man**, **mown**) preserve lateral growth. For an *active* (i.e. *non-inert*) *multiplier* λ_i, of course, both sides $\pm\lambda_i\mathbf{N}$ are equally "bad". For details, see [8] and [9].

6 Third application : simplifying the proof of the non-accumulation theorem for limit-cycles.

For most mathematicians concerned with the subject, the prime motivation seems to lie with Hilbert's 16^{th} problem, which asks for an optimal bound $B(d)$ on the number of possible limit-cycles for a polynomial vector field of degree d over $\mathbf{R}^2$. The non-accumulation theorem does indeed reinforce the conjecture that each $B(d)$ is $< \infty$, but its proper setting is that of *real-analytic* (rather than *real-polynomial*) *vector fields* on $\mathbf{R}^2$. Since the limit-cycles of such a field X might accumulate only to an invariant polycycle $\mathcal{C}$ (possibly degenerate and reduced to a point) and since the accumulating limit-cycles would correspond to isolated fixed points of the so-called *return map* F associated with the polycycle, everything reduces to disproving the accumulation of isolated fixed points of F. A geometric, non-constructive proof was given in [10], while a resummation-theoretical and constructive one is available in [6]. For a lively survey, see [1].

What we propose to show here is how the *well-behaved averages* of the present paper may be used to simplify (and beautify) the resummation-theoretical proof. It is in fact enough (at the cost of repeated blow-ups) to consider the case of a *reduced polycycle* $\mathcal{C}$ with r *summits* S_1, ..., S_r where the vector field X turns singular, but retains a *non-zero linear part* with real eigenvalues λ_1, λ_2. These summits S_i can be of three types :

Type I : Non-resonant hyperbolic : $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_1/\lambda_2 \notin \mathbf{Q}$.

Type II : Resonant hyperbolic : $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_1/\lambda_2 \in \mathbf{Q}$.

Type III : Semihyperbolic : $(\lambda_1, \lambda_2) = (0, *)$ or $(*, 0)$.

The *return map* F attached to the polycycle $\mathcal{C}$ factors into r so-called *transit-maps* G_i attached to the individual summits S_i :

$$F = G_r \circ G_{r-1} \circ \cdots \circ G_2 \circ G_1 \tag{105}$$

It being technically more convenient to work with an *infinitely large* reference variable z, F and G_i are actually *germs of maps* from $[\ldots, +\infty[$ to $[\ldots, +\infty[$. They are *real-analytic* (except at $+\infty$), and have an analytic continuation to some neighbourhood of $\mathbf{R}^+$ tapering off at $+\infty$. The resummation-theoretical scheme is to associate formal objects $\widetilde{F}$ and $\widetilde{G}_i$ to these maps :

$$\widetilde{F} = \widetilde{G}_r \circ \widetilde{G}_{r-1} \circ \cdots \circ \widetilde{G}_2 \circ \widetilde{G}_1 \tag{106}$$

and to show that the trivial *formal trichotomy* :

$$\textit{either } \{\widetilde{F}(z) - z \equiv 0\} \textit{ or } \{\widetilde{F}(z) - z > 0\} \textit{ or } \{\widetilde{F}(z) - z < 0\} \tag{107}$$

translates, for z large enough, in an *effective*, i.e. *geometric trichotomy* :

$$\textit{either } \{F(z) - z \equiv 0\} \textit{ or } \{F(z) - z > 0\} \textit{ or } \{F(z) - z < 0\} \tag{108}$$

which rules out any accumulation of isolated fixed points at infinity.

For all three types of summit, the formal maps $\widetilde{G}_i$ may be divergent, but they are always resummable under the general resummation scheme of section 1, with at most one "critical time" for each summit. The summits of Type I and II offer no special difficulty, but those of Type III (semihyperbolic) do, because, for them, the formal map $\widetilde{G}_i$ cannot be a plain *series* (with one order of infinitesimals) but a so-called *transseries*, which associates several orders of infinitesimals - in this case, only two, namely plain powers and exponentials, or plain powers and logarithms. Indeed, depending on the transit direction at a given summit of Type III (either expanding or contracting), any formal object $\widetilde{G}_i$ that encodes the whole information necessary to reconstruct G_i, must necessarily be of the form :

Type III^+ (expanding) : $\widetilde{G}_i = \widetilde{K}_i \circ E \circ \widetilde{U}_i^*$ (with $E(z) = \exp(z)$)

Type III^- (contracting) : $\widetilde{G}_i = {}^*\widetilde{U}_i \circ L \circ \widetilde{H}_i$ (with $L(z) = \log(z)$)

with ordinary formal power series at both ends and an exponential E or a logarithm L as middle factor. As for the formal return map $\widetilde{F}$ constructed by composing the transit maps $\widetilde{G}_i$, it may be a general transseries of awesome complexity, since it may involve several orders of infinitesimals (plain powers ; exponentials and logarithms ; iterated exponentials and logarithms). Nonetheless, it can always be written down, in a unique way, as a *well-ordered sum of pairwise comparable, irreducible expressions $A_\alpha(z)$ known as transmonomials* :

$$\widetilde{F}(z) = \sum c_\alpha A_\alpha(z) \quad (\textit{with} \ \ 1 \le \alpha < \gamma < \omega^\omega \ \ ; \ \ c_\alpha \in \mathbf{R}) \tag{109}$$

with a natural indexation α running through a transfinite subinterval $[1, \gamma[$ of $[1, \omega^\omega[$, in Cantor's standard notations. For the transit maps $\widetilde{G}_i$ of semi-hyperbolic summits, which involve only *two* orders of magnitude, γ is admittedly smaller (indeed, $\gamma = \omega^2$) :

$$\widetilde{G}_i = \sum_{1 \le \alpha < \omega} c_{i,\alpha} A_{i,\alpha}(z) + \sum_{\omega \le \alpha < \omega^2} c_{i,\alpha} A_{i,\alpha}(z) \quad (c_{i,\alpha} \in \mathbf{R}) \tag{110}$$

but still large enough to create both an *asymptotic part* (with finite ordinals α as indices) and a *transasymptotic part* (with transfinite indices α). As a consequence, *the very definition of the formal map $\widetilde{G}_i$ becomes a non-trivial affair*, and

involves three distinct steps, which we detail, for definiteness, in the expanding case $(TypeIII^+)$:
Step one : We obtain the formal map $\widetilde{U}_i^*$ as the *asymptotic part* of the geometric map $L \circ G_i$. That formal map $\widetilde{U}_i^*$ turns out to be the *normalizing map* of an *identity-tangent, unitary map* U_i (see section 4), which is none other than the holonomy map of the field X at S_i. Therefore (see section 4) the Borel transform of $\widetilde{U}_i^*$ is convergent, with singularities over $\mathbf{Z}$.
Step two : We resum $\widetilde{U}_i^*$ to a germ U_i^* by Borel-Laplace, *relative to some convolution preserving average* **m** *of our own choosing, but which must be the same for all summits.*
Step three : We calculate $\widetilde{K_i}$ as the asymptotic part of the germ $G_i \circ {}^*U_i \circ L$ (where ${}^*U_i$ is of course reciprocal to U_i^*).

If we now move on to the study of $\widetilde{F}$ and F, we find that this latitude in the choice of the convolution-preserving average **m** can lead to three different methods : *crude* ; *smarter* ; *smartest.*

First method (crude).

We select the *right-lateral* or *left-lateral* average (**mur** or **mul**). Then of course we have no problem with preserving lateral growth, but we get sums U_i^* which carry *imaginary parts.* The other factor, namely $\widetilde{K_i}$, will be *convergent.* Both $\widetilde{K_i}$ and its trivial sum K_i will have their own imaginary parts, which will cancel out that of U_i^*, so that the product $K_i \circ E \circ G_i$ will indeed yield the *real germ* G_i.

Still, the procedure introduces imaginary parts in the transasymptotic coefficients c_α of the transseries $\widetilde{F}$ and, even worse, inside some of the transmonomials $\widetilde{A}_\alpha$ - namely "upstairs", inside the towers of piled-up exponentials. This is a severe drawback for two reasons. First, imaginary parts are *personae non gratae* in the formalization of an inherently *real object* like F. Second, the imaginary numbers tucked away upstairs inside the exponential towers might create oscillations in the *sums* A_α of some of the transmonomials, and so in $F(z) - z$ itself. By a careful induction, we may satisfy ourselves *that this is not the case*, because the imaginary parts "sitting upstairs" are always neutralized by larger infinitesimals which are *purely real.* Nonetheless, the presence of imaginary parts is an aesthetic irritant and a practical nuisance. It robs the non-oscillation of $F(z)-z$ of the intuitive obviousness which it ought to possess, and which it acquires in the second and third methods.

Second method (smarter).

We select (as in [6]) the "uniform" median average **mun** (which in [6] was denoted by **med**). This does away with all imaginary numbers, but introduces faster-than-exponential growth in the "uniformized" or "averaged" Borel transform of $\widetilde{U}_i^*$. This is offset, fortunately, by the phenomenon of "*emanated resurgence*" (analyzed at great length in [6]) which induces *divergence* and *resurgence* inside the factor $\widetilde{K}_i$ (defined relatively to **mun** by the standard procedure : see Step Three earlier on in this section). This time, both $\widetilde{U}_i^*$ *and* $\widetilde{K}_i$ are *real* and *divergent-resurgent*, and the faster-than-exponential growth *disappears* in the (uniformized) *joint Borel transform* of $\widetilde{K}_i \circ E \circ \widetilde{U}_i^*$, so that $\widetilde{G}_1$, ..., $\widetilde{G}_r$ and $\widetilde{F}$ may be accelero-summed to G_1, ..., G_r and F.

There does remain, however, a slight flaw : unlike $\widetilde{G}_i$ and $\widetilde{F}$ taken as a whole, *some partial sums of these transseries may not always be resummed exactly, but only up to arbitrarily small ideals.* Due once again to "emanation resurgence", these ideals may be chosen as small as :

$$(111) \qquad 1/\exp\exp\ldots\exp(z) \qquad (n\ \textit{times}\ ;\ n\ \textit{arbitrary}\ ;\ z \sim +\infty)$$

that is to say, *smaller than any term present in a given transseries.* This is sufficient for all intents and purposes, and in particular more than sufficient for proving the non-oscillation of F. But the impossibility of resumming *exactly* (rather than modulo some ideals) certain subtransseries of our transseries is slightly irksome. This last remaining imperfection vanishes in the third method.

Third method (smartest).

We select a well-behaved average, like **mon**, **man**, **mown**, which respects *convolution*, *realness* and *lateral growth.* Then the formal factors $\widetilde{K}_i$ (defined according to the standard scheme) will automatically be *convergent* (like in the first method) but also *real* (like in the second method). There will be no faster-than-exponential growth to worry about, nor any need for any compensation of any sort. And not only will all our transseries be exactly resummable to their correct sums, but so will all their subtransseries (whose sums are *beyond the reach of geometry*, and definable only by *resummation*).

We conclude this section with a short table listing the main differences between the three methods :

	$\widetilde{U}_i^*$	$\widetilde{K}_i$	U_i^*	K_i
First method	real divergent resurgent	complex convergent	complex exact	complex exact
Second method	real divergent resurgent	real divergent resurgent	real approximate	real approximate
Third method	real divergent resurgent	real convergent	real exact	real exact

We note that $\widetilde{U}_i^*$ and (of course) G_i are independent of the method, but not so U_i^*, K_i, $\widetilde{K}_i$ and $\widetilde{G}_i$.

7 Conclusion. Looking ahead .

Let us review our main averages in relation to the six crucial properties P_1, ..., P_6 of section 1.

name of average	abbrev.	P_1	P_2	P_3	P_4	P_5	P_6
right-lat. av.	**mur**	yes	no	yes	yes	yes	yes
left-lat. av.	**mul**	yes	no	yes	yes	yes	yes
uniform av.	**mun**	yes	yes	no	yes	no	yes
organic av.	**mon**	yes	yes	yes	yes	no	yes
Catalan av.	**man**	yes	yes	yes	yes	yes	no
Brownian av.	**mown**	yes	yes	yes	yes	no	yes
Cauchy av.	**myn**	yes	yes	?	yes	no	yes
homogeneous av.	${}^{\tau}$**moun**	yes	yes	?	yes	no	yes

There are many open questions left, apart from the two question-marks which pock-mark the above table. What is the distinctiveness of the *diffusion-induced averages* ? *Just how exceptional* are the averages that reconcile the properties P_1, P_2, P_3 ? What is the description of the space $AVER^+$ spanned by the averages which are *strongly positive* (P_4), but *not necessarily convolution-preserving* (P_1) ? $AVER^+$ being a convex compact set, what are its *extremal elements* ? How do the fundamental averages (**mun**, **mon**, **man**, **mown**, **myn**, etc...) relate to each other ? How do the probability measures which they induce on the ultrametric space $\{+,-\}^{\mathbf{N}}$ compare with one another ? Most pairs would seem to be *mutually singular*, but not all of them - for instance, not **man** and **mown**. When the latter is the case, what is the *mutual density* of our measures ? And so on and so forth.

The subject is in fact quite new. The only non-trivial average investigated so far was the *"uniform" median average* **mun**, which was introduced in [6] (under the label **med**) for some special application. None of the other averages (*organic*, *Catalan*, etc...) seems to have been defined, let alone investigated, prior to resurgence theory. However, two recent articles, [3] and [2], by M. Kruskal and his PhD student O. Costin, must be mentioned in this context. These interesting papers (yet to appear), which show that their authors are quite alive to the need for transasymptotics and transseries, also introduce (especially [2]) a certain realness-preserving average in order to resum the formal solutions of a special differential system. That average, however, draws only on those paths which cross $\mathbf{R}^+$ *at most once*, and so it cannot preserve convolution in general (see our Lemma 5). It works alright, though, in the particular case considered by O. Costin (which essentially amounts to resumming a resonant vector field on the "good side" $-\lambda_i\mathbf{N}$; see section 5 above) due to the very special resurgence structure of the objects involved.

The space $AVER$ of *averages* from $RAMIF$ to $UNIF$ does not exhaust the structural richness of the convolution algebra $RAMIF$: there is also the space $ALIEN$ consisting of the so-called *alien operators* (chiefly : *alien derivations* and *alien automorphisms*) which act *internally* on $RAMIF$ (and commute with the *natural derivation* of $RAMIF$). Clearly, $AVER$ and $ALIEN$ are closely interlinked, and to each remarkable family of averages ("uniform", "organic", "Catalan", "Brownian", etc...), there answers a related family of *alien derivations* and *alien automorphisms.* For lack of space, $ALIEN$ was given short shrift in this paper, but we caught a glimpse of its usefulness for *establishing* the properties of the averages, and even for *formulating* some of these properties, like P_3 ("respecting lateral growth"). And then, of course, *alien operators* are the bone and marrow of *resurgence theory*, which in turn is the proper tool for investigating *Stokes phenomena*, linear or non-linear.

As for the field of applications of *well-behaved averages*, it is potentially quite vast, since it factually covers the whole range of situations characterized by a combination of (1) non-linearity, (2) divergence, (3) realness. Our three examples (in sections 4, 5 and 6) merely scratch the ground open. Still, they typify two quite different situations :

In the *first applications* (sections 4 and 5), the emphasis lies squarely on the passage from *formal* to *geometric.* The formal objects there are uniquely and simply defined, but they have no obvious geometric counterparts, and it takes resummation to *define these counterparts* (i.e. the unitary fractional iterates in section 4 ; the real normalizations in section 5) *unambiguously.* Different averages, generally speaking, lead to different sums, and thus to different geometric objects, but this is perfectly in order because, once again, there does not seem to exist any purely geometric criterion for selection. And once the convolution average is *fixed*, the correspondence from *formal* to *geometric* becomes perfectly intrinsical, i.e. chart-invariant.

In the last application, however (see section 6), the priorities are partially reversed. The emphasis there lies on the *two-way shuttle* between *formal* and *geometric.* The geometric objects, namely the return map F and the transit maps G_i, are *unambiguously given by geometry*, and their formalizations $\widetilde{F}$ and $\widetilde{G}_i$ are helpful (even indispensible) for *establishing the properties of F and G_i, not for defining them.* In fact, it is exactly the reverse : due to the "non-archimedeanness" (i.e. the coexistence of infinitesimals of different orders of smallness, like inverse powers and inverse exponentials), it is the *formal objects* $\widetilde{F}$ and $\widetilde{G}_i$ which, in this case, *require a construction and depend on the selection of the uniformizing average.* But once a proper average is chosen, and adhered to, we get an unambiguous, constructive shuttle $\widetilde{F} \leftrightarrow F$ between *formal* and *geometric*, which completely illuminates the latter side (i.e. geometric) by transposing to it all the properties (like, in our case, the non-oscillation of F) which are directly obvious on the formal side.

Most of the instances where resummation is of service to geometry would seem to partake of one or the other of these two situations, exemplified by sections 4,5 and section 6 respectively. Moreover, the applicability to all cases of the same method (namely the general scheme of accelero-summation ; see section 1) irrespective of the causes of divergence, underscores the remarkable unity which pervades modern resummation theory.

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