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Aims of the talk:

- One variable proofs of the holonomy bounds (Thm. 7.0.1)

  based on [CDT24] § 7 + [B20] + [BC22]

- Emphasis on the geometry behind these bounds

  ⇝ " arithmetic Italian geometry"
  namely inequalities relating intersection numbers
  attached to projective surfaces, in the spirit of
  Segre's proof of Hodge Index Theorem

  ⇝ use of a minimal amount of Arakelov geometry,
  which will be discussed in the first talk

  ⇝ [auxiliary functions and Siegel's lemma]
  becomes the so-called method of slopes
  which will be discussed in the second talk.

Two basic principles underlying the method of slopes:

A Focus on evaluation morphisms which "encode" the object under study

$\Sigma = \varinjlim \Sigma_i \hookrightarrow X$ say $\Sigma = \widehat{C}_P$, $\Sigma_i = P_{i-1}$

$\left\{ \begin{array}{c} \text{auxiliary polynomial} \\ \text{of degree } D \end{array} \right\} \longrightarrow \left\{ \text{formal functions on } \Sigma = \varinjlim_i \Sigma_i \right\}$

$$\begin{array}{ccc} E_D := \Gamma(X, L^{\otimes D}) & \xrightarrow{\eta_D} & F = L^{\otimes D}_{|\Sigma} \\ & \searrow^{\eta_{D,i}} & \downarrow \\ & & F_i = L_{|P_{i-1}} \end{array}$$

Here This week / [CDT24]

$X = C = \mathbb{P}^1_{\mathbb{Q}}$ $L = \mathcal{O}(1)$ $P = (1:0)$ $\widehat{C}_P \simeq \widehat{\mathbb{P}}^1_{\mathbb{Q},(1:0)}$

$E_D = \mathbb{Q}[X_0, X_1]_D$ $F \simeq \mathbb{Q}[[X]]$

B Consider the slopes invariants attached to η_D, for $D \gg 0$ and the inequality they satisfy

"relative geometry of number" = "Arakelov geometry over Spec $\mathbb{Z}$"

First part of the joint talk:

1. Green functions on Riemann surfaces
2. Green formula and $*$-products
3. Poisson - Jensen - Nevanlinna $\Longrightarrow$ Schwarz lemma
4. Arakelov intersection numbers on arithmetic curves and surfaces
5. Arithmetic Hilbert - Samuel on arithmetic surfaces

1 Green functions on Riemann surfaces

- M Riemann surface

  $\partial, \bar\partial$ $\qquad\qquad$ $i\partial\bar\partial f = i \dfrac{\partial^2 f}{\partial z \partial \bar z} \, dz \wedge d\bar z$

  $z = x + iy$ $\qquad\qquad$ $= \dfrac{1}{2}\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$

  f subharmonic $\Longleftrightarrow$ $i\partial\bar\partial f \geq 0$

- $D = \sum_{\alpha \in A} n_\alpha P_\alpha$ divisor on M

  $\rightsquigarrow \quad \delta_D := \sum_{\alpha \in A} n_\alpha \delta_{P_\alpha}$ $\qquad$ $\int_M f \, \delta_D := \sum_{\alpha \in A} n_\alpha f(P_\alpha)$

  if $f \in C^0_c(M)$

Poincaré formula:

for any f meromorphic $(\neq 0)$ on M connected

$$i\partial\bar\partial \underbrace{\log |f|}_{\text{locally } L^1} = \pi \, \delta_{\operatorname{div} f}$$

- Green functions in Arakelov geometry

  D divisor on M; a Green function for D on M is a C^∞ function

  $$g : M \setminus |D| \longrightarrow \mathbb{R}$$

  such that, if on $U \hookrightarrow M$, $D = \operatorname{div} f$, then $\quad g_{|U} - \log|f|^{-1} \in C^\infty(U)$

Equivalently :

$\underline{\underline{\alpha}}$ g is a real valued distribution on M such that

$$\omega(g) := \frac{i}{\pi} \partial\bar\partial g + \delta_D \quad \text{is } C^\infty \quad \text{(ellipticity of } \partial\bar\partial)$$

$\underline{\underline{\beta}}$ there exists a C^∞ Hermitian metric $\|\cdot\|_g$ on the line bundle $\mathcal{O}(D)$ on M such that :

$$\| \underline{\underline{1}}_{\mathcal{O}(D)} \|_g = e^{-g} \quad \text{on } M \setminus |D|$$

$\underline{\underline{N}}.\underline{\underline{B}}.$ • s meromorphic section of L analytic line bundle over M

$\|\cdot\|$ C^∞ Hermitian metric on L

$$\frac{1}{2\pi i} \partial\bar\partial \log \|s\|^2 = c_1(L, \|\ \|) - \delta_{\operatorname{div} s}$$

• notation of $\underline{\underline{\beta}}$ $\qquad \omega(g) = c_1(\mathcal{O}(D), \|\ \|_g)$

• Green functions in potential theory

$\underline{\underline{\text{Def}}}. - \underline{\underline{\text{Prop}}}.$ M compact connected Riemann surface, with C^∞ boundary $\partial M \neq \emptyset$; $\quad O \in \mathring{M} := M \setminus \partial M$

a classical Green function for O in M is a continuous / C^∞ function $g : M \setminus \{O\} \longrightarrow \mathbb{R}$ such that :

$\underline{\underline{G_1}}$ g is harmonic on $\mathring{M} \setminus \{O\}$

$\underline{\underline{G_2}}$ near O, $\quad g = \log |z - z(O)|^{-1} + C^\infty$

$\underline{\underline{G_3}}$ $g_{|\partial M} = 0.$

Such a function $g =: g_{M,0}$ exists and is unique

It is positive on $\mathring{M} \setminus \{0\}$

N.B. $G_1 + G_2 \iff$ on $\mathring{M}$, g is locally L^1 and $\frac{i}{\pi}\partial\bar{\partial} g + \delta_p = 0$

Example : When M is simply connected, there exists $\varphi : \overline{D}(0,1) \xrightarrow{\sim} M$ diffeo
such $\varphi(0) = 0$, $\mathbb{C}$-analytic from $\mathring{D}(0,1)$ onto $\mathring{M}$.
The conformal representation φ is unique "up to rotation"
Moreover :

$$g_{M,0} = -\log|\varphi^{-1}|$$

compositional inverse of φ

Indeed
$$g_{\overline{D}(0,1),0} = \log|z|^{-1}$$
$$g_{\overline{D}(0,R),0} = \log|R/z|$$

- **Further definitions** $\quad O \in \mathring{M} \subset M$, $\partial M \neq \phi$ as above

may assume $M \subset M^+$: M^+ (germ of) Riemann surface without boundary

1 Harmonic measure

extend $g_{M,O}$ to M^+ by 0 on $M^+ \setminus M$

Then $g_{M,O}$ is continuous on $M^+ \setminus \{O\}$, and satisfies

$$\frac{i}{\pi} \partial\bar\partial g_{M,O} + \delta_P = \mu_{M,O}$$

where $\mu_{M,O}$ is a probability measure supported by ∂M, defined by a positive C^∞ 1-form on ∂M.

Example : $\quad g_{\overline{D}(0,R),0} = \log^+ |R/z|$

$\mu_{\overline{D}(0,R),0}$ = rotation invariante probabil. measure on $\partial \overline{D}(0,R)$ $\quad$ "$\frac{1}{2\pi} d\theta$"

2 Capacitary metric

- $g_{M,O} \rightsquigarrow \| \cdot \|$ on $\mathcal{O}(O)$

  $\| 1_{\mathcal{O}(O)} \| := \exp(-g_{M,O})$

- "adjunction" $\quad \mathcal{O}(P)_{|P} \xrightarrow{\sim} T_{M,O}$

  $\| \cdot \|_P \longleftrightarrow \| \cdot \|^{cap}_{M,O}$

N.B. z local coordinate on M near O, $z(O)=0$

near O, $g_{M,O} = \log|z|^{-1} + h$

Then: $$\left\| \frac{\partial}{\partial z} \right\|^{cap}_{M,O} = \exp(-h(O))$$

Example: $$\varphi : \overline{D}(0,1) \xrightarrow{\sim} M \hookrightarrow \mathbb{C}$$
$$0 \longmapsto O$$

$$\left\| \frac{\partial}{\partial z} \right\|^{cap}_{M,O} = |\varphi'(0)|^{-1}$$

= inverse of the conformal radius of $\mathring{M}$ at O.

2 Green formula and *-products

- **Green formula** = Stokes × 2

  M Riemann surface (without boundary)

  f_1, f_2 distributions on M

  s.t. $\quad \mathrm{supp}\, f_1 \cap \mathrm{supp}\, f_2$ is compact

  $\quad \mathrm{sing\,supp}\, f_1 \cap \mathrm{sing\,supp}\, f_2 = \emptyset$

$$\int_M f_1 \, i\partial\bar\partial f_2 = \int f_2 \, i\partial\bar\partial f_1$$

- **Dirichlet scalar product**

  M compact connected Riemann surface $\qquad \partial M = \emptyset$

  $f_1, f_2 : M \longrightarrow \mathbb{C} \qquad C^1$

$$\langle f_1, f_2 \rangle_{Dir} = i \int_M \partial f_1 \wedge \overline{\partial f_2} = - i \int \bar\partial f_1 \wedge \partial \overline{f_2}$$

$$= \int_M \underbrace{\frac{i}{2}\left(\partial f_1 \wedge \overline{\partial f_2} - \bar\partial f_1 \wedge \partial \overline{f_2} \right)}_{\frac{1}{2}\left(\frac{\partial f_1}{\partial x}\frac{\partial \overline{f_2}}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial \overline{f_2}}{\partial y}\right) dx \wedge dy}$$

defines a Hermitian scalar product on $\mathcal{E}^1(M)/\mathbb{C}$

more generally on $L^2_1(M)/\mathbb{C}$

- $*$ - product

M compact Riemann surface ($\partial M = \phi$)

D_1, D_2 divisors on M ; g_1, g_2 Green functions for D_1, D_2

$\alpha = 1, 2 \qquad \omega_\alpha := \frac{i}{\pi} \partial\bar{\partial} g_\alpha + \delta_{D_\alpha}$

when $|D_1| \cap |D_2| = \phi$, let $g_1 * g_2 := g_2\, \delta_{D_1} + g_1\, \omega_2$

Green formula $\Longrightarrow \int_M g_1 * g_2 = \int_M g_2 * g_1$

computation rules :

- $\operatorname{supp} g_1 \cap \operatorname{supp} g_2 = \phi \Longrightarrow \int_M g_1 * g_2 = 0$
- for $f_1, f_2 \in C^\infty(M, \mathbb{R})$,

$$\int_M (g_1 + f_1) * (g_2 + f_2) = \int_M g_1 * g_2 + \int_M (f_1 \omega_2 + f_2 \omega_1) - \frac{1}{\pi} \langle f_1, f_2 \rangle_{Dir}$$

actually extends to more general situation, with M possibly not compact, provided $\operatorname{supp} g_1 \cap \operatorname{supp} g_2$ is compact

3 Poisson – Jensen – Nevanlinna $\Longrightarrow$ Schwarz Lemma

data $\quad 0 \in \mathring{M} \hookrightarrow M \hookrightarrow M^+ \qquad M$ compact connected, $\partial M \neq \emptyset$

$\rightsquigarrow g_{M,0}$

$\overline{L} := (L, \| \ \|)$ Hermitian line bundle over M^+

s meromorphic section of L over M^+, $s \neq 0$

$\rightsquigarrow \frac{i}{\pi} \partial\bar\partial \log \|s\|^{-1} + \delta_{\operatorname{div} s} = c_1(\overline{L})$

$\underline{\underline{\alpha}}$ When $0 \notin |\operatorname{div} s|$, may consider the following "Green functions" on M^+ with disjoint divisors

$$\begin{cases} g_1 = g_{M,0}, & D_1 = 0 \qquad \omega_1 = \mu_{M,0} \\ g_2 = \log \|s\|^{-1}, & D_2 = \operatorname{div} s \qquad \omega_2 = c_1(\overline{L}) \end{cases}$$

get :

$$\int_M g_1 * g_2 = \log \|s(0)\|^{-1} + \int_M g_{0,M} \, c_1(\overline{L})$$

$$\|$$

$$\int_M g_2 * g_1 = \int_M g_{M,0} \, \delta_{\operatorname{div} s} + \int_{\partial M} \log \|s\|^{-1} \mu_{M,0}$$

Poisson – Jensen – Nevanlinna

$$\log \|s(0)\| = \int_{\partial M} \log \|s\| \, \mu_{M,0} + \int_M g_{M,0} \, c_1(\overline{L}) - \int_M g_{0,M} \, \delta_{\operatorname{div} s}$$

Corollary : when s is holomorphic

$$\log\|s(0)\| \leq \int_{\partial M} \log\|s\|\, \mu_{M,0} + \int_M g_{M,0}\, c_1(\overline{L})$$

$$\leq \log\|s\|_{L^\infty(\partial M)} + \int_M g_{M,0}\, c_1(\overline{L})$$

$\underline{\beta}$ More generally, when s holomorphic $\neq 0$, $\mathrm{mult}_0 s =: n$

$$\Longleftrightarrow \begin{cases} 0 \leq \operatorname{div} s - n.0 \\ 0 \notin |\operatorname{div} s - n.0| \end{cases}$$

n-th jet of s at 0 $\quad \underbrace{j_0^n s}_{\neq 0} \in \underbrace{L_{|0}}_{\|\cdot\|_{\overline{L},0}} \otimes \underbrace{T_{M,0}^{\vee\otimes n}}_{\|\cdot\|_{M,0}^{\mathrm{cap}\,\vee\otimes n}} \rightsquigarrow \|\ \|_n^{\mathrm{cap}}$

$$\begin{cases} g_1 = g_{M,0}, & D_1 = 0 & \omega_1 = \mu_{M,0} \\ g_2 = \log\|s\|^{-1} - n\, g_{M,0}, & D_2 = \operatorname{div} s - n\,0 & \omega_2 = c_1(\overline{L}) - n\,\mu_{M,0} \end{cases}$$

$$g_2(0) = -\log\|j_0^n s\|_n^{\mathrm{cap}}$$

$$\int g_1 * g_2 = -\log\|j_0^n s\|_n^{\mathrm{cap}} + \int_M g_{M,0}\,\cancel{(c_1(\overline{L}) - n\,\mu_{M,0})}$$

$$\|$$

$$\int g_2 * g_1 = \int_M \big(\underbrace{g_{M,0}\,\delta_{\operatorname{div} s - n0}}_{\geq 0} + \cancel{(-\log\|s\|^{-1} - n\, g_{M,0})\,\mu_{M,0}}\big)$$

Basic estimate on jets of sections

$$\log \| j_0^n s \|_n^{cap} \leq \int_M \log \|s\| \, \mu_{M,0} + \int_M g_{0,M} \, c_1(\overline{L})$$

$$\leq \log \|s\|_{L^\infty(\partial M)} + \int_M g_{0,M} \, c_1(\overline{L})$$

$\underline{\gamma}$ Relation with Nevanlinna Theory:

$$0 \in \mathbb{A}^1(\mathbb{C}) \xrightarrow{\varphi} \mathbb{P}^N(\mathbb{C}) \qquad \mathbb{C}\text{-analytic}$$

$$\overline{\mathcal{O}(1)} \qquad c_1(\overline{\mathcal{O}(1)}) =: \omega_{FS} \quad \text{"Fubini-Study"}$$

$$\begin{cases} \overline{L} := \varphi^* \overline{\mathcal{O}(1)}^{\otimes D}, & D \in \mathbb{Z}_{>0} \\ M := \overline{D}(0,R), & R > 0 \end{cases}$$

$$\rightsquigarrow \int_M g_{M,0} \, c_1(\overline{L}) = D \int_{\mathbb{C}} \log^+ \frac{R}{|z|} \, \varphi^* \omega_{FS}$$

$$=: D \; T_\varphi(R)$$ ⇜ Nevanlinna characteristic function convex in $\log R$

$\| \frac{\partial}{\partial z} \|_{st} := 1$

$$\log \| j_0^n s \|_{st} \leq \log \|s\|_{L^\infty(\partial D(0,R))} - n \log R + D \, T_\varphi(R)$$

4 Arithmetic intersection numbers on arithmetic curves and surfaces

4 a Arakelov geometry in dimension 0 and 1

α $\mathcal{X}$ scheme of finite type over $\mathbb{Z}$

$Z \in Z_0(\mathcal{X})$

$\| $

$\sum_\alpha n_\alpha P_\alpha$ $\quad$ P_α closed point of $\mathcal{X}$; $\kappa(P_\alpha)$ is a finite field

Arakelov degree of Z $\quad$ $\widehat{\deg}\, Z := \sum_\alpha n_\alpha \log|\kappa(P_\alpha)|$

Examples: 1) $q \in \mathbb{Q}^\times$ $\quad$ $\mathcal{X} = \operatorname{Spec}\mathbb{Z}$

$$Z := \operatorname{div} q = \sum_p v_p(q)\,[p]$$

$$\widehat{\deg}\, \operatorname{div} q = \sum_p v_p(q) \log p = \log|q|$$

2) $\mathcal{X} / \mathbb{F}_p$ $\quad$ $\widehat{\deg}\, Z = \deg_{\mathbb{F}_p} Z \,\log p$

β Basic observation $\quad$ Arakelov degree of 0-cycles linearly equivalent to 0

$\mathcal{E} \xrightarrow{\pi} \operatorname{Spec}\mathbb{Z}$ proper

integral of dim. 1

$r \in \kappa(\mathcal{E})^\times$ $\quad$ $\operatorname{div} r \in Z_0(\mathcal{E})$

dichotomy : $\pi : \underset{\llcorner 1 \text{ dim.}}{\mathcal{E}} \to \operatorname{Spec} \mathbb{Z}$ proper

either π finite and flat

$\kappa(\mathcal{E})$ is a number field K ; $\operatorname{Spec} \mathcal{O}_K$ is the normalisation of $\mathcal{E}$

$\rightsquigarrow \mathcal{E}(\mathbb{C}) = \{ \sigma : K \hookrightarrow \mathbb{C} \}$

$q := N_{K/\mathbb{Q}}(r) \in \mathbb{Q}^\times$

$\pi_* \operatorname{div} r = \operatorname{div} q$

$$\widehat{\deg} \operatorname{div} r = \widehat{\deg} \operatorname{div} q = \log |q| = \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log |\sigma(r)|$$

$$= \sum_{x \in \mathcal{X}(\mathbb{C})} \log |r(x)|$$

"product formula"

or $\mathcal{E}$ is a proper integral curve over $\mathbb{F}_p$, p prime

$$\widehat{\deg} \operatorname{div} r = \deg_{\mathbb{F}_p} \operatorname{div} r \cdot \log p = 0$$

I Arakelov degree and heights

$\mathcal{X}$ finite type over $\mathbb{Z}$

$\overline{\mathcal{L}} := (\mathcal{L}, \| \ \|)$ Hermitian line bundle over $\mathcal{X}$

continuous / C^∞ metric on $\mathcal{L}_{\mathbb{C}}$ over $\mathcal{X}(\mathbb{C})$, c.c. invariant

line bundle over $\mathcal{X}$

$\mathcal{C} \hookrightarrow \mathcal{X}$ 1-dimensional, integral, proper over $\mathbb{Z}$

s non-zero rational section of $\mathcal{L}$ over $\mathcal{C}$

Prop. Def. $$\mathrm{ht}_{\overline{\mathcal{L}}}(\mathcal{C}) := \widehat{\deg}\,\mathrm{div}\, s - \sum_{x \in \mathcal{C}(\mathbb{C})} \log \|s(x)\| \in \mathbb{R}$$

does not depend on s; it is the height of $\mathcal{C}$ wrt $\overline{\mathcal{L}}$

Notation: $$\widehat{\deg}(\overline{\mathcal{L}} \mid \mathcal{C}) = \mathrm{ht}_{\overline{\mathcal{L}}}(\mathcal{C})$$

extends by linearity to $\mathcal{C} \in Z_1(\mathcal{X})$ proper over $\mathbb{Z}$

"height pairing" : $\mathrm{Pic}(\mathcal{X}) \times Z_1(\mathcal{X})_{\text{proper}} \longrightarrow \mathbb{R}$

$$(\overline{\mathcal{L}}, Z) \longmapsto \widehat{\deg}(\overline{\mathcal{L}} \mid Z)$$

functoriality: $$\widehat{\deg}(f^*\overline{\mathcal{L}} \mid Z) = \widehat{\deg}(\overline{\mathcal{L}} \mid f_* Z)$$

<u>Example</u> relation with classical heights

$$\mathcal{X} = \mathbb{P}^N_{\mathbb{Z}} \qquad \overline{\mathcal{L}} = \overline{\mathcal{O}(1)} \qquad \overline{\mathcal{O}}_{\mathbb{P}^N}^{\oplus(N+1)} \twoheadrightarrow \overline{\mathcal{O}(1)}$$

$$\begin{array}{ccc} \mathbb{P}^N(\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{P}^N(\mathbb{Q}) \\ \Psi & & \Psi \\ \mathcal{P} & & P = (p_0 : \cdots : p_N) \end{array}$$

$\llcorner \in \mathbb{Z}$, prime together

$$ht_{\overline{\mathcal{O}(1)}}(P) := \widehat{\deg}(\overline{\mathcal{O}(1)} \mid \mathcal{P}) = \widehat{\deg}\, \mathcal{P}^* \overline{\mathcal{O}(1)} = \log\Big(\sum_{i=0}^N p_i^2\Big)^{1/2}$$

§ Geometry of numbers.

may consider $\quad \overline{\mathcal{E}} = (\mathcal{E}, \| \ \|)$ Hermitian vector bundle over $\mathcal{X}$

vector bundle over $\mathcal{X}$ — continuous Hermitian metric on $\mathcal{E}_{\mathbb{C}}$ over $\mathcal{X}(\mathbb{C})$ invariant under c.c.

tensor operations : $\oplus, \otimes, \cdot^\vee, \Lambda^i \ldots$

<u>Exercise</u> :

when $\mathcal{X} = \operatorname{Spec} \mathbb{Z} \qquad \overline{\mathcal{E}} = (\mathcal{E}, \| \ \|)$

Euclidean metric on $E_{\mathbb{R}}$

$\llcorner$ free $\mathbb{Z}$-module of finite rank r — Euclidean lattice

$$\widehat{\deg}\, \overline{\mathcal{E}} := \widehat{\deg}(\Lambda^r \overline{\mathcal{E}} \mid \operatorname{Spec} \mathbb{Z})$$
$$= -\log \operatorname{covol}(\mathcal{E}, \| \ \|) \qquad \text{“Arakelov degree of } \overline{\mathcal{E}}\text{”}$$

$$\hat{\mu}(\overline{\mathcal{E}}) := \frac{1}{r} \widehat{\deg}\, \overline{\mathcal{E}} = -\log \operatorname{covol}(\mathcal{E}, \| \cdot \|)^{1/r} \leftarrow \text{“slope of } \overline{\mathcal{E}}\text{”}$$

4 b Arakelov geometry in dimension 2

main object of study : projective arithmetic surfaces

$$\mathcal{X} \xrightarrow{\pi} \operatorname{Spec} \mathbb{Z}$$

integral

projective + flat

regular (or normal...) of dimension 2 (relative dimension 1)

N.B. . $\mathcal{X}_{\mathbb{Q}}$ is a smooth projective geometrically irreducible curve C over the number field $K = \operatorname{Frac} \underbrace{\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})}_{= \mathcal{O}_K}$.

... and conversely (...)

. $\mathcal{X}(\mathbb{C}) = \coprod_{\sigma : K \hookrightarrow \mathbb{C}} \underbrace{C_\sigma(\mathbb{C})}$ complex conjugation (c.c.)

compact connected Riemann surface

Definitions :

. Arakelov divisor on $\mathcal{X}$:= (D, g)

Green function for $D_{\mathbb{C}}$ on $\mathcal{X}(\mathbb{C})$, c.c. invariant

$\in Z^1(\mathcal{X})$

. $\overline{Z}^1(\mathcal{X}) := \{ \text{Arakelov divisors on } \mathcal{X} \}$

. $f \in \kappa(\mathcal{X})^\times = K(C)^\times \rightsquigarrow \widehat{\operatorname{div}} f := (\operatorname{div} f, \log |f_{\mathbb{C}}|^{-1})$

. $\widehat{CH}^1(\mathcal{X}) := \overline{Z}^1(\mathcal{X}) / \widehat{\operatorname{div}}\, \kappa(\mathcal{X})^\times$ "linear equivalence"

Definitions (cont$^{\underline{d}}$) :

- $\overline{\mathcal{L}} := (\mathcal{L}, \|\ \|)$ on $\mathcal{X}$

  $\qquad\qquad\qquad \uparrow C^\infty$

  $s \in \mathcal{L}_{K(\mathcal{X})} \setminus \{0\} \rightsquigarrow \overline{\mathrm{div}}_{\overline{\mathcal{L}}}(s) := (\mathrm{div}_{\mathcal{L}}\, s, \log \|s_{\mathbb{C}}\|^{-1}) \in \overline{Z}^1(\mathcal{X})$

- $(D, g) \in \overline{Z}^1(\mathcal{X}) \rightsquigarrow \overline{\mathcal{O}}_{\mathcal{X}}(D, g) := (\mathcal{O}_{\mathcal{X}}(D), \|\cdot\|_g)$

  $\qquad\qquad\qquad \|\mathbb{1}_{\mathcal{O}(D)}\| := e^{-g}$

Observation

$$\overline{\mathrm{Pic}}_{C^\infty}(\mathcal{X}) \xrightarrow{\sim} \overline{CH}^1(\mathcal{X})$$

$$[\overline{\mathcal{L}}] \longmapsto [\overline{\mathrm{div}}_{\overline{\mathcal{L}}}\, s] \quad , \quad s \in \mathcal{L}_{K(X)} \setminus \{0\}$$

$$[\overline{\mathcal{O}}_{\mathcal{X}}(D, g)] \longleftarrow\!\shortmid [(D, g)]$$

Def + Thm : Arakelov, Deligne, Gillet – Soulé

1) $\begin{cases} \overline{\mathcal{L}} \text{ Hermitian line bundle over } \mathcal{X} \\ (D, g) \in \overline{Z}^1(\mathcal{X}) \end{cases}$

$$\overline{\mathcal{L}} \cdot (D, g) := \widehat{\deg}(\overline{\mathcal{L}} \mid D) + \int_{\mathcal{X}(\mathbb{C})} g_1 \, c_1(\overline{\mathcal{L}}_{\mathbb{C}})$$

$(D, g), (D', g') \in \overline{Z}^1(\mathcal{X})$

$$(D', g') \cdot (D, g) := \overline{\mathcal{O}}_{\mathcal{X}}(D', g) \cdot (D, g)$$

2) The so-defined Arakelov intersection pairing $\overline{Z}_1(\mathcal{X}) \times \overline{Z}_1(\mathcal{X}) \longrightarrow \mathbb{R}$ is compatible with linear equivalence, and therefore descends to $\overline{CH}^1(\mathcal{X}) \times \overline{CH}^1(\mathcal{X}) \longrightarrow \mathbb{R}$, and is symmetric.

3) $|D_{\mathbb{Q}}| \cap |D'_{\mathbb{Q}}| = \phi \implies (D', g') \cdot (D, g) = \widehat{\deg}\, D' \cdot D + \int_{\mathcal{X}(\mathbb{C})} g' * g$

Examples : "vertical cycles" negativity properties of Arakelov intersection

1) $\underline{\underline{q}} \in \operatorname{Spec} \mathcal{O}_K$; $\begin{array}{ccc} \mathcal{X} & \hookleftarrow & \mathcal{X}_{\underline{\underline{q}}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_K & \hookleftarrow & \underline{\underline{q}} \end{array}$ $D', D \in Z_1(\mathcal{X}_{\underline{\underline{q}}})$

$q := |\mathcal{O}_K / \underline{\underline{q}}|$

$$(D', 0) \cdot (D, 0) = \deg_{\mathbb{F}_q} \underbrace{D \cdot D'}_{\in CH_0(\mathcal{X}_{\underline{\underline{q}}})} \log q$$

2) $f, f' : \mathcal{X}(\mathbb{C}) \longrightarrow \mathbb{R}$, C^∞ + c.c. invariant

$$(0, f) \cdot (0, f') = -\pi^{-1} \langle f, f' \rangle_{\mathrm{Dir}}$$

5 The arithmetic Hilbert-Samuel formula

$$\begin{array}{c}\mathcal{X}\\ \downarrow\\ \operatorname{Spec}\mathbb{Z}\end{array}\quad \text{as above}\quad \overline{\mathcal{L}} \text{ on } \mathcal{X} \qquad \overline{\mathcal{L}} := (\mathcal{L}, \|\ \|)$$

λ C^∞ volume form on $\mathcal{X}(\mathbb{C})$, c.c. invariant

$$\rightsquigarrow \pi_* \overline{\mathcal{L}} := (\pi_* \mathcal{L}, \|\cdot\|_{L^2(\lambda)})$$

Euclidean lattice

where $\pi_* \mathcal{L} := \Gamma(\mathcal{X}, \mathcal{L})$ free $\mathbb{Z}$-module of rank $[K:\mathbb{Q}] \dim_K \Gamma(C, \mathcal{L}_{\mathbb{Q}})$

$$\|s\|^2_{L^2(\lambda)} := \int_{\mathcal{X}(\mathbb{C})} \|s(x)\|^2 \, d\lambda(x) \qquad \text{for any } s \in (\pi_* \mathcal{L})_{\mathbb{C}} = \Gamma(\mathcal{X}(\mathbb{C}), \mathcal{L}_{\mathbb{C}})$$

Theorem (Faltings, Bismut-Vasserot-Gillet-Soulé, S.W. Zhang...)

When $\deg_C \mathcal{L}_{|C} > 0$ (i.e. $\mathcal{L}_{\mathbb{Q}}$ ample on $\mathcal{X}_{\mathbb{Q}}$) and

and $\overline{\mathcal{L}}$ relatively semi-positive

namely $c_1(\overline{\mathcal{L}}_{\mathbb{C}}) \geq 0$ on $\mathcal{X}(\mathbb{C})$

$\forall\, \mathcal{C} \hookrightarrow \mathcal{X}$ vertical integral curve, $\widehat{\deg}(\mathcal{L} \mid \mathcal{C}) \geq 0$

$[\mathcal{C}$ component of $\mathcal{X}_q$ $\deg_{\mathbb{F}_q} \mathcal{L}_{|\mathcal{C}} \geq 0]$,

we have:

$$\text{when } D \to +\infty, \quad \widehat{\deg}\, \pi_* \overline{\mathcal{L}}^{\otimes D} = \frac{1}{2} (\overline{\mathcal{L}} \cdot \overline{\mathcal{L}}) D^2 + o(D^2)$$

N.B. compare to $\operatorname{rk} \pi_* \overline{\mathcal{L}}^{\otimes D} = (\deg_{\mathbb{Q}} \mathcal{L}_{\mathcal{X}_{\mathbb{Q}}}) D + c^{te}$ $D \gg 0$.

reformulation
$$\frac{1}{D} \hat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) = \frac{(\overline{\mathcal{L}} \cdot \overline{\mathcal{L}})}{2 \deg_{\mathbb{Q}} \mathcal{L}_{\mathcal{X}_{\mathbb{Q}}}} \qquad \text{(AHS)}$$

Comments

1) The measure λ does not appear in the RHS of formula.

Actually, replacing λ by $e^{-t}\lambda$ replaces $\hat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D})$ by $\hat{\mu}(\pi_* \overline{\mathcal{L}}^{\otimes D}) + t$

Variant : instead of $\|\cdot\|_{L^2(\lambda)}$, use $\|\cdot\|_{John}$ norm on $\Gamma(\mathcal{X}(\mathbb{C}), \mathcal{L}_{\mathbb{C}})$

"optimal" Hermitian norm s.t. $\| \ \|_{L^\infty} \leq \|\cdot\|_{John}$.

2) All proofs require some non-trivial analysis on complex manifolds

3) The conclusion still makes sense — and holds! — for Hermitian line bundles $\overline{L}$ with possibly not C^∞ metrics.

For instance : continuous metrics $\| \ \|$ on $\mathcal{L}_{\mathbb{C}}$ s.t. $c_1(\mathcal{L}_{\mathbb{C}}, \|\cdot\|)$ is a ≥ 0 measure.

Working with such metrics / Green functions (continuous + $i\partial\bar{\partial}$ measure) allows one to establish the expected functoriality properties of $\widehat{CH}^1(\mathcal{X})$ under direct images by generically finite morphisms.

Homework

To be ~~completed before~~ used in the next lecture.

1. Compute both sides of (AHS) when $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and $\overline{\mathcal{L}} = \overline{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)}$.
   [And check that they coincide :) ...]

2. Consider $\pi: \mathcal{E} \to \operatorname{Spec} \mathbb{Z}$ a regular projective surface as above, and $\mathcal{P} \in \mathcal{E}(\mathbb{Z})$ a section of π. Let moreover $0 \in \mathring{M} \hookrightarrow M$ as in the definition of classical Green functions, and let $\varphi : (M, 0) \longrightarrow (\mathcal{E}(\mathbb{C}), \mathcal{P}_{\mathbb{C}})$ be an analytic map such that $D\varphi(0) : T_0 M \xrightarrow{\sim} T_{\mathcal{P}_{\mathbb{C}}} \mathcal{E}(\mathbb{C}) = (N_{\mathcal{P}} \mathcal{E})_{\mathbb{C}}$, the norm $\|\cdot\|^{cap} := \| D\varphi(0)(\cdot) \|^{cap}_{M,0}$ on $(N_{\mathcal{P}}\mathcal{E})_{\mathbb{C}}$.

   Define :

$$\overline{N_{\mathcal{P}}\mathcal{E}} := (N_{\mathcal{P}}\mathcal{E}, \|\cdot\|^{cap}) \quad \leftarrow \text{Hermitian line bundle over } \mathcal{P} = \operatorname{Spec}\mathbb{Z}$$

$$\overline{\mathcal{L}} := \overline{\mathcal{O}}_{\mathcal{E}}(\mathcal{P}, \underbrace{\varphi_* g_{M,0}}_{(}) \quad \leftarrow \text{Hermitian line bundle over } \mathcal{X}$$

   Establish the identities:

$$\varphi_* g_{M,0}(y) := \sum_{x \in \varphi^{-1}(y)} g_{M,0}(x)$$

$$\overline{\mathcal{L}} \cdot \overline{\mathcal{L}} = h t_{\overline{\mathcal{L}}}(\mathcal{P}) + \int_M g_{M,0}\, \varphi^* c_1(\overline{\mathcal{L}}_{\mathbb{C}})$$

$$= \widehat{\deg}\, \overline{N_{\mathcal{P}}\mathcal{E}} + \operatorname{Ex}(\varphi : (M,0) \to \mathcal{E}(\mathbb{C}))$$

   where $\operatorname{Ex}(\varphi : (M,0) \to \mathcal{E}(\mathbb{C})) := \int_M g_{M,0} \left(\delta_{\varphi^*(\mathcal{P}_{\mathbb{C}}) - 0} + \varphi^* \varphi_* \mu_{M,0} \right)$. "overflow"