A counterexample to the Jacobian conjecture in dimension three

Thomas Mordant

Seed Seminar
IHES, November 4th, 2026

1. Statements of the conjecture and its counterexample

2. Geometric interpretation of the counterexample

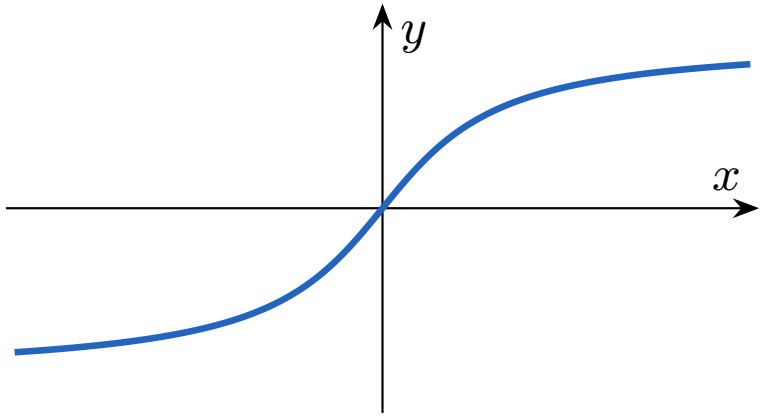
3. Concerning the history and the significance of the conjecture and its counterexample

I. Statements of the conjecture and its counterexample

Derivative and invertibility of functions over $\mathbb{R}$

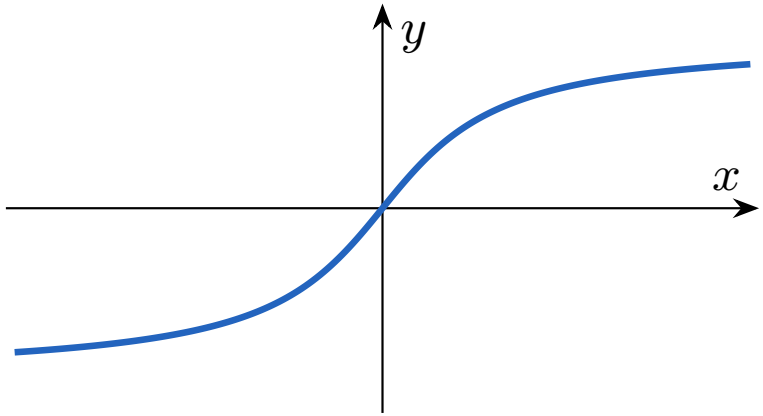
Derivative and invertibility of functions over $\mathbb{R}$

Well-known that a C^∞ function $\mathbb{R} \to \mathbb{R}$ whose derivative does not vanish is injective.

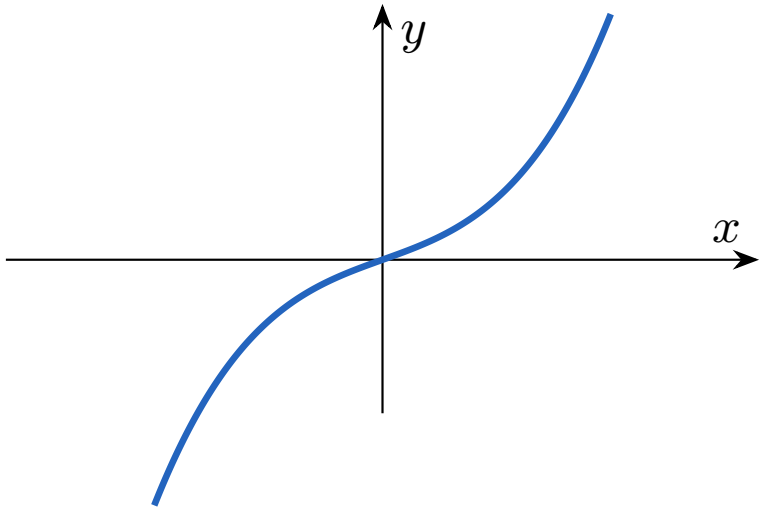


Derivative and invertibility of functions over $\mathbb{R}$

Well-known that a C^∞ function $\mathbb{R} \to \mathbb{R}$ whose derivative does not vanish is injective.



If moreover this function is a polynomial, then it is bijective.



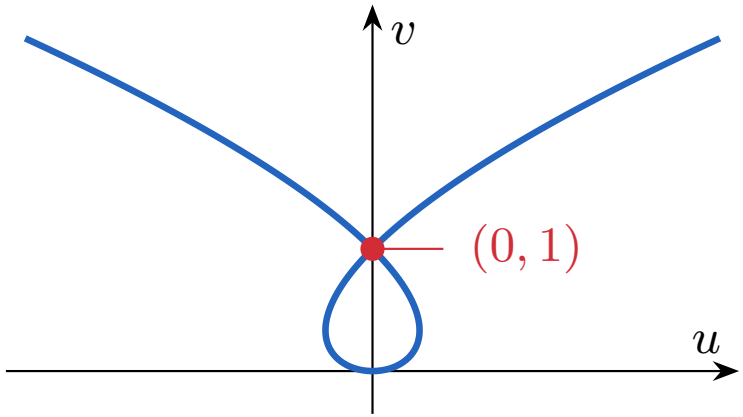
Derivative and invertibility in higher dimensions

Derivative and invertibility in higher dimensions

This does not remain true for functions $\mathbb{R} \to \mathbb{R}^2$.

Derivative and invertibility in higher dimensions

This does not remain true for functions $\mathbb{R} \to \mathbb{R}^2$.



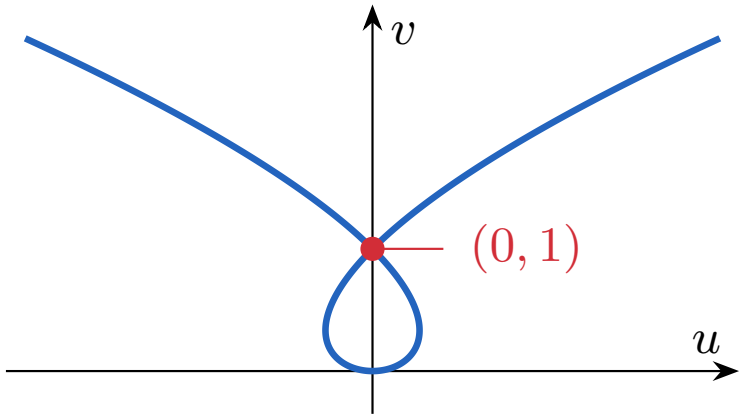
The polynomial

$$f : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto (f_1(x) = x^3 - x, f_2(x) = x^2)$$

has an everywhere non-vanishing derivative but is not injective.

Derivative and invertibility in higher dimensions

This does not remain true for functions $\mathbb{R} \to \mathbb{R}^2$.



The polynomial

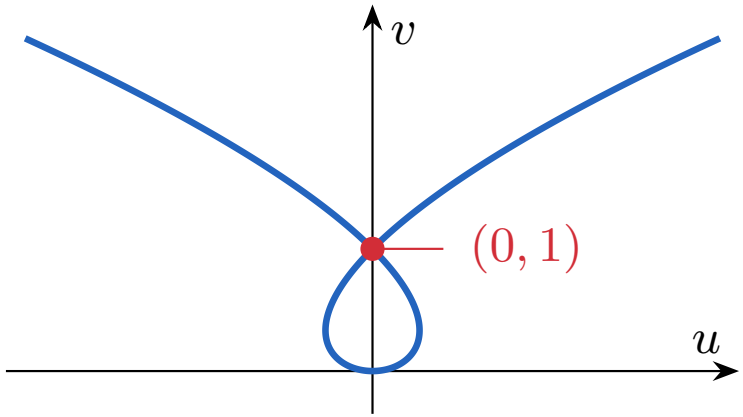
$$f : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto (f_1(x) = x^3 - x, f_2(x) = x^2)$$

has an everywhere non-vanishing derivative but is not injective.

So we must consider maps between spaces of same (higher) dimension.

Derivative and invertibility in higher dimensions

This does not remain true for functions $\mathbb{R} \to \mathbb{R}^2$.



The polynomial

$$f : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto (f_1(x) = x^3 - x, f_2(x) = x^2)$$

has an everywhere non-vanishing derivative but is not injective.

So we must consider maps between spaces of same (higher) dimension.

But what does "non-vanishing derivative" mean in higher dimension ?

Jacobian matrix of a map between vector spaces

Jacobian matrix of a map between vector spaces

If

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, ..., x_n) \mapsto (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$$

is a differentiable map, its Jacobian matrix at a point $(x_1, ..., x_n) \in \mathbb{R}^n$ is the $m \times n$ matrix:

$$\mathrm{Jac}(f)(x_1, \ldots, x_n) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_m}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

Jacobian matrix of a map between vector spaces

If

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, ..., x_n) \mapsto (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$$

is a differentiable map, its Jacobian matrix at a point $(x_1, ..., x_n) \in \mathbb{R}^n$ is the $m \times n$ matrix:

$$\mathrm{Jac}(f)(x_1, \ldots, x_n) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_m}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

- $f(x_1 + \delta x_1, ..., x_n + \delta x_n)$

$$= f(x_1, \ldots, x_n) + \mathrm{Jac}(f)(x_1, \ldots, x_n) \cdot \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix} + O(\|\delta x\|^2);$$

Jacobian matrix of a map between vector spaces

If

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, ..., x_n) \mapsto (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$$

is a differentiable map, its Jacobian matrix at a point $(x_1, ..., x_n) \in \mathbb{R}^n$ is the $m \times n$ matrix:

$$\mathrm{Jac}(f)(x_1, \ldots, x_n) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_m}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

- $f(x_1 + \delta x_1, ..., x_n + \delta x_n)$

  $= f(x_1, \ldots, x_n) + \mathrm{Jac}(f)(x_1, \ldots, x_n) \cdot \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix} + O(\|\delta x\|^2);$

- $\mathrm{Jac}(\mathrm{Id}_{\mathbb{R}^n})(x_1, ..., x_n) = \mathrm{Id}_{\mathbb{R}^n}$;

Jacobian matrix of a map between vector spaces

If

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, ..., x_n) \mapsto (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$$

is a differentiable map, its Jacobian matrix at a point $(x_1, ..., x_n) \in \mathbb{R}^n$ is the $m \times n$ matrix:

$$\mathrm{Jac}(f)(x_1, \ldots, x_n) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial f_m}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

- $f(x_1 + \delta x_1, ..., x_n + \delta x_n)$

  $= f(x_1, \ldots, x_n) + \mathrm{Jac}(f)(x_1, \ldots, x_n) \cdot \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix} + O(\|\delta x\|^2);$
- $\mathrm{Jac}(\mathrm{Id}_{\mathbb{R}^n})(x_1, ..., x_n) = \mathrm{Id}_{\mathbb{R}^n}$;
- $\mathrm{Jac}(g \circ f)(x_1, ..., x_n) = \mathrm{Jac}(g)(f(x_1, ..., x_n)) \cdot \mathrm{Jac}(f)(x_1, ..., x_n)$.

Jacobian determinant

Jacobian determinant

Idea: if $m = n$, a matrix is invertible iff its determinant is non-zero.

Jacobian determinant

Idea: if $m = n$, a matrix is invertible iff its determinant is non-zero.

If $m = n$, the Jacobian determinant of f at $(x_1, ..., x_n)$ is $\det \mathrm{Jac}(f)(x_1, ..., x_n)$.

Jacobian determinant

Idea: if $m = n$, a matrix is invertible iff its determinant is non-zero.

If $m = n$, the Jacobian determinant of f at $(x_1, ..., x_n)$ is $\det \mathrm{Jac}(f)(x_1, ..., x_n)$.

Inverse function theorem: if the Jacobian determinant of f is non-zero at $(x_1, ..., x_n)$, then f is locally a diffeomorphism near $(x_1, ..., x_n)$.

Jacobian determinant

Idea: if $m = n$, a matrix is invertible iff its determinant is non-zero.

If $m = n$, the Jacobian determinant of f at $(x_1, ..., x_n)$ is $\det \mathrm{Jac}(f)(x_1, ..., x_n)$.

Inverse function theorem: if the Jacobian determinant of f is non-zero at $(x_1, ..., x_n)$, then f is locally a diffeomorphism near $(x_1, ..., x_n)$.

Question: local diffeomorphism everywhere $\Rightarrow$ global diffeomorphism, for polynomials ?

Jacobian of polynomials over the complex numbers

Jacobian of polynomials over the complex numbers

Complex setting:

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$.

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$. (If $n = 1$, fundamental theorem of algebra/d'Alembert–Gauss theorem. In general...)

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$. (If $n = 1$, fundamental theorem of algebra/d'Alembert–Gauss theorem. In general...)

Consequence: when $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, $\det \mathrm{Jac}(f)$ is a polynomial, and therefore $\det \mathrm{Jac}(f)$ does not vanish $\Leftrightarrow$ $\det \mathrm{Jac}(f)$ is a non-zero constant.

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$. (If $n = 1$, fundamental theorem of algebra/d'Alembert–Gauss theorem. In general...)

Consequence: when $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, $\det \mathrm{Jac}(f)$ is a polynomial, and therefore $\det \mathrm{Jac}(f)$ does not vanish $\Leftrightarrow$ $\det \mathrm{Jac}(f)$ is a non-zero constant.

Jacobian conjecture: if $\det \mathrm{Jac}(f)$ does not vanish, then f admits a polynomial global inverse.

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$. (If $n = 1$, fundamental theorem of algebra/d'Alembert–Gauss theorem. In general...)

Consequence: when $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, $\det \mathrm{Jac}(f)$ is a polynomial, and therefore $\det \mathrm{Jac}(f)$ does not vanish $\Leftrightarrow$ $\det \mathrm{Jac}(f)$ is a non-zero constant.

Jacobian conjecture: if $\det \mathrm{Jac}(f)$ does not vanish, then f admits a polynomial global inverse.

True in dimension 1: the only polynomials whose derivative is a non-zero constant are the non-constant affine functions (...)

Jacobian of polynomials over the complex numbers

Complex setting:

- The inverse function theorem remains true when $\mathbb{R}$ is replaced by $\mathbb{C}$, and "differentiable" by "holomorphic."
- Every non-constant polynomial in $\mathbb{C}[X_1, ..., X_n]$ has a zero in $\mathbb{C}^n$. (If $n = 1$, fundamental theorem of algebra/d'Alembert–Gauss theorem. In general...)

Consequence: when $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, $\det \mathrm{Jac}(f)$ is a polynomial, and therefore $\det \mathrm{Jac}(f)$ does not vanish $\Leftrightarrow$ $\det \mathrm{Jac}(f)$ is a non-zero constant.

Jacobian conjecture: if $\det \mathrm{Jac}(f)$ does not vanish, then f admits a polynomial global inverse.

True in dimension 1: the only polynomials whose derivative is a non-zero constant are the non-constant affine functions (...)

S. S.-S. Wang 1980: true when the components of f are polynomials of degree ≤ 2.

Claude and Alpöge's counterexample: "raw" expression

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Consider

$$f : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Consider

$$f : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

where

$$f_1(x, y, z) := (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy),$$
$$f_2(x, y, z) := y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy),$$

and

$$f_3(x, y, z) := 2x - 3x^2 y - x^3 z.$$

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Consider

$$f : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

where

$$f_1(x, y, z) := (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy),$$
$$f_2(x, y, z) := y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy),$$

and

$$f_3(x, y, z) := 2x - 3x^2 y - x^3 z.$$

Then

$$\det \mathrm{Jac}(f)(x, y, z) = -2 \neq 0$$

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Consider

$$f : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

where

$$f_1(x, y, z) := (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy),$$
$$f_2(x, y, z) := y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy),$$

and

$$f_3(x, y, z) := 2x - 3x^2 y - x^3 z.$$

Then

$$\det \mathrm{Jac}(f)(x, y, z) = -2 \neq 0$$

but

$$f(0, 0, -1/4) = f(1, -3/2, 13/2) = f(-1, 3/2, 13/2) = (-1/4, 0, 0).$$

Claude and Alpöge's counterexample: "raw" expression

"Tweet" from Levent Alpöge, July 20th, 2026

Consider

$$f : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

where

$$f_1(x, y, z) := (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy),$$
$$f_2(x, y, z) := y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy),$$

and

$$f_3(x, y, z) := 2x - 3x^2 y - x^3 z.$$

Then

$$\det \mathrm{Jac}(f)(x, y, z) = -2 \neq 0$$

but

$$f(0, 0, -1/4) = f(1, -3/2, 13/2) = f(-1, 3/2, 13/2) = (-1/4, 0, 0).$$

Therefore the conjecture is false in dimension 3 over $\mathbb{C}$.

Consequence

Consequence

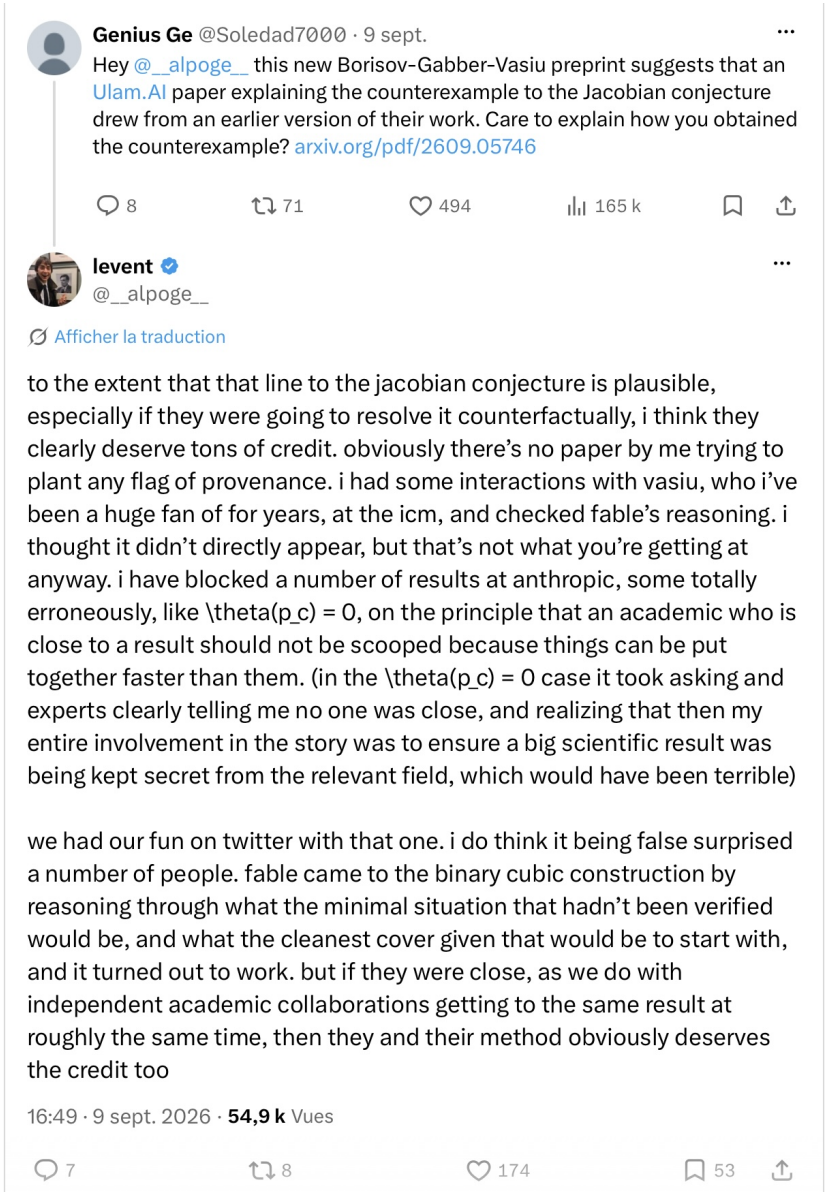
The conjecture is also false in dimension ≥ 4:

Consequence

The conjecture is also false in dimension ≥ 4:

$$f(x_1, ..., x_n) := (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3), x_4, ..., x_n).$$

Another “tweet” from Levent Alpöge, September 9th, 2026



Genius Ge @Soledad7000 · 9 sept.

Hey @__alpoge__ this new Borisov-Gabber-Vasiu preprint suggests that an Ulam.AI paper explaining the counterexample to the Jacobian conjecture drew from an earlier version of their work. Care to explain how you obtained the counterexample? arxiv.org/pdf/2609.05746

8 71 494 165 k

levent @__alpoge__

Afficher la traduction

to the extent that that line to the jacobian conjecture is plausible, especially if they were going to resolve it counterfactually, i think they clearly deserve tons of credit. obviously there’s no paper by me trying to plant any flag of provenance. i had some interactions with vasiu, who i’ve been a huge fan of for years, at the icm, and checked fable’s reasoning. i thought it didn’t directly appear, but that’s not what you’re getting at anyway. i have blocked a number of results at anthropic, some totally erroneously, like \theta(p_c) = 0, on the principle that an academic who is close to a result should not be scooped because things can be put together faster than them. (in the \theta(p_c) = 0 case it took asking and experts clearly telling me no one was close, and realizing that then my entire involvement in the story was to ensure a big scientific result was being kept secret from the relevant field, which would have been terrible)

we had our fun on twitter with that one. i do think it being false surprised a number of people. fable came to the binary cubic construction by reasoning through what the minimal situation that hadn’t been verified would be, and what the cleanest cover given that would be to start with, and it turned out to work. but if they were close, as we do with independent academic collaborations getting to the same result at roughly the same time, then they and their method obviously deserves the credit too

16:49 · 9 sept. 2026 · **54,9 k** Vues

7 8 174 53

II. Geometric interpretation of the counterexample

II. Geometric interpretation of the counterexample

- "Tweet" from Andy Jiang, July 20th, 2026

II. Geometric interpretation of the counterexample

- "Tweet" from Andy Jiang, July 20th, 2026
- Terence Tao, *A digestion of the Jacobian conjecture counterexample*, blog post, July 21st, 2026

II. Geometric interpretation of the counterexample

- "Tweet" from Andy Jiang, July 20th, 2026
- Terence Tao, *A digestion of the Jacobian conjecture counterexample*, blog post, July 21st, 2026
- Remy van Dobben de Bruyn, *Divisors in projective bundles over the projective line whose complement is affine space*, ArXiv: 2608.27341v2

II. Geometric interpretation of the counterexample

- "Tweet" from Andy Jiang, July 20th, 2026
- Terence Tao, *A digestion of the Jacobian conjecture counterexample*, blog post, July 21st, 2026
- Remy van Dobben de Bruyn, *Divisors in projective bundles over the projective line whose complement is affine space*, ArXiv: 2608.27341v2
- Many more...

The counterexample in a nutshell

The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:



The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:



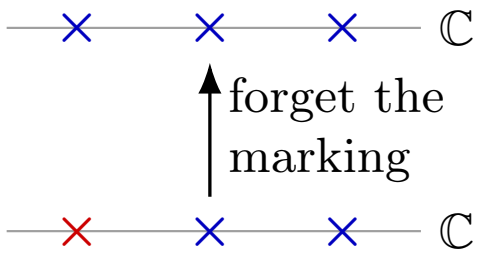
Space of 'unordered' triplets of points with one marked point:



The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:

Space of 'unordered' triplets of points with one marked point:

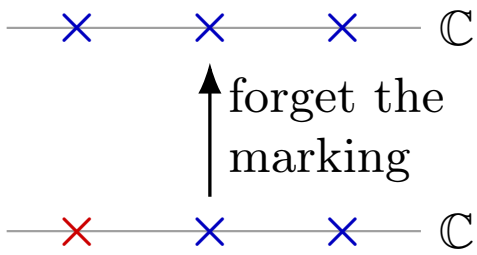


The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:

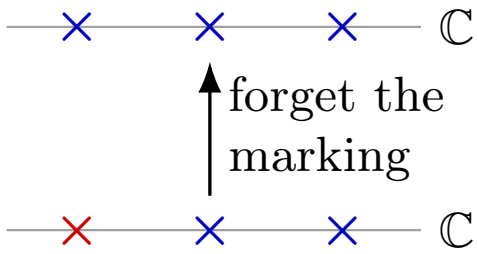
Space of 'unordered' triplets of points with one marked point:

Three-card monte/bonneteau:



The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:



Space of 'unordered' triplets of points with one marked point:

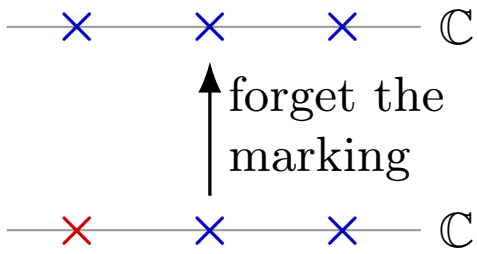
Three-card monte/bonneteau:

- when the points are all distinct, then nothing changes when they move a little (aka local diffeomorphism!).

The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:

Space of 'unordered' triplets of points with one marked point:



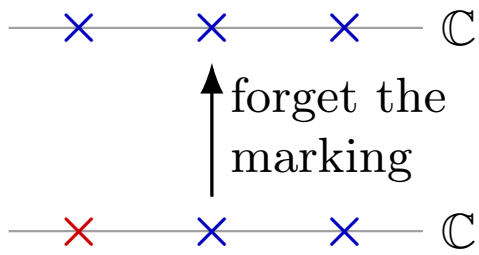
Three-card monte/bonneteau:

- when the points are all distinct, then nothing changes when they move a little (aka local diffeomorphism!).
- but no global invertibility: a general triplet has three marked triplets as preimages.

The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:

Space of 'unordered' triplets of points with one marked point:



Three-card monte/bonneteau:

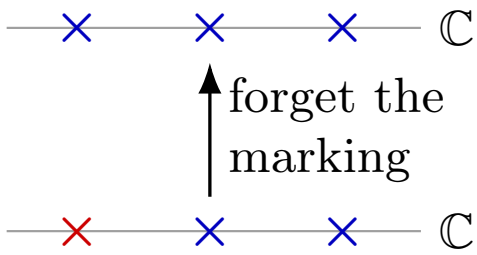
- when the points are all distinct, then nothing changes when they move a little (aka local diffeomorphism!).
- but no global invertibility: a general triplet has three marked triplets as preimages.



The counterexample in a nutshell

Space of unordered triplets of points in the complex line $\mathbb{C}^1$:

Space of 'unordered' triplets of points with one marked point:



Three-card monte/bonneteau:

- when the points are all distinct, then nothing changes when they move a little (aka local diffeomorphism!).
- but no global invertibility: a general triplet has three marked triplets as preimages.



Problem: Why does this indeed give a map on $\mathbb{C}^3$?

Projective spaces

Projective spaces

Let $n \geq 1$ be an integer.

Projective spaces

Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$.

Projective spaces

Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. In other terms, $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \quad \text{iff} \quad \exists \lambda \in \mathbb{C}^*, \ (x_0, ..., x_n) = (\lambda x'_0, ..., \lambda x'_n).$$

Projective spaces

Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. In other terms, $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \quad \text{iff} \quad \exists \lambda \in \mathbb{C}^*, \ (x_0, ..., x_n) = (\lambda x'_0, ..., \lambda x'_n).$$

For $(x_0, ..., x_n) \in \mathbb{C}^{n+1} \setminus 0$, we denote $[x_0, ..., x_n] := \mathbb{C} \cdot (x_0, ..., x_n) \in \mathbb{P}^n(\mathbb{C})$.

Projective spaces

Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. In other terms, $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \quad \text{iff} \quad \exists \lambda \in \mathbb{C}^*, \ (x_0, ..., x_n) = (\lambda x'_0, ..., \lambda x'_n).$$

For $(x_0, ..., x_n) \in \mathbb{C}^{n+1} \setminus 0$, we denote $[x_0, ..., x_n] := \mathbb{C} \cdot (x_0, ..., x_n) \in \mathbb{P}^n(\mathbb{C})$.

Examples:

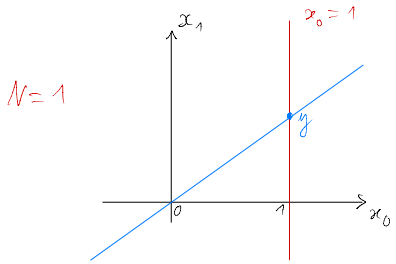
Projective spaces

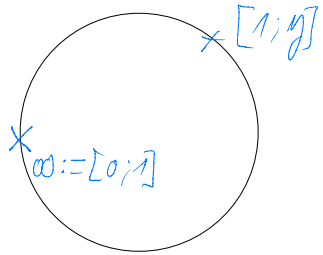
Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. In other terms, $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \quad \text{iff} \quad \exists \lambda \in \mathbb{C}^*, \ (x_0, ..., x_n) = (\lambda x'_0, ..., \lambda x'_n).$$

For $(x_0, ..., x_n) \in \mathbb{C}^{n+1} \setminus 0$, we denote $[x_0, ..., x_n] := \mathbb{C} \cdot (x_0, ..., x_n) \in \mathbb{P}^n(\mathbb{C})$.

Examples:





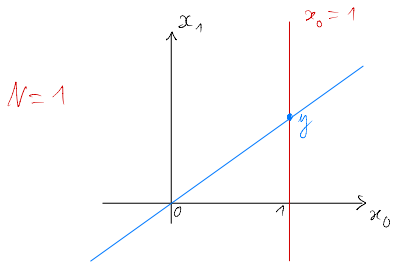
Projective spaces

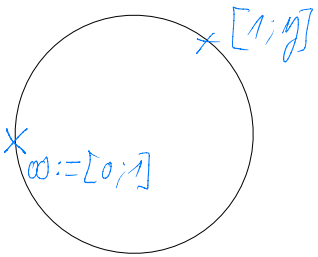
Let $n \geq 1$ be an integer. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. In other terms, $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus 0)/\sim$, where

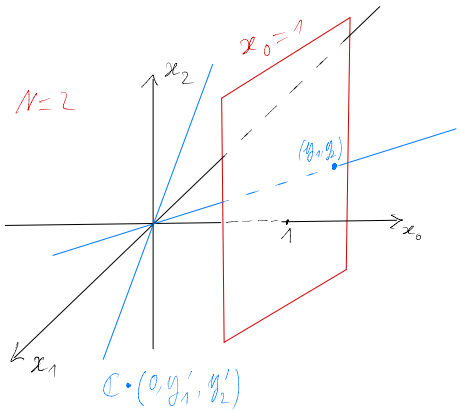
$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \quad \text{iff} \quad \exists \lambda \in \mathbb{C}^*,\ (x_0, ..., x_n) = (\lambda x'_0, ..., \lambda x'_n).$$

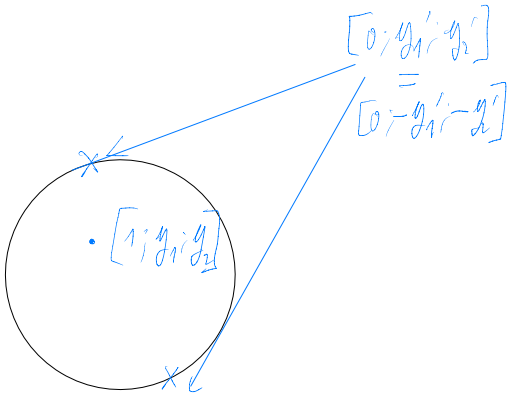
For $(x_0, ..., x_n) \in \mathbb{C}^{n+1} \setminus 0$, we denote $[x_0, ..., x_n] := \mathbb{C} \cdot (x_0, ..., x_n) \in \mathbb{P}^n(\mathbb{C})$.

Examples:









Symmetric powers

Symmetric powers

For every set M and $d \geq 1$, the symmetric power $\mathrm{Sym}^d M$ is the quotient $M^d/\sim$ where:

$$\forall s \in \mathfrak{S}_d, \forall (p_1, ..., p_d) \in M^d, \quad (p_{s(1)}, ..., p_{s(d)}) \sim (p_1, ..., p_d).$$

Symmetric powers

For every set M and $d \geq 1$, the symmetric power $\mathrm{Sym}^d M$ is the quotient $M^d/\sim$ where:

$$\forall s \in \mathfrak{S}_d, \forall (p_1, ..., p_d) \in M^d, \quad (p_{s(1)}, ..., p_{s(d)}) \sim (p_1, ..., p_d).$$

We denote by $\{p_1, ..., p_d\}$ the class of $(p_1, ..., p_d)$.

Symmetric powers

For every set M and $d \geq 1$, the symmetric power $\mathrm{Sym}^d M$ is the quotient $M^d/\sim$ where:

$$\forall s \in \mathfrak{S}_d, \forall (p_1, ..., p_d) \in M^d, \quad (p_{s(1)}, ..., p_{s(d)}) \sim (p_1, ..., p_d).$$

We denote by $\{p_1, ..., p_d\}$ the class of $(p_1, ..., p_d)$.

When M is a complex manifold (resp. complex algebraic variety), then $\mathrm{Sym}^d M$ is a complex analytic space (resp. an algebraic space), possibly singular even when M is smooth.

Symmetric powers of the complex line

Symmetric powers of the complex line

Definition: if $1 \leq j \leq d$, the j-th elementary symmetric function $\sigma_j \in \mathbb{C}[Y_1, ..., Y_d]$ is given by

$$\sigma_j(Y_1, ..., Y_d) := \sum_{\substack{I \subset \{1,...,d\} \\ |I|=j}} \prod_{i \in I} Y_i.$$

Symmetric powers of the complex line

Definition: if $1 \leq j \leq d$, the j-th elementary symmetric function $\sigma_j \in \mathbb{C}[Y_1, ..., Y_d]$ is given by

$$\sigma_j(Y_1, ..., Y_d) := \sum_{\substack{I \subset \{1,...,d\} \\ |I|=j}} \prod_{i \in I} Y_i.$$

They satisfy the equality:

$$\prod_{i=1}^{d} (T - Y_i) = T^d + \sum_{j=1}^{d} (-1)^j \sigma_j(Y_1, ..., Y_d) T^{d-j}.$$

Symmetric powers of the complex line

Definition: if $1 \leq j \leq d$, the j-th elementary symmetric function $\sigma_j \in \mathbb{C}[Y_1, ..., Y_d]$ is given by

$$\sigma_j(Y_1, ..., Y_d) := \sum_{\substack{I \subset \{1,...,d\} \\ |I|=j}} \prod_{i \in I} Y_i.$$

They satisfy the equality:

$$\prod_{i=1}^{d}(T - Y_i) = T^d + \sum_{j=1}^{d}(-1)^j \sigma_j(Y_1, ..., Y_d) T^{d-j}.$$

Theorem: $\mathrm{Sym}^d\mathbb{C}^1$ is a smooth complex manifold isomorphic to $\mathbb{C}^d$ by:

$$\begin{array}{ccc} \{y_1, ..., y_d\} & \mapsto & (\sigma_1(y_1, ..., y_d), ..., \sigma_d(y_1, ..., y_d)) \\ \{\text{roots of } T^d + \sum_{j=1}^{d}(-1)^j u_j T^{d-j} = 0\} & \leftarrow\!\shortmid & (u_1, ..., u_d). \end{array}$$

Symmetric powers of projective lines

Symmetric powers of projective lines

Theorem: $\mathrm{Sym}^d\mathbb{P}^1(\mathbb{C})$ is a smooth complex manifold isomorphic to $\mathbb{P}^d(\mathbb{C})$.

Symmetric powers of projective lines

Theorem: $\mathrm{Sym}^d\mathbb{P}^1(\mathbb{C})$ is a smooth complex manifold isomorphic to $\mathbb{P}^d(\mathbb{C})$.

Proof: We define for $[x_1 : y_1], ..., [x_d : y_d]$ in $\mathbb{P}^1(\mathbb{C})$ and $0 \leq j \leq d$:

$$\sigma_j([x_1 : y_1], ..., [x_d : y_d]) := \sum_{\substack{I \subset \{1,\dots,d\} \\ |I|=j}} \prod_{i \notin I} x_i \prod_{i \in I} y_i.$$

Symmetric powers of projective lines

Theorem: $\mathrm{Sym}^d\mathbb{P}^1(\mathbb{C})$ is a smooth complex manifold isomorphic to $\mathbb{P}^d(\mathbb{C})$.

Proof: We define for $[x_1 : y_1], ..., [x_d : y_d]$ in $\mathbb{P}^1(\mathbb{C})$ and $0 \leq j \leq d$:

$$\sigma_j([x_1 : y_1], ..., [x_d : y_d]) := \sum_{\substack{I \subset \{1,...,d\} \\ |I|=j}} \prod_{i \notin I} x_i \prod_{i \in I} y_i.$$

The isomorphism is:

$$\begin{array}{ccc} \{[x_1 : y_1], ..., [x_d : y_d]\} & \mapsto & \left[\sigma_0(...) : ... : \sigma_d(...)\right] \\ \{\text{roots of } \sum_{j=0}^{d}(-1)^j u_j X^j Y^{d-j} = 0\} & \leftarrow\!\shortmid & [u_0 : ... : u_d]. \end{array}$$

Definition of the desired map on projective spaces

Definition of the desired map on projective spaces

We may define the map

$$\pi : \begin{cases} \mathbb{P}^1(\mathbb{C}) \times \mathrm{Sym}^2\mathbb{P}^1(\mathbb{C}) \to \mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \\ \big([x_1 : y_1], \{[x_2 : y_2], [x_3 : y_3]\}\big) \mapsto \{[x_1 : y_1], [x_2 : y_2], [x_3 : y_3]\}. \end{cases}$$

Definition of the desired map on projective spaces

We may define the map

$$\pi : \begin{cases} \mathbb{P}^1(\mathbb{C}) \times \mathrm{Sym}^2\mathbb{P}^1(\mathbb{C}) \to \mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \\ \left([x_1 : y_1], \{[x_2 : y_2], [x_3 : y_3]\}\right) \mapsto \{[x_1 : y_1], [x_2 : y_2], [x_3 : y_3]\}. \end{cases}$$

Modulo the isomorphisms $\mathrm{Sym}^2\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^2(\mathbb{C})$ and $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$, it is given by:

Definition of the desired map on projective spaces

We may define the map

$$\pi : \begin{cases} \mathbb{P}^1(\mathbb{C}) \times \mathrm{Sym}^2\mathbb{P}^1(\mathbb{C}) \to \mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \\ \left([x_1 : y_1], \{[x_2 : y_2], [x_3 : y_3]\}\right) \mapsto \{[x_1 : y_1], [x_2 : y_2], [x_3 : y_3]\}. \end{cases}$$

Modulo the isomorphisms $\mathrm{Sym}^2\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^2(\mathbb{C})$ and $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$, it is given by:

$$([x : y], [u_0 : u_1 : u_2]) \mapsto [u_0x : u_0y + u_1x : u_1y + u_2x : u_2y].$$

Ramification divisor of π

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

$$R := \left\{ ([x:y],[u_0:u_1:u_2]) \mid u_0y^2 - u_1xy + u_2x^2 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).$$

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

$$R := \left\{ ([x:y],[u_0:u_1:u_2]) \mid u_0y^2 - u_1xy + u_2x^2 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).$$

R is called the ramification divisor of π.

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

$$R := \left\{ ([x : y], [u_0 : u_1 : u_2]) \;\middle|\; u_0 y^2 - u_1 xy + u_2 x^2 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).$$

R is called the ramification divisor of π. It may be identified with:

$$\left\{ ([x_1 : y_1], \{[x_2 : y_2], [x_3 : y_3]\}) \;\middle|\; [x_1 : y_1] = [x_2 : y_2] \text{ or } [x_1 : y_1] = [x_3 : y_3] \right\}.$$

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

$$R := \left\{ ([x:y],[u_0:u_1:u_2]) \;\middle|\; u_0y^2 - u_1xy + u_2x^2 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).$$

R is called the ramification divisor of π. It may be identified with:

$$\left\{ ([x_1:y_1],\{[x_2:y_2],[x_3:y_3]\}) \;\middle|\; [x_1:y_1] = [x_2:y_2] \text{ or } [x_1:y_1] = [x_3:y_3] \right\}.$$

In order to find a local diffeomorphism, we therefore need to remove R from $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$.

Ramification divisor of π

Theorem (computation): The morphism π is a local diffeomorphism precisely outside of the set:

$$R := \left\{ ([x:y],[u_0:u_1:u_2]) \;\middle|\; u_0y^2 - u_1xy + u_2x^2 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).$$

R is called the ramification divisor of π. It may be identified with:

$$\left\{ ([x_1:y_1],\{[x_2:y_2],[x_3:y_3]\}) \;\middle|\; [x_1:y_1] = [x_2:y_2] \text{ or } [x_1:y_1] = [x_3:y_3] \right\}.$$

In order to find a local diffeomorphism, we therefore need to remove R from $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$.

But this is not enough to find $\mathbb{C}^3$.

Tangent hyperplanes

Tangent hyperplanes

- $n \geq 1$, and $H := \{\varphi = 0\} \subset \mathbb{C}^n$, where φ is an affine linear form;

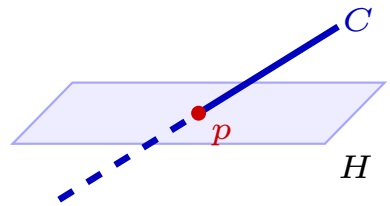
Tangent hyperplanes

- $n \geq 1$, and $H := \{\varphi = 0\} \subset \mathbb{C}^n$, where φ is an affine linear form;
- $0 \in U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}^n$ analytic, $C := f(U)$ parametrized curve, $p := f(0) \in C$.

Tangent hyperplanes

- $n \geq 1$, and $H := \{\varphi = 0\} \subset \mathbb{C}^n$, where φ is an affine linear form;
- $0 \in U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}^n$ analytic, $C := f(U)$ parametrized curve, $p := f(0) \in C$.

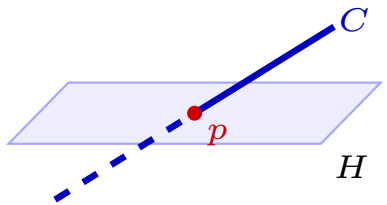
1. The hyperplane H meets the curve C at p iff $(\varphi \circ f)(0) = 0$.



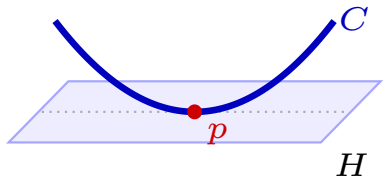
Tangent hyperplanes

- $n \geq 1$, and $H := \{\varphi = 0\} \subset \mathbb{C}^n$, where φ is an affine linear form;
- $0 \in U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}^n$ analytic, $C := f(U)$ parametrized curve, $p := f(0) \in C$.

1. The hyperplane H meets the curve C at p iff $(\varphi \circ f)(0) = 0$.



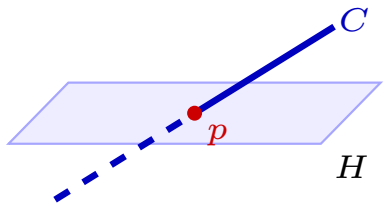
2. The hyperplane H is tangent to C at p if moreover $\frac{d}{dt}(\varphi \circ f)(0)$ vanishes at 0.



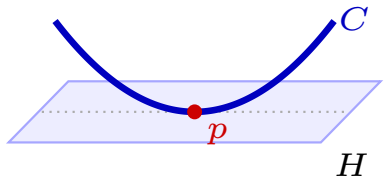
Tangent hyperplanes

- $n \geq 1$, and $H := \{\varphi = 0\} \subset \mathbb{C}^n$, where φ is an affine linear form;
- $0 \in U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}^n$ analytic, $C := f(U)$ parametrized curve, $p := f(0) \in C$.

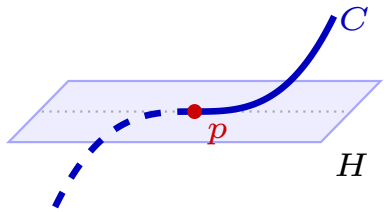
1. The hyperplane H meets the curve C at p iff $(\varphi \circ f)(0) = 0$.



2. The hyperplane H is tangent to C at p if moreover $\frac{d}{dt}(\varphi \circ f)(0)$ vanishes at 0.



3. The hyperplane H is osculating to C at p if moreover $(\frac{d}{dt})^2(\varphi \circ f)(0)$ vanishes at 0.



Small diagonal in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C})$

Small diagonal in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C})$

Definition: the small diagonal D of $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C})$ is the image of the embedding:

$$\mathbb{P}^1(\mathbb{C}) \to \mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}), \quad p \mapsto \{p, p, p\}.$$

Small diagonal in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C})$

Definition: the small diagonal D of $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C})$ is the image of the embedding:

$$\mathbb{P}^1(\mathbb{C}) \to \mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}), \quad p \mapsto \{p, p, p\}.$$

Modulo the isomorphism $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$, this embedding may be identified with the embedding of the twisted cubic:

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C}), \quad [x : y] \mapsto [x^3 : 3x^2y : 3xy^2 : y^3].$$

Tangent non-osculating hyperplanes to the small diagonal

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Theorem: H is tangent to the small diagonal D at at most one point.

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Theorem: H is tangent to the small diagonal D at at most one point.
Moreover the following are equivalent:

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Theorem: H is tangent to the small diagonal D at at most one point.
Moreover the following are equivalent:

1. H is tangent but not osculating to the small diagonal D,

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Theorem: H is tangent to the small diagonal D at at most one point. Moreover the following are equivalent:

1. H is tangent but not osculating to the small diagonal D,
2. the open subset $U := \left(\mathbb{P}^1(\mathbb{C}) \times \mathrm{Sym}^2\mathbb{P}^1(\mathbb{C})\right) \setminus \left(R \cup \pi^{-1}(H)\right)$ is diffeomorphic to $\mathbb{C}^3$, and the diffeomorphism is algebraic.

Tangent non-osculating hyperplanes to the small diagonal

Let H be a projective hyperplane in $\mathrm{Sym}^3\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^3(\mathbb{C})$.

Easy to check: the complement $\mathbb{P}^3(\mathbb{C}) \setminus H$ may be identified with $\mathbb{C}^3$.

Theorem: H is tangent to the small diagonal D at at most one point. Moreover the following are equivalent:

1. H is tangent but not osculating to the small diagonal D,
2. the open subset $U := \left(\mathbb{P}^1(\mathbb{C}) \times \mathrm{Sym}^2\mathbb{P}^1(\mathbb{C})\right) \setminus \left(R \cup \pi^{-1}(H)\right)$ is diffeomorphic to $\mathbb{C}^3$, and the diffeomorphism is algebraic.

For example, 1. holds for the projective hyperplane

$$H = \{[u_0 : u_1 : u_2 : u_3] \in \mathbb{P}^3(\mathbb{C}) \mid u_2 = 0\}$$

at the point $p = [1 : 0 : 0 : 0]$. (exercise)

End of the construction

End of the construction

When 1. and 2. hold, the restriction:

$$\pi_{|U} : \mathbb{C}^3 \simeq U \longrightarrow \mathbb{P}^3(\mathbb{C}) \setminus H \simeq \mathbb{C}^3$$

is algebraic,

End of the construction

When 1. and 2. hold, the restriction:

$$\pi_{|U} : \mathbb{C}^3 \simeq U \longrightarrow \mathbb{P}^3(\mathbb{C}) \setminus H \simeq \mathbb{C}^3$$

is algebraic, a local diffeomorphism,

End of the construction

When 1. and 2. hold, the restriction:

$$\pi_{|U} : \mathbb{C}^3 \simeq U \longrightarrow \mathbb{P}^3(\mathbb{C}) \setminus H \simeq \mathbb{C}^3$$

is algebraic, a local diffeomorphism, but not injective.

End of the construction

When 1. and 2. hold, the restriction:

$$\pi_{|U} : \mathbb{C}^3 \simeq U \longrightarrow \mathbb{P}^3(\mathbb{C}) \setminus H \simeq \mathbb{C}^3$$

is algebraic, a local diffeomorphism, but not injective.

This gives precisely Claude and Alpöge's counterexample! (after a linear change of variables)

III. Concerning the history and the significance of the conjecture and its counterexample

Timeline of the conjecture

Timeline of the conjecture

Kraus 1884: first appearance in dimension two, with faulty proof

Timeline of the conjecture

Kraus 1884: first appearance in dimension two, with faulty proof

Keller 1939: first appearance in arbitrary dimension (for integer coefficients)

Timeline of the conjecture

Kraus 1884: first appearance in dimension two, with faulty proof

Keller 1939: first appearance in arbitrary dimension (for integer coefficients)

Miyanishi 1973: earliest-known use of the name "Jacobian conjecture" in the literature

Timeline of the conjecture

Kraus 1884: first appearance in dimension two, with faulty proof

Keller 1939: first appearance in arbitrary dimension (for integer coefficients)

Miyanishi 1973: earliest-known use of the name "Jacobian conjecture" in the literature

Bass, Connell, Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. A.M.S., Vol. 7, 287-330, 1982: popularized the conjecture

Timeline of the conjecture

Kraus 1884: first appearance in dimension two, with faulty proof

Keller 1939: first appearance in arbitrary dimension (for integer coefficients)

Miyanishi 1973: earliest-known use of the name "Jacobian conjecture" in the literature

Bass, Connell, Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. A.M.S., Vol. 7, 287-330, 1982: popularized the conjecture

1998: Appears as the 16th in Smale's list of 18 *Mathematical problems for the next century*

Outstanding number of purported positive solutions, very few expected consequences

Outstanding number of purported positive solutions, very few expected consequences

- A non-exhaustive list of faulty proofs:

  Kraus 1884 (dimension 2), Engel 1955 (dimension 2), Segre 1956 (twice), Segre 1960, Oda 1980 (preprint)

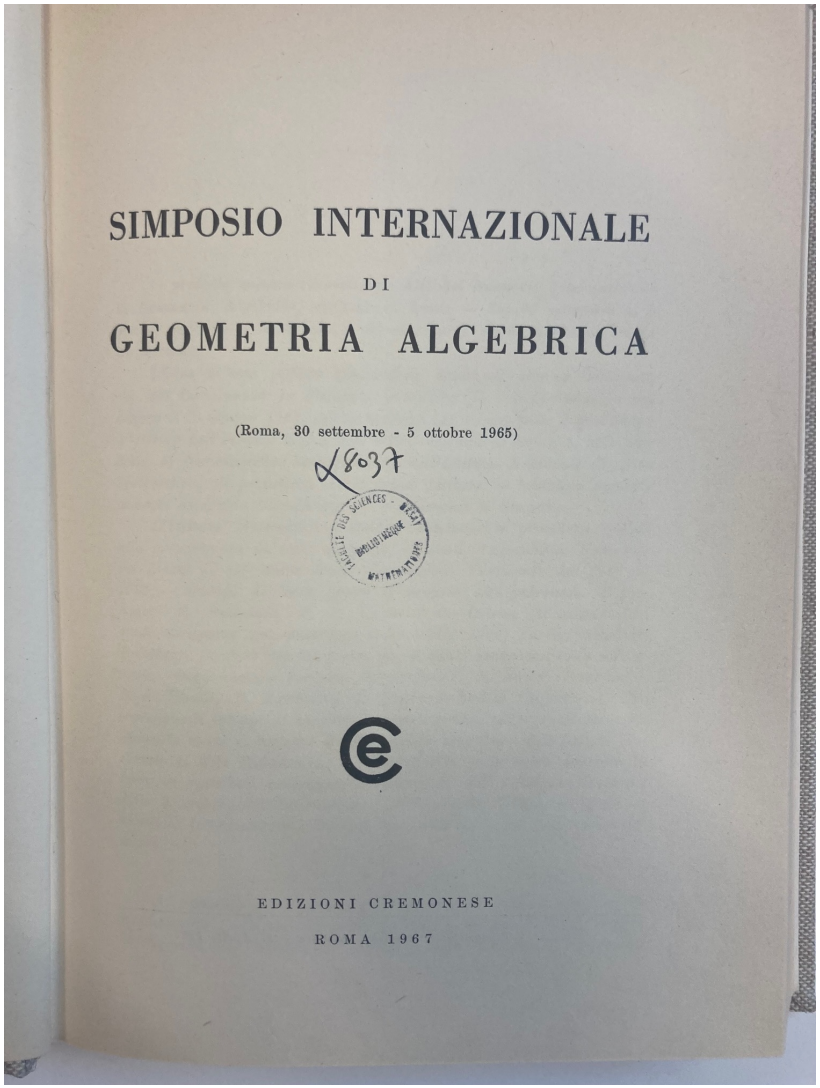
Outstanding number of purported positive solutions, very few expected consequences

- A non-exhaustive list of faulty proofs:

  Kraus 1884 (dimension 2), Engel 1955 (dimension 2),
  Segre 1956 (twice), Segre 1960, Oda 1980 (preprint)

- Shafarevich 1967: only significant consequence of the Jacobian conjecture I am aware of.

Shafarevich's 1967 paper I



SIMPOSIO INTERNAZIONALE

DI

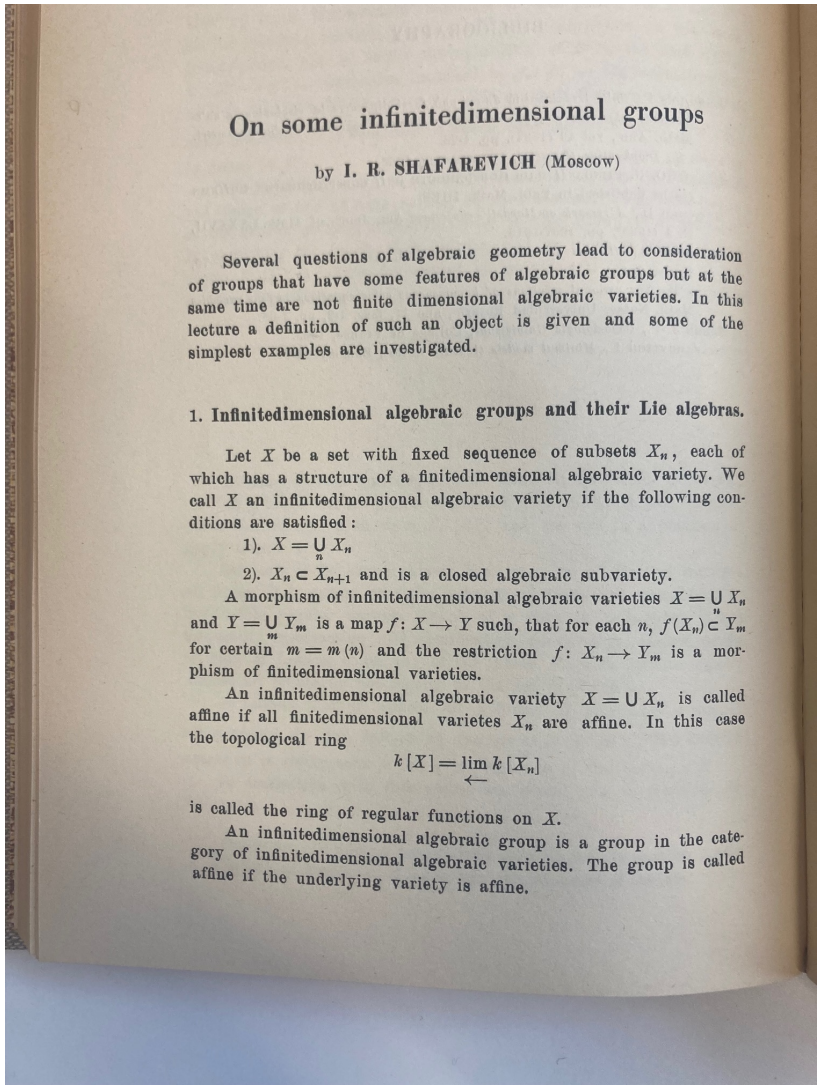
GEOMETRIA ALGEBRICA

(Roma, 30 settembre - 5 ottobre 1965)

A 8037

EDIZIONI CREMONESE

ROMA 1967



On some infinitedimensional groups

by I. R. SHAFAREVICH (Moscow)

Several questions of algebraic geometry lead to consideration of groups that have some features of algebraic groups but at the same time are not finite dimensional algebraic varieties. In this lecture a definition of such an object is given and some of the simplest examples are investigated.

1. Infinitedimensional algebraic groups and their Lie algebras.

Let X be a set with fixed sequence of subsets X_n, each of which has a structure of a finitedimensional algebraic variety. We call X an infinitedimensional algebraic variety if the following conditions are satisfied:

1). $X = \bigcup_n X_n$

2). $X_n \subset X_{n+1}$ and is a closed algebraic subvariety.

A morphism of infinitedimensional algebraic varieties $X = \bigcup_n X_n$ and $Y = \bigcup_m Y_m$ is a map $f: X \to Y$ such, that for each n, $f(X_n) \subset Y_m$ for certain $m = m(n)$ and the restriction $f: X_n \to Y_m$ is a morphism of finitedimensional varieties.

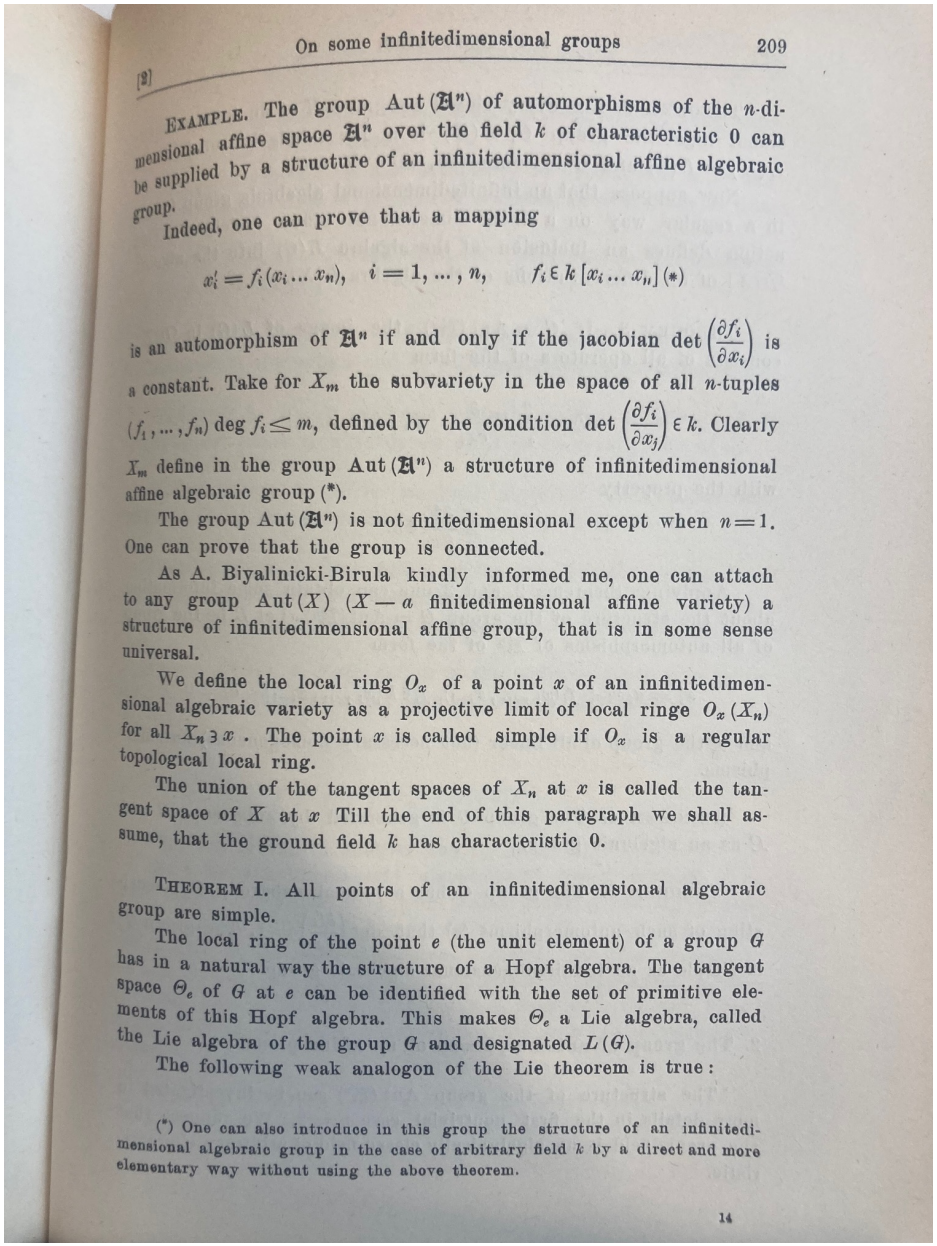
An infinitedimensional algebraic variety $X = \bigcup X_n$ is called affine if all finitedimensional varietes X_n are affine. In this case the topological ring

$$k[X] = \lim_{\leftarrow} k[X_n]$$

is called the ring of regular functions on X.

An infinitedimensional algebraic group is a group in the category of infinitedimensional algebraic varieties. The group is called affine if the underlying variety is affine.

Shafarevich's 1967 paper II



[2] On some infinitedimensional groups 209

EXAMPLE. The group $\mathrm{Aut}(\mathfrak{A}^n)$ of automorphisms of the n-dimensional affine space $\mathfrak{A}^n$ over the field k of characteristic 0 can be supplied by a structure of an infinitedimensional affine algebraic group.

Indeed, one can prove that a mapping

$$x_i' = f_i(x_i \dots x_n), \quad i = 1, \dots, n, \qquad f_i \in k[x_i \dots x_n] \; (*)$$

is an automorphism of $\mathfrak{A}^n$ if and only if the jacobian $\det\left(\frac{\partial f_i}{\partial x_i}\right)$ is a constant. Take for X_m the subvariety in the space of all n-tuples $(f_1, \dots, f_n)$ $\deg f_i \leq m$, defined by the condition $\det\left(\frac{\partial f_i}{\partial x_j}\right) \in k$. Clearly X_m define in the group $\mathrm{Aut}(\mathfrak{A}^n)$ a structure of infinitedimensional affine algebraic group (*).

The group $\mathrm{Aut}(\mathfrak{A}^n)$ is not finitedimensional except when $n=1$. One can prove that the group is connected.

As A. Biyalinicki-Birula kindly informed me, one can attach to any group $\mathrm{Aut}(X)$ (X — a finitedimensional affine variety) a structure of infinitedimensional affine group, that is in some sense universal.

We define the local ring O_x of a point x of an infinitedimensional algebraic variety as a projective limit of local ringe $O_x(X_n)$ for all $X_n \ni x$. The point x is called simple if O_x is a regular topological local ring.

The union of the tangent spaces of X_n at x is called the tangent space of X at x Till the end of this paragraph we shall assume, that the ground field k has characteristic 0.

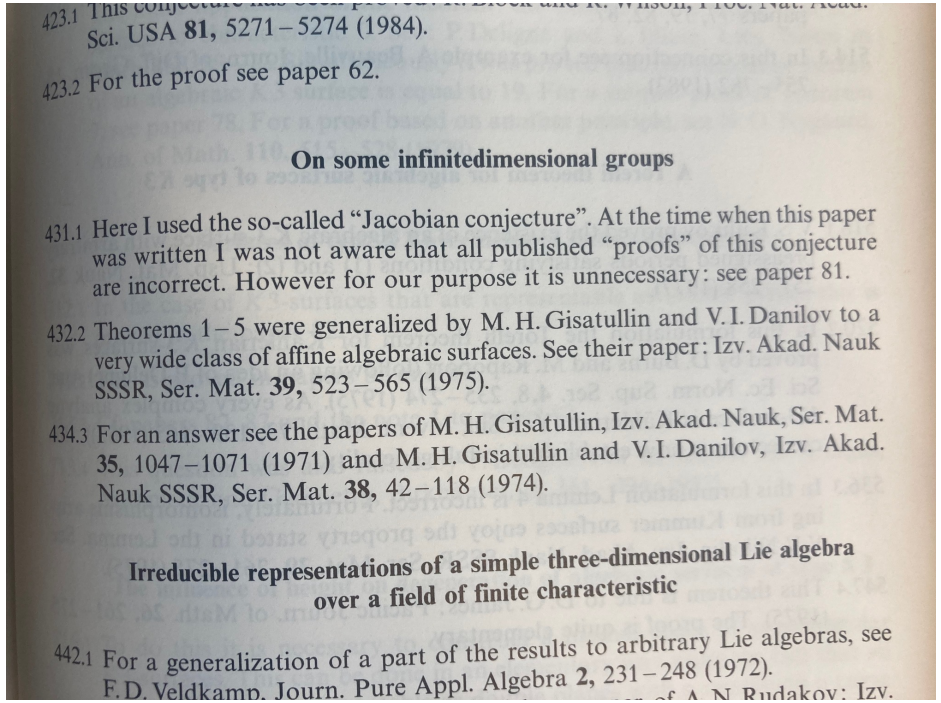
THEOREM I. All points of an infinitedimensional algebraic group are simple.

The local ring of the point e (the unit element) of a group G has in a natural way the structure of a Hopf algebra. The tangent space Θ_e of G at e can be identified with the set of primitive elements of this Hopf algebra. This makes Θ_e a Lie algebra, called the Lie algebra of the group G and designated $L(G)$.

The following weak analogon of the Lie theorem is true:

(*) One can also introduce in this group the structure of an infinitedimensional algebraic group in the case of arbitrary field k by a direct and more elementary way without using the above theorem.

14



423.1 This conjecture has been proved: H. Bass, E. H. Connell and D. Wright, Bull. Amer. Math. Soc. (N.S.) **7**, 287–330 (1982); A. Sathaye, Proc. Nat. Acad. Sci. USA **81**, 5271–5274 (1984).

423.2 For the proof see paper 62.

On some infinitedimensional groups

431.1 Here I used the so-called "Jacobian conjecture". At the time when this paper was written I was not aware that all published "proofs" of this conjecture are incorrect. However for our purpose it is unnecessary: see paper 81.

432.2 Theorems 1–5 were generalized by M. H. Gisatullin and V. I. Danilov to a very wide class of affine algebraic surfaces. See their paper: Izv. Akad. Nauk SSSR, Ser. Mat. **39,** 523–565 (1975).

434.3 For an answer see the papers of M. H. Gisatullin, Izv. Akad. Nauk, Ser. Mat. **35,** 1047–1071 (1971) and M. H. Gisatullin and V. I. Danilov, Izv. Akad. Nauk SSSR, Ser. Mat. **38,** 42–118 (1974).

Irreducible representations of a simple three-dimensional Lie algebra over a field of finite characteristic

442.1 For a generalization of a part of the results to arbitrary Lie algebras, see F. D. Veldkamp, Journ. Pure Appl. Algebra **2,** 231–248 (1972).

Why so many faulty proofs ?

Why so many faulty proofs ?

- Addictive character of the conjecture, due to its elementary character.

Why so many faulty proofs ?

- Addictive character of the conjecture, due to its elementary character. Compare Fermat's Last Theorem...

Why so many faulty proofs ?

- Addictive character of the conjecture, due to its elementary character. Compare Fermat's Last Theorem...

- Conjecture is easily established if one make some apparently innocuous extra hypothesis on the map $f : \mathbb{C}^n \to \mathbb{C}^n$ with non-vanishing Jacobian; for instance:

Why so many faulty proofs ?

- Addictive character of the conjecture, due to its elementary character. Compare Fermat's Last Theorem...

- Conjecture is easily established if one make some apparently innocuous extra hypothesis on the map $f : \mathbb{C}^n \to \mathbb{C}^n$ with non-vanishing Jacobian; for instance:
  - the map f is proper;

Why so many faulty proofs ?

- Addictive character of the conjecture, due to its elementary character. Compare Fermat's Last Theorem...

- Conjecture is easily established if one make some apparently innocuous extra hypothesis on the map $f : \mathbb{C}^n \to \mathbb{C}^n$ with non-vanishing Jacobian; for instance:

  - the map f is proper;

  - the cardinality of the fibers $f^{-1}(z)$, which is necessarily finite, is constant for z in a "big" open subset of $\mathbb{C}^n$.

The Jacobian conjecture in dimension 2

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

Known for infinitely many (pairs of) degrees:

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

Known for infinitely many (pairs of) degrees:

- Heitmann 1990: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) < 16$

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

Known for infinitely many (pairs of) degrees:

- Heitmann 1990: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) < 16$
- Abhyankar 2008 (Lemma 6.1): True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q)$ prime

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

Known for infinitely many (pairs of) degrees:

- Heitmann 1990: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) < 16$
- Abhyankar 2008 (Lemma 6.1): True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q)$ prime
- Valqui, Guccione, Guccione 2017: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) = 2p$ for p prime

The Jacobian conjecture in dimension 2

Computer-assisted proofs for (pairs of) polynomials of "small" degrees:

- Moh 1983 + L.C. Wang 2005: True for polynomials of degree ≤ 100
- Nguyen 2025: True for polynomials of degree ≤ 104
- Guccione, Guccione, Horruitiner, Valqui 2022 (preprint): True for polynomials of degree < 108, and for polynomials of degree ≤ 124 except for pairs of degrees $(72, 108)$ or $(108, 72)$

Known for infinitely many (pairs of) degrees:

- Heitmann 1990: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) < 16$
- Abhyankar 2008 (Lemma 6.1): True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q)$ prime
- Valqui, Guccione, Guccione 2017: True for pairs of polynomials (P, Q) with $\gcd(\deg P, \deg Q) = 2p$ for p prime

The Jacobian conjecture in dimension 2 seems robust.
After Claude-Alpöge's counterexample, solving it one way or the other becomes an intriguing issue.

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

It is widely seen, among algebraic geometers, as one of the most positive applications of AI.

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

It is widely seen, among algebraic geometers, as one of the most positive applications of AI.

Well-described by Geordie Williamson, in the paper *The Year Math Changed Forever* by Rivka Galchen, published in *The New Yorker*, November 1st, 2026.

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

It is widely seen, among algebraic geometers, as one of the most positive applications of AI.

Well-described by Geordie Williamson, in the paper *The Year Math Changed Forever* by Rivka Galchen, published in *The New Yorker*, November 1st, 2026.

> *The mathematician Geordie Williamson, of the University of Sydney, in Australia, described one A.I.-assisted solution to me as "like this beautiful thing for the whole math community" and another as "like the weed killer Roundup being sprayed on a garden." [...]*

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

It is widely seen, among algebraic geometers, as one of the most positive applications of AI.

Well-described by Geordie Williamson, in the paper *The Year Math Changed Forever* by Rivka Galchen, published in *The New Yorker*, November 1st, 2026.

> *The mathematician Geordie Williamson, of the University of Sydney, in Australia, described one A.I.-assisted solution to me as "like this beautiful thing for the whole math community" and another as "like the weed killer Roundup being sprayed on a garden." [...]*

The Good:

The status of Claude-Alpöge's counterexample among AI contributions to mathematics — or The Good, the Bad, and the Ugly.

It is widely seen, among algebraic geometers, as one of the most positive applications of AI.

Well-described by Geordie Williamson, in the paper *The Year Math Changed Forever* by Rivka Galchen, published in *The New Yorker*, November 1st, 2026.

> *The mathematician Geordie Williamson, of the University of Sydney, in Australia, described one A.I.-assisted solution to me as "like this beautiful thing for the whole math community" and another as "like the weed killer Roundup being sprayed on a garden." [...]*

The Good:

> *The beautiful thing concerned an eighty-seven-year-old problem, the Jacobian conjecture. [...]*
> *"Within thirty-six hours, mathematicians had kind of understood where the counterexample comes from and why it's natural." [...]*
> *The proof had opened up thinking, not decapitated it.*

The Good, the Bad, and the Ugly. II

The Good, the Bad, and the Ugly. II

The Bad: ...

The Good, the Bad, and the Ugly. II

The Bad: ...

The Ugly:

The Good, the Bad, and the Ugly. II

The Bad: ...

The Ugly:

> *But then there was Williamson's Roundup story. [...] The first few Hadamard matrices had been described in 1867, and the matrices have fascinated mathematicians ever since, with breakthroughs and new insights turning up every couple of decades or so, the most recent in 2004. At a talk not long before, Williamson had been encouraging mathematicians to find the next one. Then there it was, on X. [...]*

The Good, the Bad, and the Ugly. II

The Bad: ...

The Ugly:

> *But then there was Williamson's Roundup story. [...] The first few Hadamard matrices had been described in 1867, and the matrices have fascinated mathematicians ever since, with breakthroughs and new insights turning up every couple of decades or so, the most recent in 2004. At a talk not long before, Williamson had been encouraging mathematicians to find the next one. Then there it was, on X. [...]*
>
> *Anthropic did not release logs showing how the solutions were reached. Williamson said that he felt as if an entire garden of mathematics had been killed. "It was the nightmare scenario, like, 'Here's the answer, it's forty-two.'"*

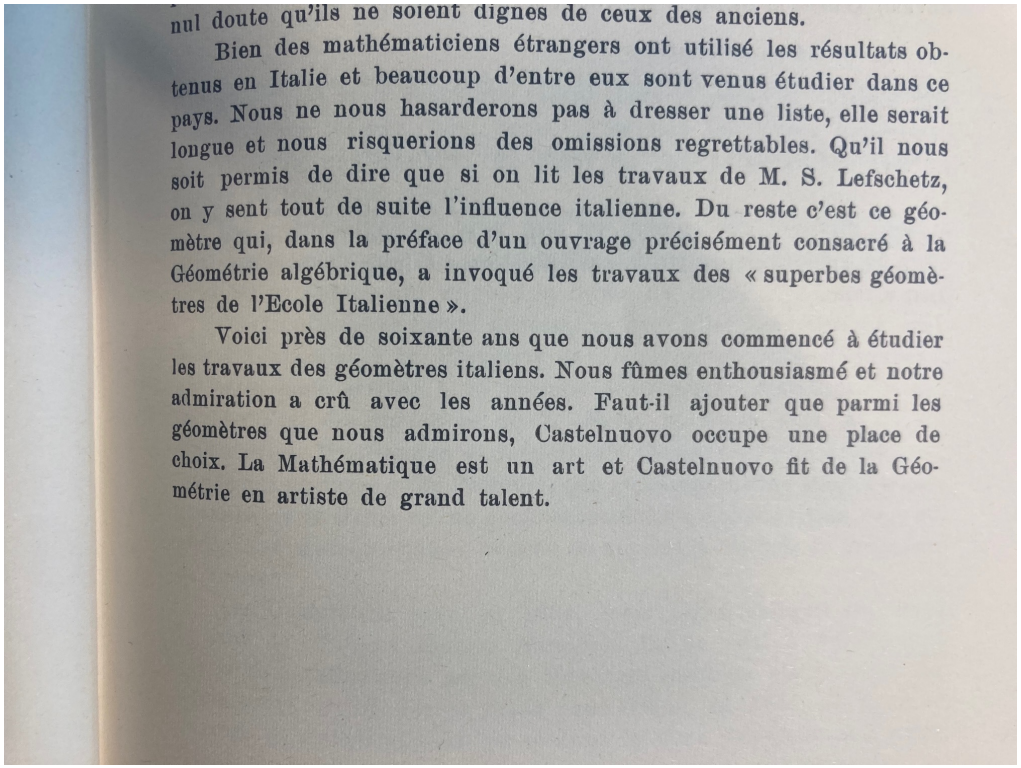
Mathematics — and algebraic geometry — as an art form ?

Mathematics — and algebraic geometry — as an art form ?

Godeaux published a paper on *La géométrie algébrique italienne* in the proceeding of the conference for the 100th anniversary of Castelnuovo's birth.

Mathematics — and algebraic geometry — as an art form ?

Godeaux published a paper on *La géométrie algébrique italienne* in the proceeding of the conference for the 100th anniversary of Castelnuovo's birth.



nul doute qu'ils ne soient dignes de ceux des anciens.

Bien des mathématiciens étrangers ont utilisé les résultats obtenus en Italie et beaucoup d'entre eux sont venus étudier dans ce pays. Nous ne nous hasarderons pas à dresser une liste, elle serait longue et nous risquerions des omissions regrettables. Qu'il nous soit permis de dire que si on lit les travaux de M. S. Lefschetz, on y sent tout de suite l'influence italienne. Du reste c'est ce géomètre qui, dans la préface d'un ouvrage précisément consacré à la Géométrie algébrique, a invoqué les travaux des « superbes géomètres de l'Ecole Italienne ».

Voici près de soixante ans que nous avons commencé à étudier les travaux des géomètres italiens. Nous fûmes enthousiasmé et notre admiration a crû avec les années. Faut-il ajouter que parmi les géomètres que nous admirons, Castelnuovo occupe une place de choix. La Mathématique est un art et Castelnuovo fit de la Géométrie en artiste de grand talent.

Thank you!

Thank you!

I am grateful to Jean-Benoît Bost and to Pablo Chenal for their help during the preparation of this talk.