Formal differential equations and renormalization

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Abstract. The study of solutions of differential equations (analytic or formal) can often be reduced to a conjugacy problem, namely the conjugation of a given equation to a much simpler one, using identity-tangent diffeomorphisms.

On one hand, following Ecalle's work (with a different terminology), such diffeomorphisms are given by characters on a given Hopf algebra (here a shuffle Hopf algebra). On the other hand, for some equations, the obstacles in the formal conjugacy are reflected in the fact that the associated characters appear to be ill-defined.

The analogy with the need for a renormalization scheme (dimensional regularization, Birkhoff decomposition) in quantum field theory becomes obvious for such equations and deliver a wide range of toy models. We discuss here the case of a simple class of differential equations where a renormalization scheme yields meaningful results.

1 Introduction.

Let us start by giving a very simple example of a differential equation that already contains all the ingredients relevant to renormalization.

1.1 A toy model for some differential equations

Let us consider the following equation

$$(E_{\alpha,d}) \qquad x^{1-d}\partial_x y = \alpha y^2$$

where $d \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Considering the right-hand side of this equation as a perturbation of the case $\alpha = 0$, we deal with the following conjugacy problem $(P_{a,d})$: does there exist a formal identity-tangent diffeomorphism

$$\Phi_{\alpha,d}(x,z) = (x, \varphi_{\alpha,d}(x,z)) \quad \varphi_{\alpha,d}(x,z) \in z + z^2\mathbb{C}[[x,z]] \tag{1}$$

such that, if z is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0 \tag{2}$$

then $y = \varphi_{\alpha,d}(x,z)$ is a solution of $(E_{d,\alpha})$. Note that, in the sequel, we will always deal with diffeomorphisms that leave the x–coordinate unchanged (as $\Phi_{\alpha,d}(x,z)$)

so we will focus on their nontrivial part : identity-tangent diffeomorphisms of the second variable whose coefficients depend on the fisrt coordinate (as $\varphi_{\alpha,d}$).

It is quite obvious to check that here,

$$\varphi_{\alpha,d}(x,z) = \frac{z}{1-\alpha\frac{x^d}{d}z} \tag{3}$$

is a very natural solution which, unfortunately, is ill-defined for $d=0$ and, in this singular case, we are led to introduce the logarithm of x so that a good candidate for the conjugacy is

$$\varphi_{\alpha,0}(x,z) = \frac{z}{1-\alpha z\log x}$$

which is no more a formal series in x and z but is connected to the regular case ($d \geq 1$) in the following way :

1. The equation ($E_{\alpha,d}$) can be solved for any $d \in \mathbb{Z}^*$, assuming that the conjugating map $\varphi_{\alpha,d}$ has coefficients in $\mathbb{C}[[x^d]]$, and even for $d \in \mathbb{R}^*$, assuming that we work with "ramified" powers of x.
2. When d is close to zero, writing $x^d = \sum_{n\geq 0} \frac{d^n \log^n x}{n!}$, the coefficients of $\varphi_{\alpha,d}$ appear as Laurent series in d.
3. One can then perform a Birkhoff decomposition of $\Phi_{\alpha,d}$: $\Phi_{\alpha,d} = \Phi^+_{\alpha,d} \circ \Phi^-_{\alpha,d}$ with
$$\Phi^+_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha\frac{(x^d-1)}{d}z}\right), \quad \Phi^-_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha\frac{1}{d}z}\right) \tag{4}$$
4. Since $\Phi^-_{\alpha,d}$ conjugates ($E_{0,d}$) to itself, $\Phi^+_{\alpha,d}$ also conjugates ($E_{0,d}$) to ($E_{\alpha,d}$) and when d goes to 0,
$$\lim_{d\to 0} \Phi^+_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha z\log x}\right) \tag{5}$$
conjugates ($E_{0,0}$) to ($E_{\alpha,0}$).

As we shall see now, this phenomenon can be generalized.

1.2 A generalization.

Let $b(x,y) \in y^2\mathbb{C}[[x,y]]$ and $d \in \mathbb{N}$. We will work on the following problem of formal conjugacy : does there exist a formal identity tangent diffeomorphism $\varphi(x,y)$ in y, with coefficients in $\mathfrak{A} = \mathbb{C}[[x]]$ such that, if y is a solution of

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x,y) \tag{6}$$

then $z = \varphi(x,y)$ is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0 \tag{7}$$

As we shall see in section 3, the answer is yes if $d \geq 1$, but rather than computing directly the coefficients of such a diffeomorphism, we will make an extensive use of Ecalle's mould-comould expansions (see [6]). As we shall see in section 2, the computation of such a diffeomorphism reduces to the computation of a character in a shuffle Hopf algebra.

In the case $d = 0$, this character happens to be ill-defined but the "dimensional regularization" suggested by the previous example gives us the final ingredient in order to perform a renormalization scheme that follows the same algebraic ideas developped in [2].

In order to to introduce this scheme, let us first remark that computing an identity-tangent diffeomorphism is the same as computing a character in the Faà di Bruno Hopf algebra (see [7]).

1.3 Identity-tangent diffeomorphisms and character in the Faà di Bruno Hopf algebra.

Let us consider the group of formal identity-tangent diffeomorphisms in one variable y, whose coefficients are in a commutative $\mathbb{C}$–algebra $\mathfrak{A}$:

$$G_{\mathfrak{A}} = \{f(y) = y + \sum_{n\geq 1} f_n y^{n+1} \in \mathfrak{A}[[y]]\}$$

with the product $\mu : G_{\mathfrak{A}} \times G_{\mathfrak{A}} \to G_{\mathfrak{A}}$:

$$\mu(f, g) = f \circ g$$

For $n \geq 0$, the functionals on $G_{\mathfrak{A}}$ defined by

$$a_n(f) = \frac{1}{(n+1)!}(\partial_y^{n+1} f)(0) = f_n \quad a_n : G_{\mathfrak{A}} \to \mathfrak{A}$$

are called the Faà di Bruno coordinates on the group $G_{\mathfrak{A}}$ and, $a_0 = 1$ being the unit, they generate a graded unital commutative algebra

$$\mathcal{H}_{\text{FdB}} = \mathbb{C}[a_1, \ldots, a_n, \ldots] \quad (\text{gr}(a_n) = n)$$

Moreover, the action of these functionals on a product in $G_{\mathfrak{A}}$ defines a coproduct on $\mathcal{H}_{\text{FdB}}$ that turns to be a graded connected Hopf algebra (see [7] for details). For $n \geq 0$, the coproduct is defined by

$$a_n \circ \mu = m \circ \Delta(a_n) \tag{8}$$

where m is the usual multiplication in $\mathfrak{A}$, and the antipode reads

$$S \circ a_n = a_n \circ \text{rec}$$

where $\text{rec}(\varphi) = \varphi^{-1}$ is the composition inverse of φ.

Note that we can forget that the Faà di Bruno coordinates are functionals and then the Hopf algebra structure $\mathcal{H}_{\text{FdB}}$ does not depend on the algebra $\mathfrak{A}$. Once we have such a Hopf algebra $\mathcal{H}$, one can define the group of characters on $\mathcal{H}$ with values in a commutative unital algebra $\mathfrak{A}$, that is to say algebra morphisms from $\mathcal{H}$ to $\mathfrak{A}$ and $\mathcal{C}(\mathcal{H}, \mathfrak{A})$ with the product :

$$\forall \varphi, \psi \in \mathcal{C}(\mathcal{H}, \mathfrak{A}), \quad \varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta \tag{9}$$

This group is obviously isomorphic to $G_{\mathfrak{A}}$ so that computing some identity-tangent diffeomorphism means computing a character on the Faà di Bruno Hopf algebra and some renormalization scheme, if needed, can be used as in quantum field theory.

1.4 The renormalization scheme.

We will now describe at an abstract level (assuming that the reader is familiar with graded Hopf algebras) what could be called a renormalization scheme in a Hopf algebra and, at each step, we will translate our first problem in terms of this scheme. Let us consider a mathematical problem (P) with the following properties :

Dimension parameter : The problem depends on a parameter $d \in \mathbb{N}$ (or $d \in \mathbb{Z}$ or else...) that can be called the dimension : $(P) = (P_d)$.

In the former conjugacy problem, d is obviously defined.

Hopf background : In the course of computing a solution to the problem (P_d), it appears that, using for example some perturbative expansions in some parameters other than d, we have to compute coefficients, with values in a commutative algebra $\mathfrak{A}$, indexed by a linear basis of a graded Hopf algebra $\mathcal{H}$ with product m and coproduct Δ (see section 4 for an example of Hopf algebra). Moreover, if such coefficients exists, they define an element of the group of characters on $\mathcal{H}$ with values in $\mathfrak{A}$ ($\mathcal{C}(\mathcal{H}, \mathfrak{A})$).

In the former problem, one looks for a formal identity-tangent diffeomorphism $\varphi_{\alpha,d}$ in the variable z, that is to say a character on the Faà di Bruno Hopf algebra of coordinates of identity-tangent diffeomorphisms with values in $\mathfrak{A} = \mathbb{C}[[x]]$.

Ill-defined character : Unfortunately, this character is ill-defined for some singular value d_0 of the dimension parameter.

In our example, this happens for $d = 0$.

Dimensional regularization : Working eventually in some extension $\mathfrak{B}$ of the algebra $\mathfrak{A}$, there is a way to to generalize our problem to complex values of $d = d_0 + \varepsilon$ such that if $\varepsilon \neq 0$, one can compute a character ψ_ε with values in $\mathfrak{B}[[\varepsilon]][\varepsilon^{-1}]$ (Laurent series with coefficients in $\mathfrak{B}$). Moreover, this gives the attempted character if $d_0 + \varepsilon = d$ is not a singular value of the parameter.

In our case, we have introduce a ramified power $x^\varepsilon = \exp(\varepsilon \log x)$ such that we can define an equation $(E_{\alpha,\varepsilon})$ and its associated character has its values in $\mathfrak{B}[[\varepsilon]][\varepsilon^{-1}]$, with $\mathfrak{B} = \mathbb{C}[[x, \log x]]$. The character ψ_ε corresponds to the diffeomorphism $\varphi_{\alpha,\varepsilon}$.

Birkhoff decomposition : As $\mathfrak{B}[[\varepsilon]][\varepsilon^{-1}]$ is a Rota-Baxter algebra (see [5], [4]) with respect to the decomposition $\mathfrak{B}[[\varepsilon]][\varepsilon^{-1}] = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}] \oplus \mathfrak{B}[[\varepsilon]]$, there exists unique Birkhoff decomposition of our character $\psi_\varepsilon = \psi_\varepsilon^+ * \psi_\varepsilon^-$ (or, depending of its pertinence in the problem, $\psi_\varepsilon = \psi_\varepsilon^- * \psi_\varepsilon^+$). And now $\psi_0^+ = \lim_{\varepsilon \to 0} \psi_\varepsilon^+$ is the renormalized value at $d = d_0$, for the given Hopf algebra and dimensional regularization.

In our first example, the Birkhoff-decomposition of the character is expressed on diffeomorphisms by $\Phi_{\alpha,\varepsilon} = \Phi^+_{\alpha,\varepsilon} \circ \Phi^-_{\alpha,\varepsilon}$ and, thanks to the choice of the Hopf algebra and of the dimensional regularization, the renormalized character ψ_0^+ is a solution to the problem for the singular dimension $d = 0$. For details on renormalization and Hopf algebras, see [2], [3], [7].

1.5 Contents.

In section 2, we explain how conjugating diffeomorphisms for the equation $(E_{b,d})$ can be computed in a very algebraic way, using Ecalle's mould-comould expansions. This gives the attempted result for $d \geq 1$ in section 3.

In the case $d = 0$, we show how the previous computations yield an ill-defined character of a shuffle Hopf algebra. Using a quite natural dimensional regularization, this character can be renormalized (see section 4).

We give then in section 5, an interesting interpretation of the renormalized character for our conjugacy problem.

2 Mould-Comould expansions and conjugacy of differential equations.

Let us go back to the study of the equation :

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x,y) \tag{1}$$

Where $b(x,y) \in y^2\mathbb{C}[[x,y]]$ and $d \in \mathbb{N}$. To compute the diffeomorphism (in the variable y) φ such that $z = \varphi(x,y)$ is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0 \tag{2}$$

we could try to compute its coefficients and thus, work in the the Faà di Bruno Hopf algebra. As we shall see now, these computations are simpler and explicit when working with mould-comould expansions.

2.1 Diffeomorphisms and substitutions automorphisms.

We are looking for identity-tangent diffeomorphisms

$$\varphi \in G_{\mathfrak{A}} = \{\varphi(x,y) \in y + y^2\mathfrak{A}[[y]]\}$$

Such a diffeomorphism defines a substitution automorphism on $\mathfrak{A}[[y]]$:

$$\forall f \in \mathfrak{A}[[y]], \quad F_\varphi(f) = f \circ \varphi$$

such that $F_\varphi(fg) = F_\varphi(f)F_\varphi(g)$. Conversely, if F is an endomorphisms on $\mathfrak{A}[[y]]$ such that $F(y) = \varphi(x, y) \in G_\mathfrak{A}$ and

$$\forall f, g \in \mathfrak{A}[[y]], \quad F(fg) = F(f)F(g)$$

then $F = F_\varphi$ (see [6]).

Moreover, using Taylor expansions, if $\varphi(y) = y + \sum_{n\geq 1} \varphi_n y^{n+1} \in G_\mathfrak{A}$, then

$$F_\varphi = \mathrm{Id} + \sum_{s\geq 1} \sum_{n_i\geq 1} \frac{1}{s!} \varphi_{n_1} \dots \varphi_{n_s} y^{n_1+\dots+n_s+s} \partial_y^s \tag{3}$$

is a differential operator.

We will now look for substitutions automorphisms (rather that diffeomorphisms) that can be computed using elementary differential operators associated to the equation $(E_{b,d})$, that is to say mould-comould expansions.

2.2 Mould-Comould expansions for the conjugacy problem.

If y is a solution of the equation $(E_{b,d})$, then for any power series $f(y)$,

$$x^{1-d}\partial_x(f(y)) = x^{1-d}(\partial_x y)f'(y) = b(x, y)f'(y) = b(x, y)\partial_y f(y)$$

This suggests to consider the right-hand term of the equation $(E_{b,d})$ as a derivation. Using the expansion in x, we get

$$x^{1-d}\partial_x y = \sum_{n\geq 0} x^n b_n(y) = \sum_n x^n \mathbb{B}_n.y$$

where

$$\mathbb{B}_n = b_n(y)\partial_y$$

so that the datas in $b(x, y)$ are encoded in the derivations $\mathbb{B}_n$. It seems reasonable to think that the conjugating diffeomorphism (or rather its associated substitution automorphism) can be expressed with the help of these operators. To do so, let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \dots, n_s), \quad s \geq 1, \quad n_i \in \mathbb{N}\}$$

and

$$\mathbb{B}_{\boldsymbol{n}} = \mathbb{B}_{n_s} \dots \mathbb{B}_{n_1} \quad (\mathbb{B}_\emptyset = \mathrm{Id}) \tag{4}$$

Now that we have a set of differential operators, which is called a cosymmetral comould in Ecalle's work (see [6]), this suggest that the attempted conjugating map $\varphi(x, y)$, or rather its associated substitution automorphism, may be expressed with the help of this "comould" :

$$F_\varphi = \mathrm{Id} + \sum_{s\geq 1} \sum_{n_1,\dots n_s\in\mathbb{N}} M^{n_1,\dots,n_s}\mathbb{B}_{n_s}\dots\mathbb{B}_{n_1} = \sum_{\boldsymbol{n}\in\mathcal{N}} M^{\boldsymbol{n}}\mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet \mathbb{B}_\bullet$$

where $M^\emptyset = 1$ (for identity diffeomorphism), $F_\varphi(y) = \varphi(x, y)$ and the collection of coefficients $M^\bullet$, which is called a mould, has its values in $\mathfrak{A} = \mathbb{C}[[x]]$. In order to manipulate such mould-comould expansions, we will now give some classical results on moulds.

2.3 Reminder on moulds.

For details see [6].

Definition 1. *A mould $M^\bullet$ on $\mathcal{N}$ with values in a commutative algebra $\mathfrak{A}$ is a map from $\mathcal{N}$ to $\mathfrak{A}$. Such a mould $M^\bullet$ is symmetral if $M^\emptyset = 1$ and*

$$\forall \boldsymbol{n}^1, \boldsymbol{n}^2 \in \mathcal{N}, \quad M^{\boldsymbol{n}^1} M^{\boldsymbol{n}^2} = \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} M^{\boldsymbol{n}}$$

where the sum is over all the possible shuffling of the sequences $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$. A mould $M^\bullet$ is alternal if $M^\emptyset = 0$ and

$$\forall \boldsymbol{n}^1, \boldsymbol{n}^2 \in \mathcal{N}, \quad \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} M^{\boldsymbol{n}} = 0$$

Provided that the series makes sense, to any mould $M^\bullet$ one can associate a differential operator

$$\mathbb{M} = \sum_{\boldsymbol{n} \in \mathcal{N}} M^{\boldsymbol{n}} \mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet \mathbb{B}_\bullet$$

For example,

$$b(x,y)\partial_y = \sum_n x^n \mathbb{B}_n = \sum_{\boldsymbol{n} \in \mathcal{N}} I^{\boldsymbol{n}} \mathbb{B}_{\boldsymbol{n}} = \sum I^\bullet \mathbb{B}_\bullet$$

where $I^\emptyset = 0$ and

$$I^{n_1, \ldots, n_s} = \begin{cases} x^{n_1} & \text{if} \quad s = 1 \\ 0 & \text{otherwise} \end{cases}$$

defines an alternal mould.

If $M^\bullet$ and $N^\bullet$ are two moulds, then

$$\begin{aligned}
\mathbb{M}.\mathbb{N} &= \left(\sum_{\boldsymbol{n}^1 \in \mathcal{N}} M^{\boldsymbol{n}^1} \mathbb{B}_{\boldsymbol{n}^1} \right) . \left(\sum_{\boldsymbol{n}^2 \in \mathcal{N}} N^{\boldsymbol{n}^2} \mathbb{B}_{\boldsymbol{n}^2} \right) \\
&= \sum_{\boldsymbol{n}^1, \boldsymbol{n}^2} M^{\boldsymbol{n}^1} N^{\boldsymbol{n}^2} \mathbb{B}_{\boldsymbol{n}^1} \mathbb{B}_{\boldsymbol{n}^2} \\
&= \sum_{\boldsymbol{n}^1, \boldsymbol{n}^2} M^{\boldsymbol{n}^1} N^{\boldsymbol{n}^2} \mathbb{B}_{\boldsymbol{n}^2 \boldsymbol{n}^1} \quad \text{(see (4))} \\
&= \sum_{\boldsymbol{n}} \left(\sum_{\boldsymbol{n}^2 \boldsymbol{n}^1 = \boldsymbol{n}} M^{\boldsymbol{n}^1} N^{\boldsymbol{n}^2} \right) \mathbb{B}_{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n}} \left(\sum_{\boldsymbol{n}^1 \boldsymbol{n}^2 = \boldsymbol{n}} N^{\boldsymbol{n}^1} M^{\boldsymbol{n}^2} \right) \mathbb{B}_{\boldsymbol{n}}
\end{aligned}$$

where the sum is over pairs $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ whose concatenation gives $\boldsymbol{n}$. These formulas define a product on moulds :

Proposition 1. *For any moulds $M^\bullet$ and $N^\bullet$, their product $P^\bullet = M^\bullet \times N^\bullet$ is defined by*

$$\forall \boldsymbol{n} \in \mathcal{N}, \quad P^{\boldsymbol{n}} = \sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} M^{\boldsymbol{n}^1} N^{\boldsymbol{n}^2}$$

Moreover the set of symmetral moulds, is a group whose unit $1^\bullet$ is given by $1^\emptyset = 1$ and $1^{\boldsymbol{n}} = 0$ otherwise. The inverse $N^\bullet$ of a given symmetral mould $M^\bullet$ is given by $N^\emptyset = 1$ and

$$N^{n_1,\dots,n_s} = (-1)^s M^{n_s,\dots,n_1}$$

Of course, specialists of Hopf algebras can already smell the flavor of a shuffle Hopf algebra here and we will see the connection in section 4.

Symmetral moulds play a central role in the search of conjugating diffeomorphisms since

Proposition 2. *If $M^\bullet$ is a symmetral mould, then its associated mould-comould expansion $\mathbb{M}$ is a substitution automorphism corresponding to the diffeomorphism $m(x,y) = \mathbb{M}.y$. Moreover if $M^\bullet$ and $N^\bullet$ are two symmetral moulds corresponding to diffeomorphisms m and n, then the mould $P^\bullet = M^\bullet \times N^\bullet$ corresponds to the diffeomorphism $m \circ n$.*

For the first part of this proposition, see [6]. For the second part,

$$m \circ n(x,y) = \mathbb{N}.\mathbb{M}.y = \sum P^\bullet \mathbb{B}_\bullet y = \mathbb{P}.y$$

With this short reminder on moulds, we are now ready to deal with the equation $(E_{b,d})$ when everything works, that is to say when $d \in \mathbb{N}^*$.

3 The case $d \in \mathbb{N}^*$.

Suppose that $z = \varphi(x,y)$ conjugates the equation

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x,y)$$

to $x^{1-d}\partial_x z$=0. One expects that the associated substitution automorphism can be written as a mould-comould expansion

$$\varphi(x,y) = \sum V_d^\bullet \mathbb{B}_\bullet y = \mathbb{V}_d.y$$

where $V_d^\bullet$ is a symmetral mould. The equation yields

$$\begin{array}{rcl} x^{1-d}\partial_x z & = & x^{1-d}\partial_x\varphi(x,y) \\ & = & \sum x^{1-d}\partial_x(V_d^\bullet \mathbb{B}_\bullet y) \\ & = & \sum(x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet(x^{1-d}\partial_x(\mathbb{B}_\bullet y)) \\ & = & \sum(x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet(x^{1-d}\partial_x y)(\partial_y \mathbb{B}_\bullet y) \\ & = & \sum(x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet b(x,y)(\partial_y \mathbb{B}_\bullet y) \\ & = & \sum(x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \left(\sum I^\bullet \mathbb{B}_\bullet\right)\left(\sum V_d^\bullet \mathbb{B}_\bullet\right).y \\ & = & \sum(x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum(V_d^\bullet \times I^\bullet)\mathbb{B}_\bullet y \\ & = & 0 \end{array}$$

This suggest to look for a symmetral mould $V_d^\bullet$ such that $V_d^\emptyset = 1$ and

$$x^{1-d}\partial_x V_d^\bullet = -V_d^\bullet \times I^\bullet \tag{1}$$

Of course the conjugacy of $x^{1-d}\partial_x z = 0$ to $(E_{b,d})$ is given by the inverse of φ, which is given by the inverse of $V_d^\bullet$, namely $U_d^\bullet$, that satisfies the equation

$$x^{1-d}\partial_x U_d^\bullet = I^\bullet \times U_d^\bullet \tag{2}$$

A straightforward computation shows that one can make the following choice :

Proposition 3. *For $d \geq 1$, the moulds defined for $(n_1, \dots, n_s) \in \mathcal{N}$ by*

$$\begin{array}{rclc} U_d^{n_1,\dots,n_s} & = & \dfrac{x^{n_1+\dots+n_s+sd}}{(\hat{n}_1+sd)(\hat{n}_2+(s-1)d)\dots(\hat{n}_s+d)} & (\hat{n}_i = n_i+\dots+n_s) \\ V_d^{n_1,\dots,n_s} & = & \dfrac{(-1)^s x^{n_1+\dots+n_s+sd}}{(\check{n}_1+d)(\check{n}_2+2d)\dots(\check{n}_s+sd)} & (\check{n}_i = n_1+\dots+n_i) \end{array}$$

are symmetral and solutions of the previous equations. Moreover the substitution automorphism defined by $U_d^\bullet$ (resp. $V_d^\bullet$) conjugates $(E_{0,d})$ to $(E_{b,d})$ (resp. $(E_{b,d})$ to $(E_{0,d})$).

Unfortunately, if $d = 0$, the mould $V_d^\bullet$ is ill-defined (for example if $n_1 = 0$). This really looks like the situation that occurs in quantum field theory and calls for some renormalization. We will now describe a renormalization scheme at $d = 0$.

4 Renormalization in a shuffle Hopf algebra.

In order to use a renormalization scheme, we will first give the quite obvious Hopf algebra settings related to such symmetral moulds. We will then describe a very natural dimensional regularization and perform the renormalization in the "mould" terminology.

4.1 The shuffle Hopf algebra $\mathrm{sh}_{\mathcal{N}}$.

Once again, let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \ldots, n_s), \quad s \geq 1, \quad n_i \in \mathbb{N}\}$$

If

$$l(n_1, \ldots, n_s) = s \quad (l(\emptyset) = 0) \quad \|(n_1, \ldots, n_s)\| = n_1 + \ldots + n_s \quad (\|\emptyset\| = 0)$$

then the linear span of $\mathcal{N}$ is a graded (for the graduation $\|.\| + l(.)$) vector space with finite dimensional graded components. This space $\mathrm{sh}_{\mathcal{N}}$ turns to be a Hopf algebra with the following definitions. The product is as follows

- $\emptyset$ is the unit,
- For $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ in $\mathcal{N}$, the product $m : \mathrm{sh}_{\mathcal{N}} \otimes \mathrm{sh}_{\mathcal{N}} \to \mathrm{sh}_{\mathcal{N}}$ is defined by

  $$m(\boldsymbol{n}^1 \otimes \boldsymbol{n}^2) = \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} \boldsymbol{n}$$

  where the sum is over all the possible shuffling of the topees $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$.

For example

$$m((n_1) \otimes (n_2, n_3)) = (n_1, n_2, n_3) + (n_2, n_1, n_3) + (n_2, n_3, n_1)$$

With this product, $\mathrm{sh}_{\mathcal{N}}$ is a graded commutative algebra and it remains to define the coproduct $\Delta : \mathrm{sh}_{\mathcal{N}} \to \mathrm{sh}_{\mathcal{N}} \otimes \mathrm{sh}_{\mathcal{N}}$:

- $\Delta\emptyset = \emptyset \otimes \emptyset$.
- For $\boldsymbol{n} \in \mathcal{N}$,

  $$\Delta(\boldsymbol{n}) = \sum_{\boldsymbol{n} = \boldsymbol{n}^1 \boldsymbol{n}^2} \boldsymbol{n}^1 \otimes \boldsymbol{n}^2$$

  where the sum is over the pairs $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ whose concatenation gives $\boldsymbol{n}$.

For example,

$$\Delta(n_1, n_2, n_3) = (n_1, n_2, n_3) \otimes \emptyset + (n_1, n_2) \otimes (n_3) + (n_1) \otimes (n_2, n_3) + \emptyset \otimes (n_1, n_2, n_3)$$

With these product and coproduct, $\mathrm{sh}_{\mathcal{N}}$ is a very classical graded connected Hopf algebra whose antipode is given by

$$S(n_1, \ldots, n_s) = (-1)^s (n_s, \ldots, n_1)$$

For details on Hopf algebras and shuffle Hopf algebras, see [1].

When one deals with Hopf algebras, one can define characters and it is now obvious that symmetral moulds and characters are strongly connected : If $\mathfrak{A}$ is a commutative unital algebra then characters (algebra morphisms) on $\mathrm{sh}_{\mathcal{N}}$ with values in $\mathfrak{A}$ form a group for the product

$$\forall \varphi, \psi \in \mathcal{C}(\mathrm{sh}_{\mathcal{N}}, \mathfrak{A}), \quad \varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta$$

This group is isomorphic to Ecalle's group of symmetral moulds with values in $\mathfrak{A}$ since a symmetral mould can be identified to the image of the basis $\mathcal{N}$ by a character.

Now we are ready to express a renormalization scheme on some examples.

4.2 Divergences for the moulds (or characters) $U_d^\bullet$ and $V_d^\bullet$.

In our case $\mathfrak{A} = \mathbb{C}[[x]]$ and, as quoted before, our "characters" $U_d^\bullet$ and $V_d^\bullet$ are unfortunately ill-defined when $d = 0$. When looking at $V_d^\bullet$, if

$$\forall (n_1, \ldots, n_s) \in \mathcal{N}, \quad D(n_1, \ldots, n_s) = \max\{0 \leq i \leq s \quad ; \quad \forall 1 \leq j \leq i, \quad \check{n}_j = 0\}$$

from the physicists point of view :

- If $D(n_1, \ldots, n_s) = 0$, $V_d^{n_1,\ldots,n_s}$ has no divergence at $d = 0$,
- If $D(n_1, \ldots, n_s) = 1$, $V_d^{n_1,\ldots,n_s}$ has an overall divergence but no subdivergence at $d = 0$,
- If $D(n_1, \ldots, n_s) > 1$, $V_d^{n_1,\ldots,n_s}$ has an overall divergence and $D(n_1, \ldots, n_s) - 1$ subdivergences at $d = 0$.

Now the formula for $V_d^\bullet$ suggest that we could define a dimensional regularization by using the same formula for $d = \varepsilon \in \mathbb{C}^*$. The price to pay is to consider now that

$$x^\varepsilon = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} \log^n x$$

so that, for $\varepsilon \in \mathbb{C}^*$ close to zero, the mould $V_\varepsilon^\bullet$ has its values in $\mathcal{A} = \mathfrak{B}[[\varepsilon]][\varepsilon^{-1}]$ where $\mathfrak{B} = \mathbb{C}[[x, \log x]]$. Using the usual Birkhoff decomposition in terms of moulds, we get

Theorem 1. *There exists a unique pair of moulds* $(C_\varepsilon^\bullet, R_\varepsilon^\bullet)$ *such that*

$$R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$$

where $C_\varepsilon^\bullet$ *(counterterms) is symmetral with values in* $\mathcal{A}_- = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}]$ *and* $R_\varepsilon^\bullet$ *(regularized) is symmetral with values in* $\mathcal{A}_+ = \mathfrak{B}[[\varepsilon]]$*. Moreover*

$$C_\varepsilon^{n_1,\ldots,n_s} = \begin{cases} \dfrac{1}{s!\varepsilon^s} & \text{if} \quad n_1 = \ldots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

For a proof see section 6. We have now a renormalization scheme for our problem but, as in quantum field theory, this would be useless if it had no meaning for our equations. It is indeed meaningful as we shall see now.

5 Interpretation of the renormalized mould $R_\varepsilon^\bullet$.

5.1 Ramified conjugacy.

On one hand, the ill-definedness of $V_d^\bullet$ at $d = 0$ suggest that the equations

$$x\partial_x y = b(x, y)$$

cannot be formally (with diffeomorphisms in $\mathbb{C}[[x,y]]$) conjugated to the equation

$$x\partial_x z = 0$$

On the other hand, we chose a quite natural dimensional regularization for our mould $V_d^\bullet$ since for $\varepsilon \in \mathbb{C}^*$, we still have the equation

$$x^{1-\varepsilon}\partial_x V_\varepsilon^\bullet = -V_\varepsilon^\bullet \times I^\bullet$$

but now as $R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$ and $C_\varepsilon^\bullet$ does not depend on x,

$$\begin{array}{rcl} x^{1-\varepsilon}\partial_x R_\varepsilon^\bullet & = & x^{1-\varepsilon}\partial_x (C_\varepsilon^\bullet \times V_\varepsilon^\bullet) \\ & = & C_\varepsilon^\bullet \times (x^{1-\varepsilon}\partial_x V_\varepsilon^\bullet) \\ & = & -C_\varepsilon^\bullet \times V_\varepsilon^\bullet \times I^\bullet \\ & = & -R_\varepsilon^\bullet \times I^\bullet \end{array}$$

The mould $R_\varepsilon^\bullet$ (as $V_\varepsilon^\bullet$) defines a diffeomorphism that also conjugates the equation

$$x^{1-\varepsilon}\partial_x y = b(x,y)$$

to $x^{1-\varepsilon}\partial_x z = 0$. The mould $R_\varepsilon^\bullet$ is regular at $\varepsilon = 0$, with a price to pay : it contains monomials in x and $\log x$. When ε goes to 0, we get :

Theorem 2. *There exists a "ramified" identity tangent diffeomorphism $\varphi(x,y) \in y + y^2\mathbb{C}[[x, \log x, y]]$ that conjugates $x\partial_x y = b(x,y)$ to $x\partial_x z = 0$*

The need for logarithms, as well as the ill-definedness of a "formal" conjugating diffeomorphism, suggest that, in the case $d = 0$, some part of the right-hand term of the equation $x\partial_x y = b(x,y)$ cannot be canceled by formal conjugacy : there should remain some formal "invariants". The next natural question becomes : If one cannot formally conjugate to $x\partial_x z = 0$, what is the simplest equation to which one can conjugate ?

The following section gives a partial answer to this.

5.2 The logarithmic-alogarithmic factorization of $R_0^\bullet$ and its interpretation.

As it shall be proved in section 6, we have

Theorem 3. *The symmetral mould $R_0^\bullet$ admits the following factorization :*

$$R_0^\bullet = L^\bullet \times S^\bullet$$

where

1. $L^\bullet$ is a purely logarithmic symmetral mould defined for $(n_1, \ldots, n_s) \in \mathcal{N}$ by

$$L^{n_1,\ldots,n_s} = \begin{cases} \dfrac{(-1)^s}{s!}\log^s x & \text{if} \quad n_1 = \ldots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. $S^\bullet$ is a symmetral mould with values in $\mathbb{C}[[x]]$.

We have then

$$\begin{array}{rcc} x\partial_x R_0^\bullet & = & x\partial_x(L^\bullet \times S^\bullet) \\ & = & L^\bullet \times (x\partial_x S^\bullet) + (x\partial_x L^\bullet) \times S^\bullet \\ & = & -R_0^\bullet \times I^\bullet \\ & = & -L^\bullet \times S^\bullet \times I^\bullet \end{array}$$

and if $-L^\bullet \times A^\bullet = x\partial_x L^\bullet$,

$$x\partial_x S^\bullet + S^\bullet \times I^\bullet = A^\bullet \times S^\bullet$$

A straightforward computation shows that the mould $A^\bullet$ is alternal ($A^\emptyset = 0$) and for $(n_1, \ldots, n_s) \in \mathcal{N}$,

$$A^{n_1,\ldots,n_s} = \begin{cases} 1 & \text{if} \quad s = 1 \quad \text{and} \quad n_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum A^\bullet \mathbb{B}_\bullet y = b(0, y)$$

but now, if φ^{nor} is the formal diffeomorphism associated to $S^\bullet$ and $z = \varphi^{\mathrm{nor}}(x, y)$ with

$$x\partial_x y = b(x, y)$$

then, as in the computations for $V_d^\bullet$,

$$\begin{array}{rcc} x\partial_x z & = & x\partial_x \varphi^{\mathrm{nor}}(x, y) \\ & = & x\partial_x \left(\sum S^\bullet \mathbb{B}_\bullet y\right) \\ & = & \sum (x\partial_x S^\bullet + S^\bullet \times I^\bullet)\mathbb{B}_\bullet y \\ & = & \sum (A^\bullet \times S^\bullet)\mathbb{B}_\bullet y \\ & = & \left(\sum S^\bullet \mathbb{B}_\bullet\right)\left(\sum A^\bullet \mathbb{B}_\bullet y\right) \\ & = & b(0, \varphi^{\mathrm{nor}}(x, y)) \\ & = & b(0, z) \end{array}$$

Until $b(0, y) = 0$, the previous results suggests that the equation $x\partial_x y = b(x, y)$ cannot be formally conjugated to $x\partial_x z = 0$ but, at least, it is formally conjugated to a “normal” equation

$$x\partial_x z = b(0, z) = b_0(z)$$

Moreover, it is easy to check that the only way to conjugate $x\partial_x z = b_0(z)$ to $x\partial_x z = 0$ is to use a diffeomorphism in $\mathbb{C}[[\log x, z]]$, that corresponds to the mould $L^\bullet$ in the factorization of $R_0^\bullet$.

6 Proofs.

It remains to prove that $R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$, with

$$C_\varepsilon^{n_1,\dots,n_s} = \begin{cases} \dfrac{1}{s!\varepsilon^s} & \text{if} \quad n_1 = \dots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

and $R_0^\bullet = L^\bullet \times S^\bullet$ with

$$L^{n_1,\dots,n_s} = \begin{cases} \dfrac{(-1)^s}{s!} \log^s x & \text{if} \quad n_1 = \dots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let us suppose that $C_\varepsilon^\bullet$ is defined as above. Since it is symmetral, with values in $\mathcal{A}_- = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}]$, it is clear that $R_\varepsilon^\bullet$ is symmetral and it remains to prove that $R_\varepsilon^\bullet$ has its values in $\mathcal{A}_+ = \mathfrak{B}[[\varepsilon]]$. Let $0^{(k)}$ be the sequence with k zeros. Any non empty sequence in $\mathcal{N}$ can be written $\boldsymbol{n}^k = (0^{(k)}, n_1, \dots, n_s) = (0^{(k)}\boldsymbol{n})$ with $k \geq 0$, $s \geq 0$ and $\boldsymbol{n} = (n_1, \dots, n_s)$ is such that

$$\boldsymbol{n} = \emptyset \quad \text{or} \quad \boldsymbol{n} \neq \emptyset \quad \text{but} \quad n_1 \neq 0$$

It is clear now that

$$R_\varepsilon^{0^{(k)}\boldsymbol{n}} = (C_\varepsilon^\bullet \times V_\varepsilon^\bullet)^{0^{(k)}\boldsymbol{n}} = \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}\boldsymbol{n}}$$

Let us consider first the case $\boldsymbol{n} = \emptyset$, $k \geq 1$,

$$\begin{aligned} R_\varepsilon^{0^{(k)}} &= \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}} \\ &= \frac{1}{\varepsilon^k} \sum_{j=0}^{k} \frac{1}{(k-j)!} . \frac{(-1)^j x^{j\varepsilon}}{j!} \\ &= \frac{1}{k!} \left(\frac{1 - x^\varepsilon}{\varepsilon} \right)^k \end{aligned}$$

It is clear that, after expansion (in ε), this coefficient belongs to $\mathcal{A}_+$ and

$$R_0^{0^{(k)}} = \frac{(-\log x)^k}{k!} = (L^\bullet \times S^\bullet)^{0^{(k)}}$$

with, for $j \geq 1$, $S^{0^{(j)}} = 0$.

Let us suppose now that $\boldsymbol{n} = (n_1, \dots, n_s)$ is non empty and $n_1 \neq 0$. If $p_i = n_1 + \dots + n_i$, then, after expansion in the variable ε, for $j \geq 0$

$$\begin{aligned} V_\varepsilon^{0^{(j)}\boldsymbol{n}} &= \frac{(-1)^{j+s} x^{n_1+\dots+n_s+(s+j)\varepsilon}}{j!\varepsilon^j (p_1 + (j+1)\varepsilon) \dots (p_s + (j+s)\varepsilon)} \\ &= \frac{(-1)^{j+s} x^{\|\boldsymbol{n}\|}}{j!\varepsilon^j} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\varepsilon^{\|\boldsymbol{l}\|} (-1)^{\|\boldsymbol{l}\| - l_0}}{l_0! p_1^{l_1+1} \dots p_s^{l_s+1}} (\log x)^{l_0} \gamma^{l_0,\dots,l_s}(j) \end{aligned}$$

Where $\|\boldsymbol{n}\| = n_1 + \ldots + n_s$, $\boldsymbol{l} = (l_0, \ldots, l_s)$, $\|\boldsymbol{l}\| = l_0 + \ldots + l_s$ and $\gamma^{l_0,\ldots,l_s}(j) = (s+j)^{l_0}(j+1)^{l_1}\ldots(j+s)^{l_s}$. We have

$$\begin{aligned} R_\varepsilon^{0^{(k)}\boldsymbol{n}} &= (C_\varepsilon^\bullet \times V_\varepsilon^\bullet)^{0^{(k)}\boldsymbol{n}} \\ &= \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}\boldsymbol{n}} \\ &= \frac{(-1)^s x^{\|\boldsymbol{n}\|}}{\varepsilon^k} \sum_{j=0}^{k} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\varepsilon^{\|\boldsymbol{l}\|}(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0! p_1^{l_1+1}\ldots p_s^{l_s+1}} (\log x)^{l_0} \sum_{j=0}^{k} \frac{(-1)^j \gamma^{l_0,\ldots,l_s}(j)}{j!(k-j)!} \end{aligned}$$

In order to prove that $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$ is in $\mathcal{A}_+$, it is sufficient to check that, for $\|\boldsymbol{l}\| = l_0 + \ldots + l_s < k$ $(k \geq 1)$,

$$\theta^{l_0,\ldots,l_s}(k) = \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!j!} \gamma^{l_0,\ldots,l_s}(j) = 0$$

Let $t_0, \ldots, t_s$ be $s+1$ variables, then

$$f_j(t_0, \ldots, t_s) = \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\gamma^{l_0,\ldots,l_s}(j)}{l_0! \ldots l_s!} t_0^{l_0} \ldots t_s^{l_s} = e^{(s+j)t_0 + (j+1)t_1 + \ldots + (j+s)t_s}$$

We have

$$\begin{aligned} g_k(t_0, \ldots, t_s) &= \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\theta^{l_0,\ldots,l_s}(k)}{l_0! \ldots l_s!} t_0^{l_0} \ldots t_s^{l_s} \\ &= \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!j!} e^{j(t_0+\ldots+t_s)} e^{st_0 + t_1 + 2t_2 + \ldots + st_s} \\ &= \frac{1}{k!} \left(1 - e^{t_0+\ldots+t_s}\right)^k e^{st_0+t_1+2t_2+\ldots+st_s} \end{aligned}$$

It becomes clear that, in this series, if $l_0 + \ldots + l_s < k$, then $\theta^{l_0,\ldots,l_s}(k) = 0$ and this proves that $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$ is regular in ε.

In the series defining $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$, the value of $R_0^{0^{(k)}\boldsymbol{n}}$ is then given by

$$\begin{aligned}
R_0^{0^{(k)}\boldsymbol{n}} &= \sum_{j=0}^{k} \frac{(-1)^{j+s}x^{\|\boldsymbol{n}\|}}{(k-j)!j!} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s \\ l_0+\ldots+l_s=k}} \frac{(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0! p_1^{l_1+1}\ldots p_s^{l_s+1}} (\log x)^{l_0} \gamma^{l_0,\ldots,l_s}(j) \\
&= (-1)^s x^{\|\boldsymbol{n}\|} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s \\ l_0+\ldots+l_s=k}} \frac{(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0! p_1^{l_1+1}\ldots p_s^{l_s+1}} (\log x)^{l_0} \theta^{l_0,\ldots,l_s}(k) \\
&= \sum_{j=0}^{k} L^{0^{(k-j)}} \cdot \left(\sum_{\substack{l_t \geq 0 \\ 1 \leq t \leq s \\ l_1+\ldots+l_s=j}} \frac{(-1)^{k+s}x^{\|\boldsymbol{n}\|}}{p_1^{l_1+1}\ldots p_s^{l_s+1}} \theta^{k-j,l_1,\ldots,l_s}(k) \right)
\end{aligned}$$

To prove the second factorization, it remains to prove that

$$\sum_{\substack{l_t \geq 0 \\ 1 \leq t \leq s \\ l_1+\ldots+l_s=j}} \frac{(-1)^k}{p_1^{l_1+1}\ldots p_s^{l_s+1}} \theta^{k-j,l_1,\ldots,l_s}(k)$$

does not depend on $k \geq j$. If $l_0 = k-j \geq 0$ and $l_1+\ldots+l_s = j$, then

$$\begin{aligned}
\theta^{k-j,l_1,\ldots,l_s}(k) &= \theta^{l_0,l_1,\ldots,l_s}(l_0+l_1+\ldots+l_s) \\
&= \frac{\partial^{l_0+\ldots+l_s}}{\partial t_0^{l_0}\ldots\partial t_s^{l_s}} (g_k(t_0,\ldots,t_s))|_{t_0=\ldots=t_s=0}
\end{aligned}$$

but, because of the valuation of $g_k(t_0,\ldots,t_s)$, it is clear that if $l_0+\ldots+l_s = k$,

$$\begin{aligned}
\theta^{k-j,l_1,\ldots,l_s}(k) &= \frac{\partial^{l_0+\ldots+l_s}}{\partial t_0^{l_0}\ldots\partial t_s^{l_s}} \left(\frac{(1-e^{t_0+\ldots+t_s})^{l_0+\ldots+l_s}}{(l_0+\ldots+l_s)!} \right)_{t_0=\ldots=t_s=0} \\
&= (-1)^k
\end{aligned}$$

This proves that $S^\bullet$ is well-defined and that, if $\boldsymbol{n} \neq \emptyset$ and $n_1 \neq 0$,

$$S^{0^{(k)}\boldsymbol{n}} = (-1)^s x^{n_1+\ldots n_s} \sum_{\substack{l_t \geq 0 \\ 1 \leq t \leq s \\ l_1+\ldots+l_s=k}} \frac{1}{p_1^{l_1+1}\ldots p_s^{l_s+1}}$$

if $p_i = n_1+\ldots+n_i$. Of course, as $R_0^\bullet$ and $L^\bullet$ are symmetral, it automatically ensures that $S^\bullet$ is symmetral.

7 Conclusion.

Our results illustrate, in a simple situation, the interactions between Ecalle's work and Hopf algebras and renormalization. The same ideas can be adapted to a wide range of problems of conjugacy of local objects (formal or analytic differential equations, vector fields, difference equations, diffeomorphisms ...). See [6] for details.

At the formal level, one tries to conjugate such objects to a more simple one, for instance their linear part. For differential equations and vector fields, the attempted conjugating map is given by a character on some shuffle Hopf algebra and, when such obstructions as resonance occur, this leads to a ill-defined character. As in our example, some renormalization scheme can be applied and gives interesting results on the existence of "normal forms" and "ramified" conjugating maps. The same holds for difference equations and diffeomorphisms except that the conjugating map is associated to a character on a quasishuffle Hopf algebra (see [6], [8]), that is to say a "symmetrel" mould. In the case of difference equations, such characters are closely related to multizeta values.

In addition to the difficulties of such formal problems, one can also look at the analytic case, that is to say analytic conjugacy of analytic objects. In this case, two new difficulties arise and interacts with renormalization.

1. Mould-comould expansions are not well suited for analytics estimates since many terms contribute to a same monomial in the power series of the conjugating map. In many cases, a solution can be found in Ecalle's work : arborification-coarborification. Roughly speaking, the mould-comould expansion can be reorganized as a series of operators indexed by trees. This often gives better estimates that lead to analyticity and points out a new interaction between Ecalle'work and Hopf algebras since such "tree" expansion are closely related to characters on the Connes-Kreimer Hopf algebra of (eventually decorated) trees.

2. The second difficulty comes from the fact that, even after arborification-coarborification, the attempted conjugating map may remain formal but with some Gevrey estimates on the coefficients. One can then obtain "sectorial" analytic diffeomorphisms, using the usual tools of resummation, and this gives rise to a wide range of mathematical problems on the interactions between renormalization and resummation.

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