An introduction to Uniform Rectifiability

Guy David, Université Paris-Saclay

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This is a series of 3 lectures planned to give an idea of the notion, and some techniques for, uniform rectifiability. Please forgive the relatively old contents, as the notion and main basic results date from roughly 1990.
The present version was slightly revisited after the lectures, but is probably still full of mistakes. The longer text should be available too on my website.

Rectifiability

We will always consider measurable (often closed) subsets E of $\mathbb{R}^n$.
Recall that $E \subset \mathbb{R}^n$ is called d-rectifiable ($d \in \mathbb{N}^*$) when

$$E \subset Z \cup \big(\cup_{j \in J} A_j \big), \tag{1}$$

with $\mathcal{H}^d(Z) = 0$, J is at most countable, and each A_j is a C^1 surface of dimension d (we could also say, the image of a subset of $\mathbb{R}^d$ by a Lipschitz mapping).
Here and below, $\mathcal{H}^d$ is the Hausdorff measure.

Countable unions of rectifiable sets are rectifiable, and, among sets of σ-finite $\mathcal{H}^d$ measure, rectifiable sets can be characterized by the existence $\mathcal{H}^d$-almost everywhere of approximate tangent planes, or the fact that they have the same density $\mathcal{H}^d$-a.e. as d-planes, or the fact that their projection on almost every d-plane has vanishing $\mathcal{H}^d$-measure.

On the opposite, the unrectifiable sets have the opposite definition ($\mathcal{H}^d(E \cap A) = 0$ for every C^1 surface of dimension d) and properties. Pictures soon (Cantor sets).

Ahlfors-regular sets (AR)

It will be convenient to restrict to Ahlfors regular (AR) sets of dimension d, i.e., closed sets $E \subset \mathbb{R}^n$ such that

$$C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C r^d \text{ for } x \in E \text{ and } 0 < r < \mathrm{diam}(E). \tag{2}$$

Here we allow all $d \in (0, n]$, and bounded and unbounded sets E. In short, $E \in AR(d)$. An Ahlfors regular measure of dimension d is a positive measure μ such that

$$C^{-1} r^d \leq \mu(E \cap B(x,r)) \leq C r^d \text{ for } x \in E \text{ and } 0 < r < \mathrm{diam}(E), \tag{3}$$

where E is the closed support of μ. In short, $\mu \in AR(d)$. It is easy to check that then $E \in AR(d)$ and $C^{-1}\mathcal{H}^d_{|E} \leq \mu \leq C\mathcal{H}^d_{|E}$ (cover $E \cap B(x,r)$ by balls of radius ηr and use (3) to count the balls).

We could often deal with non AR measures (and not even doubling), but often this is more complicated. Some times it is needed though.

Comment: AR is about size, not regularity.

Uniformly rectifiable sets (UR) of dimension 1

For $d = 1$ the definition is simpler. We say that $z : I \to \mathbb{R}^n$, 1-Lipschitz, is an Ahlfors regular parameterization (with $I \subset \mathbb{R}$ an interval) when

$$\left|z^{-1}(B(x,r))\right| \leq Cr \quad \text{for } x \in \mathbb{R}^n \text{ and } r > 0. \tag{4}$$

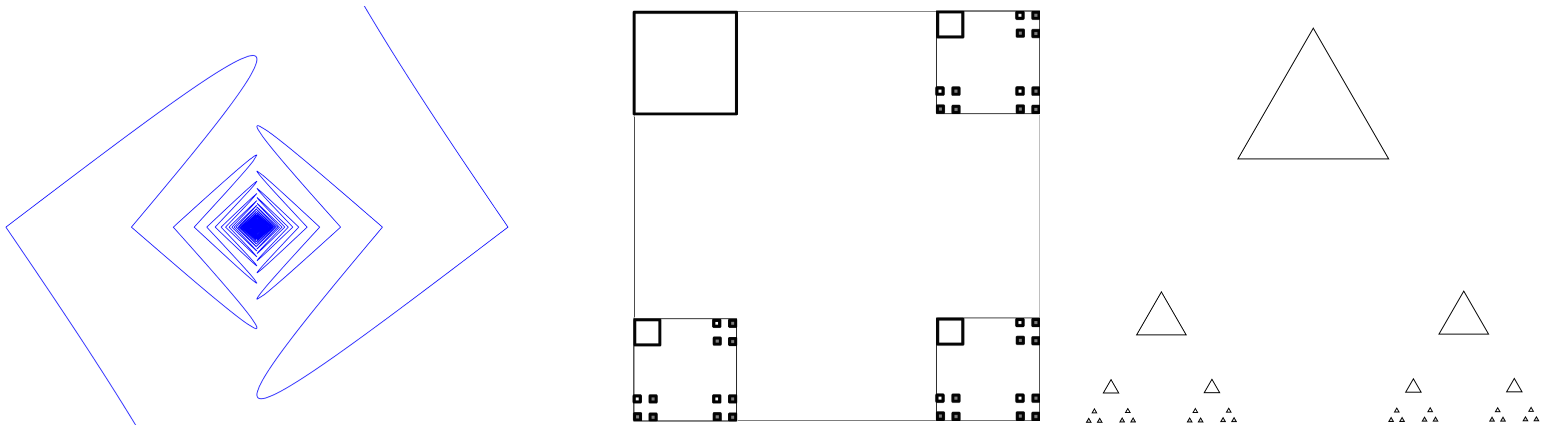
And $E = z(I)$ is called an Ahlfors regular curve. [Note, the lower bound for AR holds : the projection on a diameter is connected.] Examples: a line, a Lipschitz curve. But not the infinite cross X. So, connected Ahlfors regular sets (of dimension 1) are a little more general. But not so much: every connected Ahlfors regular sets of dimension 1 is contained in a regular curve (a simple parameterization lemma for connected graphs).

Definition

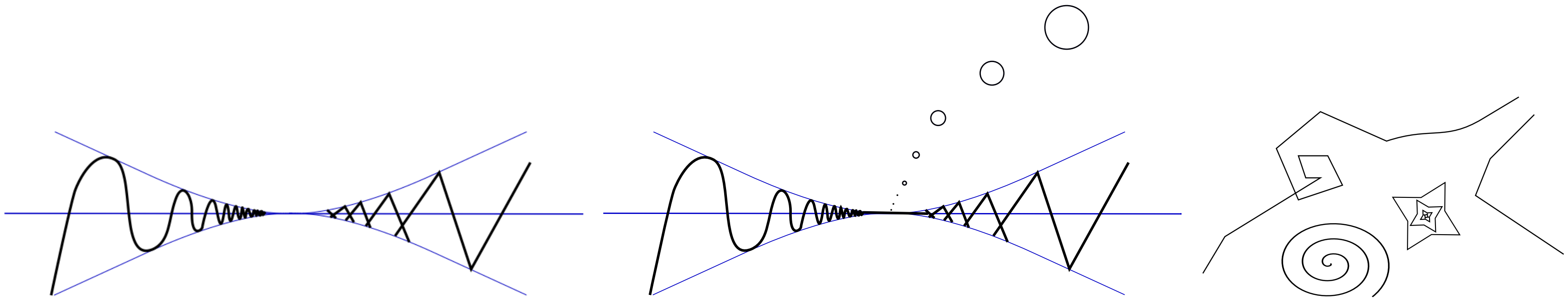
The set $E \in AR(1)$ is called uniformly rectifiable when there is an Ahlfors regular curve Γ (or, equivalently, a connected set $\Gamma \in AR(1)$) such that $E \subset \Gamma$.

Other equivalent definitions below. But this one is the simplest.

Some Pictures



Left: This set is not AR (but countably rectifiable).
Right: Rectifiable but not UR sets (if we add more detail correctly)



Left: A true tangent line
Middle: Only an approximate tangent line (but the set is not AR)
Right: an UR set (logarithmic spirals are allowed).

UR sets of dimension $d \geq 1$, definition by BPLI

The simplest definition for $d > 1$ is probably by BPLI:

Definition

We say that the set $E \in AR(d)$ contains big pieces of Lipschitz images of $\mathbb{R}^d$, in short $E \in BPLI(d)$) when there exist constants $\theta > 0$ and $M \geq 1$ such that, for $x \in E$ and $0 < r < \mathrm{diam}(E)$, we can find an M-Lipschitz mapping $\Phi : rQ_0^d \to \mathbb{R}^n$ such that

$$\mathcal{H}^d(E \cap B(x,r) \cap \Phi(rQ_0^d)) \geq \theta r^d. \tag{5}$$

Here Q_0^d denotes the unit cube in $\mathbb{R}^d$.

And let us say that $E \in UR(d)$ when ($E \in AR(d)$ and) $E \in BPLI(d)$.

Other equivalent definitions exist; see later.

We mention one with parameterizations because it looks tempting, but it is not as useful as we may think.

ω-regular parameterizations (a tempting definition but...)

It is possible to parameterize UR sets as follows. For any weight $\omega \in A_1(\mathbb{R}^d)$ (see below), define ω-regular parameterizations as follows. These are the mappings $z : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$|z(x) - z(y)| \leq C \int_{B((x+y)/2,|x-y|)} \omega^{1/d} \quad \text{for } x, y \in \mathbb{R}^d \tag{6}$$

and, for $x \in \mathbb{R}^n$ and $r > 0$,

$$\left|z^{-1}(B(x,r))\right|_{\omega du} := \int \mathbb{1}_{z(u)\in B(x,r)}\omega(u)du \leq Cr^d. \tag{7}$$

In dimension $d = 1$, we can content ourselves with $\omega = 1$ and regular parameterizations, but in general the best we can do is use A_1-weights, i.e., weights ω such that $\int_B \omega \leq C|B| \inf_B \omega$ for all B. It turns out that (with some work):

- If z is an ω-regular parameterization, then $E = z(\mathbb{R}^d)$ is UR
- For $E \in UR(d)$ there existe $z : \mathbb{R}^d \to \mathbb{R}^{n^*}$ such that $E \subset z(\mathbb{R}^d)$.

Here $n^* = \min(n+1, 2d)$ (some space is needed to avoid too much crossing for the extensions).
Local versions with E of finite diameter exist too.

Why should we care? (1)

Our initial motivation (with S. Semmes) was to find necessary and sufficient conditions on $E \in AR(d)$ for our usual singular integrals to define bounded operators on $L^2(E, d\mu)$.
We did this, but for the converse part we used a large class of SIO's. The more natural question is rather: suppose that the Riesz kernels **alone** define bounded operators on $L^2(E, d\mu)$, can you say that $E \in UR$? Because of relations with harmonic functions.
Riesz kernels means, all the kernels $K_j(x-y)$, with $K_j(x)\frac{x_j}{||x||^{d-1}}$.
And we ask for uniform bounds for the truncated operators T_ε defined by

$$T_\varepsilon f(x) = \int_{E \setminus B(x,\varepsilon)} K(x-y) d\mu(y). \tag{8}$$

Positive answers by P. Mattila, M. Melnikov, J. Verdera (96) when $d=1$ and $n=2$ (the Cauchy kernel) and by F. Nazarov, X. Tolsa, and A. Volberg (2014) when $d=n-1$. Still unknown in general!

Why should we care? (2)

More generally, interested in relations between various purely geometric properties (as for rectifiability), and Analysis on E or near E (the boundedness of the Riesz transforms is an example),

There are also circumstances where interesting sets are UR. For instance, minimizers of the Mumford-Shah functional, perimeter almost minimizers, almost minimal sets (as in Plateau's problem), other free boundaries (as in the Alt Caffarelli-Friedman functionals). Although we often expect better regularity from these objects.

Also, UR is the right condition for mutual quantitative (A_∞) absolute continuity of harmonic measure and σ, in the context of Non Tangential Access domains with $AR(n-1)$ boundaries.

We probably won't develop, but S. Semmes has a proof of A_∞ absolute continuity under "Condition B" that relies on a Corona Construction (and there is another one with D. Jerison based on "big pieces" and the maximum principle).

Big projections

Unrectifiable sets have vanishing projections in almost all directions (as for the Cantor set above).

Not the UR sets, since they are rectifiable, but the following big projection property is then more precise.

Let $E \in AR(d)$. We say that E has big projections when there exists $\theta > 0$ such that such that for all $x \in E$ and $0 < r < \mathrm{diam}(E)$ we can find a d-plane P such that

$$\mathcal{H}^d(\pi_P(E \cap B(x,r))) \geq \theta r^d. \tag{9}$$

Here π_P is the orthogonal projection from $\mathbb{R}^n$ onto P.

Big projections alone do not imply that $E \in UR$.

If $E \in UR$ has big projections, then $E \in BPLG$, a condition strictly stronger than UR (counterexamples with venitian blinds).

Very nice recent results (T. Orponen, A. Chang, D. Dąbrowski, M. Villa, and others) on sufficient conditions for UR involving more big projections. But we skip.

Flatness 1: Peter Jones numbers

Recall rectifiable d-sets have approximate tangent d-planes almost everywhere. Here for AR sets, approximate tangent planes are true tangent planes. We want to quantify this. The simplest is to use

$$\beta(x,r) = \beta_{E,\infty}(x,r) = \frac{1}{r} \inf_P \sup_{y \in E \cap B(x,r)} \mathrm{dist}(y,P), \tag{10}$$

where the infimum is taken over all d-planes P. But for characterizations, we some times need to use the less precise L^p-variants, $1 \le p < +\infty$, defined (for $x \in E$) by

$$\beta_p(x,r) = \beta_{E,p}(x,r) = \frac{1}{r} \inf_P \left\{ r^{-d} \int_{y \in E \cap B(x,r)} \mathrm{dist}(y,P)^p d\mu \right\}^{1/p}$$

where μ is a given AR-measure on E (such as $\mathcal{H}^d_{|E}$) so the inside bracket looks like an average of $\mathrm{dist}(y,P)^p$. We normalized both to be invariant by dilations.

Jones initially used those numbers to compare the Cauchy operator for a Lipschitz graph, or a regular curve, to the same for the straight line. With errors controlled by the $\beta(x,r)^2$.

Intermission: Carleson measures

Many scale invariant quantities involve Carleson measure (because they are nicely invariant under translations and dilations).
Let $E \in AR(d)$, and fix an AR measure μ on E.
A Carleson measure on $E \times (0, \mathrm{diam}(E))$ (the set of balls) is a positive measure $\mathcal{M}$ on $E \times (0, \mathrm{diam}(E))$ such that

$$\mathcal{M}(B(X,R) \times (0,R]) \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \mathrm{diam}(E).$$

We also say that a function f on $E \times (0, \mathrm{diam}(E))$ satisfies a square Carleson estimate when

$$|f(x,r)|^2 \frac{d\mu(x)dr}{r} \quad \text{is a Carleson measure.} \tag{11}$$

That is, when

$$\int_{(B(X,R)\times(0,R]} |f(x,r)|^2 \frac{d\mu(x)dr}{r} \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \mathrm{diam}(E).$$

The "invariant" measure $\frac{d\mu(x)dr}{r}$ is NOT Carleson, so the condition says that $f(x,r)$ is often small! And the square is convenient because often (11) comes from some sort of orthogonality.

Geometric lemmas (often useful)

A necessary and sufficient condition for $E \in AR(d)$ to be uniformly rectifiable is that $E \in WL(p)$, i.e.,

$$(x, r) \mapsto \beta_p(x, r) \text{ satisfies a square Carleson condition.} \qquad (12)$$

More precisely: this is the case for all $1 \leq p \leq +\infty$ when $d = 1$, but only for $1 \leq p < \frac{2d}{d-2}$ when $d \geq 2$. Otherwise, counterexample by Xiang Fang that does not satisfy the $GL(p)$ for p large (with many small little tents).

Jones and Okikiolu had a characterization by the β_∞ of "$E \subset \Gamma$ for some curve Γ of finite length", even without AR.

Many more results exist, including in Carnot groups, metric spaces.

Main reason: for the graph of $A : \mathbb{R}^d \to \mathbb{R}^{n-d}$, this amounts to controlling the good approximation of A by affine functions, in L^p-norms. This uses a result of Dorronsoro with Littlewood-Payley decompositions.

And a major ingredient for general AR sets is a Corona Construction (see later)!

Density and Tolsa's α-numbers (here d is an integer)

Recall that for rectifiable sets $\mathcal{H}^d_{|E}$ has a density at almost all points $x \in E$. What about $f(x)\mathcal{H}^d_{|E}$, or in our case μ, where $\mu \in AR(d)$? We will measure how close μ is to flat measures. Call $\mathcal{F} = \{\lambda \mathcal{H}^d_{|P}\,;\, \lambda > 0 \text{ and } P \text{ is an affine } d\text{-plane}\}$ (the set of nontrivial flat measures). Then set

$$\alpha(x,r) = \inf_{\nu \in \mathcal{F}} d_{x,r}(\mu,\nu), \tag{13}$$

with (the variant of Wasserstein distance)

$$d_{x,r}(\mu,\nu) = r^{-d-1} \sup_{\varphi} \left| \int_{B(x,r)} \varphi d\mu - \int_{B(x,r)} \varphi d\nu \right|, \tag{14}$$

where the supremum is over 1-Lipschitz functions φ supported in $B(x,r)$ Thus $\alpha(x,r) \leq C$ and α has the usual scale invariance. Notice that $\beta_1(x,r) \leq C\alpha(x,2r)$ (try a truncated $\mathrm{dist}(y,P)$), but α also controls the irregularity of μ! And...

X. Tolsa 2009: Let $E \in AR(d)$, d integer. Then $E \in UR(d)$ if and only if $(x,r) \mapsto \alpha(x,r)$ satisfies a square Carleson condition. Proof by Littlewood-Payley and stopping times.

The Weak geometric lemma (WGL)

We say that $E \in AR(d)$, d integer satisfies the Weak Geometric Lemma ($E \in WGL$) if for each $\varepsilon > 0$ the bad set

$$\mathcal{B}(\varepsilon) = \{(x, r) \in E \times (0, \mathrm{diam}(E))\,;\ \beta(x, r) > \varepsilon\}$$

satisfies a Carleson packing condition, i.e.,

$$\int_{x \in B(X,R)} \int_{0<r<R} \mathbb{1}_{\mathcal{B}(\varepsilon)}(x, r) \frac{d\mu(x)dr}{r} \leq CR^d \tag{15}$$

for $X \in E$ and $0 < R < \mathrm{diam}(E)$. Equivalently, if $(x, r) \mapsto \mathbb{1}_{\mathcal{B}(\varepsilon)}(x, r)$ satisfies a (square) Carleson condition.
Easy: using any $\beta_p(x, r)$, $1 \leq p \leq +\infty$, yields an equivalent number (only $C = C(\varepsilon)$ changes).
And any $WG(p)$ implies the WGL by Chebyshev ($=$ Markov). But the condition seems much weaker.

Why did we care then? Because if $E \in WGL$ also has big projections (as in (9)), then $E \in UR$ and even $E \in BPLG$.

Some snowflakes

Use the Van Koch snowflake construction, except that at stage k we replace each of the 4^k segments I of length 4^{-k} that compose K_k with 4 segments of length 4^{-k-1} that make angles 0, α_k, $-\alpha_k$, 0 with I [Partial picture below].

We leave small holes of size about $\alpha_k^2 4^{-k-2}$ between the segments.

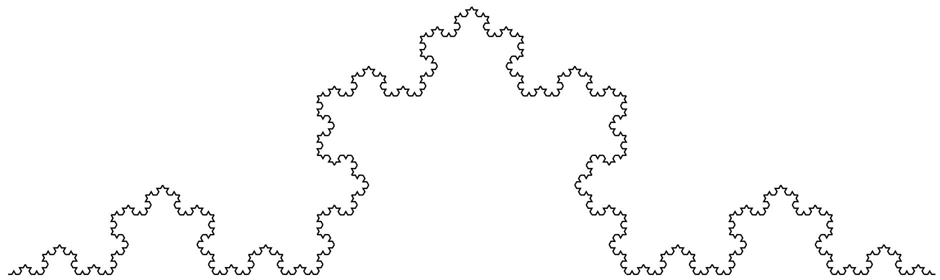
And we may ask about UR for the limiting set K.

Easy: if $\sum_k \alpha_k^2 < +\infty$, $K \in UR$ (and the regular curve that contains K is easy to find, that fills the holes). Converse true and logical (how else could you fill?)

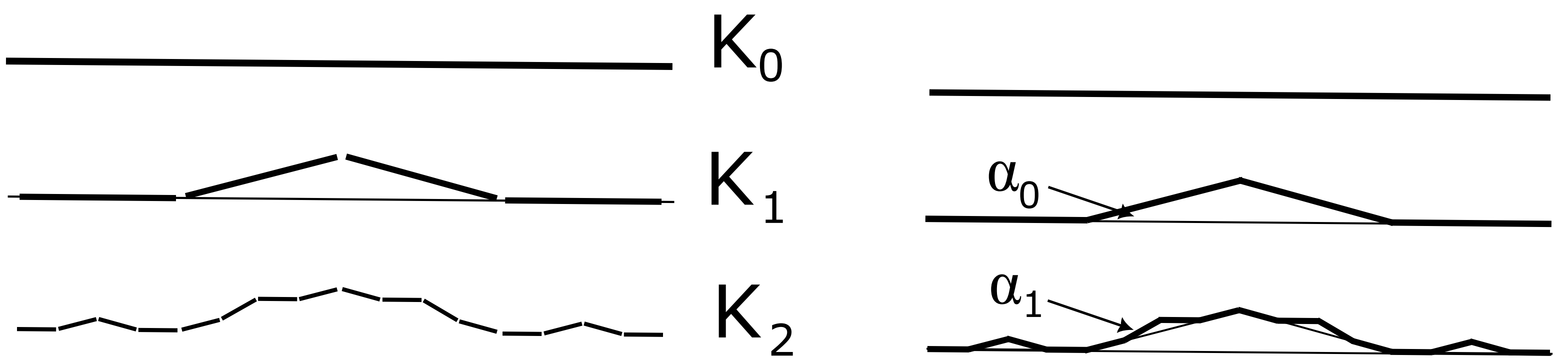
Straightforward: for $1 \leq p \leq +\infty$, $K \in GL(p)$ iff $\sum_k \alpha_k^2 < +\infty$.
But $E \in WGL$ iff $\lim_{k \to +\infty} \alpha_k = 0$.
And E satisfies the "weak bilateral geometric lemma" (see below) iff $\sum_k \alpha_k^2 < +\infty$ again (the bad balls are those that see a hole well, near a tip for instance).
Tangent planes everywhere when $\sum \alpha_k < +\infty$ (so, not always).

Pictures of three Koch Snowflakes

Picture of the Van Koch construction, and angles of 60 degrees:



But here we do lots of small holes between the segments, to preserve the fact that each approximation K_k has exactly the same length 1.
Also, we use different angles in the construction (hinted on the right). And then we discuss the condition above and UR happens iff $\sum_k \alpha_k^2 < +\infty$.



Weak conditions (the BWGL)

We were surprised that apparently much weaker conditions are also equivalent to *UR*. We just give one example, the *BWGL*, but there are variants (weak symmetry, weak convexity). We use the bilateral Jones number

$$b\beta(x,r) = \inf_P d_{x,r}(E,P) \tag{16}$$

where the supremum is over all affine d-planes P and $d_{x,r}$ is the normalized local Hausdorff distance

$$d_{x,r}(E,F) = \frac{1}{r} \sup_{y \in E \cap B(x,r)} \mathrm{dist}(y,F) + \frac{1}{r} \sup_{z \in F \cap B(x,r)} \mathrm{dist}(z,E) \tag{17}$$

(minor adaptations with the empty sets).

We say that E satisfies a bilateral weak geometric lemma (BWGL) when for $\varepsilon > 0$, $\mathcal{B}(\varepsilon) = \{(x,r) \in E \times (0, \mathrm{diam}(E))\,;\, b\beta(x,r) > \varepsilon\}$ satisfies a Carleson packing condition.

Thm [D., Semmes 93] For $E \in AR(d)$, $E \in UR$ iff $E \in BWGL$.

Proof by Corona construction and Reifenberg.

Dyadic (pseudo)cubes [D, C, DM]

These are often needed when we do stopping time arguments.
But the details are not interesting or surprising.
Let $E \in AR(d)$ be given (any $d > 0$).
We can find, for $k \in \mathbb{Z}$ such that $2^{-k} < \mathrm{diam}(E)$, say, partitions of E into "cubes" Q, $Q \in \Delta_k$, (measurable subsets of E) with the following properties:

- For each $Q \in \Delta_k$, there is a center $x_Q \in E$ such that
  $E \cap B(x_Q, C^{-1}2^{-k}) \subset Q \subset E \cap B(x_Q, C2^{-k})$;
- If $R \in \Delta_k$ and $Q \in \Delta_{k+1}$, then either $Q \subset R$ or $Q \cap R = \emptyset$;

(thus, the usual nesting properties of dyadic cubes in $\mathbb{R}^n$) and a quite useful "small boundary" property:

- for $Q \in \Delta_k$ and $\tau < 1$,

$$\mu(\{x \in Q\,;\, \mathrm{dist}(x, E \setminus Q) \leq \tau 2^{-k}\}) + \mu(\{x \in E \setminus Q\,;\, \mathrm{dist}(x, Q) \leq \tau 2^{-k}\}) \leq C\tau^a 2^{-kd}.$$

Here $a > 0$ is a constant that depends on the AR constants. We'll use those sets like one uses dyadic cubes.

Call $\Delta = \cup_k \Delta_k$ our set of (pseudo)cubes in $E \in AR(d)$.

Let $\mathcal{B} \subset \Delta$. We say that $\mathcal{B}$ satisfies a (Carleson) packing condition (in short, $\mathcal{B} \in Capac$) when there is $C \geq 0$ such that

$$\sum_{Q \in \mathcal{B}; Q \subset B(X,R)} \mu(Q) \leq CR^d \quad \text{for } X \in E \text{ and } 0 < R < \mathrm{diam}(E). \tag{18}$$

This is the analogue for cubes as our smallness Carleson condition on sets of bad balls (for the WGL and the such).

So for instance if $E \in WGL$, then for every $\varepsilon > 0$, the set

$$\mathcal{B}(\varepsilon) = \left\{ Q \in \Delta \,;\, \beta(c_Q, 10\,\mathrm{diam}(Q)) > \varepsilon \right\} \tag{19}$$

satisfies a Capac.

Stopping time regions 1: initiation

The corona construction (in general) consists in picking any starting cube $Q_0 \in \Delta$, and decomposing $\Delta(Q_0)$, the set of subcubes of Q_0, into (a disjoint union of) stopping time regions $\mathcal{S}$, $\mathcal{S} \in \Sigma$. Named after Carleson's proof of the absence of a corona.
There are definitions and rules (coherent regions, etc) on how we are allowed to do this, but here is the standard way.
We actually define, for each $Q \in \Delta$, a region $\mathcal{S}(Q)$ below Q.

Pick an initial class $\mathcal{B}$ of "bad cubes". For instance, assuming the *WGL*, pick ε very small and take the set $\mathcal{B}(\varepsilon)$ of (19). And for $Q \notin \mathcal{B}$, choose a d-plane $P(Q)$ that approximates E well in $B(c_Q, 10\,\mathrm{diam}(Q))$.
Then also choose various reasons, possibly depending on Q, to stop at a cube $R \subset Q$. We always stop at $R \subset Q$ when $R \in \mathcal{B}$, but we could decide to stop for other reasons. For instance, the standard one, if $P(R)$ makes an angle $\geq C\varepsilon$ with $P(Q)$.
Or when $b\beta(c_R, C\,\mathrm{diam}(R)) \geq C\varepsilon$ (when we assume the *BWGL*).

Stopping time regions 2: the region $\mathcal{S}(Q)$

Call $Stop_0(Q)$ all the cubes $R \subset Q$ such that we decided to stop at R. For organizational reasons, we use the larger $Stop_1(Q)$, the set of $R \subset Q$ such that R, or one of its sibblings (the cubes of the same generation as R, which have the same parent in the previous generation) lies in $Stop_0(Q)$.
And, to have the same notation as others, just call $Stop(Q)$ the maximal cubes like this (this does not change the line below).

The region $\mathcal{S}(Q)$ is just $\{Q\}$ alone if $Q \in \mathcal{B}$ or we stop at Q for some other reason.
And otherwise, it is the set of subcubes R of Q such that none of the cubes R', $R \subset R' \subset Q$ lie in $Stop_1(Q)$.
We thus exclude the cubes of $Stop(Q)$ and their descendants, that will have to be included in next regions.

It could be that $Stop(Q)$ is empty (we never stop) and then we will be happy. Or that the cubes of $Stop(Q)$ cover Q (then we are less happy, but we don't despair yet). But in general the cubes of $Stop(Q)$ only cover part of Q, and we often control the rest nicely.

Added pictures (since I did them for the final text)

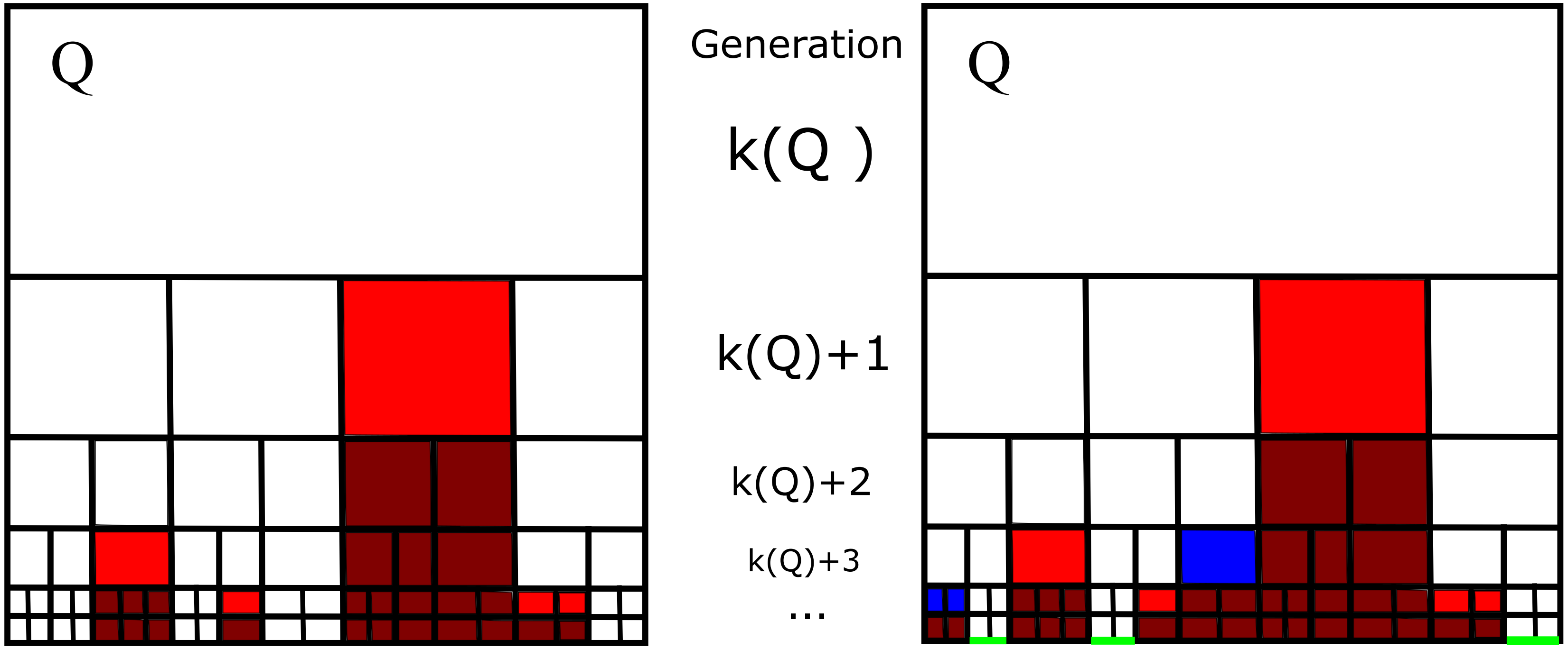


Figure: Symbolic pictures of $\mathcal{S}(Q) \subset \Delta(Q)$. Left: With only red stopped cubes (we also remove the subcubes, in brown). Right: With additional blue stopped cubes. I also marked the access region $A(Q)$ in green.

Stopping time regions 3: the decomposition of $\Delta(Q_0)$

Finally here is the decomposition of $\Delta(Q_0)$ into regions $\mathcal{S}$, $\mathcal{S} \in W$.

First we announce that all the cubes $Q \subset Q_0$ which lie in the bad set $\mathcal{B}$ will be associated to a region $\mathcal{S}(Q)$ composed of Q alone.

We start from Q_0, and define $\mathcal{S}(Q_0)$ as above. When this is done, we still have to cover the union of all the collections of subcubes of R, $R \in Stop(Q_0)$. So we add all the $\mathcal{S}(R)$, $R \in Stop(Q_0)$.
Now we just need to add the subcubes of cubes S of $Stop(R)$, $R \in Stop(Q_0)$. We add the corresponding regions $\mathcal{S}(S)$, and continue like this. At the end we get our disjoint covering of Δ.

Notice that all our regions $\mathcal{S}$ are of the form $\mathcal{S}(Q_0)$, or $\mathcal{S}(Q)$ for some $Q \in Stop(R)$ for some previous R. This we can see W as a collection of stopped cubes, plus Q_0.
I add a picture about this in the next slide.

We can always do all that, but what is the point?

Added picture (since I did them for the final text)

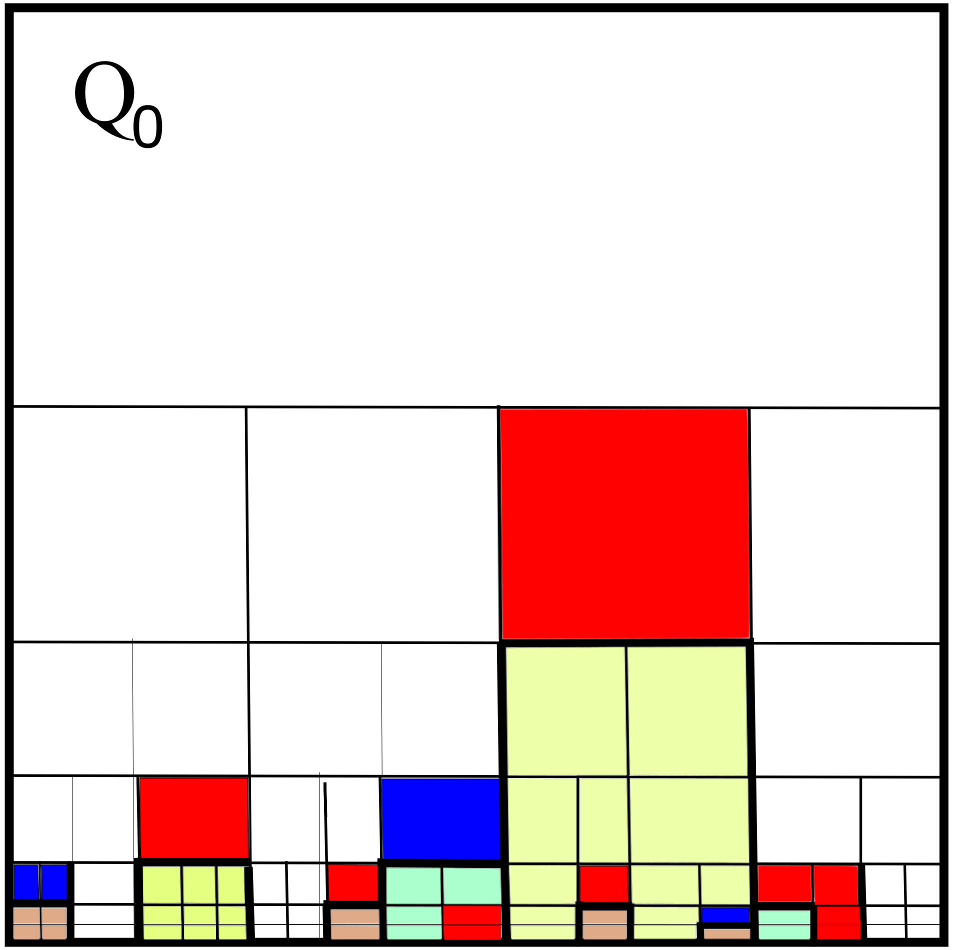


Figure: Symbolic picture of the covering of $\Delta(Q_0)$ by regions $\mathcal{S}(Q)$; we keep the colors of the red and blue cubes, which are thus at the bottom of their region. The main region is in white, the subsequent ones in colors.

Corona decompositions

Let $E \in AR(d)$, d integer. We give a last equivalent definition of $E \in UR$.

We say that E has a corona decomposition when (for every small $\varepsilon > 0$, and then for all Q_0) it is possible to construct regions $\mathcal{S}$ as above (with $\mathcal{B} \supset \mathcal{B}(\varepsilon)$), so that for each of our region $\mathcal{S} = \mathcal{S}(Q)$, $Q \in W$

$$Angle(P(R), P(Q)) \leq C\varepsilon \text{ for } R \in \mathcal{S} \tag{20}$$

and (important because stopping all the time won't help)

$$\text{the top cubes } Q,\ Q \in W \text{ satisfy a Packing condition} \tag{21}$$

(with a constant that depends only on ε and C).

The original definition is a little more flexible, but this is OK.

With (21), then by Chebyshev, almost every $x \in Q_0$ lies in at most a finite number stopped cubes. That is, all the small enough cubes that contain x lie in a single region $\mathcal{S}$. This is good to get geometric control at x.

Corona decompositions 2: a statement

Theorem [DS 93]: Let $E \in AR(d)$, d integer. Then $E \in UR$ if and only if E has a corona decomposition.

Comments: Only one (very small) ε is needed. And, as hinted above, (21) is the true important condition.
Very useful definition because it is easy to use both ways.

We start with a general observation: with only (20), we can prove that "in a given $\mathcal{S} = \mathcal{S}(Q)$", E is well approximated by a (rotated) $C\varepsilon$-Lipschitz graph $\Gamma = \Gamma(\mathcal{S})$: for $x \in Q$ (and even near Q),

$$\mathrm{dist}(x, \Gamma) \leq C\varepsilon d(x), \tag{22}$$

where $d(x) = \inf_{R \in \mathcal{S}}(\mathrm{diam}(R) + \mathrm{dist}(x, R))$.
[Unless $Q \in \mathcal{B}$, and then we say nothing.]

In particular, $d(x) = 0$ and $x \in \Gamma$ in the good access set $Q \setminus \bigcup_{S \in Stop(Q)} S$.
Proved by easy geometry and partitions of unity. And then we can use Γ to prove results on E (near $\mathcal{S}$).

How to use a Corona decompositions 1

• First example: if we have a corona decomposition, how do we construct a good parameterization of a set $G \supset E$?
We won't do this here, because it is technical, but you will hopefully believe that this makes sense, because...
We have a **not too large** collection of small Lipschitz graphs $\Gamma(\mathcal{S})$, $\mathcal{S} \in W$ (by the Capac for the top cubes), and almost every point of Q_0 connects to at least one of them. We "just" need to glue them, with added pieces to connect them, and find coherent parameterizations.
A bit more detail was given in the lecture and in the notes.
When $2d > n$, we need a bit of extra rom so that the added pieces don't cross too much, whence n^* above.
And we have to make the parameterization go a little faster each time we change to a smaller version, whence the weight $\omega \in A_1$.
• Assuming $BWGL$, we can build regions where (20) is replaced by bilateral approximation by d-planes that are allowed to turn, replace the $\Gamma(\mathcal{S})$ above by Reifenberg-flat Ahlfors-regular sets, and eventually prove that E has a corona decomposition (if I recall correctly).

How to use a Corona decompositions 2

• Similar argument have been used to prove estimates for solutions of elliptic equations on Ω when $\partial\Omega \in UR$, reducing via corona decompositions like the one above to the case of Lipschitz graphs. To my knowledge the first time this was done is by S. Semmes, who proved the A_∞-absolute continuity of harmonic measure under non-tangential access, $AR(n-1)$ boundary, and Condition B (large balls in the complement too). [There was also a proof by D-Jerison, using the earlier more basic Big Pieces tool.]

Other works by Hofmann, Martell, Mayboroda, Dabrowski, Mourgoglou, Poggi, Tolsa, and others.

• Other types of work mixing geometry of $\partial\Omega$ and analysis on Ω, where even non-AR-sets (and non- doubling measures) are used. Maybe see Tolsa's book.

How do we prove the existence of a Corona construction?

Only one example, from the geometric lemma to the corona condition.
Let $E \in GL(p)$ be given. Choose the $\mathcal{S}$, $\mathcal{S} \in W$, as explained above; we just need to show (21).

In the mean time we can use the graphs $\Gamma(\mathcal{S})$ to prove estimates near each $\mathcal{S}$!
This way, the corona construction acts like a linearization, that allows us to go from estimates that we know on small Lipschitz graphs to estimates that work on any set $E \in AR(d)$.

Anyway, how do we prove (21), i.e., control the number of stopped cubes (recall that every top cube, except Q_0, is a stopped cube of the previous step)?

We need to show that each $\mathcal{S}$ contributes in some controlled way to some quantity that we control, with "bounded overlap". We will distinguish 3 types of regions $\mathcal{S} = \mathcal{S}(Q)$, that we control with 3 different quantities (from the assumptions).

How do we prove the existence of a Corona construction 2

• First type of $\mathcal{S} = \mathcal{S}(Q)$: those for which $Q \in \mathcal{B}$ (a bad cube). Call this $Q \in W_1$ Then use the Carleson Packing Condition on the bad cubes (see (19)): $\sum_{Q \in W_1} \mu(Q) \leq Cdiam(Q_0)^d$.

• Second type: those for which the accessible set $Q_{acc} = Q \setminus \bigcup_{R \in Stop(Q)} R$ is large, i.e., $\mu(Q_{acc}) \geq \frac{1}{2}\mu(Q)/2$. Call this $Q \in W_2$. Then use the fact that $\mu \in AR$, and that the Q_{acc} are disjoint:

$$\sum_{Q \in W_2} \mu(Q) \leq 2 \sum_{Q} \mu(Q_{acc}) \leq 2\mu(Q_0) \leq Cdiam(Q_0)^d.$$

• Remains the most interesting set W_3 of top cubes such that

$$\mu\Big(\bigcup_{R \in RStop(Q)} R\Big) = \sum_{R \in RStop(Q)} \mu(R) \geq \frac{1}{4}\mu(Q), \tag{23}$$

where $RStop$ corresponds to stop for the "red" reason: large angle with $P(Q)$. We want to associate, to each cube $Q \in W_3$, a region $\mathcal{R}(Q)$ of $E \cap B(x_{Q_0}, C\,\mathrm{diam}(Q_0)) \times (0, C\,\mathrm{diam}(Q_0))$, so that....

How do we prove the existence of a Corona construction 3

[Still assuming (23)] ...

$$\text{The regions } \mathcal{R}(Q), Q \in W_3 \text{ have bounded overlap} \tag{24}$$

and, for each Q,

$$\int_{\mathcal{R}(Q)} \beta_p(x,r)^2 \frac{d\mu(x)dr}{r} \geq c\mu(Q). \tag{25}$$

If we do this, we get the desired control on $\sum_Q \mu(Q)$ because

$$\int_{E \cap B(x_{Q_0}, C\,\mathrm{diam}(Q_0) \times (0, C\,\mathrm{diam}(Q_0))} \beta_p(x,r)^2 \frac{d\mu(x)dr}{r} \leq C\,\mathrm{diam}(Q_0)^d$$

by assumption. The last thing to do is now transfer information from E to the Lipschitz graph $\Gamma(\mathcal{S}(Q))$ and back!

End of proof (hints)

We start from the definition (23) of W_3, which says that we often stopped because (20) fails, i.e., at the scale of $R \in Stop(Q)$, E looks like a d-plane, but with a relatively large angle with $P(Q)$.

Set $d(Q) = \mathrm{diam}(Q)$ to save space.

We deduce from this that, if $\Gamma(\mathcal{S})$ is the graph of A over $P(Q)$, we have $|\nabla A| \geq C\varepsilon$ on a significant part of $P \cap B(x_Q, 2d(Q))$.

Then by Littlewood Payley (the desired estimate on small Lipschitz graphs), and because A is flat at the large scale,

$$\int\int_{P\cap B(x_Q,3d(Q))\times(0,Cd(Q)} \gamma_p(x,r)^2 \frac{dxdr}{r} \geq c(\varepsilon)d(Q)^d,$$

where γ measure the approximation of A by affine functions, or equivalently the approximation of $\Gamma(\mathcal{S})$ by planes.

Then by geometry (and the comparison with E; see (22)), we obtain a similar result on E:

$$\int\int_{E\cap B(x_Q,Cd(Q))\times(0,Cd(Q)} \beta_p(x,r)^2 \frac{dxdr}{r} \geq c(\varepsilon)d(Q)^d.$$

As needed.

Big Pieces of other sets and the Riesz Rising Sun Lemma

The following could have existed, with even more time,
because it is amusing, connected to earlier stories because the Big Piece technique is used to extend analytic results (on SIO's or harmonic measure) from one class of sets E to the next.
And because the lemma is nice.

And in the next slides I keep the same very incomplete list of reference (a little more in the lecture notes).

Some references 1

L. Ahlfors, Zur Theorie der Überlagerungsfläschen, Acta Math. 65 (1935), 157–194. First paper with Ahlfor regular curves

M. Christ, Lectures on Singular Integral Operators, Regional Conference series in Mathematics 77, AMS 1990. Construction of "Dyadic cubes" for general spaces of homogeneous type

D. Dabrowski and M. Villa, Analytic capacity and dimension of sets with plenty of big projections, preprint. Also for references and history of projections

G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complexe, Ann. Sci. Ec. Norm. Sup. 17 (1984), 157–189. One could say that the use of the "Big Pieces" functor starts here, although Coiman had a good λ construction much before, for commutators

G. David, Morceaux de graphes lipschitziens et intégrales singulières sur une surface, Revista Matematica Iberoamericana, vol. 4, 1 (1988), 73–114. I think the first "dyadic cubes" on AR sets.

References 2

G. David and D. Jerison, Lipschitz approximations to hypersurfaces, harmonic measure, and singular integrals, Indiana U. Math. Journal. 39, 3 (1990), 831–845. A_∞ absolute continuity (under NTA, AR, and big balls in the complement) with Big pieces of Lipschitz graph and the maximum principle

G. David and S. Semmes, Singular integrals and rectifiable sets in $\mathbb{R}^n$: au-delà des graphes lipschitziens, Astérisque 193, SMF 1991. First collection of results specifically on UR, with the Corona construction

G. David and S. Semmes, Analysis of and on uniformly rectifiable sets, A.M.S. series of Mathematical surveys and monographs, Volume 38, 1993. Second large collection of results on UR, with more Corona constructions and weak conditions

J. R. Dorronsoro, A characterization of potential spaces, Proc. A.M.S. 95 (1985), 21–31. Contains many Littlewood-Payley results on the approximation by affine functions.

References 3

Xiang Fang, The Cauchy integral, analytic capacity and subsets of quasicircles, PhD. Thesis, Yale university. For the counterexample about $\beta_\infty(x,r)$ when $d \geq 2$.

P. Jones, Rectifiable sets and the traveling salesman problem, Inventiones Mathematicae 102, 1 (1990), 1–16. The first half of the characterization by the $\beta_\infty(x,r)$ of sets contained in a curve of finite length.

P. Mattila, M. Melnikov, and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. 144, 1 (1996), 127–136. Where the boundedness of the Cauchy transform on $E \in AR(1)$ is shown to imply UR.

F. Nazarov, X. Tolsa, and A. Volberg. The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions, *Publ. Mat.*, 58(2):517–532, 2014. Where the boundedness of the Riesz transforms on $E \in AR(n-1)$ is shown to imply UR.

References 4

F. Nazarov, S. Treil, and A. Volberg, Cauchy integral and Calderón-Zygmund operators on non homogeneous spaces, International Math. Res. Notices 1997, 15, 703–726. Singular integrals with non-doubling measures. There is also a paper of David-Mattila in Revista.

K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbb{R}^n$, J. of the London Math. Soc. 46 (1992), 336-348. The second half of the characterization by the $\beta_\infty(x, r)$ of sets contained in a curve of finite length.

T. Orponen. Plenty of big projections imply big pieces of Lipschitz graphs. *Invent. Math.*, 226(2):653–709, 2021. I think, the beginning of the new era of big projections.

E. R. Reifenberg, Solution of the Plateau problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960), 1–92. Because I mention the Reifenberg theorem.

References 5

S. Semmes, Differentiable function theory on hypersurfaces in $\mathbb{R}^n$ (without bounds on their smoothness), Indiana Univ. Math. Journal 39 (1990), 985–1004. This one, or the next one, should contain the first proof of A_∞-absolute continuity of harmonic measure using the Corona construction.

S. Semmes, Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbb{R}^n$, Indiana Univ. Math. Journal 39 (1990), 1005–1036.

X. Tolsa, Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory, volume 307 of *Progr. Math.*, Birkhäuser 2014. Should contain lots of corona constructions and clever stopping time arguments.

A. Zygmund, Trigonometric series, Cambridge University Press 1968. For the Rising sun lemma.