Nakayama - Zariski decomposition.

References :
- Bădescu, Algebraic surfaces
- BCHM
- Lazarsfeld, Positivity in algebraic geometry I.
- Nakayama, Zariski-decomposition and Abundance

I - The case of surfaces

$k = \bar{k}$, X smooth projective surface $/k$.

motivation : D divisor on X. Define the D-dimension as follows:

- If $H^0(X, \mathcal{O}_X(mD)) = 0$ $\forall m \geq 1$, $\kappa(X, D) =_{def} -\infty$
- Else, $\kappa(X,D) = \mathrm{Tr.deg}_k R(X,D) - 1$ where

$$R(X,D) := \bigoplus_{m \geq 1} H^0(X, \mathcal{O}_X(mD)).$$

Iitaka: If $\kappa(X,D) \geq 0$, then

$$\kappa(X,D) = \max \dim(\varphi_{|mD|}(X))$$

where $\varphi_{|mD|} : X \dashrightarrow \mathbb{P}^{N(m)} = \mathbb{P}(H^0(X, \mathcal{O}_X(mD))^\vee)$

In particular, $\kappa(X,D) \leq 2$.

Zariski : analog question to Hilbert 14th problem:

When is $R(X,D)$ a f.g k-algebra?

If (X,D) is such that $\kappa(X,D)=2$, then there exists an algebraic space of dim 2 and a m. φ

$f: X \to Y$ that is minimal in some sense.

If $R(X,D)$ is f.g over k, $Y = \operatorname{Proj} R(X,D)$ is a normal projective surface.

Zariski decomposition was developped to study $R(X,D)$.

Theorem (Fujita - Zariski): Let D be an integral pseudo-effective divisor on X. Then D can be written uniquely as a sum $D = P + N$ of $\mathbb{Q}$-div with

$\boxed{\forall A \text{ ample},\ D \cdot A \geq 0}$

(i) P is nef

(ii) $N = \sum a_i E_i$ is effective, and if $N \neq 0$, the intersection matrice $(E_i \cdot E_j)_{i,j}$ is negative definite

(iii) $\forall i$, $P \cdot E_i = 0$.

Remark: Zariski decomposition gives the "nef part" of the divisor.

example:

x_1, x_2, x_3 colinear points in $\mathbb{P}^2$. Let σ be the blowup

$$\sigma : Bl_{x_1,x_2,x_3} \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$ of $\mathbb{P}^2$ at the x_i's.

$$Bl_{x_1,x_2,x_3} \mathbb{P}^2 = X$$

Denote by
- E_i the exceptional divisor over x_i
- L line in $\mathbb{P}^2$ that do not meet x_i, $H := \sigma^* L$
- E_{123} the proper transform of the line through x_i

We have $E_{123} \sim H - E_1 - E_2 - E_3$.

We can compute the intersections

- $E_i \cdot E_j = -\delta_{i,j}$, $H^2 = 1$, $H \cdot E_i = 0$.
- $E_{123}{}^2 = (H - E_1 - E_2 - E_3)^2 = 1 - 3 = -2$

Now consider

$$D := 3H - 2E_1 - 2E_2 - 2E_3 \sim H + 2E_{123} .$$

Then, D is not effective since

$$D \cdot E_{123} = (H + 2E_{123}) \cdot E_{123} = 1 - 2 \cdot 2 = -3 < 0$$

The coefficient 2 in $H + 2E_{123}$ is too high, $2E_{123}$ brings too much negativity for D to be nef.

But:
$$D = \underbrace{\left(D - \tfrac{3}{2} E_{123}\right)}_{=P} + \underbrace{\tfrac{3}{2} E_{123}}_{=N}$$
$$\sim \left(H + \tfrac{1}{2} E_{123}\right) + \tfrac{3}{2} E_{123}$$

Fact: this is zariski decomposition of D.

(i) P is nef: - $P \cdot E_i = \left(H + \frac{1}{2} E_{123}\right) \cdot E_i = -\frac{1}{2} E_i{}^2 = \frac{1}{2} > 0$

- $P \cdot E_{123} = \left(H + \frac{1}{2} E_{123}\right) \cdot E_{123} = 1 + \frac{1}{2} E_{123}{}^2 = 1 - 1 = 0$

(ii) $\left(\frac{3}{2} E_{123}\right)^2 = -\frac{9}{2} < 0$ is negative definite

(iii) $P \cdot \frac{3}{2} E_{123} = 0$

Remark: P, N are $\mathbb{Q}$-divisors in this example.

example: X' surface of non-negative canonical dimension.

$\exists!$ $f : X' \to X$ where X is minimal. If K is a canonical divisor on X, it is nef, and f^*K as well.

By the construction of the minimal model X, there exists an effective divisor N with $Supp(N) \subset Exc(f)$

st $K' = f^*K + N$ is a canonical divisor on X'.

Then, $K' = f^*K + N$ is the Zariski-decomposition of K'.

Proof of Fujita-Zariski theorem:

Existence: Let D be a pseudo-effective divisor on X.

If D is nef, $P = D$ and $N = 0$. ✓

Else, let $C_1, \dots, C_{q_1}$ be the integral curves on X that satisfies $C_i \cdot D < 0$ (finite since $NS(X)$ finite rank).

The intersection matrice $I_1 = (C_i \cdot C_j)_{i,j}$ is negative definite.

$\Rightarrow \exists!$ $\mathbb{Q}$-div $N_1 \in Vect(C_1, \dots, C_{q_1})$ st $(N_1 \cdot C_i) = (D \cdot C_i)$.

$N_1 \geq 0$ since I_1 neg def and $(N_1 \cdot C_i) < 0$ $\forall i$.

Now, let $D_1 = D - N_1$. If D_1 is nef, $D = D_1 + N_1$ ✓

Else, let $C_{q_1+1}, \dots, C_{q_2}$ int curves st $C_i \cdot D_1 < 0$.
different from $C_1, \dots, C_{q_1}$ since $C_1 \cdot D_1 = 0$.

As before, $\exists!$ N_2 $\mathbb{Q}$-div st $N_2 \in Vect(C_{q_1+1}, \dots, C_{q_2})$, and $N_2 \geq 0$. Let $D_2 = D_1 - N_2$, ie

$$D = (D - N_1 - N_2) + (N_1 + N_2).$$

Then this process has to stop since $NS(X)$ of finite rank and the C_i are a free family inside $NS(X)$, hence the existence.

Uniqueness:

Let $D = P+N = P'+N'$ two decompositions.

Denote by (C_i) the support of N. We have

$(P \cdot C_i) = 0$ and $(P' \cdot C_i) \geq 0$ since P' is nef.
Hence,
$$(N \cdot C_i) = (N+P)\cdot C_i = (N'+P')\cdot C_i \geq N' \cdot C_i$$
so we have $\forall i$, $(N-N')\cdot C_i \geq 0$.
This implies that $N - N' \geq 0$. Similarly, $N' - N \geq 0$, so $N = N'$ and $P = P'$. $\square$

Remark: - we see that we retract from D effective divisors with negative definite intersection matrice.
- we can't have the decomposition with $\mathbb{Z}$-coeff since we resolve systems over $\mathbb{Q}$-vector spaces.
$\rightarrow$ in the example, want $D \cdot E_{123} = N \cdot E_{123}$

The following proposition shows the utility of Zariski decomposition to the study of $R(X,D)$.

Proposition: Let $D = P+N$ the Zariski decomposition of a pseudo-effective divisor D on X. Then,
$$\forall n \geq 1, \quad H^0(X, \mathcal{O}_X(nD)) = H^0(X, \mathcal{O}_X(nP)).$$
In particular, $R(X,D) = R(X,P)$ and $\kappa(X,D) = \kappa(X,D)$.

proof: $nD = nP + nN$ is a Zariski-decomp for nD so we are reduced to the case
$$H^0(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(P)).$$

Let $f \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}$.

Can write $(((f)+D)-\nu) \cdot C = (f) \cdot C + P \cdot C = 0 + 0$ (A)

$\forall C$ curve in the support of ν

and since $(f)+D \geq 0$, propos(A) $\Rightarrow (f)+D-\nu \geq 0$,

ie $(f)+P \geqslant 0$.

We deduce the following <u>corollary</u>:

With the same notations, $\kappa(X,D) = 2 \Leftrightarrow P^2 > 0$.

<u>proof</u>: $\kappa(X,D) = 2 \Leftrightarrow \kappa(X,P) = 2 \Leftrightarrow P^2 > 0$
where the second equivalence uses P nef and Riemann-Roch.

We can finally state Zariski theorem about finite generation of $R(X,D)$.

<u>Theorem (Zariski)</u>: Let $D \in \mathrm{Div}(X)$.

(i) If $\kappa(X,D) \leq 1$, then $R(X,D)$ is f.g over k.

(ii) If $\kappa(X,D) = 2$, $R(X,D)$ is a f.g k-algebra <u>iff</u> $\exists n > 0$ st $nP \in \mathrm{Div}(X)$ and $|nP|$ has no base points.

<u>Remark</u>: If $|nP|$ has no base point, the rational map from Iitaka thm $\varphi_{|mD|}: X \dashrightarrow \mathbb{P}^n$ is a morphism. that ~~can be factorized as~~

II - Generalization to higher dim varieties

X smooth proj $/ k = \bar{k}$, dim n.

First restriction on the extension:

There exists D big divisor on X st $Vol(D) \notin \mathbb{Q}$.

If $D = P + N$ is a Zariski-decomposition for D,

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \rightsquigarrow$ ask for this.

then $Vol(D) = Vol(P)$ since $(*)$ $H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP))$

but P is nef so $Vol(P) = P^2 \in \mathbb{Q}$ since P is a $\mathbb{Q}$-divisor.

Therefore, D can't have a Zariski-decomposition in the sense of surfaces.

What we can do is working with $\mathbb{R}$-divisors.

But even so, Nakayama showed that there exist divisors that don't admit decomposition with $\mathbb{R}$-divisors.

What Nakayama do instead:

define for all pseudo effective D a

σ-decomposition $\quad D = P_\sigma(D) + N_\sigma(D)$

(where $P_\sigma(D)$, $N_\sigma(D)$ $\mathbb{R}$-divisors)

and call it Nakayama-Zariski decomposition when $P_\sigma(D)$ is nef.

III - Nakayama-Zariski decomposition in the general case.

X smooth projective variety $/\mathbb{C}$.

Def: Let B be a big divisor on X. We define for all prime C, $\sigma_C(B) := \inf\{\mathrm{mult}_C(B'), B' \sim_{\mathbb{Q}} B, B' \geq 0\}$.

proposition: Fix C a prime divisor.

The function $\sigma_C : \mathrm{Big}(X) \to \mathbb{R}^+$ is continuous.

proof: Let $B, (B_n)$ big divisors st $c_1(B_n) \to c_1(B)$ in the $\mathbb{R}$-vector space $N^1(X)$. Let A ample on X.

Let $\|\cdot\|$ be a norm on $N^1(X)$, and $\forall R > 0$, $B(R) := \{|B| < R\} \subset N^1(X)$.

The proof consists of 3 steps:

(i) $\cancel{\forall\, 0 < \varepsilon \ll A}$, $\sigma_C(B) \leq \lim_{\varepsilon \to 0} \sigma_C(B + \varepsilon A) \leq \liminf \sigma_C(B_n)$

(ii) $\forall\, 0 < \delta \ll 1$, $\limsup \sigma_C(B_n) \leq \sigma_C(B - \delta A)$

(iii) $\forall\, 0 < \delta \ll 1$, $\lim_{t \to 0} \sigma_C(B - t\delta A) \leq \sigma_C(B)$.

(i)+(ii)+(iii) $\Rightarrow$ $\sigma_C(B) = \lim_{n\infty} \sigma_C(B_n)$.

(i) $\mathrm{Amp}(X)$ is open in $N^1(X)$ so

$$\exists\, r_0 > 0,\ B(r_0) + \varepsilon A \subset \mathrm{Amp}(X). \quad (1)$$

As $c_1(B_n) \to c_1(B)$, $\exists\, n_0$ st $\forall n \geq n_0$, $c_1(B_n - B) \in B(r_0)$.

Therefore, $B + \varepsilon A = \underbrace{(B - B_n + \varepsilon A)}_{\in \mathrm{Amp}(X) \text{ by } (1)} + B_n$

so $\sigma_C(B + \varepsilon A) \leq \sigma_C(B - D_m + \varepsilon A) + \sigma_C(B_m)$
$= \sigma_C(B_n)$

And therefore, $\sigma_C(B+\varepsilon A) \leq \liminf \sigma_C(B_m)$

But B is big so $\exists\, \Delta$ effective $\mathbb{R}$-divisor st $B \sim_{\mathbb{Q}} \delta' A + \Delta$ for small δ', and we then have

$$(1+\varepsilon)\sigma_C(B) \leq \sigma_C(B+\varepsilon\delta' A) + \varepsilon \operatorname{mult}_C \Delta$$

$$\underset{\varepsilon \to 0}{\downarrow} \sigma_C(B) \qquad \downarrow \lim_{\varepsilon\to 0} \sigma_C(B+\varepsilon A)$$

so we get $\sigma_C(B) \leq \liminf \sigma_C(B_m)$.

(ii) For $0 < \delta \ll 1$, $B - \delta A$ is big.

$\mathrm{Big}(X)$ is open in $N^1(X)$, so

$$\exists\, r_1 > 0 \text{ st } B - \delta A + B(r_1) \subset \mathrm{Big}(X) \quad (2).$$

As before, $\forall \varepsilon' > 0$, $\exists\, 0 < r_2 < r_1$ st

$$B(r_2) + \varepsilon' A \subset \mathrm{Amp}(X) \quad (3)$$

Thus, $\forall \varepsilon' < \delta$,

$$B_m + (\varepsilon' - \delta)A = \underbrace{(B_m - B + \varepsilon' A)}_{} + (B - \delta A)$$

$\underbrace{B_m - B + \varepsilon' A}_{}$ $\cap$ $\mathrm{Amp}(X)$ by (3); $B_m - B \in B(r_1)$ for $m \gg 1$, $\varepsilon' \ll \delta$

so $B_m + (\varepsilon' - \delta)A \in \mathrm{Big}(X)$ by (2)

and $\sigma_C(B_m + (\varepsilon' - \delta)A) \leq \sigma_C(B_m - B + \varepsilon' A) + \sigma_C(B - \delta A)$
$= \sigma_C(B - \delta A)$

but since $\varepsilon' - \delta < 0$, $\sigma_C(B_m) \leq \sigma_C(B_m + (\varepsilon' - \delta)A) + \sigma_C(\delta - \varepsilon')A)$
$= \sigma_C(B_m + (\varepsilon' - \delta)A)$

Therefore, $\limsup \sigma_C(B_m) \leq \sigma_C(B - \delta A)$.

(iii) $B - \delta A$ big $\Rightarrow B - \delta A \sim_{\mathbb{Q}} \Delta$ effective $\mathbb{R}$ div so $B - t\delta A \sim (1-t)B + t\Delta$ and $\sigma_C(B - t\delta A) \leq (1-t)\sigma_C(B) + t \operatorname{mult}_C \Delta$

def: an $\mathbb{R}$-divisor D is pseudo-effective if $\forall \varepsilon > 0$, $\forall A$ ample, $D + \varepsilon A$ is Big.

Let C be a prime divisor. Define

$$\sigma_C(D) := \lim_{\varepsilon \to 0} \sigma_C(D + \varepsilon A).$$

Remarks:

* the limit exists since $\sigma_C(D+\varepsilon A)$ grows when $\varepsilon \to 0$ and is bounded by $\frac{D \cdot A^{n-1}}{C \cdot A^{n-1}}$.
* If A' is an other ample divisor, $\exists\, \Delta$ effective $\mathbb{R}$-div and $\delta > 0$ st

$$A' \equiv_{Num} \delta A + \Delta.$$ We get

$$\sigma_C(D + \varepsilon A') = \sigma_C(D + \varepsilon \delta A + \varepsilon \Delta)$$
$$\leq \sigma_C(D + \varepsilon \delta A) + \varepsilon \, \mathrm{mult}_C \Delta$$

and we have $\lim_{\varepsilon \to 0} \sigma_C(D + \varepsilon A') \leq \lim_{\varepsilon \to 0} \sigma_C(D + \varepsilon A)$.
Exchange $A \leftrightarrow A'$ and get the same limit.

Proposition/Facts: Let D be a pseudo-effective $\mathbb{R}$-divisor.

* There are finitely many C with $\sigma_C(D) > 0$.
  ↳ same as surfaces, bounded by Picard number $\rho(X)$.
* We define $N_\sigma(D) := \sum_C \sigma_C(D)\, C$.
* $D - N_\sigma(D)$ is pseudo-effective
* $N_\sigma(D - N_\sigma(D)) = 0$

Def: Write $D = P_\sigma(D) + N_\sigma(D)$, where $P_\sigma(D) := D - N_\sigma(D)$.
If $P_\sigma(D)$ is nef, then the σ-decomposition is called Nakayama-Zariski

The goal is now to prove the following theorem:

Thm (BCHM, Prop 3.3.2)

Let X be a projective, smooth variety $/\mathbb{C}$, and let D be a pseudo effective $\mathbb{R}$-divisor. Let B be any big $\mathbb{R}$-divisor. If $D \not\equiv N_\sigma(D)$, then

$$\exists k \in \mathbb{N}^*, \beta \in \mathbb{Q}_+^*, \ h^0(X, \mathcal{O}_X(\lfloor mD\rfloor + \lfloor kB\rfloor)) > \beta m \quad \forall m \gg 0.$$

Proof: A integral divisor. B is big so

$$\exists k \gg 0 \text{ st } h^0(X, \mathcal{O}_X(\lfloor kB\rfloor - A)) \geq 1.$$

let $s \in H^0(X, \mathcal{O}_X(\lfloor kB\rfloor - A)) \setminus \{0\}$. We have

$$H^0(X, \lfloor mD\rfloor + A) \xhookrightarrow{\times s} H^0(X, \lfloor mD\rfloor + \lfloor kB\rfloor)$$

so we are reduced to prove the existence of A ample and $\beta \in \mathbb{Q}_+^*$ st

$$h^0(X, \lfloor mD\rfloor + A) > \beta m \quad \forall m \gg 0.$$

Replacing D by $D - N_\sigma(D)$, we can assume $N_\sigma(D) = 0$. Then we can apply the following theorem. $\square$

Thm (Nakayama, V.1.11)

X smooth projective variety $/\mathbb{C}$, D pseudo-effective $\mathbb{R}$-divisor. Suppose that $D \not\equiv 0$ and $N_\sigma(D) = 0$. Then $\exists A$ ample and $\beta \in \mathbb{Q}_+^*$ st $h^0(X, \lfloor mD\rfloor + A) > \beta m \quad \forall m \gg 0$.

To prove the theorem, we will need the following proposition about restriction and extension sections along a curve.
We will prove the theorem by looking at the restriction along a curve, and apply asymptotic Riemann-Roch.

Proposition: X smooth proj $/\mathbb{C}$, $\mathcal{C}$ smooth curve, projective.
Let L be a $\mathbb{Z}$-divisor on X, and denote by

$\rho : Z \to X$ the blow-up of X along $\mathcal{C}$.

Suppose that:
- $\exists\, \Delta$ effective $\mathbb{R}$-divisor st (X, Δ) is log-terminal around $\mathcal{C}$.
- $\mathcal{C} \not\subset \operatorname{Supp} \Delta$ (*)
- $\rho^*(L - (K_X + \Delta)) - (n-1) E_C$ is ample. (*)

Then $H^0(X, L) \twoheadrightarrow H^0(C, L|_C)$ is surjective.

proof:

idea: $Y \xrightarrow{f} X$ find a birational map $\mathcal{C}$ (*)
st
- $E := \operatorname{Exc}(f) \cup \operatorname{Supp}(f^*\Delta)$ is normal crossing
- $\exists R$ on Y st $f^*L + R - K_Y$ nef and big (*)
- $f_* R \subset \mathcal{I}_{\mathcal{C}}$ the ideal sheaf of $\mathcal{C}$

And then apply Kawamata-Viehweg vanishing to $f^*L + R - K_Y$ to get annulation of H^1 and surjectivity on H^0.

We have $K_Z = \rho^*(K_X + \Delta) + (n-2) E_C + \Delta_Z$ ← proper transform of Δ

We can choose a resolution $\mu : Y \to Z$ st

$E := \sum a_i E_i$ is a normal crossing divisor
$= \operatorname{Exc}(f) \cup \operatorname{Supp}(f^*\Delta)$. We have

$$K_Y = f^*(K_X + \Delta) + \underbrace{(n-2)\mu^* E_{\mathcal{C}} - \mu^* \Delta_Z}_{= E}$$

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[rr, bend left, "f"] & Z \arrow[r, "\rho"] & X \\ E \arrow[u, hook] & E_{\mathcal{C}} \arrow[u, hook] \arrow[r] & \mathcal{C} \arrow[u, hook] \end{tikzcd}$$

$$R = \sum r_i E_i = E - (n-1)\nu^* E_{\mathcal{C}}.$$

Now consider the divisor

$$D = f^*L + E - (n-1)\nu^* E_{\mathcal{C}} - K_Y$$

$$= \nu^*\big(\rho^*(L - (K_X + \Delta)) - (n-1)E_{\mathcal{C}}\big)$$

so D is nef and big $\underline{(A2)}$ ample by (*)

Write $D = f^*L + R - K_Y$, and E_0 the proper transform of $E_{\mathcal{C}}$.

Then (X, Δ) log-terminal around $\mathcal{C}$

$\Downarrow$

* if $f(E_i) \cap \mathcal{C} \neq \emptyset$ and $f(E_i) \not\subset \mathcal{C}$, $r_i = a_i > -1$.
* $r_0 = a_0 - (n-1) = (n-2) - (n-1) = -1$

So if we consider $\lceil R \rceil$, $\lceil R \rceil = -E_0 + \sum r_i' E_i$, $r_i' \geq 0$.

Therefore, $\nu_* \mathcal{O}_Y(\lceil R \rceil) = \mathcal{J}_0 \cap \mathcal{J}_1$ with

* $\operatorname{Supp} \mathcal{O}_Z / \mathcal{J}_0 = E_{\mathcal{C}}$
* $\operatorname{Supp} \mathcal{O}_Z / \mathcal{J}_0 \cap \operatorname{Supp} \mathcal{O}_Z / \mathcal{J}_1 = \emptyset$
* $\operatorname{Supp} \mathcal{O}_Z(-E_{\mathcal{C}}) / \mathcal{J}_0$ does not dominate $\mathcal{C}$ $\underline{(A1)}$

Get 2 annulations: (A1) $\mathcal{I}_{\mathcal{C}} / \rho_* \mathcal{J}_0$ skyscraper sheaf, $H^1(\mathcal{C}, \mathcal{I}_{\mathcal{C}} / \rho_* \mathcal{J}_0) = 0$

(A2) Kawamata-Viehweg: $H^1(X, \mathcal{O}_X(L) \otimes f_* \mathcal{O}_Y(\lceil R \rceil)) = 0$

We have the ~~B~~ following exact sequences:

$$0 \to \mathcal{O}_X(L)\otimes p_*\mathcal{O}_H(\lceil R\rceil) \to \mathcal{O}_X(L) \to \mathcal{O}_X(L)\otimes \mathcal{O}_X/p_*(\mathcal{J}_0\cap\mathcal{J}_1) \to 0$$

[associated to the closed subscheme $\mathrm{Supp}\left(\mathcal{O}_X/p_*(\mathcal{J}_0\cap\mathcal{J}_1)\right)$]

and

$$0 \to \mathcal{O}_X(L)\otimes \mathcal{I}_\mathcal{C}/p_*\mathcal{J}_0 \to \mathcal{O}_X(L)\otimes \mathcal{O}_X/p_*\mathcal{J}_0 \to \mathcal{O}_X(L)\otimes \mathcal{O}_X/\mathcal{I}_\mathcal{C} \to 0$$

[associated to the curve $\mathcal{C}\subset \mathrm{Supp}\,\mathcal{O}_X/p_*\mathcal{J}_0$]

so by (A2)

$$H^0(X,L)\twoheadrightarrow H^0(X,\mathcal{O}_X(L)\otimes\mathcal{O}_X/p_*\mathcal{J}_0)$$

and by (A1)

$$H^0(X,\mathcal{O}_X(L)\otimes\mathcal{O}_X/p_*\mathcal{J}_0)\twoheadrightarrow H^0(\mathcal{C},L|_\mathcal{C}) \quad \square$$

<u>proof of the theorem:</u>

<u>notations</u>: • $\forall x\in X$, $\sigma_x(D)=\sup_{\mathcal{C}\ni x}\sigma_C(D)$

• $NBs(D) := \{x\in X,\ \sigma_x(D)>0\}$.

<u>Fact</u>: $N_\sigma(D)=0 \Rightarrow NBs(D)=\bigcup_{i\in\mathbb{N}} V_i$, codim $V_i\geq 2$.

Therefore, $\exists\, \mathcal{C}$ curve in X, complete intersection of ample divisors st $\begin{cases} C\cap NBs(D)=\emptyset \\ C\not\subset \mathrm{Supp}\langle D\rangle. \end{cases}$

Let $p: Z\to X$ blowup of $\mathcal{C}$. Fix $0<\alpha<1$.

There exists A ample st $\forall m$,

(*) $p^*((1-\alpha)A - K_X - \langle mD\rangle) - (n-1)E_\mathcal{C}$ is ample

Let $L_m = \lfloor mD \rfloor + A$. As $\sigma_x(D) = 0 \ \forall x \in \mathcal{C}$,

$\exists \ \Delta_m \sim mD + \alpha A$ effective divisor st

* (X, Δ_m) log-terminal around $\mathcal{C}$
* $\mathcal{C} \not\subset \operatorname{Supp} \Delta_m$.

As
$$L_m - (K_X + \Delta_m) \sim \lfloor mD \rfloor + A - K_X - mD - \alpha A$$
$$\sim (1-\alpha) A - (K_X + \langle mD \rangle),$$

the divisor

$$p^*(L_m - (K_X + \Delta_m)) - (n-1) E_{\mathcal{C}}$$
$$\sim p^*((1-\alpha)A - (K_X + \langle mD \rangle)) - (n-1) E_{\mathcal{C}}$$

is ample by (*).

We apply the proposition and we have

$$H^0(X, \lfloor mD \rfloor + A) \twoheadrightarrow H^0(\mathcal{C}, (\lfloor mD \rfloor + A)|_{\mathcal{C}})$$

and as $D \cdot C > 0$, Riemann-Roch formula for curves gives

$$h^0(\mathcal{C}, (\lfloor mD \rfloor + A)|_{\mathcal{C}}) = m\,(D \cdot C) + cst$$
$$> \beta m \quad \forall m \gg 1 \quad \square$$

<u>Conclusion</u>: BCHM uses prop 3.3.2
to do the case $h^0(X, \ldots) \begin{cases} \text{bounded} \\ \text{unbounded} \end{cases} \Rightarrow N_\sigma(D) \equiv D$.