

C 7 - 8 Luigi Martinelli (Bielefeld) and Thomas Mordant (Cergy)

Aims of the talk:

- One variable proofs of the holonomy bounds (Thm. 7.0.1)

based on [CDT24] § 7 + [B20] + [BC22]

- Emphasis on the geometry behind these bounds

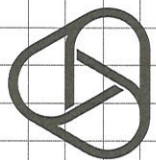
⇒ "arithmetic Italian geometry"

namely inequalities relating intersection numbers attached to projective surfaces, in the spirit of Segre's proof of Hodge Index Theorem

⇒ use of a minimal amount of Arakelov geometry, which will be discussed in the first talk

⇒ [auxiliary functions and Siegel's lemma]

becomes the so-called method of slopes which will be discussed in the second talk.



Two basic principles underlying the method of slopes:

A Focus on evaluation morphisms which "encode" the object under study

$$\Sigma = \varinjlim \Sigma_i \hookrightarrow X \quad \text{say } \Sigma = \hat{C}_P, \Sigma_i = P_{i-1}$$

{ auxiliary polynomial of degree D } \longrightarrow { formal functions on $\Sigma = \varinjlim \Sigma_i$ }

$$E_D := \Gamma(X, L^{\otimes D}) \xrightarrow{\eta_D} F = L^{\otimes D}_{|\Sigma}$$

$$\eta_{D,i} \searrow \quad \downarrow$$

$$F_i = L^{\otimes D}_{|P_{i-1}}$$

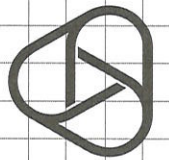
Here This week / [CDT24]

$$X = \mathbb{C} = \mathbb{P}^1_{\mathbb{Q}} \quad L = \mathcal{O}(1) \quad P = (1:0) \quad \hat{C}_P \simeq \mathbb{P}^1_{\mathbb{Q}, (1:0)}$$

$$E_D = \mathbb{Q}[X_0, X_1]_D \quad F \simeq \mathbb{Q}[[X]]$$

B Consider the slopes invariants attached to η_D , for $D \gg 0$ and the inequality they satisfy

"relative geometry of number" = "Arakelov geometry over $\text{Spec } \mathbb{Z}$ "



First part of the joint talk:

1 Green functions on Riemann surfaces

2 Green formula and $*$ -products

3 Poisson-Jensen - Nevanlinna \Rightarrow Schwarz lemma

4 Arakelov intersection numbers on arithmetic curves and surfaces

5 Arithmetic Hilbert - Samuel on arithmetic surfaces



Green functions on Riemann surfaces

- M Riemann surface

$$\partial, \bar{\partial}$$

$$z = x + iy$$

$$i \partial \bar{\partial} f = i \frac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$$

$$= \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy$$

$$f \text{ subharmonic} \iff i \partial \bar{\partial} f \geq 0$$

- $D = \sum_{\alpha \in A} m_\alpha P_\alpha$ divisor on M

$$\implies \delta_D := \sum_{\alpha \in A} m_\alpha \delta_{P_\alpha}$$

$$\int_M f \delta_D := \sum_{\alpha \in A} m_\alpha f(P_\alpha)$$

$$\text{if } f \in C_c^0(M)$$

Poincaré formula:

for any f meromorphic ($\neq 0$) on M connected

$$i \partial \bar{\partial} \log |f| = \pi \delta_{\text{div} f}$$

locally L^1

- Green functions in Arakelov geometry

D divisor on M ; a Green function for D on M is a C^∞ function

$$g: M \setminus |D| \rightarrow \mathbb{R}$$

such that, if on $U \hookrightarrow M$, $D = \text{div} f$, then $g|_U - \log |f|^{-1} \in C^\infty(U)$



Equivalently:

\Leftrightarrow g is a real valued distribution on M such that $\omega(g) := \frac{i}{\pi} \partial \bar{\partial} g + \delta_D$ is C^∞ (ellipticity of $\partial \bar{\partial}$)

\Leftrightarrow there exists a C^∞ Hermitian metric $\langle \cdot, \cdot \rangle_g$ on the line bundle $\mathcal{O}(D)$ on M such that:

$$\| \mathbb{1}_{\mathcal{O}(D)} \|_g = e^{-g} \text{ on } M \setminus |D|$$

\Leftrightarrow \exists meromorphic section of L analytic line bundle over M
 $\| \cdot \|$ C^∞ Hermitian metric on L

$$\frac{1}{2\pi i} \partial \bar{\partial} \log \|s\|^2 = c_1(L, \| \cdot \|) - \delta_{\text{div } s}$$

notation of β_3 $\omega(g) = c_1(\mathcal{O}(D), \| \cdot \|_g)$

Green functions in potential theory

Def - Prop. M compact connected Riemann surface, with C^∞ boundary $\partial M \neq \emptyset$; $0 \in \mathring{M} := M \setminus \partial M$

a classical Green function for 0 in M is a continuous / C^∞ function $g : M \setminus \{0\} \rightarrow \mathbb{R}$ such that:

\underline{G}_1 g is harmonic on $\mathring{M} \setminus \{0\}$

\underline{G}_3 $g|_{\partial M} = 0$.

\underline{G}_2 near 0 , $g = \log |z - z(0)|^{-1} + C^\infty$



Such a function $g =: g_{M,0}$ exists and is unique.

It is positive on $\mathring{M} \setminus \{0\}$.

NB $\underline{G}_1 + \underline{G}_2 \iff$ on \mathring{M} , g is locally L^1 and $\frac{i}{\pi} \partial \bar{\partial} g + \delta_p = 0$

Example: When M is simply connected, there exists $\varphi: \mathring{D}(0,1) \xrightarrow{\sim} M$ diffeomorphism such $\varphi(0) = 0$, \mathbb{C} -analytic from $\mathring{D}(0,1)$ onto \mathring{M} .
The conformal representation φ is unique "up to rotation".

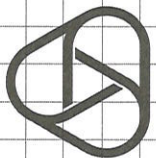
Moreover:

$$g_{M,0} = -\log |\varphi^{-1}|$$

\uparrow compositional
inverse of φ

Indeed $g_{\mathring{D}(0,1),0} = \log |z|^{-1}$

$$g_{\mathring{D}(0,R),0} = \log |R/z|$$



Further definitions

$0 \in \dot{M} \subset M, \partial M \neq \emptyset$ as above

may assume $M \subset M^+$ on (germ of) Riemann surface without boundary

Harmonic measure

extend $g_{M,0}$ to M^+ by 0 on $M^+ \setminus M$

Then $g_{M,0}$ is continuous on $M^+ \setminus \{0\}$, and satisfies

$$\frac{i}{\pi} \partial \bar{\partial} g_{M,0} + \delta_P = \mu_{M,0}$$

where $\mu_{M,0}$ is a probability measure supported by ∂M , defined by a positive C^∞ 1-form on ∂M .

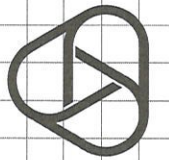
Example : $g_{\overline{D}(0,R),0} = \log^+ |R/z|$

$\mu_{\overline{D}(0,R),0} =$ rotation invariante probabil. measure on $\partial \overline{D}(0,R)$ " $\frac{1}{2\pi} d\theta$ "

Capacity metric

$g_{M,0} \rightsquigarrow \|\cdot\|$ on $\mathcal{O}(0)$
 $\|\mathbb{1}_{\mathcal{O}(0)}\| := \exp(-g_{M,0})$

"adjunction" $\mathcal{O}(P)_{|P} \xrightarrow{\sim} T_{M,0}$
 $\|\cdot\|_P \leftrightarrow \|\cdot\|_{M,0}^{cap}$



N.B. z local coordinate on M near 0 , $z(0) = 0$
near 0 , $g_{M,0} = \log|z|^{-1} + h$

Then: $\left\| \frac{\partial}{\partial z} \right\|_{M,0}^{\text{cap}} = \exp(-h(0))$

Example: $\varphi: \overline{D}(0,1) \xrightarrow{\sim} M \hookrightarrow \mathbb{C}$
 $0 \mapsto 0$

$$\left\| \frac{\partial}{\partial z} \right\|_{M,0}^{\text{cap}} = |\varphi'(0)|^{-1}$$

= inverse of the conformal radius of M at 0 .



2. Green formula and *-products

• Green formula = Stokes $\times 2$

M Riemann surface (without boundary)

f_1, f_2 distributions on M

s.t.

$\text{supp } f_1 \cap \text{supp } f_2$ is compact

$\text{sing supp } f_1 \cap \text{sing supp } f_2 = \emptyset$

$$\int_M f_1 i \partial \bar{\partial} f_2 = \int_M f_2 i \partial \bar{\partial} f_1$$

• Dirichlet scalar product

M compact connected Riemann surface $\partial M = \emptyset$

$f_1, f_2 : M \rightarrow \mathbb{C} \quad \mathbb{C}^2$

$$\langle f_1, f_2 \rangle_{\text{Dir}} = i \int_M \partial f_1 \wedge \bar{\partial} f_2 = -i \int_M \bar{\partial} f_1 \wedge \partial f_2$$

$$= \int_M \frac{i}{2} (\partial f_1 \wedge \bar{\partial} f_2 - \bar{\partial} f_1 \wedge \partial f_2)$$

$$= \int_M \left(\frac{\partial f_1}{\partial x} \frac{\bar{\partial} f_2}{\partial y} + \frac{\partial f_1}{\partial y} \frac{\bar{\partial} f_2}{\partial x} - \left(\frac{\bar{\partial} f_1}{\partial x} \frac{\partial f_2}{\partial y} + \frac{\bar{\partial} f_1}{\partial y} \frac{\partial f_2}{\partial x} \right) \right) dx \wedge dy$$

defines a Hermitian scalar product on $\mathcal{E}^1(M)/\mathbb{C}$

more generally on $L^2(M)/\mathbb{C}$



• * - product

M compact Riemann surface ($\partial M = \emptyset$)

D_1, D_2 divisors on M ; g_1, g_2 Green functions for D_1, D_2

$$\alpha = 1, 2 \quad \omega_\alpha := \frac{i}{\pi} \partial \bar{\partial} g_\alpha + \delta_{D_\alpha}$$

when $|D_1| \cap |D_2| = \emptyset$, let $g_1 * g_2 := g_2 \delta_{D_1} + g_1 \omega_2$

Green formula $\Rightarrow \int_M g_1 * g_2 = \int_M g_2 * g_1$

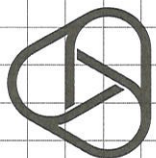
computation rules:

• $\text{supp } g_1 \cap \text{supp } g_2 = \emptyset \Rightarrow \int_M g_1 * g_2 = 0$

• for $f_1, f_2 \in C^\infty(M, \mathbb{R})$,

$$\int_M (g_1 + f_1) * (g_2 + f_2) = \int_M g_1 * g_2 + \int_M (f_1 \omega_2 + f_2 \omega_1) - \frac{i}{\pi} \langle f_1, f_2 \rangle_{D_1}$$

actually extends to more general situation, with M possibly not compact,
provided $\text{supp } g_1 \cap \text{supp } g_2$ is compact



3 Poisson - Jensen - Nevanlinna \Rightarrow Schwarz Lemma

data $0 \in \overset{\circ}{M} \hookrightarrow M \hookrightarrow M^+$ M compact connected, $\partial M \neq \emptyset$

$$\hookrightarrow g_{M,0}$$

$\bar{L} := (L, \|\cdot\|)$ Hermitian line bundle over M^+

s meromorphic section of L over M^+ , $s \neq 0$

$$\hookrightarrow \frac{i}{\pi} \partial \bar{\partial} \log \|s\|^{-1} + \delta_{\text{div } s} = c_1(\bar{L})$$

ix When $0 \notin |\text{div } s|$, may consider the following "Green functions" on M^+ with disjoint divisors

$$\begin{cases} g_1 = g_{M,0}, & D_1 = 0 & \omega_1 = \mu_{M,0} \\ g_2 = \log \|s\|^{-1}, & D_2 = \text{div } s & \omega_2 = c_1(\bar{L}) \end{cases}$$

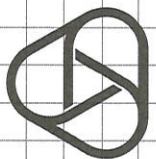
get:

$$\int_M g_1 * g_2 = \log \|s(0)\|^{-1} + \int_M g_{0,M} c_1(\bar{L})$$

$$\int_M g_2 * g_1 = \int_M g_{M,0} \delta_{\text{div } s} + \int_{\partial M} \log \|s\|^{-1} \mu_{M,0}$$

Poisson - Jensen - Nevanlinna

$$\log \|s(0)\| = \int_{\partial M} \log \|s\| \mu_{M,0} + \int_M g_{M,0} c_1(\bar{L}) - \int_M g_{0,M} \delta_{\text{div } s}$$



Corollary: when s is holomorphic

$$\log \|s(0)\| \leq \int_{\partial M} \log \|s\| \mu_{M,0} + \int_M g_{M,0} c_1(\bar{L})$$

$$\leq \log \|s\|_{L^\infty(\partial M)} + \int_M g_{M,0} c_1(\bar{L})$$

β More generally, when s holomorphic $\neq 0$, $\text{mult}_0 s =: m$

$$\Leftrightarrow \begin{cases} 0 \leq \text{div} s - m \cdot 0 \\ 0 \notin |\text{div} s - m \cdot 0| \end{cases}$$

m -th jet of s at 0 $j_0^m s \in L_{10} \otimes T_{M,0}^{\vee \otimes m}$

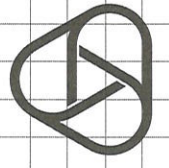
$\neq 0$ \hookrightarrow $\|\cdot\|_{L,0}$ \hookrightarrow $\|\cdot\|_{M,0}^{\text{cap } \vee \otimes m}$ $\parallel \parallel_m^{\text{cap}}$

$$\begin{cases} g_1 = g_{M,0}, \quad D_1 = 0 & \omega_1 = \mu_{M,0} \\ g_2 = \log \|s\|^{-1} - m g_{M,0}, \quad D_2 = \text{div} s - m \cdot 0 & \omega_2 = c_1(\bar{L}) - m \mu_{M,0} \end{cases}$$

$$g_2(0) = - \log \|j_0^m s\|_m^{\text{cap}}$$

$$\int g_1 * g_2 = - \log \|j_0^m s\|_m^{\text{cap}} + \int_M g_{M,0} (c_1(\bar{L}) - m \mu_{M,0})$$

$$\int g_2 * g_1 = \int_M (g_{M,0} \delta_{\text{div} s - m \cdot 0} + (-\log \|s\|^{-1} - m g_{M,0}) \mu_{M,0})$$



Basic estimate on jets of sections

$$\begin{aligned} \log \|j_0^m s\|_m^{\text{cap}} &\leq \int_M \log \|s\| \mu_{M,0} + \int_M g_{0,M} c_1(\bar{L}) \\ &\leq \log \|s\|_{L^\infty(\partial M)} + \int_M g_{0,M} c_1(\bar{L}) \end{aligned}$$

γ Relation with Nevanlinna Theory:

$$0 \in \mathbb{A}^1(\mathbb{C}) \xrightarrow{\varphi} \mathbb{P}^n(\mathbb{C}) \quad \mathbb{C}\text{-analytic}$$

$$\overline{\partial(\mathbb{C})} \quad c_1(\overline{\partial(\mathbb{C})}) =: \omega_{\text{FS}} \quad \text{"Fubini-Study"}$$

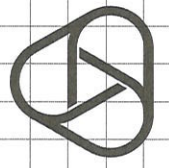
$$\begin{cases} \bar{L} := \varphi^* \overline{\partial(\mathbb{C})}^{\otimes D} & , \quad D \in \mathbb{Z}_{>0} \\ M := \overline{\mathbb{D}}(0, R) & , \quad R > 0 \end{cases}$$

$$\implies \int_M g_{M,0} c_1(\bar{L}) = D \int_{\mathbb{C}} \log^+ \frac{R}{|z|} \varphi^* \omega_{\text{FS}}$$

$=: D T_\varphi(R)$ \leftarrow Nevanlinna characteristic function convex in $\log R$

$$\|\frac{\partial}{\partial \bar{z}}\|_{\text{st}} := 1$$

$$\log \|j_0^m s\|_{\text{st}} \leq \log \|s\|_{L^\infty(\partial \mathbb{D}(0,R))} - m \log R + D T_\varphi(R)$$



Arithmetic intersection numbers on arithmetic curves and surfaces

Arakelov geometry in dimension 0 and 1

\mathcal{X} scheme of finite type over \mathbb{Z}

$$Z_i \in Z_0(\mathcal{X})$$

$$\sum_{\alpha} m_{\alpha} P_{\alpha} \quad P_{\alpha} \text{ closed point of } \mathcal{X}; K(P_{\alpha}) \text{ is a finite field}$$

Arakelov degree of Z

$$\widehat{\deg} Z := \sum_{\alpha} m_{\alpha} \log |K(P_{\alpha})|$$

Examples:

1) $q \in \mathbb{Q}^{\times} \quad \mathcal{X} = \text{Spec } \mathbb{Z}$

$$Z := \text{div } q = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(q) [P]$$

$$\widehat{\deg} \text{div } q = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(q) \log p = \log |q|$$

2) $\mathcal{X} / \mathbb{F}_p$

$$\widehat{\deg} Z = \deg_{\mathbb{F}_p} Z \log p$$

Basic observation

Arakelov degree of 0-cycles linearly equivalent to 0

$$\mathcal{E} \xrightarrow{\pi} \text{Spec } \mathbb{Z} \text{ proper}$$

\hookrightarrow integral of dim. 1

$$\omega \in K(\mathcal{E})^{\times}$$

$$\text{div } \omega \in Z_0(\mathcal{E})$$



dichotomy :

$$\pi : \mathcal{E} \rightarrow \text{Spec } \mathbb{Z} \quad \text{proper} \\ \subset 1 \text{ dim.}$$

either

π finite and flat

$X(\mathcal{E})$ is a number field K ; $\text{Spec } \mathcal{O}_K$ is the normalisation of \mathbb{Z}

$$\mapsto \mathcal{E}(\mathbb{C}) = \{ \sigma : K \hookrightarrow \mathbb{C} \}$$

$$q := N_{K/\mathbb{Q}}(\alpha) \in \mathbb{Q}^\times$$

$$\pi_* \text{div } \alpha = \text{div } q$$

$$\widehat{\deg} \text{div } \alpha = \widehat{\deg} \text{div } q = \log |q| = \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log |\sigma(\alpha)|$$

$$= \sum_{\alpha \in X(\mathbb{C})} \log |\alpha|$$

or

\mathcal{E} is a proper integral curve over \mathbb{F}_p , p prime

"product formula"

$$\widehat{\deg} \text{div } \alpha = \deg_{\mathbb{F}_p} \text{div } \alpha - \log p = 0$$



Arakelov degree and heights

\mathcal{X} finite type over \mathbb{Z}

$\bar{L} := (L, \|\cdot\|)$ Hermitian line bundle over \mathcal{X}

$\left\{ \begin{array}{l} \uparrow \\ \text{line bundle} \\ \text{over } \mathcal{X} \end{array} \right.$
 $\left\{ \begin{array}{l} \uparrow \\ \text{continuous / } C^\infty \text{ metric on } L_x \text{ over } \mathcal{X}(\mathbb{C}), \text{ c.c. invariant} \end{array} \right.$

$\mathcal{O} \hookrightarrow \mathcal{X}$ 1-dimensional, integral, proper over \mathbb{Z}

s non-zero rational section of L over \mathcal{O}

Prop. Def.

$$ht_{\bar{L}}(\mathcal{O}) := \hat{\deg} \operatorname{div} s - \sum_{x \in \mathcal{O}(\mathbb{C})} \log \|s(x)\| \in \mathbb{R}$$

does not depend on s ; it is the height of \mathcal{O} wrt \bar{L}

Notation:

$$\hat{\deg}(\bar{L} | \mathcal{O}) = ht_{\bar{L}}(\mathcal{O})$$

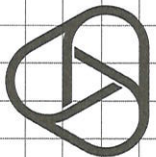
$\hat{\deg}$ extends by linearity to $\mathcal{O} \in \mathbb{Z}, (\mathcal{X})$ proper over \mathbb{Z}

"height pairing" : $\operatorname{Pic}(\mathcal{X}) \times \mathbb{Z}, (\mathcal{X})_{\text{proper}} \rightarrow \mathbb{R}$

$$(\bar{L}, \mathcal{Z}) \mapsto \hat{\deg}(\bar{L} | \mathcal{Z})$$

functoriality:

$$\hat{\deg}(f^* \bar{L} | \mathcal{Z}) = \hat{\deg}(\bar{L} | f_* \mathcal{Z})$$



Example relation with classical heights

$$\mathcal{X} = \mathbb{P}_{\mathbb{Z}}^n \quad \overline{\mathcal{L}} = \overline{\mathcal{O}(1)} \quad \mathcal{O}_{\mathbb{P}^n}^{\oplus (n+1)} \rightarrow \mathcal{O}(1)$$

$$\mathbb{P}^n(\mathbb{Z}) \xrightarrow{\sim} \mathbb{P}^n(\mathbb{Q})$$

$$\downarrow \quad \downarrow$$

$$\mathcal{P} \quad \mathcal{P} = (p_0 : \dots : p_n)$$

$L \in \mathbb{Z}$, glue together

$$hL_{\overline{\mathcal{O}(1)}}(\mathcal{P}) := \deg \hat{\mathcal{O}(1)}|_{\mathcal{P}} = \deg \mathcal{P}^* \overline{\mathcal{O}(1)} = \log \left(\sum_{i=0}^n p_i^2 \right)^{1/2}$$

Geometry of numbers.

may consider

$$\overline{\mathcal{E}} = (\mathcal{E}, \|\cdot\|) \quad \text{Hermitian vector bundle over } \mathcal{X}$$

vector bundle over \mathcal{X} continuous Hermitian metric on $\mathcal{E}_{\mathbb{C}}$ over $\mathcal{X}(\mathbb{C})$ invariant under c.c.

tensor operations: $\oplus, \otimes, \vee, \wedge, \dots$

Exercise:

when $\mathcal{X} = \text{Spec } \mathbb{Z}$ $\overline{\mathcal{E}} = (\mathcal{E}, \|\cdot\|)$

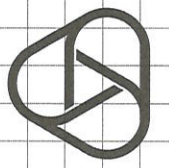
Euclidean metric on $E_{\mathbb{R}}$

free \mathbb{Z} -module of finite rank r Euclidean lattice

$$\hat{\deg} \overline{\mathcal{E}} := \hat{\deg} (\mathbb{Z}^r \overline{\mathcal{E}} | \text{Spec } \mathbb{Z})$$

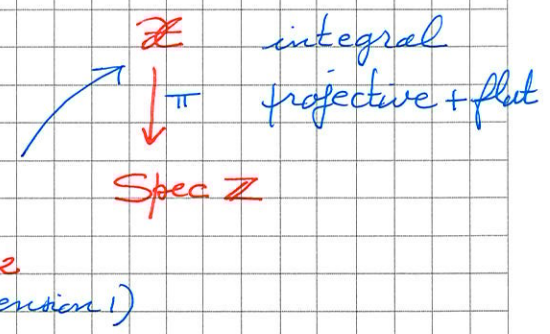
$$= -\log \text{caval}(\mathcal{E}, \|\cdot\|) \quad \text{"Arakelov degree of } \overline{\mathcal{E}}"$$

$$\hat{\mu}(\overline{\mathcal{E}}) := \frac{1}{r} \hat{\deg} \overline{\mathcal{E}} = -\log \text{caval}(\mathcal{E}, \|\cdot\|)^{1/r} \quad \text{"slope of } \overline{\mathcal{E}}"$$



4b Arakelov geometry in dimension 2

main object of study : projective arithmetic surfaces



N.B. $X_{\mathbb{Q}}$ is a smooth projective geometrically irreducible curve C over the number field $K = \text{Frac} \Gamma(X, \mathcal{O}_X) = \mathcal{O}_K$.
 ... and conversely (...)

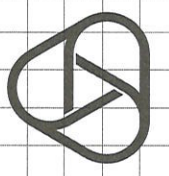
$X(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} C_{\sigma}(\mathbb{C})$ \hookrightarrow complex conjugation (c.c.)

\uparrow compact connected Riemann surface

Green function for $D_{\mathbb{C}}$ on $X(\mathbb{C})$, c.c. invariant

Definitions :

- Arakelov divisor on $X := (D, g)$
- $\overline{Z}^1(X) := \{ \text{Arakelov divisors on } X \}$ $\uparrow \in Z^1(X)$
- $f \in K(X)^{\times} = K(\mathbb{C})^{\times} \implies \overline{\text{div}} f := (\text{div} f, \log |f_{\sigma}|^{-1})$
- $\overline{CH}^1(X) := \overline{Z}^1(X) / \overline{\text{div}} K(X)^{\times}$ "linear equivalence"



Definitions (cont'd):

$$\cdot \bar{\mathcal{L}} := (\mathcal{L}, \|\cdot\|) \text{ on } \mathcal{X}$$

$\uparrow C^\infty$

$$s \in \mathcal{L}_{\mathcal{X}(\mathcal{X})} \setminus \{0\} \rightsquigarrow \overline{\text{div}}_{\bar{\mathcal{L}}}(s) := (\text{div}_{\mathcal{L}} s, \log \|s_c\|^{-1}) \in \overline{\mathcal{Z}}'(\mathcal{X})$$

$$\cdot (\mathcal{D}, g) \in \overline{\mathcal{Z}}'(\mathcal{X}) \rightsquigarrow \overline{\mathcal{O}}_{\mathcal{X}}(\mathcal{D}, g) := (\mathcal{O}_{\mathcal{X}}(\mathcal{D}), \|\cdot\|_g)$$

$\downarrow \|\cdot\|_{\mathcal{O}(\mathcal{D})} := e^{-g}$

Observation

$$\overline{\text{Pic}}_{C^\infty}(\mathcal{X}) \xrightarrow{\sim} \overline{\text{CH}}'(\mathcal{X})$$

$$[\bar{\mathcal{L}}] \mapsto [\overline{\text{div}}_{\bar{\mathcal{L}}} s], \quad s \in \mathcal{L}_{\mathcal{X}(\mathcal{X})} \setminus \{0\}$$

$$[\overline{\mathcal{O}}_{\mathcal{X}}(\mathcal{D}, g)] \longleftarrow [(\mathcal{D}, g)]$$



Def + Thm :

Arakelov, Deligne, Gillet - Soulé

- 1) $\begin{cases} \overline{L} \text{ Hermitian line bundle over } X \\ (\mathcal{D}, g) \in \overline{Z}^1(X) \end{cases}$

$$\overline{L} \cdot (\mathcal{D}, g) := \widehat{\deg}(\overline{L} | \mathcal{D}) + \int_{X(\mathbb{C})} g \cdot c_1(\overline{L}_{\mathbb{C}})$$

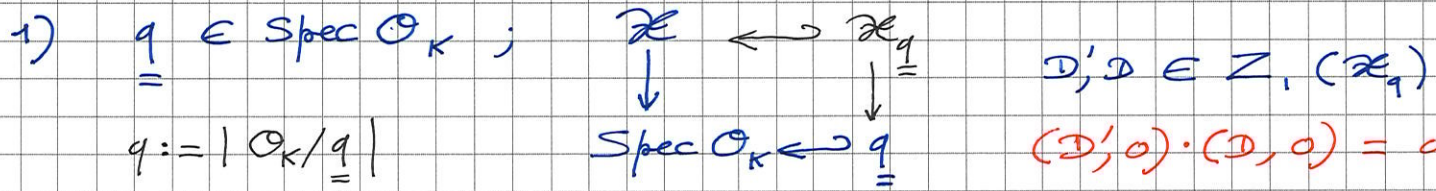
$$(\mathcal{D}, g), (\mathcal{D}', g') \in \overline{Z}^1(X)$$

$$(\mathcal{D}', g') \cdot (\mathcal{D}, g) := \mathcal{O}_X(\mathcal{D}', g) \cdot (\mathcal{D}, g)$$

2) The so-defined Arakelov intersection pairing $\overline{Z}_1(X) \times \overline{Z}_1(X) \rightarrow \mathbb{R}$ is compatible with linear equivalence, and therefore descends to $\overline{CH}^1(X) \times \overline{CH}^1(X) \rightarrow \mathbb{R}$, and is symmetric.

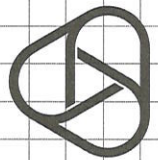
3) $|\mathcal{D}_q| \cap |\mathcal{D}'_q| = \emptyset \Rightarrow (\mathcal{D}', g') \cdot (\mathcal{D}, g) = \widehat{\deg} \mathcal{D}' \cdot \mathcal{D} + \int_{X(\mathbb{C})} g' * g$

Examples : "vertical cycles" negativity properties of Arakelov intersection



$$q := |\mathcal{O}_K / \mathfrak{q}|$$

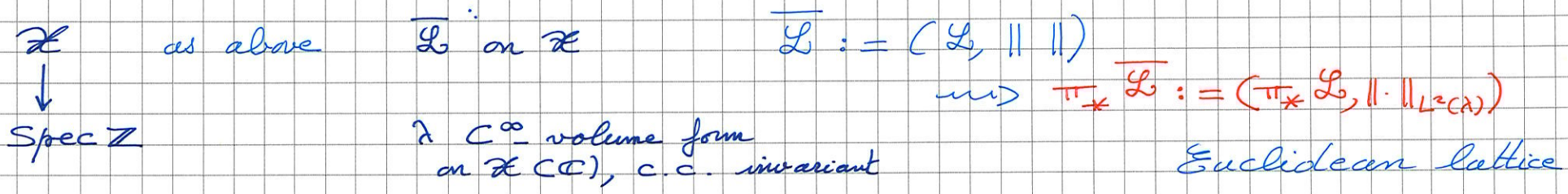
$$(\mathcal{D}', 0) \cdot (\mathcal{D}, 0) = \widehat{\deg}_{\mathbb{F}_q} \mathcal{D} \cdot \mathcal{D}' \cdot \log q \in \text{CH}_0(X_q)$$



2) $f, f' : X(\mathbb{C}) \rightarrow \mathbb{R}$, $C^\infty + c.c.$ -invariant

$$(0, f) \cdot (0, f') = -\pi^{-1} \langle f, f' \rangle_{\text{Dir}}$$

The arithmetic Hilbert - Samuel formula



where $\pi_* \mathcal{L}_i := \Gamma(\mathcal{X}, \mathcal{L}_i)$ free \mathbb{Z} -module of rank $[K:\mathbb{Q}] \dim_K \Gamma(\mathbb{C}, \mathcal{L}_i)$
 $\|s\|_{L^2(\mathcal{X})}^2 := \int_{\mathcal{X}(\mathbb{C})} \|s(x)\|^2 d\lambda(x)$ for any $s \in (\pi_* \mathcal{L})_{\mathbb{C}} = \Gamma(\mathcal{X}(\mathbb{C}), \mathcal{L}_{\mathbb{C}})$

Theorem (Faltings, Bismut-Vasserot-Gillet-Soulé, S.W. Zhang...)

When $\deg_{\mathbb{C}} \mathcal{L}_{1\mathbb{C}} > 0$ (i.e. $\mathcal{L}_{\mathbb{Q}}$ ample on $\mathcal{X}_{\mathbb{Q}}$) and
 and $\overline{\mathcal{L}}$ relatively semi-positive

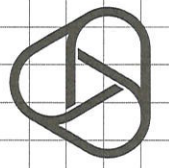
namely $c_1(\overline{\mathcal{L}}_{\mathbb{C}}) \geq 0$ on $\mathcal{X}(\mathbb{C})$
 $\forall \mathbb{C} \subset \mathbb{A}^1_{\mathbb{Q}} \rightarrow \mathcal{X}$ vertical, $\widehat{\deg}(\mathcal{L}|_{\mathbb{C}}) \geq 0$
integral curve
 $[G \text{ component of } \mathcal{X}_q \text{ } \deg_{\mathbb{F}_q} \mathcal{L}_{1\mathbb{C}} \geq 0]$

we have:

when $D \rightarrow +\infty$, $\widehat{\deg} \pi_* \overline{\mathcal{L}}^{\otimes D} = \frac{1}{2} (\overline{\mathcal{L}} \cdot \overline{\mathcal{L}}) D^2 + o(D^2)$

$\frac{N \cdot B}{=} \text{compare to } \text{rk } \pi_* \overline{\mathcal{L}}^{\otimes D} = (\deg_{\mathbb{Q}} \mathcal{L}_{\mathcal{X}_{\mathbb{Q}}}) D + o(D) \quad D \gg 0.$

reformulation $\frac{1}{D} \widehat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{(\overline{\mathcal{L}} \cdot \overline{\mathcal{L}})}{2 \deg_{\mathbb{Q}} \mathcal{L}_{\mathcal{X}_{\mathbb{Q}}}} \quad (\text{AHS})$



Comments

1) The measure λ does not appear in the RHS of formula.

Actually, replacing λ by $e^{-t} \lambda$ replaces $\hat{\mu}(\pi_* \bar{L}^{\otimes D})$ by $\hat{\mu}(\pi_* \bar{L}^{\otimes D}) + t$

Variant: instead of $\|\cdot\|_{L^2(\lambda)}$, use $\|\cdot\|_{\text{John}}$ norm on $\Gamma(\mathcal{X}(\mathbb{C}), \mathcal{L}_{\mathbb{C}})$

↑ "optimal" Hermitian norm
s.t. $\|\cdot\|_{L^\infty} \leq \|\cdot\|_{\text{John}}$

2) All proofs require some non-trivial analysis
on complex manifolds

3) The conclusion still makes sense — and holds! — for Hermitian
line bundles \bar{L} with possibly not C^∞ -metrics.

For instance: continuous metrics $\|\cdot\|$ on $\mathcal{L}_{\mathbb{C}}$ s.t. $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|)$
is a ≥ 0 measure.

Working with such metrics / Green functions (continuous + $i \partial \bar{\partial}$ measure)
allows one to establish the expected functoriality properties of $\overline{CH}^1(\mathcal{X})$
under direct images by generically finite maps.



Homework

To be ~~completed before~~ used in the next lecture.

1. Compute both sides of (AHS) when $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$.
 [And check that they coincide ; ...]

2. Consider $\pi: \mathcal{G} \rightarrow \text{Spec } \mathbb{Z}$ a regular projective surface as above, and $\mathcal{P} \in \mathcal{G}(\mathbb{Z})$ a section of π . Let moreover $o \in \mathfrak{M} \hookrightarrow M$ as in the definition of classical Green functions, and let $\varphi: (M, o) \rightarrow (\mathcal{G}(\mathbb{C}), \mathcal{P}_o)$ be an analytic map such that $D\varphi(o): T_o M \xrightarrow{\sim} T_{\mathcal{P}_o} \mathcal{G}(\mathbb{C}) = (N_{\mathcal{P}} \mathcal{G})_o$, and define the norm $\|\cdot\|^{cap} := \|\cdot\|_{M,o}^{cap}$ on $(N_{\mathcal{P}} \mathcal{G})_o$.

Define:

$$\begin{aligned} \overline{N_{\mathcal{P}} \mathcal{G}} &:= (N_{\mathcal{P}} \mathcal{G}, \|\cdot\|^{cap}) \leftarrow \text{Hermitian line bundle over } \mathcal{P} = \text{Spec } \mathbb{Z} \\ \overline{\mathcal{L}} &:= \overline{\mathcal{O}_{\mathcal{G}}(\mathcal{P}, \varphi_* g_{M,o})} \leftarrow \text{Hermitian line bundle over } \mathcal{X} \end{aligned}$$

Establish the identities:

$$\varphi_* g_{M,o}(y) := \sum_{x \in \varphi^{-1}(y)} g_{M,o}(x)$$

$$\begin{aligned} \overline{\mathcal{L}} \cdot \overline{\mathcal{L}} &= \text{ht}_{\mathbb{Z}}(\mathcal{P}) + \int_M g_{M,o} \varphi^* c_1(\overline{\mathcal{L}}_o) \\ &= \text{deg } \overline{N_{\mathcal{P}} \mathcal{G}} + \text{Ex}(\varphi: (M, o) \rightarrow \mathcal{G}(\mathbb{C})) \end{aligned}$$

where $\text{Ex}(\varphi: (M, o) \rightarrow \mathcal{G}(\mathbb{C})) := \int_M g_{M,o} (\delta_{\varphi^*(\mathcal{P}_o)-o} + \varphi^* \varphi_* \mu_{M,o})$. "weylow"

