CRITICAL AVALANCHE TAILS IN THE ONE-DIMENSIONAL ANISOTROPIC BAK–SNEPPEN MODEL

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Abstract. We study avalanche ranges in the anisotropic Bak–Sneppen model on $\mathbb{Z}$. In exponential coordinates, the right endpoint X_t of an avalanche started at the origin has a nonlinear jump evolution whose tails satisfy a closed triangular system. Write $m(t) = \mathbb{E}[X_t]$ and let t_c be the threshold for finite $m(t)$. Below t_c, the endpoint has exponential moments and $m(t) \sim 2/(t_c - t)$. At criticality it is finite almost surely, but

$$\mathbb{P}(X_{t_c} \geq k) \geq ck^{-1/2}, \qquad \mathbb{E}[X_{t_c}^{1/3}] = \infty.$$

We propose two approaches for the critical estimates. First, a simple compensation argument, based on an extremal inequality for decreasing probability masses, yields a $k^{-2/3}$ lower bound. This already proves equality of the susceptibility and infinite-avalanche thresholds. We then exploit the additional fact that the exact evolution preserves geometric mixtures for the tails, which sharpens the bound to $k^{-1/2}$. Finally, by freezing the grafting law at criticality, we prove that the avalanche is infinite with positive probability for every $t > t_c$.

1. Introduction and main result

The Bak–Sneppen model [1] is a classical example of self-organized criticality driven by extremal dynamics. Each site carries a fitness; at each update, the site of minimum fitness and a prescribed set of its neighbours receive fresh independent values. We consider the one-dimensional anisotropic, or one-sided, rule, which refreshes the minimum site and its right neighbour. For a fixed fitness threshold, an avalanche is the sequence of updates triggered from a prescribed site and continued until all fitnesses are again above that threshold. In the one-sided model, an avalanche started at the origin has range $\{0, \dots, X_t\}$, and its right endpoint admits a closed scalar evolution as the fitness level t varies. It is this closure, rather than the interval shape of the range alone, that is special to the one-sided model; Figure 1 illustrates this evolution.

We work in exponential coordinates: the order-preserving transformation $u \mapsto -\log(1-u)$ turns uniform refresh variables into independent Exp(1) variables. Throughout, $t \geq 0$ denotes a fitness level, not the discrete time of the dynamics, and $\mathbb{N} = \{1, 2, \dots\}$. More precisely, a t-avalanche starts with all fitnesses above t, forces an update of sites 0 and 1, and then runs while some fitness is at most t. Its endpoint $X_t \in \mathbb{N} \cup \{\infty\}$ is infinite exactly when it does not terminate. The number of sites in its range is $X_t + 1$, and the avalanche family can be coupled increasingly in t.

Set $q_k(t) = \mathbb{P}(X_t \geq k)$ and

$$m(t) := \mathbb{E}[X_t] = \sum_{k \geq 1} q_k(t).$$

The susceptibility threshold is

$$t_c := \sup\{t \geq 0 : m(t) < \infty\}. \tag{1}$$

Key words and phrases. Bak–Sneppen model, avalanche ranges, self-organized criticality, anisotropic model, nonlinear jump processes, geometric mixtures, Hausdorff moments, critical tails.

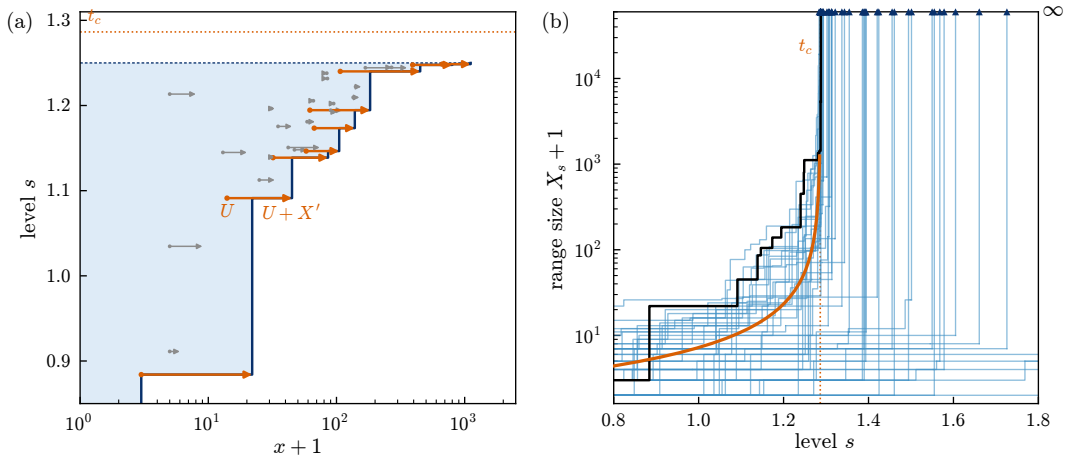


FIGURE 1. Avalanche growth in the one-sided Bak–Sneppen model (exponential coordinates; dotted lines: $t_c \approx 1.2864$). (a) One avalanche in the level–space plane, up to level 1.25 (dashed line). The shaded region is $\{(x,s) : x \le X_s\}$; arrows indicate grafts $U \mapsto U + X'$, in orange when the range grows and grey otherwise. This is the largest of 3000 sampled avalanches (most avalanches are still small at criticality). (b) Range size $X_s + 1$ for 50 independent avalanches (blue) and for the avalanche in (a), continued (black). The orange curve is the mean $m(s) + 1$.

Our main result describes the subcritical, critical and postcritical laws.

Theorem 1.1 (Quantitative avalanche phase transition)**.** *One has*

$$1 \le t_c \le 2\log 2, \qquad m(t) \sim \frac{2}{t_c - t} \quad \text{as } t \uparrow t_c.$$

Moreover the following hold.

(i) *For every $t < t_c$, there exists $\lambda_t > 0$ such that*

$$\mathbb{E}\left[e^{\lambda_t X_t}\right] < \infty.$$

(ii) *At the critical level,*

$$\mathbb{P}(X_{t_c} < \infty) = 1, \qquad \mathbb{E}[X_{t_c}^{1/3}] = \infty.$$

and there exists $c > 0$ such that

$$\mathbb{P}(X_{t_c} \ge k) \ge ck^{-1/2}, \qquad k \ge 1.$$

(iii) *For every $t > t_c$,*

$$\mathbb{P}(X_t = \infty) > 0.$$

Moreover, there exist constants $c_* > 0$ and $s_0 > 0$ such that

$$\mathbb{P}(X_{t_c+s} = \infty) \ge c_* s, \qquad 0 < s \le s_0.$$

Consequently,

$$t_c = \inf\{t \ge 0 : \mathbb{P}(X_t = \infty) > 0\}.$$

The pointwise exponent $1/2$ is not expected to be sharp. Numerical integration of the exact spatial-size master equation in [8] gives a critical probability-mass exponent $\tau_r = 1.299(3)$ for the range size $X_t + 1$, corresponding formally to a tail exponent $\tau_r - 1 \simeq 0.299$. Our fractional-moment estimate implies that, if the logarithmic critical-tail exponent exists, then it is at most $1/3$; we do not prove that existence.

Exact evolution and earlier work. At a finite endpoint x, proposed updates occur at rate $x + 1$, with new endpoint

$$\max\{x, U_x + X'_t\}, \tag{2}$$

where U_x is uniform on $\{0, \dots, x\}$, and X'_t is an independent copy of X_t, independent of U_x. Consequently $q_1 \equiv 1$ and, for $k \geq 2$,

$$q'_k = \sum_{u=1}^{k-1} q_u(q_{k-u} - q_k) + q_k(1 - q_k). \tag{3}$$

The equation at level k involves only $q_1, \dots, q_k$.

Exact avalanche equations go back to Maslov [7]. Maslov, De Los Rios, Marsili and Zhang derived the triangular master equation for the anisotropic spatial-size distribution [8]; (3) is its equivalent tail formulation, with range size $X_t + 1$. The graphical avalanche construction was developed rigorously by Meester and Znamenski [9, 10]. Other rigorous approaches include branching and percolation bounds on avalanche critical values [2], maximal avalanches [3], rank-driven variants [4, 5], and, more recently, extinction and survival criteria for the Bak–Sneppen model on arbitrary graphs [11]. Here we combine the exact one-sided evolution with moment inequalities and a geometric-mixture representation to obtain quantitative control at and near criticality.

A simple sharpness argument and its quantitative refinement. Summing the tail system gives

$$m' = \frac{1}{2}m^2 + m - \frac{1}{2}\sum_{k\geq 1} q_k^2. \tag{4}$$

This identity yields the upper bound on t_c and, together with a finite-mean continuation argument, divergence of $m(t)$ at t_c. It then gives the stated susceptibility asymptotic, a rigorous version of the asymptotic derived in [8]. The gain part of the system gives $t_c \geq 1$ and, through a generating-function estimate, subcritical exponential moments.

For the critical tail, write $A(t) = \sum_{k\geq 1} k q_k(t)$. We proceed in two stages. First, the normalized tail law $\mathbb{P}(K = k) = q_k/m$ has a decreasing probability mass function. Representing such laws as *mixtures of discrete uniform distributions* gives

$$\mathbb{E}[K \vee K'] \geq \frac{4}{3}\mathbb{E}[K] - \frac{1}{3}$$

for independent copies K, K'. Its insertion into the exact equation for A, followed by Paley–Zygmund, yields

$$A(t) \leq m(t)^{8/3}, \qquad q_k(t_c) \geq ck^{-2/3}.$$

The second layer is the structural result of Proposition 4.7: for every $t \geq 0$, there is a unique probability measure ρ_t on $[0, 1]$ such that

$$q_k(t) = \int_{[0,1]} z^{k-1}\, \rho_t(dz), \qquad k \geq 1.$$

Thus X_t is a mixture of geometric laws, and an atom at $z = 1$ represents an infinite avalanche. Below t_c, the normalized tail law is itself a *geometric mixture*, and the sharper maximum inequality

$$\mathbb{E}[K \vee K'] \geq \frac{3}{2}\mathbb{E}[K] - \frac{1}{4} - \frac{1}{4}\mathbb{P}(K = 1) \tag{5}$$

improves the same compensation argument to

$$A(t) \leq m(t)^{5/2}, \qquad t < t_c. \tag{6}$$

Paley–Zygmund now gives the $k^{-1/2}$ bound, while a truncated interpolation argument proves the fractional-moment conclusion, including the endpoint $p = 1/3$. A separate diagonal argument proves almost sure finiteness at t_c.

From critical tails to postcritical explosion. Freeze the graft law in (2) at its critical law. More generally, if its tail is at least $ck^{-\alpha}$, with $0 < \alpha < 1$, then, while the frozen process lies between x and $2x$, grafts of size at least $2x$ occur at rate at least a constant times $x^{1-\alpha}$. The expected crossing time is therefore at most a constant times $x^{\alpha-1}$. Summing over dyadic scales gives explosion, and a triangular comparison transfers it to the true postcritical law. Quantitatively, this argument gives a lower bound proportional to $s^{\alpha/(1-\alpha)}$ on the probability of explosion by time s. Thus the elementary exponent $2/3$ already proves sharpness, with a quadratic postcritical lower bound; the geometric-mixture exponent $1/2$ improves that bound to a linear one.

Section 2 develops the avalanche hierarchy and exact endpoint evolution. Section 3 treats subcritical susceptibility and exponential moments. Section 4 develops the two compensation arguments and proves critical finiteness, and Section 5 establishes the postcritical bounds.

Version note. This paper replaces the earlier manuscript entitled *Sharpness of the avalanche phase transition in the Bak–Sneppen model.* The isotropic and stationary-limit results are now treated separately in the companion work [6], which considers the nearest-neighbour model on $\mathbb{Z}^d$ by dimension-independent softer arguments. The present paper restricts to the stronger quantitative results that rely on the closed one-sided endpoint evolution, and adds in particular the improved bound for the tail exponent.

Disclosure on the use of AI. A detailed account of the role played by AI is given in Appendix A. In view of the extent of the AI contribution, we do not intend to submit this work for publication.

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2. The avalanche hierarchy and its exact endpoint evolution

By one-sidedness, an avalanche started at the origin never updates a negative site, so there is no loss in working on $\mathbb{Z}_+ := \{0, 1, \dots\}$. We use exponential coordinates. Thus every refreshed fitness has law $\mathrm{Exp}(1)$, and a threshold $t \geq 0$ corresponds to the level $1 - e^{-t}$ in the usual uniform coordinates.

Fix $t \geq 0$. Start from a configuration whose fitnesses all exceed t, force one update of the sites 0 and 1, and then, for as long as some fitness is at most t, update the site carrying the smallest such fitness together with its right neighbour. Since all untouched fitnesses exceed t, their actual values play no role. The law of the resulting procedure therefore does not depend on the initial configuration above level t.

Let $\xi(t)$ be the set of sites refreshed at least once during this t-avalanche, including the forced update, and let

$$X_t := \sup \xi(t) \in \mathbb{N} \cup \{\infty\},$$

with the convention that $X_t = \infty$ when the range is unbounded.

Lemma 2.1. *For every $t \geq 0$, almost surely,*

$$\xi(t) = \{0, \dots, X_t\}$$

when $X_t < \infty$, whereas $\xi(t) = \mathbb{Z}_+$ when $X_t = \infty$. Moreover, the t-avalanche terminates if and only if $X_t < \infty$.

Proof. After the forced update the range is $\{0, 1\}$. If the current range is $\{0, \dots, \ell\}$ and the selected site is $i \leq \ell$, then the refreshed pair is $\{i, i+1\} \subseteq \{0, \dots, \ell+1\}$. Consequently the range remains an interval with left endpoint 0. In particular, an unbounded range must equal $\mathbb{Z}_+$.

It remains to rule out infinite duration with bounded range. Fix $L \geq 1$. On the event that the range never leaves $\{0, \dots, L\}$, there are at most $L+1$ sites whose current fitness can be at most t. Consider the next $L+1$ updates and the event that, at each of them, both refreshed fitnesses exceed t. Its conditional probability is $e^{-2t(L+1)}$. On this event, either the range leaves $\{0, \dots, L\}$, or no new site with fitness at most t is created and at least one such site disappears at every update; in the latter case the avalanche terminates within $L+1$ steps. Let E be the event that the avalanche continues for the next $L+1$ updates while its range stays in $\{0, \dots, L\}$. Thus, conditional on any past history,

$$\mathbb{P}(E \mid \text{past}) \leq 1 - e^{-2t(L+1)}.$$

Applying this estimate to successive blocks gives

$$\mathbb{P}\big(\text{the avalanche never terminates, } X_t \leq L\big) = 0.$$

Taking the union over L completes the proof. □

We next couple the avalanche ranges as the level varies. To make threshold continuation unambiguous, fix a finite upper level $T > 0$ and start with all untouched fitnesses equal to $T + 1$. They remain inactive at every level in $[0, T]$. Since the fixed-level avalanche law does not depend on the initial values above its threshold, this choice leaves all the required marginal laws unchanged.

The fixed-level endpoint laws form a measurable probability kernel in s. Indeed, using a sequence of independent pairs of refresh variables, the event that an s-avalanche reaches a given site within its first n updates is a Borel function of s and of finitely many refresh variables. Taking the union over n and then integrating shows that every endpoint tail is Borel in s. The finite-state masses and the mass at infinity are then Borel as well. This argument does not use the tail equations derived below.

During the exploration of a fixed t-avalanche, reveal a refresh variable exactly when it is at most t, and otherwise reveal only that it exceeds t. Let $\mathcal{H}_t$ be the sigma-field generated by the update history, the identities of the refresh variables used, and this partially revealed information. Values above t need not be compared with one another, since they are inactive.

Lemma 2.2 (Regeneration at a level)**.** *Fix $0 \leq t \leq T$, and suppose that the t-avalanche terminates with range $I = \{0, \dots, x\}$. Conditionally on $\mathcal{H}_t$, the terminal fitnesses on I are independent and have the form*

$$F_y = t + E_y, \qquad y \in I,$$

where the E_y are independent $\mathrm{Exp}(1)$ *variables.*

Consequently, the next terminal fitness above t occurs after an exponential level increment of rate $|I| = x+1$, at a uniform site of I, independent of that increment. If this crossing occurs at a site y and a level $s \le T$, the range of the grafted avalanche, conditionally on the past, s and y, has the law of an s-avalanche translated by y; it is generated using fresh refresh variables independent of the past.

Proof. At termination, the last refresh variables assigned to the sites of I are distinct. The exploration has revealed each only through the event that it exceeds t. Since variables above t are never compared with one another, the transcript imposes no further restriction on their values. Independence of the refresh variables and the lack-of-memory property therefore give the stated conditional product law.

The minimum of the excesses is exponential with rate $|I|$; its location is uniform on I and independent of the minimum. At the corresponding level $s \le T$, all other old terminal fitnesses exceed s, as do the untouched initial fitnesses. Updating the crossed site and its right neighbour therefore initiates an s-avalanche. Conditionally on s and its origin, its fresh refresh variables and translation invariance give the asserted graft law.

If the graft terminates, the old terminal variables that it did not refresh still have independent $s + \mathrm{Exp}(1)$ laws by lack of memory. The last variables on the sites it refreshed have the same conditional laws by the preceding exploration argument. Thus regeneration is preserved after each graft, including grafts that do not enlarge the range. □

Lemma 2.2 is the one-sided specialization of the hierarchical avalanche decomposition used for the nearest-neighbour model in [6]. The common mechanism is regeneration whenever a terminal fitness is crossed; here the interval geometry makes each graft reducible to a scalar endpoint update.

We now construct the graphical family from these fixed-level endpoint laws. Independently for each $y \in \mathbb{Z}_+$, take a marked Poisson point process on $(0,\infty) \times (\mathbb{N} \cup \{\infty\})$ with intensity

$$ds\, \mathbb{P}(X_s \in dz).$$

Equivalently, the heights on each line form a rate-one Poisson process; conditionally on all heights, the marks are independent, and a mark at height s has law $\mathcal{L}(X_s)$. Thus the construction couples fixed-level laws, rather than defining them recursively.

At a point (s,z) on line y, draw horizontal arrows from (y,s) to $(y+j,s)$ for $0 \le j \le z$, or to every site on its right when $z = \infty$. Vertical motion is allowed upwards. From the seed $\{0,1\}$ at level 0, let $\xi^{\mathrm{gr}}(t)$ be the set of sites reachable at height t, and let $X_t^{\mathrm{gr}} = \sup \xi^{\mathrm{gr}}(t)$.

For comparison, continue the original avalanche by raising its threshold on $[0,T]$, using Lemma 2.2 whenever a terminal fitness is crossed. Denote this endpoint family by X_t^{con}. At any fixed target level $t \le T$, its concatenated updates are exactly those of a t-avalanche: when a graft ends, either all terminal fitnesses exceed t, or the next one to be crossed is the minimum and initiates the next graft. If infinitely many updates are required, the endpoint is set to infinity, consistently with Lemma 2.1.

To justify the comparison without any non-explosion assumption, for $n \ge 2$ set

$$\sigma_n^{\mathrm{gr}} := \inf\{t \ge 0 : X_t^{\mathrm{gr}} \ge n\}, \qquad \sigma_n^{\mathrm{con}} := \inf\{t \ge 0 : X_t^{\mathrm{con}} \ge n\},$$

with the second infimum taken over $[0,T]$, and $\inf \varnothing = \infty$. Before either stopping level, only the fixed lines $\{0,\dots,n-1\}$ can be active, and the total proposed-update rate is at most n. There are therefore almost surely only finitely many relevant marks

on $[0,T]$ before this stopping level. Lemma 2.2 identifies successively the conditional law of the next level increment, origin and graft in the two constructions. It follows that

$$\big(X^{\mathrm{gr}}_{t\wedge\sigma_n^{\mathrm{gr}}}\big)_{0\le t\le T} \overset{\mathrm{d}}{=} \big(X^{\mathrm{con}}_{t\wedge\sigma_n^{\mathrm{con}}}\big)_{0\le t\le T},$$

including the update that crosses n. Letting $n\to\infty$ identifies the endpoint laws on $[0,T]$, with infinity absorbing. Since T is arbitrary, the graphical construction gives a simultaneous increasing coupling of all fixed-level avalanche laws.

This is the one-sided specialization of the graphical representation of Meester and Znamenski [10, Section 4]. We henceforth use this coupling and again write $\xi(t)$ and X_t for its range and endpoint; in particular, $X_0=1$.

The graphical construction immediately gives an exact nonlinear evolution in the level parameter.

Proposition 2.3 (Exact endpoint evolution)**.** *The process $(X_t)_{t\ge0}$ is increasing. At level t, conditionally on the current finite state $X_t=x$, marks affecting the range arrive at instantaneous rate $x+1$. At such a mark the endpoint is replaced by*

$$\max\{x, U_x + X'_t\},$$

where U_x is uniform on $\{0,\dots,x\}$, X'_t is an independent copy of X_t, and U_x and X'_t are independent. Null changes are allowed.

Equivalently, for every bounded function f on $\mathbb{N}\cup\{\infty\}$, the time-dependent jump operator at a finite state x is

$$\mathcal{L}_t f(x) = (x+1)\,\mathbb{E}\big[f\big(\max\{x,U_x+X'_t\}\big) - f(x)\big]\,.$$

Proof. When the current range is $\{0,\dots,x\}$, exactly $x+1$ vertical lines are active. Their independent rate-one Poisson processes superpose to a process of rate $x+1$, and the line carrying the next mark is uniform on the current range. Write its position as U_x. A mark at level t carries an independent t-avalanche, whose range is $\{0,\dots,X'_t\}$. Translating it by U_x and taking the union with the existing interval gives

$$\{0,\dots,\max\{x,U_x+X'_t\}\}.$$

This proves both assertions. □

For $k\ge1$, set

$$q_k(t) := \mathbb{P}(X_t\ge k), \qquad q_\infty(t) := \mathbb{P}(X_t=\infty) = \lim_{k\to\infty} q_k(t).$$

The exact endpoint evolution closes at the level of these tail probabilities.

Proposition 2.4 (Triangular tail system)**.** *For every $k\ge1$, q_k is non-decreasing and continuously differentiable. Moreover $q_1\equiv1$, $q_k(0)=0$ for $k\ge2$, and for every $k\ge2$,*

$$q'_k = \sum_{u=1}^{k-1} q_u(q_{k-u}-q_k) + q_k(1-q_k). \tag{7}$$

In particular, the marginal law of X_t is uniquely determined by this triangular system.

Proof. Monotonicity of X_t gives monotonicity of every q_k, while $q_1\equiv1$ and the initial conditions follow from $X_0=1$.

Fix $k\ge2$. While $X_t=x<k$, Proposition 2.3 shows that the instantaneous rate at which an update crosses level k is

$$(x+1)\mathbb{P}(U_x+X'_t\ge k) = \sum_{j=0}^{x} q_{k-j}(t).$$

Writing

$$p_x(t) := \mathbb{P}(X_t = x) = q_x(t) - q_{x+1}(t),$$

the compensator identity therefore gives, for $0 \le s \le t$,

$$q_k(t) - q_k(s) = \int_s^t \sum_{x=1}^{k-1} (q_x - q_{x+1}) \sum_{j=0}^{x} q_{k-j}\, du,$$

where all tail probabilities in the integrand are evaluated at u. A summation by parts yields

$$\sum_{x=1}^{k-1} (q_x - q_{x+1}) \sum_{j=0}^{x} q_{k-j} = \sum_{u=1}^{k-1} q_u (q_{k-u} - q_k) + q_k(1 - q_k).$$

Hence

$$q_k(t) - q_k(s) = \int_s^t \left[\sum_{u=1}^{k-1} q_u (q_{k-u} - q_k) + q_k(1 - q_k) \right] du. \tag{8}$$

The integrand is bounded, so q_k is absolutely continuous. Inductively in k, $q_1, \dots, q_{k-1}$ are continuous; since q_k is itself continuous, the integrand in (8) is continuous. Thus q_k is continuously differentiable and satisfies (7) pointwise.

Finally, once $q_1, \dots, q_{k-1}$ are known, the equation for q_k is a scalar locally Lipschitz differential equation with prescribed initial condition. Uniqueness therefore follows recursively in k. □

If $p_r(t) := \mathbb{P}(X_t + 1 = r)$ denotes the range-size law, the two descriptions are related by

$$p_r = q_{r-1} - q_r \quad (r \ge 2), \qquad q_k = 1 - \sum_{r=2}^{k} p_r \quad (k \ge 1),$$

where an empty sum is zero. These identities remain valid above criticality: the missing mass $1 - \sum_{r \ge 2} p_r$ is precisely q_∞. Applying this change of variables shows that (7) is the tail form of the spatial-size master equation in [8], with the endpoint shifted by one relative to the range size. The tail formulation is particularly convenient for the moment estimates below.

Equation (7) is the only input from the graphical avalanche construction needed in the quantitative analysis below.

3. Subcritical susceptibility and exponential moments

Recall that

$$m(t) := \mathbb{E}[X_t] = \sum_{k \ge 1} q_k(t), \qquad t_c := \sup\{t \ge 0 : m(t) < \infty\}.$$

We first determine the evolution of the susceptibility and then strengthen finite susceptibility to exponential integrability.

It is convenient to rewrite (7) in gain–loss form: for $k \ge 2$,

$$q_k' = \sum_{u=1}^{k-1} q_u q_{k-u} - q_k \sum_{u=2}^{k} q_u. \tag{9}$$

Since $k \mapsto q_k(t)$ is non-increasing, every term in the original form (7) is non-negative. In particular,

$$0 \le q_k'(t) \le \sum_{u=1}^{k-1} q_u(t) q_{k-u}(t). \tag{10}$$

We shall repeatedly sum the tail equations. The following elementary justification will also be used in Section 4.

Lemma 3.1 (Termwise differentiation)**.** *Fix $t_0 < t_c$. Then $\sum_{k\ge1} q_k'(t)$ converges uniformly on $[0, t_0]$, and*

$$m'(t) = \sum_{k\ge1} q_k'(t), \qquad 0 \le t \le t_0.$$

If in addition $\sum_{k\ge1} k q_k(t_0) < \infty$, then $\sum_{k\ge1} k q_k'(t)$ also converges uniformly on $[0, t_0]$, and

$$\frac{d}{dt} \sum_{k\ge1} k q_k(t) = \sum_{k\ge1} k q_k'(t).$$

Proof. For $0 \le t \le t_0$, monotonicity in the level and (10) give

$$q_k'(t) \le \sum_{u=1}^{k-1} q_u(t_0) q_{k-u}(t_0).$$

The sum of the right-hand side over $k \ge 2$ equals $m(t_0)^2$, so the Weierstrass test gives uniform convergence and permits termwise differentiation.

For the weighted statement, writing $A(t_0) := \sum_{k\ge1} k q_k(t_0)$,

$$\sum_{k\ge2} k \sum_{u=1}^{k-1} q_u(t_0) q_{k-u}(t_0) = \sum_{u,v\ge1} (u+v) q_u(t_0) q_v(t_0) = 2 m(t_0) A(t_0) < \infty,$$

and the same argument applies. □

Proposition 3.2 (Susceptibility identity)**.** *For every $t < t_c$,*

$$m'(t) = \frac{1}{2} m(t)^2 + m(t) - \frac{1}{2} \sum_{k\ge1} q_k(t)^2. \tag{11}$$

Consequently,

$$m'(t) \ge \frac{1}{2} m(t)(m(t)+1), \tag{12}$$

so that

$$1 \le t_c \le 2 \log 2.$$

Moreover,

$$\int_0^t (m(s)+1)\, ds \le 2 \log m(t), \qquad t < t_c, \tag{13}$$

and, more precisely,

$$m(t) \sim \frac{2}{t_c - t} \qquad \text{as } t \uparrow t_c. \tag{14}$$

Proof. We first prove $t_c \geq 1$. Set $m_n(t) := \sum_{k=1}^{n} q_k(t)$. By (10),

$$m_n'(t) \leq \sum_{\substack{u,v\geq 1\\ u+v\leq n}} q_u(t)q_v(t) \leq m_n(t)^2.$$

Since $m_n(0) = 1$, comparison with $y' = y^2$, $y(0) = 1$, gives $m_n(t) \leq (1-t)^{-1}$ for $t < 1$. Letting $n \to \infty$ proves $m(t) < \infty$ for $t < 1$.

Now fix $t < t_c$. Lemma 3.1 permits summation of (9). The gain term is m^2. For the loss term, writing $a_k = q_k$ for $k \geq 2$,

$$\sum_{k\geq 2} a_k \sum_{u=2}^{k} a_u = \frac{1}{2}\left[\left(\sum_{k\geq 2} a_k\right)^2 + \sum_{k\geq 2} a_k^2\right].$$

Since $q_1 = 1$, this gives (11). As $0 \leq q_k \leq 1$, one has $\sum_k q_k^2 \leq m$, and therefore (12).

The solution of $y' = y(y+1)/2$ with $y(0) = 1$ blows up at $2\log 2$, hence $t_c \leq 2\log 2$. Dividing (12) by m and integrating gives (13).

It remains to prove divergence at t_c. Since m is non-decreasing, suppose that $m(t) \uparrow M < \infty$ as $t \uparrow t_c$. Monotone convergence then gives $m(t_c) = M$. For every n, the estimate $m_n' \leq m_n^2$, now integrated from t_c, yields

$$m_n(t_c + s) \leq \frac{m_n(t_c)}{1 - s m_n(t_c)} \leq \frac{M}{1 - sM}, \qquad 0 \leq s < M^{-1}.$$

Letting $n \to \infty$ shows that $m(t_c+s) < \infty$ for $0 < s < M^{-1}$, contradicting the definition of t_c. Hence $m(t) \to \infty$ as $t \uparrow t_c$.

To obtain the asymptotic, set $S_2(t) := \sum_{k\geq 1} q_k(t)^2$. Since $q_1 = 1$ and $0 \leq q_k \leq 1$, we have $1 \leq S_2(t) \leq m(t)$. Dividing (11) by $m(t)^2$ therefore gives

$$\frac{1}{2} + \frac{1}{2m(t)} \leq \frac{m'(t)}{m(t)^2} \leq \frac{1}{2} + \frac{1}{m(t)} - \frac{1}{2m(t)^2}.$$

Thus $m'(t)/m(t)^2 \to 1/2$ as $t \uparrow t_c$. Writing $h(t) := 1/m(t)$, we have $h(t) \to 0$ and $h'(t) \to -1/2$. Hence

$$h(t) = -\int_t^{t_c} h'(s)\, ds \sim \frac{t_c - t}{2},$$

which proves (14). □

The gain estimate contains more information than finite susceptibility: strictly below t_c, it propagates an exponential moment.

Proposition 3.3 (Subcritical exponential moments)**.** *For every $t_0 < t_c$, there exists $\lambda_{t_0} > 0$ such that*

$$\sup_{0\leq t\leq t_0} \mathbb{E}[e^{\lambda_{t_0} X_t}] < \infty.$$

If $t_0 > 0$, one may take

$$\lambda_{t_0} = \log\left(1 + \frac{1}{2t_0 m(t_0)^4}\right). \tag{15}$$

Consequently, for every $t < t_c$ there exist $c_t, C_t > 0$ such that

$$\mathbb{P}(X_t \geq k) \leq C_t e^{-c_t k}, \qquad k \geq 1,$$

and X_t has moments of every positive order.

Proof. Fix $t_0 < t_c$ and $a > 1$, and set

$$\Psi_n(t) := \sum_{k=1}^{n} (a^k - 1) q_k(t).$$

By (10),

$$\Psi_n' \le \sum_{u,v=1}^{n} (a^{u+v} - 1) q_u q_v \le 2m\Psi_n + \Psi_n^2,$$

where we used

$$a^{u+v} - 1 = (a^u - 1) + (a^v - 1) + (a^u - 1)(a^v - 1).$$

Let

$$B(t) := \exp\left\{ 2 \int_0^t m(s)\, ds \right\}, \qquad \delta := a - 1.$$

Since $\Psi_n(0) = \delta$,

$$\left(\frac{\Psi_n}{B} \right)' \le B \left(\frac{\Psi_n}{B} \right)^2,$$

and therefore, as long as the denominator is positive,

$$\Psi_n(t) \le \frac{\delta B(t)}{1 - \delta \int_0^t B(s)\, ds}. \tag{16}$$

By (13),

$$2 \int_0^t m(s)\, ds \le 4 \log m(t) - 2t,$$

so $B(t) \le m(t)^4 \le m(t_0)^4$ on $[0, t_0]$. If $t_0 > 0$, choose

$$\delta = \frac{1}{2 t_0 m(t_0)^4}.$$

Then $\delta \int_0^{t_0} B(s)\, ds \le 1/2$, and (16) is bounded uniformly in n and $t \in [0, t_0]$. Letting $n \to \infty$ gives

$$\sup_{0 \le t \le t_0} \sum_{k \ge 1} (a^k - 1) q_k(t) < \infty.$$

For any $W \in \mathbb{N} \cup \{\infty\}$, Tonelli's theorem gives, in the extended sense,

$$\mathbb{E}[a^W] - 1 = (a - 1) \sum_{k \ge 1} a^{k-1} \mathbb{P}(W \ge k).$$

Since $(a - 1)a^{k-1} \le a^k - 1$, the preceding bound proves the exponential moment estimate with $\lambda_{t_0} = \log a$. The case $t_0 = 0$ is immediate. Markov's inequality gives the exponential tail bound, and hence all polynomial moments are finite. □

For later use set

$$A(t) := \sum_{k \ge 1} k q_k(t), \qquad t < t_c. \tag{17}$$

Proposition 3.3 gives $A(t) < \infty$. Since $x^2 = \sum_{k=1}^{x} (2k - 1)$, Tonelli's theorem yields

$$\mathbb{E}[X_t^2] = 2A(t) - m(t) < \infty, \qquad t < t_c. \tag{18}$$

In particular, the weighted part of Lemma 3.1 applies on every compact subinterval of $[0, t_c)$.

4. Critical tails

4.1. **Decreasing-mass compensation.** We first bound the critical tail from below using only the fact that the normalized tail law has a decreasing probability mass function. The argument will later be sharpened by exploiting additional structure of the exact evolution.

Lemma 4.1 (Mixtures of discrete uniform laws)**.** *Let $(p_k)_{k\geq 1}$ be a non-increasing probability mass function and set $a_n := n(p_n - p_{n+1})$. Then $(a_n)_{n\geq 1}$ is a probability mass function and*

$$p_k = \sum_{n\geq k} \frac{a_n}{n}, \qquad k \geq 1.$$

Equivalently, if $\mathbb{P}(N = n) = a_n$ and, conditionally on $N = n$, K is uniform on $\{1, \dots, n\}$, then K has law $(p_k)_{k\geq 1}$.

Proof. Monotonicity and summability imply $np_n \to 0$. Indeed,

$$np_n \leq 2 \sum_{j=\lceil n/2 \rceil}^{n} p_j \longrightarrow 0.$$

Hence

$$\sum_{n=1}^{M} a_n = \sum_{n=1}^{M} p_n - Mp_{M+1} \longrightarrow 1,$$

while

$$\sum_{n\geq k} \frac{a_n}{n} = \sum_{n\geq k} (p_n - p_{n+1}) = p_k.$$

□

Lemma 4.2 (Maximum under a decreasing mass function)**.** *Let K, K' be independent and identically distributed random variables with a non-increasing probability mass function on $\mathbb{N}$ and finite mean. Then*

$$\mathbb{E}[K \vee K'] \geq \frac{4}{3}\mathbb{E}[K] - \frac{1}{3}. \tag{19}$$

Proof. Let U_a, U_b be independent and uniform on $\{1, \dots, a\}$ and $\{1, \dots, b\}$, with $a \leq b$. Since

$$\mathbb{E}[U_a \wedge U_b] = \sum_{j=1}^{a} \mathbb{P}(U_a \geq j, U_b \geq j) = \frac{1}{ab} \sum_{j=1}^{a} (a - j + 1)(b - j + 1),$$

and $U_a \vee U_b = U_a + U_b - U_a \wedge U_b$, summing the quadratic polynomial gives

$$\mathbb{E}[U_a \vee U_b] = \frac{a + b + 1}{3} + \frac{(b - a)^2 + b - 1}{6b} \geq \frac{a + b + 1}{3}.$$

Apply Lemma 4.1, and let J, J' be independent with law (a_n). Conditionally on (J, J'), represent K, K' by independent uniforms on $\{1, \dots, J\}$ and $\{1, \dots, J'\}$. Then

$$\mathbb{E}[K \vee K'] \geq \frac{2\mathbb{E}[J] + 1}{3} = \frac{4}{3}\mathbb{E}[K] - \frac{1}{3},$$

because $\mathbb{E}[K] = (\mathbb{E}[J] + 1)/2$. □

Lemma 4.3 (Weighted-tail identity)**.** *For $t < t_c$, let K, K' be independent with*

$$\mathbb{P}(K = k) = \frac{q_k(t)}{m(t)}, \qquad k \geq 1.$$

Then $\mathbb{E}[K] = A(t)/m(t)$ *and*

$$A' = 2mA + A - \frac{1}{2}m^2\mathbb{E}[K \vee K'] - \frac{1}{2}\sum_{k\geq 1} kq_k^2. \tag{20}$$

Proof. Fix $t < t_c$ and suppress t from the notation. By Proposition 3.3 and Lemma 3.1, the weighted tail equation may be summed term by term. From (9),

$$A' = 2mA - \sum_{k\geq 2}\sum_{u=2}^{k} kq_u q_k.$$

Clearly $\mathbb{E}[K] = A/m$. Moreover,

$$m^2\mathbb{E}[K \vee K'] = 2\sum_{1\leq u<k} kq_u q_k + \sum_{k\geq 1} kq_k^2,$$

$$\sum_{k\geq 2}\sum_{u=2}^{k} kq_u q_k = \sum_{1\leq u<k} kq_u q_k + \sum_{k\geq 1} kq_k^2 - A.$$

Substitution gives (20). □

Proposition 4.4 (First compensation estimate)**.** *For every* $t < t_c$*,*

$$A(t) \leq m(t)^{8/3}. \tag{21}$$

Proof. With the notation of Lemma 4.3, the mass function q_k/m is non-increasing. Lemma 4.2 and (20) give

$$A' \leq \frac{4}{3}mA + A + \frac{1}{6}m^2 - \frac{1}{2}\sum_{k\geq 1} kq_k^2.$$

By Jensen's inequality and (18), $m^2 \leq 2A - m$, hence

$$A' \leq \frac{4}{3}mA + \frac{4}{3}A - \frac{1}{6}m - \frac{1}{2}\sum_{k\geq 1} kq_k^2 \leq \frac{4}{3}(m+1)A.$$

Together with (12),

$$\frac{A'}{A} \leq \frac{4}{3}(m+1) \leq \frac{8}{3}\frac{m'}{m}.$$

Since $A(0) = m(0) = 1$, integration yields (21). □

Corollary 4.5 (A first lower bound on the critical tail)**.** *There exists* $c > 0$ *such that*

$$q_k(t_c) \geq ck^{-2/3}, \qquad k \geq 1. \tag{22}$$

Proof. By (18) and Proposition 4.4, $\mathbb{E}[X_t^2] \leq 2m(t)^{8/3}$ for $t < t_c$. Paley–Zygmund gives

$$\mathbb{P}\left(X_t \geq \frac{1}{2}m(t)\right) \geq \frac{1}{4}\frac{m(t)^2}{\mathbb{E}[X_t^2]} \geq \frac{1}{8}m(t)^{-2/3}.$$

For every $k \geq 1$, choose $t_k < t_c$ with $m(t_k) = 2k$. Monotonicity in the level then yields

$$q_k(t_c) \geq q_k(t_k) \geq \frac{1}{8}(2k)^{-2/3},$$

which proves the claim. □

The preceding argument uses only that $k \mapsto q_k/m$ is non-increasing. It already gives a lower bound on the critical tail with exponent strictly below 1, which will be enough for the postcritical explosion argument. We now use the exact triangular system to uncover a stronger structure and improve the quantitative bounds.

4.2. **Geometric-mixture refinement.** The first layer used only that the tails decrease with their index. We now identify a stronger property that is preserved by the exact evolution. A sequence $(M_n)_{n\geq 0}$, with $M_0 = 1$, is a *Hausdorff moment sequence* if

$$M_n = \int_{[0,1]} z^n \, \rho(dz), \qquad n \geq 0,$$

for a probability measure ρ. Applied to $M_n = q_{n+1}$, this is exactly a geometric-mixture representation of the endpoint law.

The proof works with finitely many moments at a time. Their admissible vectors form a compact convex set. A supporting hyperplane of this set can be encoded by a polynomial that is nonnegative on $[0, 1]$ and vanishes on the support of a representing measure. We shall check that the tail vector field points into every such supporting halfspace, then pass to all moments by compactness. We first record the finite-dimensional invariance criterion needed for this argument.

Lemma 4.6 (Nagumo criterion for convex sets)**.** *Let $K \subset \mathbb{R}^N$ be a non-empty compact convex set and let $F : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz. Assume that, for every $x \in K$ and every $a \in \mathbb{R}^N$ satisfying*

$$a \cdot (y - x) \leq 0 \qquad \text{for all } y \in K,$$

one has $a \cdot F(x) \leq 0$. Then every solution of $\dot{x} = F(x)$ starting in K remains in K throughout its interval of existence.

Proof. Let $P_K z$ denote the metric projection onto K. For every $z \in \mathbb{R}^N$,

$$(z - P_K z) \cdot (y - P_K z) \leq 0, \qquad y \in K,$$

and $d_K(z)^2 := \operatorname{dist}(z, K)^2$ is continuously differentiable with $\nabla(d_K^2/2) = z - P_K z$. On a compact neighbourhood of K, let L be a Lipschitz constant for F. If $x(\cdot)$ is a solution and $y = P_K x$, the assumption applied at y to $a = x - y$ gives $(x - y) \cdot F(y) \leq 0$. Hence, as long as the solution stays in this neighbourhood,

$$\frac{d}{dt}\frac{1}{2} d_K(x(t))^2 = (x - y) \cdot F(x) \leq L|x - y|^2 = L d_K(x(t))^2.$$

If $x(0) \in K$, Gronwall's inequality gives $d_K(x(t)) = 0$. Thus the solution cannot leave K, and continuation proves the claim. □

Proposition 4.7 (Geometric-mixture invariance)**.** *For every $t \geq 0$, there exists a unique probability measure ρ_t on $[0, 1]$ such that*

$$q_k(t) = \int_{[0,1]} z^{k-1} \, \rho_t(dz), \qquad k \geq 1. \tag{23}$$

Moreover $t \mapsto \rho_t$ is weakly continuous and

$$q_\infty(t) = \rho_t(\{1\}). \tag{24}$$

Equivalently, conditionally on a mixing parameter $z < 1$, X_t is geometric on $\{1, 2, \ldots\}$ with tail z^{k-1}, while $z = 1$ represents $X_t = \infty$.

Proof. We prove invariance of each finite-dimensional moment set before constructing ρ_t.

Step 1: the closed system for finite moment vectors. Set $M_n(t) := q_{n+1}(t)$ for $n \geq 0$, so $M_0 \equiv 1$. The gain–loss form of the tail system reads

$$M_n' = \sum_{j=0}^{n-1} M_j M_{n-1-j} - M_n \sum_{j=1}^{n} M_j, \qquad n \geq 1. \tag{25}$$

Fix $N \geq 1$. Write $\mathcal{P}([0,1])$ for the probability measures on $[0,1]$, and define

$$\mathcal{M}_N := \left\{ \left(\int z \, d\rho, \ldots, \int z^N \, d\rho \right) : \rho \in \mathcal{P}([0,1]) \right\}.$$

This subset of $\mathbb{R}^N$ is convex and compact, being the image of the weakly compact set $\mathcal{P}([0,1])$ under a continuous affine map. The first N equations in (25) form a closed polynomial system $\dot{\mathbf{m}} = F_N(\mathbf{m})$. It remains to show that $\mathcal{M}_N$ is invariant under this system.

Step 2: expressing the vector field on polynomials. Fix $\mathbf{m} \in \mathcal{M}_N$, and choose any representing measure ρ. For a polynomial g, put

$$\mathcal{B}_\rho g := \iint \left[\frac{g(x) - g(y)}{x - y} + \frac{y}{1-y}\big(g(xy) - g(x)\big) \right] \rho(dx)\rho(dy). \tag{26}$$

The first term is extended to $g'(x)$ on $x = y$, and the second to $-xg'(x)$ on $y = 1$. Both are continuous on $[0,1]^2$; for polynomial g, these extensions are themselves polynomials. Thus atoms on the diagonal or at the endpoints cause no integrability issue.

For $g(z) = z^n$, the two terms are

$$\frac{x^n - y^n}{x - y} = \sum_{j=0}^{n-1} x^{n-1-j} y^j, \qquad \frac{y}{1-y}(x^n y^n - x^n) = -x^n \sum_{j=1}^{n} y^j.$$

Integrating reproduces the two terms of (25). Consequently, if $g(z) = g_0 + \sum_{n=1}^N g_n z^n$, then

$$\mathcal{B}_\rho g = \sum_{n=1}^{N} g_n \big(F_N(\mathbf{m})\big)_n. \tag{27}$$

In particular, constants are annihilated, and for $\deg g \leq N$ this quantity depends only on $\mathbf{m}$, not on the chosen representing measure.

Step 3: checking the inward-pointing condition. Let $a = (a_1, \ldots, a_N)$ be an outward normal at $\mathbf{m}$, so

$$a \cdot (\mathbf{v} - \mathbf{m}) \leq 0, \qquad \mathbf{v} \in \mathcal{M}_N.$$

Define

$$p(z) := \sum_{n=1}^{N} a_n z^n, \qquad c := \int p \, d\rho, \qquad f(z) := c - p(z).$$

The moment vector of δ_z is in $\mathcal{M}_N$ for every $z \in [0,1]$, so the supporting inequality says $p(z) \leq c$. Thus $f \geq 0$ on $[0,1]$, whereas $\int f \, d\rho = 0$. By continuity, f vanishes on $S := \operatorname{supp} \rho$. Formula (27) reduces the required inequality to

$$a \cdot F_N(\mathbf{m}) = \mathcal{B}_\rho p = -\mathcal{B}_\rho f \leq 0.$$

We therefore check that the integrand defining $\mathcal{B}_\rho f$ is nonnegative on $S \times S$.

At an interior zero of a nonnegative polynomial the derivative is zero. At boundary zeros, its sign satisfies

$$f'(x) = 0 \quad (x \in S \cap (0,1)), \qquad f'(0) \geq 0 \quad (0 \in S), \qquad f'(1) \leq 0 \quad (1 \in S).$$

For $x, y \in S$ with $y < 1$, the second term of (26) is $yf(xy)/(1-y) \geq 0$, because $xy \in [0,1]$. The first term is zero when $x \neq y$. On the diagonal it is $f'(x)$, which is zero for $0 < x < 1$ and nonnegative at $x = 0$.

If $y = 1$ and $x < 1$, the first term is zero, and the second is $-xf'(x) = 0$: an interior support point has zero derivative, while $x = 0$ supplies a zero factor. Finally, at $(x, y) = (1, 1)$, the two extended terms cancel exactly:

$$f'(1) - f'(1) = 0.$$

This cancellation is essential: $f'(1)$ alone need not be nonnegative. We have proved $\mathcal{B}_\rho f \geq 0$, and hence the inward-pointing condition at every supporting hyperplane. By Lemma 4.6, $\mathcal{M}_N$ is invariant.

Step 4: constructing and identifying the mixing measure. The initial moments $M_0(0) = 1$, $M_n(0) = 0$ for $n \geq 1$, are those of δ_0. Invariance therefore implies $(M_1(t), \ldots, M_N(t)) \in \mathcal{M}_N$ for every N and $t \geq 0$. For fixed t, consider

$$\mathcal{C}_N(t) := \left\{ \rho \in \mathcal{P}([0,1]) : \int z^n \, d\rho = M_n(t), \ 1 \leq n \leq N \right\}.$$

These sets are nonempty, weakly compact, and nested as N increases. Their intersection is nonempty, and any measure in it satisfies (23). Two such measures agree on all polynomials and hence, by uniform approximation, on all continuous functions on $[0, 1]$. The representing measure ρ_t is therefore unique.

If $t_j \to t$, compactness gives a weakly convergent subsequence of any subsequence of (ρ_{t_j}). Continuity of each M_n shows that its limit represents the moments at t, so uniqueness identifies it as ρ_t. This proves weak continuity.

Finally, for each fixed t, dominated convergence gives

$$q_\infty(t) = \lim_{k\to\infty} \int z^{k-1} \, \rho_t(dz) = \rho_t(\{1\}).$$

For finite k, subtracting consecutive tails gives $\mathbb{P}(X_t = k) = \int_{[0,1)} (1 - z) z^{k-1} \, \rho_t(dz)$. This proves both the atom identity and the stated geometric-mixture interpretation. □

Lemma 4.8 (Maximum of two geometric mixtures)**.** *Let K, K' be independent and identically distributed random variables on $\mathbb{N}$. Assume that their common law is a mixture of geometric laws on $\{1, 2, \ldots\}$ and that $\mathbb{E}[K] < \infty$. Then*

$$\mathbb{E}[K \vee K'] \geq \frac{3}{2}\mathbb{E}[K] - \frac{1}{4} - \frac{1}{4}\mathbb{P}(K = 1). \tag{28}$$

Proof. Condition on the two mixing parameters, parametrizing a geometric law by its mean. Thus, conditionally on $a, b \geq 1$, let G_a, G_b be independent geometric random variables on $\{1, 2, \ldots\}$ with means a, b. Their tail parameters are $1 - a^{-1}$ and $1 - b^{-1}$, so

$$\mathbb{E}[G_a \wedge G_b] = \frac{ab}{a + b - 1}.$$

The identity

$$\frac{ab}{a+b-1} = \frac{a+b+1}{4} + \frac{1 - (a-b)^2}{4(a+b-1)}$$

and $a, b \geq 1$ imply

$$\frac{ab}{a+b-1} \leq \frac{a+b+1}{4} + \frac{1}{8}\left(\frac{1}{a} + \frac{1}{b}\right),$$

because $2ab \leq (a + b)(a + b - 1)$, equivalently $a(a - 1) + b(b - 1) \geq 0$. Since $G_a \vee G_b + G_a \wedge G_b = G_a + G_b$,

$$\mathbb{E}[G_a \vee G_b] \geq \frac{3}{4}(a + b) - \frac{1}{4} - \frac{1}{8}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Averaging over two independent copies of the mixing parameter gives

$$\mathbb{E}[K \vee K'] \geq \frac{3}{2}\mathbb{E}[K] - \frac{1}{4} - \frac{1}{4}\mathbb{E}[a^{-1}].$$

Conditionally on a, the probability that the geometric variable equals 1 is a^{-1}. Hence $\mathbb{E}[a^{-1}] = \mathbb{P}(K = 1)$, proving (28). □

Proposition 4.9 (Improved compensation estimate)**.** *For every $t < t_c$,*

$$A(t) \leq m(t)^{5/2}. \tag{29}$$

Consequently,

$$\mathbb{E}[X_t^2] \leq 2m(t)^{5/2}. \tag{30}$$

Proof. Fix $t < t_c$ and suppress t from the notation. By Proposition 4.7, $q_k = \int z^{k-1}\rho(dz)$. Since $m < \infty$, Tonelli's theorem gives $m = \int (1-z)^{-1}\rho(dz) < \infty$, so $\rho(\{1\}) = 0$. Hence

$$\widehat{\rho}(dz) := \frac{\rho(dz)}{m(1-z)}$$

is a probability measure and

$$\frac{q_k}{m} = \int_{[0,1)} (1-z)z^{k-1}\,\widehat{\rho}(dz). \tag{31}$$

Thus the random variable K from Lemma 4.3 is a mixture of geometric laws, with

$$\mathbb{E}[K] = \frac{A}{m}, \qquad \mathbb{P}(K = 1) = \frac{1}{m}.$$

Returning to the exact identity (20) and applying Lemma 4.8,

$$A' \leq \frac{5}{4}mA + A + \frac{1}{8}(m^2 + m) - \frac{1}{2}\sum_{k\geq 1} kq_k^2.$$

Since $m^2 \leq 2A - m$, one has $m^2 + m \leq 2A$, and therefore

$$A' \leq \frac{5}{4}(m+1)A - \frac{1}{2}\sum_{k\geq 1} kq_k^2 \leq \frac{5}{4}(m+1)A.$$

Using (12),

$$\frac{A'}{A} \leq \frac{5}{4}(m+1) \leq \frac{5}{2}\frac{m'}{m}.$$

As $A(0) = m(0) = 1$, integration proves (29). The second assertion follows from $\mathbb{E}[X_t^2] = 2A(t) - m(t)$. □

Proposition 4.10 (Lower bound on the critical tail)**.** *There exists $c > 0$ such that*

$$q_k(t_c) = \mathbb{P}(X_{t_c} \geq k) \geq ck^{-1/2}, \qquad k \geq 1. \tag{32}$$

One may take $c = (8\sqrt{2})^{-1}$. In particular, $\mathbb{E}[X_{t_c}] = \infty$.

Proof. By Proposition 4.9, $\mathbb{E}[X_t^2] \leq 2m(t)^{5/2}$ for $t < t_c$. Paley–Zygmund gives

$$\mathbb{P}\left(X_t \geq \frac{1}{2}m(t)\right) \geq \frac{1}{4}\frac{m(t)^2}{\mathbb{E}[X_t^2]} \geq \frac{1}{8}m(t)^{-1/2}.$$

Since m is continuous on $[0, t_c)$ and tends to infinity at t_c, for every $k \geq 1$ there exists $t_k < t_c$ with $m(t_k) = 2k$. Monotonicity in the level yields

$$q_k(t_c) \geq q_k(t_k) \geq \frac{1}{8\sqrt{2}}\,k^{-1/2}.$$

Summing in k gives $\mathbb{E}[X_{t_c}] = \infty$. □

Corollary 4.11 (Critical fractional moments)**.** *For every $p \geq 1/3$,*

$$\mathbb{E}[X_{t_c}^p] = \infty. \tag{33}$$

Consequently, if the logarithmic critical-tail exponent

$$\alpha := \lim_{k\to\infty} \frac{-\log q_k(t_c)}{\log k}$$

exists, then $\alpha \leq 1/3$.

Proof. We prove the endpoint case $p = 1/3$ by a truncation argument. Suppose, for a contradiction, that $\mathbb{E}[X_{t_c}^{1/3}] < \infty$. For $R \geq 1$, set

$$H_R := \mathbb{E}\Big[X_{t_c}^{1/3}\mathbf{1}_{\{X_{t_c}>R\}}\Big].$$

Then $H_R \to 0$ as $R \to \infty$. Under the monotone coupling, $X_t \leq X_{t_c}$ for $t < t_c$, so

$$\mathbb{E}\Big[X_t^{1/3}\mathbf{1}_{\{X_t>R\}}\Big] \leq H_R.$$

For every $t < t_c$, Hölder's inequality with conjugate exponents $5/3$ and $5/2$, followed by (30), gives

$$\begin{aligned} m(t) &\leq R + \mathbb{E}\big[X_t\mathbf{1}_{\{X_t>R\}}\big] \\ &\leq R + \mathbb{E}\Big[X_t^{1/3}\mathbf{1}_{\{X_t>R\}}\Big]^{3/5}\,\mathbb{E}\big[X_t^2\mathbf{1}_{\{X_t>R\}}\big]^{2/5} \\ &\leq R + H_R^{3/5}\big(2m(t)^{5/2}\big)^{2/5} \\ &= R + 2^{2/5}H_R^{3/5}m(t). \end{aligned}$$

Dividing by $m(t)$ and letting $t \uparrow t_c$, using $m(t) \to \infty$, yields

$$1 \leq 2^{2/5}H_R^{3/5}, \qquad R \geq 1.$$

Letting $R \to \infty$ gives a contradiction. Hence $\mathbb{E}[X_{t_c}^{1/3}] = \infty$. Since $X_{t_c} \geq 1$, it follows that $\mathbb{E}[X_{t_c}^p] = \infty$ for every $p \geq 1/3$.

If $\alpha > 1/3$, choose $p \in (1/3, \min\{\alpha, 1\})$ and then $\beta \in (p, \alpha)$. By the definition of α, $q_k(t_c) \leq k^{-\beta}$ for all sufficiently large k. Since

$$\mathbb{E}[X_{t_c}^p] = \sum_{k\geq 1}\big(k^p - (k-1)^p\big)q_k(t_c)$$

and $k^p - (k-1)^p = O(k^{p-1})$, this would make the p-th moment finite, a contradiction. Thus $\alpha \leq 1/3$. □

4.3. **Critical finiteness.**

The quantitative estimates above leave open whether some mass has already escaped to infinity at the critical level. The triangular structure rules this out.

Proposition 4.12 (Almost sure finiteness at criticality)**.** *At the susceptibility threshold,*

$$q_\infty(t_c) = \mathbb{P}(X_{t_c} = \infty) = 0.$$

Thus $X_{t_c} < \infty$ almost surely.

Proof. Set

$$a := q_\infty(t_c).$$

Since (12) implies $m' \geq m^2/2$ and $m(t) \to \infty$ as $t \uparrow t_c$, integration of $(1/m)' \leq -1/2$ gives

$$m(t) \leq \frac{2}{t_c - t}, \qquad t < t_c. \tag{34}$$

Fix $C > 0$, and for all sufficiently large n set

$$t_n := t_c - \frac{C}{n}.$$

Since $k \mapsto q_k(t_n)$ is non-increasing,

$$n q_n(t_n) \leq m(t_n) \leq \frac{2n}{C},$$

and hence

$$q_n(t_n) \leq \frac{2}{C}. \tag{35}$$

We claim that

$$q_n(t_n) \longrightarrow a. \tag{36}$$

The upper bound is immediate from monotonicity in the level: $q_n(t_n) \leq q_n(t_c) \to a$. For the reverse bound, let $s \in [t_n, t_c]$. For $1 \leq u \leq n-1$, monotonicity in the spatial index and in the level gives

$$q_n(s) \leq q_{n-u}(s) \leq q_{n-u}(t_c).$$

Using (7), $q_u(s) \leq 1$, and $q_n(s)(1 - q_n(s)) \leq 1$, we obtain

$$\begin{aligned} q_n'(s) &\leq \sum_{u=1}^{n-1} \big(q_{n-u}(t_c) - q_n(s)\big) + 1 \\ &= (n-1)\big(b_n - q_n(s)\big), \end{aligned}$$

where

$$b_n := \frac{1 + \sum_{j=1}^{n-1} q_j(t_c)}{n-1}.$$

Gronwall's inequality over $[t_n, t_c]$ yields

$$q_n(t_c) \leq e^{-(n-1)C/n} q_n(t_n) + b_n\big(1 - e^{-(n-1)C/n}\big).$$

Now $q_n(t_c) \to a$, while $b_n \to a$ by Cesàro convergence. Rearranging the preceding inequality gives

$$\liminf_{n\to\infty} q_n(t_n) \geq a.$$

Together with the already proved upper bound, this establishes (36).

Finally, (35) and (36) imply $a \leq 2/C$. Since $C > 0$ is arbitrary, $a = 0$. □

Combining Proposition 4.10, Corollary 4.11, and Proposition 4.12, the critical avalanche is finite almost surely but has infinite mean. The elementary bound (22) already has exponent below 1, whereas the geometric-mixture refinement gives the stronger pointwise and fractional-moment information. The next section shows that either bound is heavy enough to force explosion at every level strictly above t_c.

5. Postcritical explosion

We now derive postcritical explosion from the critical tail estimates. The elementary estimate proves sharpness, while the refined estimate improves the quantitative lower bound. We first isolate the mechanism in a time-homogeneous process with a frozen graft law.

Let Y be a random variable taking values in $\mathbb{N}$, and write

$$Q_k := \mathbb{P}(Y \geq k), \qquad k \geq 1.$$

Given an initial state $Z_0 \in \mathbb{N}$, define the increasing process Z by independent driving variables: let $(E_j)_{j\geq 0}$ be unit exponentials, $(V_j)_{j\geq 0}$ uniforms on $(0,1)$, and $(Y_j)_{j\geq 0}$ copies of Y, with all variables mutually independent and independent of Z_0. Set $T_0 = 0$, $z_0 = Z_0$, and, recursively,

$$T_{j+1} = T_j + \frac{E_j}{z_j + 1},$$
$$U_j = \lfloor (z_j + 1)V_j \rfloor, \qquad z_{j+1} = \max\{z_j, U_j + Y_j\}.$$

Put $Z_s = z_j$ on $[T_j, T_{j+1})$, and let $\zeta = \lim_{j\to\infty} T_j$. At a finite accumulation time ζ, send the process to ∞, which is absorbing. Thus, conditionally on the current finite state x, the waiting time to the next proposed update is exponential of rate $x+1$, its origin is uniform on $\{0, \dots, x\}$, and its graft has law Y. The actual waiting times depend on the successive states; it is the driving variables that are independent. Proposed updates that leave the endpoint unchanged are retained in this construction.

Proposition 5.1 (Frozen heavy-tail explosion)**.** *Suppose that*

$$Q_k \geq ck^{-\alpha}, \qquad k \geq 1,$$

for some $c > 0$ and $0 < \alpha < 1$. If Z_0 has the law of Y, then

$$\mathbb{P}(\zeta \leq s) > 0, \qquad s > 0.$$

More precisely, there are constants $c_0 > 0$ and $s_0 > 0$, depending only on c and α, such that

$$\mathbb{P}(\zeta \leq s) \geq c_0 s^{\alpha/(1-\alpha)}, \qquad 0 < s \leq s_0. \tag{37}$$

Proof. Fix $x \geq 1$, and let $\tau_{2x} := \inf\{s \geq 0 : Z_s \geq 2x\}$. While $x \leq Z_s < 2x$, proposed updates whose graft is at least $2x$ occur at rate at least xQ_{2x}, and each such update reaches level $2x$. Consequently, uniformly in $z \in [x, 2x)$,

$$\mathbb{E}_z[\tau_{2x}] \leq \frac{1}{xQ_{2x}} \leq Cx^{\alpha-1}. \tag{38}$$

Fix $K \geq 1$ and $z \geq K$, and choose $j \geq 0$ such that $2^j K \leq z < 2^{j+1}K$. For $i \geq j+1$, let $\sigma_i := \inf\{s \geq 0 : Z_s \geq 2^i K\}$. On every bounded set of states the proposed-update rates are bounded, so update times cannot accumulate while Z remains bounded. Consequently $\zeta = \lim_{i\to\infty} \sigma_i$. At time σ_i, either the process has already crossed $2^{i+1}K$, or its state lies in $[2^i K, 2^{i+1}K)$; in the latter case (38) and the strong Markov property give

$$\mathbb{E}_z[\sigma_{i+1} - \sigma_i \mid \mathcal{F}_{\sigma_i}] \leq C(2^i K)^{\alpha-1}.$$

The analogous estimate for the first crossing from z to $2^{j+1}K$, followed by monotone convergence, yields

$$\mathbb{E}_z[\zeta] \leq C \sum_{i\geq j} (2^i K)^{\alpha-1} \leq C' K^{\alpha-1}.$$

The series converges because $\alpha < 1$. Given $s > 0$, choose K large enough that $C'K^{\alpha-1} \le s/2$. Markov's inequality then implies

$$\inf_{z \ge K} \mathbb{P}_z(\zeta \le s) \ge \frac{1}{2}.$$

Since Z_0 has the law of Y,

$$\mathbb{P}(\zeta \le s) \ge Q_K \inf_{z \ge K} \mathbb{P}_z(\zeta \le s) \ge \frac{1}{2} Q_K > 0.$$

For the quantitative bound, keep the dependence of K on s. The constants C and C' above depend only on c and α. Set $s_0 = \min\{1, 2C'\}$ and, for $0 < s \le s_0$, take

$$a_s := \left(\frac{2C'}{s}\right)^{1/(1-\alpha)}, \qquad K = \lceil a_s \rceil.$$

Then $a_s \ge 1$, $a_s \le K \le 2a_s$, and $C'K^{\alpha-1} \le s/2$. The same Markov bound therefore yields

$$\mathbb{P}(\zeta \le s) \ge \frac{c}{2} K^{-\alpha} \ge \frac{c}{2^{1+\alpha}} (2C')^{-\alpha/(1-\alpha)} s^{\alpha/(1-\alpha)}.$$

This proves (37). □

We apply the proposition with the critical avalanche law. Set

$$Q_k := q_k(t_c), \qquad k \ge 1.$$

By Proposition 4.12, $Q_k \downarrow 0$, so there is a finite $\{1, 2, \ldots\}$-valued random variable Y with $\mathbb{P}(Y \ge k) = Q_k$. Let Z be the corresponding frozen process, started with the law of Y, and set

$$r_k(s) := \mathbb{P}(Z_s \ge k).$$

Proposition 5.2 (Postcritical domination)**.** *For every $k \ge 1$ and $s \ge 0$,*

$$q_k(t_c + s) \ge r_k(s).$$

Consequently,

$$q_\infty(t_c + s) \ge \mathbb{P}(\zeta \le s).$$

Proof. The same crossing computation as in Proposition 2.4, now with the graft law frozen at Q, gives $r_1 \equiv 1$ and, for $k \ge 2$,

$$r_k' = \sum_{u=1}^{k-1} Q_u (r_{k-u} - r_k) + Q_k(1 - r_k), \qquad r_k(0) = Q_k. \tag{39}$$

Set $\widetilde{q}_k(s) := q_k(t_c + s)$. By monotonicity in the level, $\widetilde{q}_u(s) \ge Q_u$. Since the tail sequence is decreasing, $\widetilde{q}_{k-u}(s) - \widetilde{q}_k(s) \ge 0$, while $1 - \widetilde{q}_k(s) \ge 0$. Hence (7) yields

$$\widetilde{q}_k' \ge \sum_{u=1}^{k-1} Q_u (\widetilde{q}_{k-u} - \widetilde{q}_k) + Q_k(1 - \widetilde{q}_k).$$

We prove $\widetilde{q}_k \ge r_k$ by induction on k. The case $k = 1$ is immediate. If the assertion holds below level k, put $\Delta_k := \widetilde{q}_k - r_k$. Subtracting (39) gives

$$\Delta_k' + \left(\sum_{u=1}^{k} Q_u\right) \Delta_k \ge \sum_{u=1}^{k-1} Q_u \Delta_{k-u} \ge 0, \qquad \Delta_k(0) = 0.$$

An integrating factor gives $\Delta_k \ge 0$, completing the induction. Letting $k \to \infty$, and recalling that $Z_s = \infty$ exactly when $\zeta \le s$, proves the second assertion. □

We can now distinguish the roles of the two critical estimates. Corollary 4.5 gives $Q_k \geq ck^{-2/3}$. Applying Propositions 5.1 and 5.2 with $\alpha = 2/3$ already yields

$$q_\infty(t_c + s) > 0, \qquad s > 0,$$

and even a lower bound proportional to s^2 for small s. Together with Proposition 4.12 and finite susceptibility below t_c, this proves

$$t_c = \inf\{t \geq 0 : \mathbb{P}(X_t = \infty) > 0\}.$$

Thus the geometric-mixture argument is not needed for sharpness itself. It strengthens the quantitative conclusion as follows.

Corollary 5.3 (Quantitative postcritical bound)**.** *There exist constants $c_* > 0$ and $s_0 > 0$ such that*

$$\mathbb{P}(X_{t_c+s} = \infty) \geq c_* s, \qquad 0 < s \leq s_0. \tag{40}$$

Proof. Proposition 4.10 gives $Q_k \geq ck^{-1/2}$. The quantitative estimate (37), with $\alpha = 1/2$, gives $\mathbb{P}(\zeta \leq s) \geq c_* s$ for $0 < s \leq s_0$. Proposition 5.2 transfers this bound to $q_\infty(t_c + s)$. □

The bound (40) is one-sided: it does not assert a linear asymptotic for the infinite-avalanche probability. Combined with Proposition 3.2, Proposition 3.3, and the critical results of Section 4, it completes the proof of Theorem 1.1.

Appendix A. Use of AI

We began the investigation of the Bak–Sneppen model two years ago by seeking soft, model-independent PDE arguments for sharpness. After several unsuccessful attempts, described in the appendix of [6], we turned to the one-dimensional one-sided, or anisotropic, model. We expected its dynamics to be more tractable than those of the isotropic model, and hoped that sharpness could subsequently be transferred from one to the other. For the anisotropic model, we first derived the triangular system governing the tails of the avalanche range.

In July 2026, taking this triangular system as input, ChatGPT 5.6 Sol proposed the compensation argument that led to our first proof of sharpness for the anisotropic model, and which also corresponds to the first argument presented here, leading to Proposition 4.4. We then observed that stochastic domination allowed us to transfer sharpness to the isotropic model, as presented in the first version of this paper.

For the present revision, we asked GPT-6 Astra to improve the critical tail exponent. The model proposed a refinement based on the preservation of geometric mixtures under the evolution, Proposition 4.7, yielding the sharper exponent.

Claude did the simulation leading to the picture.

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