<u>Q1</u> : $13,9938.. < 14$

<u>setup</u> : we have 14 dimensional $\mathbb{Q}(x)$-vector space that is the span $/\mathbb{Q}(x)$ of 14 functions G_i on $X_0(2)$ denoted $\mathcal{H}$.

For a suitable $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ holomorphic, we have bounds

$$\dim_{\mathbb{Q}(x)} \mathcal{H} \leq \frac{\overbrace{\iint_{T^2} \log|\varphi(z) - \varphi(w)| \mu(z) \mu(w)}^{=: I}}{\log|\varphi'(0)| - \tau}$$

$$\dim_{\mathbb{Q}(x)} \mathcal{H} \leq \frac{\overbrace{\int_0^1 2t \left(\log|\varphi(e^{2i\pi t})|\right)^* dt}^{=: J}}{\log|\varphi'(0)| - \tau}$$

where ~~τ is~~ * μ Haar measure on $T := \{|z| = 1\}$

* τ is a real positive fixed
* $(\log|\varphi|)^*$ rearrangement integral from C4

<u>Goal</u> : Optimize the quotient $\frac{I}{\log|\varphi'(0)| - \tau}$

so that it get lower than 14 by choosing a good function φ.

<u>Plan</u> : Use an intermediate function $h : \mathbb{D} \to \mathbb{C}$ so that we only have to choose a conformal map ψ

such that
$$\begin{tikzcd} \mathbb{D} \arrow[r, "\psi", "\sim"'] \arrow[rrr, bend right, "\varphi"'] & \Omega \subset & \mathbb{D} \arrow[r, "h"] & \mathbb{C} \end{tikzcd}$$

with numerical conditions on γ that make this optimization problem independent from the rest of irrationality proof.

I _ Some context

A _ Potential theory and Greenflow

Def: A pointed Riemann surface with boundary (V, V^+, P) is a Riemann surface V^+ and an embedded domain $V \hookrightarrow V^+$ with C^∞ boundary and $P \in \mathring{V}$.

Let (V, P) compact, connected Riemann surface with boundary.

Prop-def: (Green function) There exists a unique $g_{V,P} : V \setminus P \to \mathbb{R}^+$ such that:

(i) $g_{V,P}$ continuous on $V \setminus P$, equal to 0 on ∂V

(ii) $g_{V,P}$ harmonic on $\mathring{V} \setminus P$

log-sing (iii) near P, $g_{V,P}(z) - \log|z-P|^{-1}$ is bounded.

example: $(V, P) = (\overline{\mathbb{D}}, 0) \rightsquigarrow g_{V,P} = \log^+|z|^{-1}$

Def: (ii) $\Rightarrow$ $g_{V,P}$ L^2_{loc}, define the harmonic measure associated to (V, P) as

$$\mu_{V,P} := \frac{i}{\pi} \partial\bar{\partial} g_{V,P} + \delta_P$$

Rmk: $\left(\frac{i}{\pi}\partial\bar\partial g_{V,P}+\delta_P\right)\Big|_{V^\circ}=0$ by (i)+(ii) and $g_{V,P}=0$ on ∂V so $\mu_{V,P}$ is supported on the smooth curve ∂V and is a probability measure.

ex: $(V,P)=(\bar{\mathbb{D}},0)$. Recall $g_{V,P}=\log^+|z|^{-1}$ and $\mu_{V,P}$ is a probability measure on $\mathbb{T}$, invariant by rotation hence the Haar measure.

Def (capacitary norm): Define the capacitary norm on $T_P V$ associated to (V,P) as

$$\left\|\frac{\partial}{\partial z}\Big|_P\right\|^{cap}_{g_{V,P}}=e^{-(g_{V,P}-\log|z-P|^{-1})(P)}$$

ex: $(V,P)=(\bar{\mathbb{D}},0)\rightsquigarrow\left\|\frac{\partial}{\partial z}\Big|_0\right\|^{cap}=e^0=1$.

Def ([Bost 20]): (V,P) compact connected with $\partial V\neq\emptyset$. N Riemann surface, $\alpha:(V,P)\to N$ analytic, étale at 0. Define the Overflow

$$\mathrm{Ex}(\alpha:(V,P)\to N)=\int_V g_{V,P}\left(\alpha^*\alpha_*\mu_{V,P}+\delta_{\alpha^{-1}(\alpha(P))-P}\right)$$

Thm ([BC 22]): Let $\varphi:(\bar{\mathbb{D}},0)\to\mathbb{C}$ analytic, étale at 0. Then

$$\mathrm{Ex}(\varphi:(\bar{\mathbb{D}},0)\to\mathbb{C})=\iint_{\mathbb{T}^2}\log|\varphi(z)-\varphi(w)|\,\mu(z)\mu(w)-\log|\varphi'(0)|.$$

Rmk: Thm true more generally: $\alpha:(V,P)\to N$, β C^∞ 2-form on N, G_N Green function for the diagonal.

Then

$$Ex(\alpha : (V,\beta) \to \mathcal{V}) = \int_V g_{V,\beta}\, \alpha^*\beta - \iint_{(\partial V)^2} g_{\mathcal{V}}(\alpha(z), \alpha(w))\, \mu_{V,\beta}(z)\, \mu_{V,\beta}(w)$$

$$- \log \|\alpha'(0)\|^{cap}_{g_{V,\beta}}$$

<u>ex</u>: apply with $\beta = 0$, $(V,\beta) = (\overline{\mathbb{D}}, 0)$, $\mathcal{V} = \mathbb{C}$, Green function for diagonal $\log|z-w|^{-1}$.

<u>Thm</u> (Nazarov, [CDT24]): $\forall \varphi : \mathbb{T} \to \mathbb{C}$ continuous

$$I \leq \iint_{\mathbb{T}^2} \log|\varphi(z) - \varphi(w)|\, \mu(z)\mu(w)$$

$$\leq \int_0^1 2t \left(\log|\varphi(e^{2i\pi t})|\right)^* dt = J.$$

<u>Rmk</u>: In the proof of theorem C (irrationality of $\log$)

$$I \sim 9{,}963 < 9{,}972 \sim J$$

<u>Rmk</u>: Principle of energy

B- ~~What is h and~~ Explicit conditions on the optimization problem.

Recall: 14 functions G_i on $\Gamma_0(2)$, want them to be holomorphic $\rightsquigarrow$ impose some conditions on φ.

Let
$$h : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0,1\}$$
$$q \longmapsto \begin{cases} -256q \prod_{n=1}^{+\infty} (1+q^n)^{24} & \text{if } q \neq 0 \\ 0 & \text{else} \end{cases}$$

convention $q = e^{2i\pi\tau}$.

[Cor 9.0.19]

This function satisfies the following property:

if $\mathbb{D} \xrightarrow[\sim]{\psi} \Omega$ is a conformal map with $\psi^{-1}(0) = \{0\}$ and $\Omega \subset \mathbb{D}$, if φ defined as

$$\begin{tikzcd} \overline{\mathbb{D}} \arrow[r, "\psi"] \arrow[rrr, bend right, "\varphi"'] & \overline{\Omega} \subset \mathbb{D} \arrow[rr, "h"] & & \mathbb{C} \setminus \{0,1\} \end{tikzcd}$$

has a univalent leaf over $\mathbb{C} \not\ni -\frac{1}{72}$

Then $\varphi^* G_i \in \mathcal{O}(\overline{\mathbb{D}})$ $\rightsquigarrow$ By this choice of φ, G_i's holomorphic.

$\rightsquigarrow$ $-\frac{1}{72}$ is a singularity of the G_i's.

Now the problem is to find a conformal map $\mathbb{D} \xrightarrow[\sim]{\psi} \Omega$ with the following conditions:

(i) $\overline{\mathbb{D}} \subset \Omega$ (to compose with h)

(ii) $\psi(\mathbb{D}) \cap h^{-1}(-\frac{1}{72}) \subset \{0{,}0000541\cdots\}$

(iii) $\log|256\,\psi'(0)| - \tau > 0$

Rmk: * condition (ii): the preimages by h of $-1/72$ are $x_0 = 0{,}0000541\cdots$ and the others are on horoball near $\mathbb{T} \subset \overline{\mathbb{D}}$.
We can't avoid x_0 as it is near from 0.

* condition (iii) $\Rightarrow$ to apply holonomy bounds.

~~By Riemann mapping theorem,~~

The bound

$$\frac{\iint_{\mathbb{T}^2} \log|h\circ\psi(z) - h\circ\psi(w)|\,\mu(z)\mu(w)}{\log|256\,\psi'(0)| - \tau}$$

only depends on the conformal radius of Ω and on the values of h along $\partial\Omega$

$\rightsquigarrow$ By Riemann mapping theorem, choosing a contour gives us a ψ that satisfies (i), (ii), (iii) if the contour is well chosen.

II – The choice of contour

We want the denominator to be as large as possible

$$\log|256\,\varphi'(0)| - \tau$$

and the only thing that we can do is to make

$\log|\varphi'(0)|$ large, i.e. $\partial\Omega$ near $\mathbb{T} \subset \overline{\mathbb{D}}$

<u>but</u> if $e^{2i\pi\alpha}$ is any cusp of $\Gamma_0(2)$, $\alpha = \frac{a}{b}$, b odd

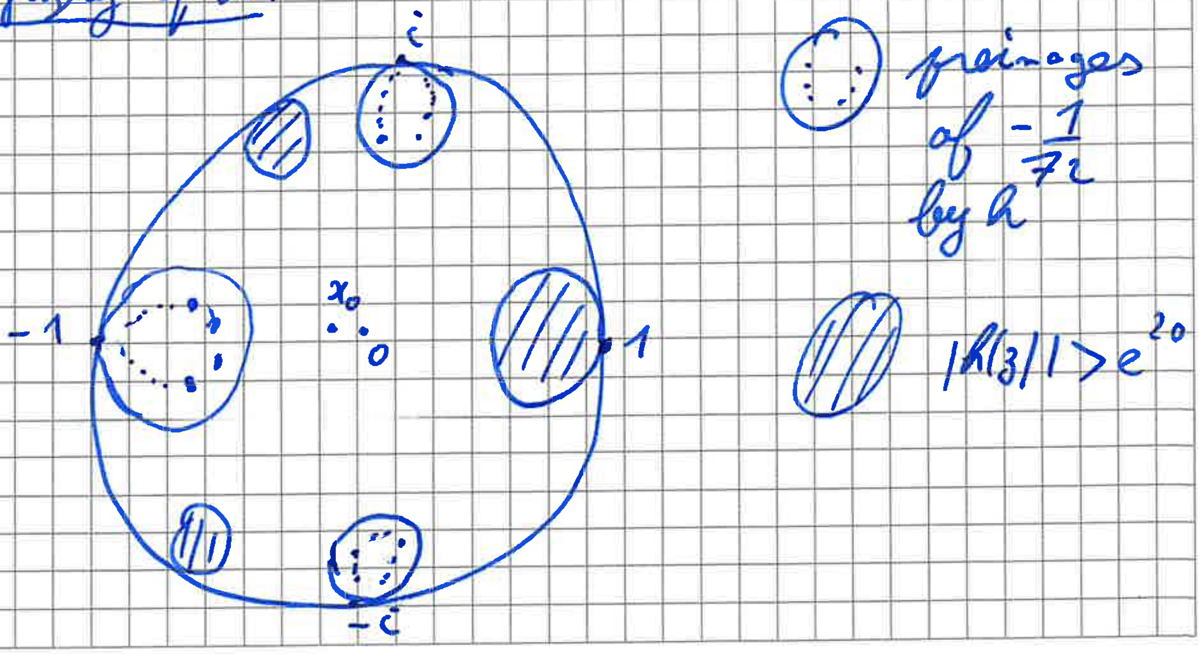
$$\log|h(z)| \sim \frac{1}{2\pi^2 b^2(1-|z|)}$$

and therefore ~~I is large~~

the numerator $\displaystyle\iint_{\mathbb{T}^2} \log|h(\varphi(z)) - h(\varphi(w))|\,\mu(z)\mu(w)$

is large near the cusps.

<u>topography of h:</u>



idea: (i) circle of center o with radius R big st $D(0,R)$ only contains preimages of $-\frac{1}{72}$ in the horoball near -1

(ii) Remove a lune near the horoball at 1 to remove large radius of h

(iii) remove slits ~~so~~ that the only preimage left is x_0.

rmk: possible to remove a second lune for the preimages near 1 but computations where ~~the~~ difficult.

def: (lunes) Let $c > 1$. Consider

$$f(z,c) = z \cdot \frac{(c^2+1)+(c^2-1)z}{(c^2-1)+(c^2+1)z}$$

It is a conformal map from the lune

$$L(c) := \mathbb{D} \setminus \mathbb{D} \cap D\left(-\frac{c^2+1}{c^2-1}, \frac{2c}{c^2-1}\right)$$

to $\mathbb{D}$

with inverse

$$\operatorname{lune}(z,c) = \frac{z(1+c^2)-1-c^2+\sqrt{(1+c^2)^2(1+z)^2-16c^2z}}{2(c^2-1)}$$

$$\rightsquigarrow c(L(c)) = \frac{c^2-1}{c^2+1}$$

def (slit) $0 < r < 1$

$\mathrm{slit}(z, r)$ conformal map

$$(\mathbb{D}, 0) \xrightarrow{\simeq} (\mathbb{D} \setminus]-1; -r], 0)$$

with $\mathrm{slit}(z, r) = \dfrac{4r}{(1+r)^2} z + \cdots$

and hence conformal radius $\dfrac{4r}{(1+r)^2}$

to see that $c(\mathbb{D} \setminus]-1; -r], 0) = \dfrac{4r}{(1+r)^2}$,

consider $z \mapsto \dfrac{z}{(1+z)^2}$ which takes conformally

$\mathbb{D} \setminus]-1; -r]$ to $i\left(\mathbb{P}^1 \setminus \left[-\frac{(1-r)^2}{r}, 4\right]\right)$

with $i : z \mapsto \dfrac{1}{z}$

and $\mathrm{capacity}\left(\left[-\frac{(1-r)^2}{r}; 4\right]\right) = \dfrac{4 - \left(-\frac{(1-r)^2}{r}\right)}{4}$

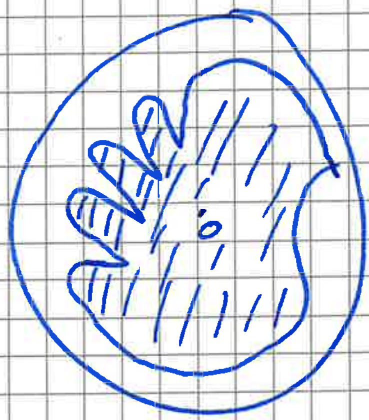
$$= \frac{(1+r)^2}{4r}$$

so $c(\Omega) = \dfrac{1}{\mathrm{cap}} = \dfrac{4r}{(1+r)^2}$.

Now define:

$$G(z) = -R\,\mathrm{Cune}\left(-e^{+2i\pi\theta_1}\, \mathrm{slit}\left(e^{2i\pi\theta_2} \cdot \mathrm{slit}\left(e^{2i\pi\theta_3} \cdot \mathrm{slit}\left(e^{2i\pi\theta_4} \cdot \mathrm{slit}(z, r_1), r_2\right), r_3\right), r_4\right), c\right)$$

for some parameters.



and $\Psi(z) := G\left(\frac{995}{1000} z\right)$.

Rmks: • compose the application but good approx of the finite operations at the same time
⇝ easier to compute
• $\frac{995}{1000}$ to ensure that we take out preimages, the slits become "open".

Estimates:

$$I = 11{,}844\ldots$$

$$\frac{I}{\log|\Psi'(0)| - \tau} = 13{,}9938\ldots < 14.$$