Griffiths-Kato heights of pencils of projective varieties

Hauteurs de Griffiths-Kato des pinceaux de variétés projectives

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Résumé :

On définit la hauteur de Griffiths d'une variation de structures de Hodge sur une courbe projective comme le degré de son fibré en droites canonique, tel qu'il est défini par Griffiths et généralisé par Peters afin de permettre des points de mauvaise réduction. On peut voir cette hauteur comme un analogue géométrique de la hauteur de Kato attachée aux motifs purs sur des corps de nombres. Dans ce mémoire, nous établissons diverses formules exprimant la hauteur de Griffiths de la cohomologie en dimension moitié d'un pinceau d'hypersurfaces projectives complexes en termes de classes caractéristiques.

En premier lieu, à l'aide de la théorie de Steenbrink et du théorème de Grothendieck-Riemann-Roch, nous exprimons en termes de classes caractéristiques la somme alternée des hauteurs de Griffiths des groupes de cohomologie des fibres d'un pinceau de variétés projectives, avec un espace total non singulier, et dont les fibres singulières sont des diviseurs à croisements normaux. Cette expression nous permet de calculer la même somme alternée des hauteurs de Griffiths associées à un pinceau de variétés projectives, avec un espace total non singulier, et dont les seules singularités des fibres sont des points doubles ordinaires.

À l'aide du théorème de Lefschetz faible et des formules ainsi établies, nous exprimons ensuite la hauteur de Griffiths de la cohomologie en dimension moitié d'un pinceau ample d'hypersurfaces dans un pinceau lisse en termes de classes caractéristiques et des hauteurs de Griffiths associées au pinceau lisse ambiant. Cela mène à des formules explicites pour la hauteur de Griffiths de la cohomologie en dimension moitié de pinceaux de variétés projectives dans les cas particuliers suivants : celui des pinceaux d'hypersurfaces dans un fibré en projectifs, et celui des pinceaux linéaires d'hypersurfaces dans une variété projective lisse — dont les pinceaux de Lefschetz sont un exemple.

Enfin, en annexe, nous établissons des résultats de transversalité qui montrent que les hypothèses sur les pinceaux d'hypersurfaces pour lesquelles sont établies plusieurs de nos formules pour les hauteurs de Griffiths — admettre un espace total lisse et des fibres singulières à points doubles ordinaires — sont satisfaites de manière générique.

Title : Griffiths-Kato heights of pencils of projective varieties
Keywords : Height, Variation of Hodge structures, Pencil of hypersurfaces

Abstract : The Griffiths height of a variation of Hodge structures over a projective curve is defined as the degree of its canonical line bundle, as defined by Griffiths and generalized by Peters to allow bad reduction points. It may be seen as a geometric analog of the Kato height attached to pure motives over number fields. In this paper, we establish various formulas expressing the Griffiths height of the middle-dimensional cohomology of a pencil of projective complex hypersurfaces in terms of characteristic classes.

Firstly, using Steenbrink's theory and the Grothendieck-Riemann-Roch theorem, we give an expression in terms of characteristic classes of the alternating sum of the Griffiths heights of the cohomology groups of the fibers of a pencil of projective varieties with non-singular total space, whose singular fibers are divisors with normal crossings. Using this expression, we may compute the same alternating sum of Griffiths heights associated to a pencil of projective varieties with non-singular total space and only ordinary double points as singularities of its fibers.

By using the weak Lefschetz theorem and the above formulas, we may express the Griffiths height of the middle-dimensional cohomology of an ample pencil of hypersurfaces in a smooth pencil in terms of characteristic classes and of the Griffiths heights associated to the ambient smooth pencil. This leads to closed formulas for the Griffiths height of the middle-dimensional cohomology of pencils of projective varieties in the following special cases : for pencils of hypersurfaces in a projective bundle, and for linear pencils of hypersurfaces in a smooth projective variety, of which Lefschetz pencils are a special instance.

Finally, as an appendix, we establish transversality results which show that the hypotheses on the pencils of hypersurfaces under which some of our formulas for Griffiths heights hold (namely, that the pencils admit smooth total spaces and singular fibers with only ordinary double points), are generically satisfied.

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Préface

La hauteur de Kato des motifs. Le point de départ de ce travail est l'article [**Kat14**], où Kato introduit une fonction de hauteur associée aux motifs purs sur un corps de nombres K, définis en termes de réalisations (p-adique, Betti, de Rham).

Si M est un tel motif, par exemple sur $K = \mathbb{Q}$, et si M_{dR} désigne sa réalisation de de Rham, $F^\bullet M_{\mathrm{dR}}$ sa filtration de Hodge, et

$$\mathrm{gr}_F^r M_{\mathrm{dR}} := F^r M_{\mathrm{dR}} / F^{r+1} M_{\mathrm{dR}}$$

les sous-quotients associés, Kato considère le K-espace vectoriel de dimension un :

$$L(M) = \bigotimes_{r \in \mathbb{Z}} (\det \mathrm{gr}_F^r M_{\mathrm{dR}})^{\otimes r} \simeq \bigotimes_{r > r_0} (\det_K F^r M_{\mathrm{dR}}), \tag{0.0.1}$$

où r_0 désigne un entier tel que $F^{r_0} M_{\mathrm{dR}} = M_{\mathrm{dR}}$.

Le $\mathbb{R}$-espace vectoriel $L(M)_{\mathbb{R}} := L(M) \otimes_{\mathbb{Q}} \mathbb{R}$ est muni d'une norme canonique, définie au moyen de la théorie de Hodge. De même, pour tout nombre premier p, le $\mathbb{Q}_p$-espace vectoriel $L(M)_{\mathbb{Q}_p} := L(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ est doté d'un $\mathbb{Z}_p$-réseau naturel, défini au moyen de la réalisation p-adique de M et de la théorie de Hodge p-adique. Une fois muni de ces structures supplémentaires, $L(M)$ devient un fibré en droites hermitien $\overline{L(M)}$ sur $\operatorname{Spec} \mathbb{Z}$, ce qui lui confère un degré d'Arakelov. Ce degré définit la *hauteur de Kato* du motif M :

$$\mathrm{ht}_K(M) := \widehat{\deg}\, \overline{L(M)} \in \mathbb{R}. \tag{0.0.2}$$

Cette définition s'inspire de la définition de la hauteur de Faltings d'une variété abélienne sur un corps de nombres — qui est un cas particulier de la définition de Kato quand M est un motif de poids un — et de la preuve de Faltings de l'invariance par isogénie de la hauteur des variétés abéliennes, sur lesquelles se fonde sa preuve de la conjecture de Tate dans le cas des variétés abéliennes sur les corps de nombres [**Fal83**].

La définition spécifique (0.0.1) de la K-droite $L(M)$ — où la droite vectorielle déterminant $\det \mathrm{gr}_F^r M_{\mathrm{dR}}$ du r-ième sous-quotient $\mathrm{gr}_F^r M_{\mathrm{dR}}$ de la filtration de Hodge apparaît à la puissance tensorielle r — s'est en fait imposée à Kato pour sa « compatibilité » avec les conjectures de Weil : comme Kato l'a montré, les conjectures de Weil entraînent que la hauteur définie comme le degré d'Arakelov (0.0.2) vérifie une propriété d'invariance qui généralise l'invariance par isogénie de la hauteur des variétés abéliennes aux motifs de poids quelconque.

À la suite de l'article fondamental de Kato [**Kat14**], les hauteurs de Kato ont été généralisées et explorées par Kato lui-même [**Kat18**], par Koshikawa [**Kos15**] qui a étudié leurs possibles applications aux conjectures classiques de finitude et de semi-simplicité concernant les motifs sur les corps de nombres, et par Venkatesh [**Ven18**], qui s'est penché sur leur lien avec les formes automorphes.

Le fibré en droites de Griffiths et les hauteurs de Griffiths d'une VSH sur une courbe projective. Comme Kato le mentionne dans [**Kat18**], un précurseur de ses définitions (0.0.1) et (0.0.2) apparaît déjà dans les célèbres travaux de Griffiths consacrés aux structures de Hodge sur la cohomologie des variétés projectives complexes et aux applications des périodes associées aux familles de telles variétés.

Rappelons qu'une variation de structures de Hodge (VSH) entière de poids $n \in \mathbb{N}$ sur une variété analytique complexe S est un couple :

$$\mathbb{V} := (V_{\mathbb{Z}}, \mathcal{F}^{\bullet}),$$

où $V_{\mathbb{Z}}$ est un système local de $\mathbb{Z}$-modules libres de type fini sur S, et

$$\mathcal{F}^{\bullet} : \mathcal{F}^0 := \mathcal{V} \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^n \supseteq \mathcal{F}^{n+1} = 0$$

est une filtration décroissante par des sous-fibrés du fibré vectoriel holomorphe

$$\mathcal{V} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S^{\mathrm{an}}$$

attaché au système local $V_{\mathbb{Z}}$. Ces données doivent vérifier la condition de décomposition de Hodge — de sorte que pour tout $s \in S$, la fibre $\mathbb{V}_s := (V_{\mathbb{Z},s}, \mathcal{F}_s^{\bullet})$ soit une structure de Hodge de poids n — et la condition d'horizontalité de Griffiths (voir par exemple [**Voi02**, III] ou [**PS08**, Chapter 10]).

À un fibré vectoriel holomorphe $\mathcal{V}$ sur une variété analytique complexe S, muni d'une filtration décroissante finie

$$F^{\bullet} := F^0 := \mathcal{V} \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0$$

par des sous-fibrés, nous pouvons associer le fibré en droites holomorphe sur S :

$$\mathcal{GK}_S(\mathcal{V}, F^{\bullet}) := \bigotimes_{i=1}^{n} \det F^i \simeq \bigotimes_{r=0}^{n} (\det F^r/F^{r+1})^{\otimes r},$$

que nous appellerons le *fibré en droites de Griffiths* associé à $(\mathcal{V}, F^{\bullet})$.

En particulier, une variation de structures de Hodge $\mathbb{V} = (V_{\mathbb{Z}}, \mathcal{F}^{\bullet})$ sur S comme ci-dessus admet un fibré en droites de Griffiths associé :

$$\mathcal{GK}_S(\mathbb{V}) := \mathcal{GK}_S(\mathcal{V}, \mathcal{F}^{\bullet}) = \bigotimes_{r=0}^{n} (\det \mathcal{F}^r/\mathcal{F}^{r+1})^{\otimes r}. \tag{0.0.3}$$

Cette définition est introduite par Griffiths dans [**Gri70**, Section 7 b)], sous la dénomination de *fibré en droites canonique* de la variation de structures de Hodge.

Le choix spécifique de puissances de fibrés en droites qui apparaît dans le membre de droite de (0.0.3) est le même que dans la définition de la « droite de Kato » dans (0.0.1). Mais il s'introduit dans le travail de Griffiths pour des raisons complètement différentes. Comme le montre Griffiths dans [**Gri70**], une polarisation sur la VSH $\mathbb{V}$ induit des métriques hermitiennes canoniques sur les fibrés vectoriels $\mathcal{F}^r/\mathcal{F}^{r+1}$, et donc sur le fibré en droites $\mathcal{GK}_S(\mathbb{V})$. L'introduction du fibré en droites (0.0.3) est motivée par le fait que la courbure [1] de $\mathcal{GK}_S(\mathbb{V})$ muni de cette structure hermitienne canonique est toujours une $(1,1)$-forme réelle *positive* sur S.

En particulier, quand S est une courbe complexe projective lisse connexe C, le degré de ce fibré en droites :

$$\mathrm{ht}_{GK}(\mathbb{V}) := \deg_C \mathcal{GK}_S(\mathbb{V}) \tag{0.0.4}$$

— que nous appellerons la *hauteur de Griffiths-Kato*, ou simplement la *hauteur de Griffiths*, de la VSH $\mathbb{V}$ sur C — est un entier positif quand $\mathbb{V}$ admet une polarisation.

La hauteur de Griffiths $\mathrm{ht}_{GK}(\mathbb{V})$ apparaît comme un analogue de la hauteur de Kato $\mathrm{ht}_K(M)$, où le corps de nombres K est remplacé par le corps de fonctions $\mathbb{C}(C)$, et le motif M par la VSH $\mathbb{V}$ sur C.

1. ou plus précisément la première forme de Chern.

La construction de Peters et la hauteur de Griffiths de la cohomologie d'un pinceau de variétés projectives. Comme le montre Peters dans [**Pet84**], il est en fait possible d'étendre la définition (0.0.4) à la situation où $\mathbb{V} := (V_{\mathbb{Z}}, \mathcal{F}^{\bullet})$ est une variation de structures de Hodge sur un ouvert Zariski dense $\mathring{C}$ de C.

La construction de Peters repose sur le travail de Deligne ([**Del70**]) sur les extensions de systèmes locaux analytiques définis sur un ouvert Zariski dense d'une variété complexe projective lisse et sur les résultats de Schmid ([**Sch73**]) sur les variations de structures de Hodge.

Cette construction apparaît naturellement sous deux variantes, faisant intervenir un choix de signe et ce qu'on appelle les extensions supérieure et inférieure. Ces variantes mènent à l'introduction de deux extensions canoniques sur C, notées $\mathcal{GK}_{C,+}(\mathbb{V})$ et $\mathcal{GK}_{C,-}(\mathbb{V})$, du fibré en droites analytique $\mathcal{GK}_{\mathring{C}}(\mathbb{V})$ sur $\mathring{C}$.[2] Les hauteurs de Griffiths associées seront notées :

$$\mathrm{ht}_{GK,+}(\mathbb{V}) := \deg_C \mathcal{GK}_{C,+}(\mathbb{V}) \quad \text{et} \quad \mathrm{ht}_{GK,-}(\mathbb{V}) := \deg_C \mathcal{GK}_{C,-}(\mathbb{V}).$$

Les fibrés en droites $\mathcal{GK}_{C,+}(\mathbb{V})$ et $\mathcal{GK}_{C,-}(\mathbb{V})$, et donc les hauteurs $\mathrm{ht}_{GK,+}(\mathbb{V})$ et $\mathrm{ht}_{GK,-}(\mathbb{V})$, coïncident de fait quand la monodromie locale de $V_{\mathbb{Z}}$ en chaque point du complémentaire $C - \mathring{C}$ (qui est automatiquement quasi-unipotente) est unipotente.

La situation étudiée par Peters recouvre les variations de structures de Hodge qui « proviennent de la géométrie », c'est-à-dire celles qui sont associées à un pinceau de variétés projectives paramétrées par C et à la cohomologie de ses fibres.

Soit en effet X une variété complexe projective lisse, et soit

$$f : X \longrightarrow C$$

un morphisme surjectif. Il existe $\mathring{C}$ comme ci-dessus tel que le morphisme :

$$f_{|X_{\mathring{C}}} : X_{\mathring{C}} := f^{-1}(\mathring{C}) \longrightarrow \mathring{C}$$

soit lisse. Pour tout $n \in \mathbb{N}$, nous pouvons considérer la VSH $\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})$ sur $\mathring{C}$, définie par la cohomologie de Betti relative en degré n, et la filtration sur la cohomologie de de Rham relative induite par la suite spectrale de Hodge-de Rham relative, puis les fibrés en droites $\mathcal{GK}_{C,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ et $\mathcal{GK}_{C,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ sur C et les hauteurs de Griffiths correspondantes :

$$\mathrm{ht}_{GK,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})) := \deg_C \mathcal{GK}_{C,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$$

et :

$$\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})) := \deg_C \mathcal{GK}_{C,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})).$$

Ces nombres ne dépendent que de la fibre générique X_η de f sur le point générique η de C, et seront aussi notés $\mathrm{ht}_{GK,+}(\mathbb{H}^n(X_\eta/C_\eta))$ et $\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_\eta/C_\eta))$.

L'objet de ce mémoire. En dépit de leur importance pour la compréhension des motifs purs sur les corps de nombres, les hauteurs de Kato ont été peu étudiées au-delà des motifs des variétés abéliennes. En particulier, aucun calcul explicite de la hauteur de Kato d'un motif de poids au moins 2 ne semble connu[3].

De manière frappante, les hauteurs de Griffiths $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$ quand $n \geq 2$, qui en sont les équivalents géométriques, ne semblent pas mieux comprises. Ce constat a été à l'origine de notre travail, dont l'objectif a été de calculer ces hauteurs de Griffiths $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$, quand $n \geq 2$, dans diverses situations significatives, en termes de classes caractéristiques et de la géométrie des

2. Nous renvoyons à la section 1.2 pour davantage de détails. En fait, Peters n'étudie que l'extension $\mathcal{GK}_{C,+}(\mathbb{V})$, pour laquelle la positivité de la hauteur de Griffiths dans le cas polarisé est encore vérifiée. La théorie de Steenbrink ([**Ste76, Ste77, PS08**]) est naturellement reliée à l'extension $\mathcal{GK}_{C,-}(\mathbb{V})$. Cette divergence entraîne certaines confusions dans la littérature publiée, qui sont clarifiées notamment dans [**Kol86**], [**Mor87**] et [**EFiMM21**].

3. en dehors des motifs construits par des opérations tensorielles à partir du motif de poids 1 associé à une variété abélienne.

fibres singulières de la famille X/C. Ces formules permettraient d'en « deviner » des analogues arithmétiques concernant les hauteurs de Kato associées à la cohomologie de certaines variétés projectives sur les corps de nombres.

Dans ce mémoire de thèse, nous établissons des formules de ce type pour les hauteurs de Griffiths $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$ quand n est la dimension relative de X/C, et quand X/C est un pinceau de surfaces semi-stable, un pinceau d'hypersurfaces dans un espace projectif, ou un pinceau de Lefschetz.

L'essentiel de ce mémoire — ses chapitres 1 à 6 — reproduit avec des modifications mineures la prépublication [**Mor22**]. Nous y avons ajouté l'annexe A, où nous établissons des énoncés de transversalité qui montrent que les hypothèses sur les pinceaux X/C pour lesquelles sont établies plusieurs de nos formules pour les hauteurs de Griffiths — admettre un espace total lisse et des fibres singulières à points doubles ordinaires — sont satisfaites de manière générique.

Une présentation détaillée de nos résultats fait l'objet du chapitre 1. Dans la suite de cette préface, nous essayons de présenter brièvement quelques-uns d'entre eux qui nous semblent particulièrement significatifs.

Sommes alternées de hauteurs de Griffiths. Nos premières tentatives de calculer les degrés des fibrés de Hodge et des fibrés de Griffiths associés aux pinceaux d'hypersurfaces dans l'espace projectif s'appuyaient sur la théorie de l'anneau jacobien et les résultats de Green sur les résolutions de Koszul (voir [**Gre84**] et [**Voi03**, Chapter 6]), mais n'ont pas abouti.

En définitive, notre calcul des hauteurs de Griffiths de pinceaux d'hypersurfaces s'est pour une large part inspirée du calcul classique par Hirzebruch des nombres de Hodge des intersections complètes dans l'espace projectif ([**Hir56**] et [**Hir95**, Appendix I]) — le théorème de Grothendieck-Riemann-Roch y joue un rôle analogue à celui du théorème de Hirzebruch-Riemann-Roch dans le calcul de Hirzebruch.

Le résultat central sur lequel reposent nos calculs est une expression pour la somme alternée de hauteurs de Griffiths :

$$\sum_{k=0}^{2n}(-1)^{k-1}\,\mathrm{ht}_{GK,-}(\mathbb{H}^k(X_\eta/C_\eta)) \tag{0.0.5}$$

associées à un pinceau de variétés projectives X/C général.

Chacun des termes de cette somme paraît difficile à calculer. En revanche, la somme (0.0.5) est le degré du fibré en droites sur C :

(0.0.6)

$$\bigotimes_{k=0}^{2n}\mathcal{GK}_{C,-}(\mathbb{H}^k(X_{\mathring{C}}/\mathring{C}))^{\otimes(-1)^{k-1}} := \bigotimes_{0\leq p\leq k\leq 2n}\det\left[\mathcal{F}^p_-\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})/\mathcal{F}^{p+1}_-\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})\right]^{\otimes(-1)^{k-1}p},$$

où $\mathcal{F}^p_-\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})$ désigne le fibré vectoriel sur C prolongeant le sous-fibré $\mathcal{F}^p\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})$ de $\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})$ sur $\mathring{C}$ obtenu par la construction de Peters inférieure. Lorsqu'en outre les fibres singulières du pinceau X/C sont des diviseurs à croisements normaux stricts, la théorie de Steenbrink permet d'identifier le sous-quotient $\mathcal{F}^p_-\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})/\mathcal{F}^{p+1}_-\mathbb{H}^k(X_{\mathring{C}}/\mathring{C})$ à l'image directe $R^{k-p}f_*\omega^p_{X/C}$, où $\omega^\bullet_{X/C}$ désigne les formes différentielles logarithmiques relatives. Ainsi la somme alternée (0.0.5) peut encore s'écrire :

$$\sum_{k=0}^{2n}(-1)^{k-1}\,\mathrm{ht}_{GK,-}(\mathbb{H}^k(X_\eta/C_\eta)) = \sum_{0\leq p\leq n}(-1)^{p-1}p\,\deg\det R^\bullet f_*\omega^p_{X/C}. \tag{0.0.7}$$

Chacun des termes $\deg\det R^\bullet f_*\omega^p_{X/C}$ peut s'exprimer en termes de classes caractéristiques grâce au théorème de Grothendieck-Riemann-Roch. Il s'avère que former la combinaison linéaire de ces termes figurant dans le membre de droite de (0.0.7) conduit à des simplifications considérables entre ces expressions en termes de classes caractéristiques, comme cela apparaîtra dans la formule pour la somme alternée (0.0.7) que nous donnons un peu plus loin dans le théorème 0.0.1. Il est remarquable que le choix spécifique de puissances de fibrés en droites intervenant dans la définition des (fibrés en) droites de Kato et Griffiths donne lieu à de telles simplifications dans le calcul de cette somme alternée.

La lecture des travaux récents d'Eriksson, Freixas et Mourougane à propos des invariants de BCOV des variétés de Calabi-Yau de dimension quelconque ([**EFiMM18, EFiMM21, EFiMM22**]) a catalysé l'obtention de nos résultats, tout particulièrement grâce à leur analyse claire et détaillée des résultats de Steenbrink et Peters en vue de calculs concrets. Leurs travaux nous ont aussi influencé en mettant en évidence l'importance de versions relatives du théorème de Riemann-Roch pour la compréhension des invariants des pinceaux de variétés projectives définis en termes de théorie de Hodge. [4]

Une application de la théorie de Steenbrink et du théorème de Grothendieck-Riemann-Roch. Voici une première formulation de notre calcul en termes de classes caractéristiques pour la somme alternée de hauteurs de Griffiths (0.0.5), qui, comme nous l'avons mentionné plus haut, repose sur la théorie de Steenbrink et le théorème de Grothendieck-Riemann-Roch.

THÉORÈME 0.0.1. *Soit C une courbe complexe projective lisse connexe de point générique η, soit Y une variété complexe projective lisse connexe de dimension pure N, et soit*

$$g : Y \longrightarrow C$$

un morphisme surjectif de variétés complexes. Supposons qu'il existe un sous-ensemble fini Δ de C tel que g soit lisse au-dessus de $C - \Delta$, et tel que le diviseur Y_Δ soit un diviseur à croisements normaux stricts dans Y.

Alors l'égalité suivante est vérifiée :

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \int_Y \rho_{N-1}(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}, \tag{0.0.8}$$

où $\omega^1_{Y/C}$ désigne le fibré vectoriel sur Y des 1-formes logarithmiques relatives [5] *:*

$$\omega^1_{Y/C} := \Omega^1_{Y/C}(\log Y_\Delta),$$

où $[T_g]$ est la classe tangente relative en K-théorie :

$$[T_g] := [T_{Y/\mathbb{C}}] - g^*[T_{C/\mathbb{C}}] \in K^0(Y),$$

et où Td *désigne la classe de Todd et ρ_{N-1} la classe caractéristique définie à partir des classes de Chern par :*

$$\rho_{N-1} := c_{N-2} - \frac{N-1}{2} c_{N-1} + \frac{1}{12} c_1 c_{N-1}.$$

Dans le membre de droite de (0.0.8), nous notons :

$$\int_Y : \mathrm{CH}^*(Y)_\mathbb{Q} \longrightarrow \mathbb{Q}$$

l'application qui envoie une classe α dans $\mathrm{CH}^*(Y)_\mathbb{Q}$ sur le degré de sa composante homogène $\alpha^{[N]}$ dans $\mathrm{CH}^N(Y)_\mathbb{Q} \simeq \mathrm{CH}_0(Y)_\mathbb{Q}$.

4. Le produit tensoriel alterné de fibrés en droites de Griffiths (0.0.6), qui joue un rôle essentiel dans notre approche, coïncide avec le fibré en droites de BCOV étudié par Eriksson, Freixas et Mourougane dans le cas particulier des pinceaux de variétés de Calabi-Yau. Eriksson, Freixas et Mourougane en étudient des propriétés métriques, mettant en jeu métriques de Quillen et torsion analytique, en s'appuyant sur le ≪ théorème de Riemann-Roch arithmétique ≫ de Bismut, Gillet et Soulé.

5. Nous utilisons la même notation que [**Ill94**]. Noter que ce fibré vectoriel ne dépend pas du choix du diviseur Δ dans C satisfaisant les conditions ci-dessus.

La simplicité de l'expression apparaissant dans le membre de droite de (0.0.8) est une conséquence de l'identité de classes caractéristiques suivante, valable pour tout fibré vectoriel E de rang $N-1$:

$$\sum_{p=0}^{N-1}(-1)^{p-1}p\,\mathrm{ch}(\Lambda^p E^\vee)\,\mathrm{Td}(E) = \rho_{N-1}(E) \qquad \mathrm{mod}\ \mathrm{CH}^{\geq N+1},$$

qui apparaît comme une version dérivée de l'identité classique :

$$\sum_{p=0}^{r}(-1)^p\mathrm{ch}(\Lambda^p E^\vee)\,\mathrm{Td}(E) = c_r(E),$$

valable pour tout fibré vectoriel de rang r ; voir par exemple [**Ful98**, Ex. 3.2.5].

Quand le morphisme g est lisse, la classe dans $K^0(Y)$ de $\omega^{1\vee}_{Y/C} = \Omega^{1\vee}_{Y/C}$ et $[T_g]$ coïncident, et (0.0.8) devient la formule plus simple :

$$\sum_{n=0}^{2(N-1)}(-1)^{n-1}\mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\Omega^{1\vee}_{Y/C})c_{N-1}(\Omega^{1\vee}_{Y/C}). \tag{0.0.9}$$

Quand le morphisme g n'est pas nécessairement lisse, les restrictions dans $K^0(Y - Y_\Delta)$ de $[T_g]$ et $[\omega^{1\vee}_{Y/C}]$ coïncident encore, et il existe un cycle à coefficients rationnels V à support dans Y_Δ tel que :

$$\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} = 1 + [V].$$

Ainsi la formule (0.0.8) peut s'écrire :

$$\sum_{n=0}^{2(N-1)}(-1)^{n-1}\mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\omega^{1\vee}_{Y/C})c_{N-1}(\omega^{1\vee}_{Y/C}) + \int_Y \rho_{N-1}(\omega^{1\vee}_{Y/C})[V]. \tag{0.0.10}$$

Classes caractéristiques des différentielles logarithmiques. La formule (0.0.10) est similaire à (0.0.9), mais son membre de droite contient un second terme qui dépend de la géométrie du morphisme g au voisinage infinitésimal des « mauvaises fibres » Y_Δ. Ce second terme peut être explicité à l'aide de calculs de classes de Chern de fibrés de différentielles logarithmiques relatives. Nous renvoyons au théorème 1.3.2 pour plus de détails.

Ces calculs reposent notamment sur le résultat ci-dessous, d'intérêt indépendant, concernant les formes différentielles logarithmiques dans le cas absolu.

Commençons par introduire des notations. Soient X un schéma complexe lisse de dimension n et E un diviseur réduit à croisements normaux stricts dans X. Notons $(E_i)_{i\in I}$ les composantes irréductibles de E, et pour chaque sous-ensemble $J \subseteq I$, posons : [6]

$$E_J := \bigcap_{i\in J} E_i.$$

C'est un sous-schéma lisse de X de codimension $|J|$, et nous notons :

$$i_{E_J} : E_J \hookrightarrow X$$

le morphisme d'inclusion.

Comme dans [**Del70**, II, 3.4], pour chaque entier positif r, nous notons E^r le sous-schéma de X de codimension r défini comme la réunion des intersections de r composantes distinctes de E :

$$E^r := \bigcup_{J\subset I, |J|=r} E_J.$$

6. En particulier, $E_\emptyset = X$.

On remarque que pour chaque sous-ensemble J de I, le sous-schéma :

$$E_J \cap E^{|J|+1} = \bigcup_{i \in I-J} E_{J \cup \{i\}}$$

est un diviseur à croisements normaux dans E_J.

Proposition 0.0.2. *Avec les notations précédentes, pour chaque sous-ensemble J de I, la relation suivante entre classes de Chern totales est vérifiée dans* $\mathrm{CH}^*(E_J)$ *:*

$$c\big(\Omega^1_X(\log E)_{|E_J}\big) = c\big(\Omega^1_{E_J}(\log E_J \cap E^{|J|+1})\big). \tag{0.0.11}$$

En outre, l'égalité suivante est vérifiée dans $\mathrm{CH}^*(X)$ *:*

$$c(\Omega^1_X(\log E)) = \sum_{J \subseteq I} i_{E_J *} c(\Omega^1_{E_J}), \tag{0.0.12}$$

et l'égalité suivante est vérifiée dans $\mathrm{CH}^*(X)_{\mathbb{Q}}$ *:*

$$\mathrm{Td}(\Omega^1_X(\log E)) = \mathrm{Td}(\Omega^1_X) \prod_{i \in I} \mathrm{td}(-[E_i])^{-1}, \tag{0.0.13}$$

où $\mathrm{td}(x)$ *désigne la série formelle* $x/(1-e^{-x})$.

Application aux pinceaux de variétés projectives de fibres singulières à points doubles ordinaires. Nous appliquerons notamment le théorème 0.0.1 au cas où Y est l'éclatement d'un schéma complexe H projectif lisse de dimension N, le long d'un nombre fini de points critiques non dégénérés d'un morphisme

$$f : H \longrightarrow C.$$

Dans ce cas, en utilisant un résultat d'Eriksson, Freixas et Mourougane concernant la comparaison d'extensions de Deligne et ce qu'on appelle les « exposants élémentaires » de fibrés de Hodge dans le cas de fibres singulières à points doubles ordinaires ([**EFiMM21**, Prop. 3.10]), et en nous appuyant sur des calculs de classes caractéristiques, nous établirons le théorème suivant.

Théorème 0.0.3. *Soient C une courbe complexe projective lisse connexe de point générique η, H un schéma complexe projectif lisse de dimension N, et :*

$$f : H \longrightarrow C$$

un morphisme de schémas complexes. Supposons qu'il existe un sous-ensemble fini Σ de H tel que f soit lisse sur $H - \Sigma$ et admette un point critique non dégénéré à chaque point de Σ.[7]

Alors les égalités suivantes sont vérifiées :

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(H_\eta/C_\eta)) = \frac{1}{12} \int_H c_1([\Omega^1_{H/C}]^\vee) c_{N-1}([\Omega^1_{H/C}]^\vee) + u_N^- \, |\Sigma|, \tag{0.0.14}$$

et

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,+}(\mathbb{H}^n(H_\eta/C_\eta)) = \frac{1}{12} \int_H c_1([\Omega^1_{H/C}]^\vee) c_{N-1}([\Omega^1_{H/C}]^\vee) + u_N^+ \, |\Sigma|, \tag{0.0.15}$$

où u_N^- et u_N^+ sont les nombres rationnels définis par :

$$u_N^- := \begin{cases} (5N-3)/24 & \textit{si } N \textit{ est impair} \\ N/24 & \textit{si } N \textit{ est pair,} \end{cases}$$

et :

$$u_N^+ := \begin{cases} -(7N-9)/24 & \textit{si } N \textit{ est impair} \\ N/24 & \textit{si } N \textit{ est pair.} \end{cases}$$

7. Ce qui signifie que la différentielle de f s'annule et que sa hessienne est non dégénérée à chaque point de Σ ; autrement dit, que les seules singularités des fibres de f sont des points doubles ordinaires.

Application aux pinceaux d'hypersurfaces projectives. Soient E un fibré vectoriel de rang $N+1$ sur une courbe complexe projective lisse connexe C, et

$$\pi : \mathbb{P}(E) := \operatorname{Proj} S^\bullet E^\vee \longrightarrow C$$

le fibré en projectifs associé. Soient de plus $\mathcal{O}_E(-1)$ le sous-fibré de rang un tautologique de $\pi^* E$, et $\mathcal{O}_E(1)$ son dual. Une hypersurface horizontale dans le fibré en projectifs $\mathbb{P}(E)$ est un diviseur de Cartier effectif H dans $\mathbb{P}(E)$ tel que le morphisme :

$$\pi_{|H} : H \longrightarrow C$$

soit plat. Alors ses fibres :

$$H_x := \pi_{|H}^{-1}(x), \quad x \in C$$

sont des hypersurfaces dans les espaces projectifs $\mathbb{P}(E_x) \simeq \mathbb{P}^N(\mathbb{C})$. Leur degré d est indépendant de $x \in C$, et définit le *degré relatif* de l'hypersurface horizontale.

Nous introduirons la *hauteur au sens de la théorie de l'intersection* d'une hypersurface horizontale H. Elle est définie par le nombre rationnel :

$$\mathrm{ht}_{int}(H/C) := \int_{\mathbb{P}(E)} c_1(\mathcal{O}_E(1))^N \cap [H] + dN\mu(E), \tag{0.0.16}$$

où

$$\mu(E) := \deg_C E/\mathrm{rk} E = \deg(c_1(E) \cap [C])/(N+1)$$

désigne la pente du fibré vectoriel E sur C.[8]

Dans cette situation, les sommes alternées de hauteurs de Griffiths (0.0.14) et (0.0.15) se réduisent à leur terme « en dimension moitié $n = N-1$ » en vertu du théorème de Lefschetz faible. Nous établirons ainsi le théorème suivant.

THÉORÈME 0.0.4. *Soient C une courbe complexe projective lisse connexe de point générique η, E un fibré vectoriel de rang $N+1$ sur C, et $H \subset \mathbb{P}(E)$ une hypersurface horizontale de degré relatif d, lisse sur $\mathbb{C}$. Si $\pi_{|H}$ n'a qu'un nombre fini de points critiques, tous non dégénérés, alors le cardinal de l'ensemble Σ des points critiques vérifie :*

$$|\Sigma| = (N+1)(d-1)^N \, \mathrm{ht}_{int}(H/C). \tag{0.0.17}$$

En outre, sous la même hypothèse, les égalités suivantes sont vérifiées :

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_+(d,N)\, \mathrm{ht}_{int}(H/C)$$

et

$$\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_-(d,N)\, \mathrm{ht}_{int}(H/C),$$

où $F_+(d,N)$ et $F_-(d,N)$ sont les éléments de $(1/12)\mathbb{Z}$ donnés quand N est impair par :

$$F_+(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(7d^2N - 7d^2 - 2dN - 2) + 2(d^2-1)\right]$$

et

$$F_-(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(-5d^2N + 5d^2 - 2dN - 2) + 2(d^2-1)\right],$$

et quand N est pair par :

$$F_+(d,N) = F_-(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N + 2d^2 - 2dN - 2) - 2(d^2-1)\right].$$

8. La normalisation additive par $dN\mu(E)$ dans le membre de droite de (0.0.16) assure que la hauteur $\mathrm{ht}_{int}(H/C)$ reste inchangée quand on remplace le fibré vectoriel E par $E \otimes L$, où L désigne un fibré en droites sur C : elle ne dépend que de H comme sous-schéma du fibré en projectifs $\mathbb{P} := \mathbb{P}(E)$, et non du choix d'un fibré vectoriel E tel que $\mathbb{P} \simeq \mathbb{P}(E)$; *cf.* la remarque 6.2.3 plus bas.

CHAPTER 1

Introduction

1.1. The Kato height of motives over number fields and the Griffiths height of variations of Hodge structures

1.1.1. The Kato height of motives. In [**Kat14**], Kato has introduced a height function associated to pure motives over a number field K, defined in terms of realizations (p-adic, Betti, de Rham).

If M is such a motive, say over $K = \mathbb{Q}$, and if M_{dR} denotes its de Rham realization, $F^\bullet M_{\mathrm{dR}}$ its Hodge filtration, and

$$\mathrm{gr}^r_F M_{\mathrm{dR}} := F^r M_{\mathrm{dR}} / F^{r+1} M_{\mathrm{dR}}$$

the associated subquotients, Kato considers the one-dimensional K-vector space:

$$L(M) = \bigotimes_{r\in\mathbb{Z}} (\det \mathrm{gr}^r_F M_{\mathrm{dR}})^{\otimes r} \simeq \bigotimes_{r>r_0} (\det_K F^r M_{\mathrm{dR}}), \tag{1.1.1}$$

where r_0 denotes an integer such that $F^{r_0} M_{\mathrm{dR}} = M_{\mathrm{dR}}$.

The $\mathbb{R}$-vector space $L(M)_\mathbb{R}$ is equipped with a canonical norm, defined by means of Hodge theory. Similarly, for every prime number p, the $\mathbb{Q}_p$-vector space $L(M)_{\mathbb{Q}_p}$ is endowed with a natural $\mathbb{Z}_p$-lattice, defined by means of the p-adic realization of M and of p-adic Hodge theory. Equipped with these additional structures, $L(M)$ becomes a Hermitian line bundle $\overline{L(M)}$ over $\operatorname{Spec}\mathbb{Z}$, and as such, admits an Arakelov degree. This degree defines the *Kato height* of the motive M:

$$\mathrm{ht}_K(M) := \widehat{\deg}\, \overline{L(M)} \in \mathbb{R}. \tag{1.1.2}$$

This definition is inspired by the definition of the Faltings height of an Abelian variety over a number field — which is a special instance of Kato's definition when M is a motive of weight one — and by Faltings' proof of the invariance by isogenies of the height of Abelian varieties, which played a key role in his proof of the Tate conjecture for Abelian varieties over number fields [**Fal83**].

The specific numerology in the definition (1.1.1) of the K-line $L(M)$ — where the determinant $\det_K \mathrm{gr}^r_F M_{\mathrm{dR}}$ of the r-th subquotient $\mathrm{gr}^r_F M_{\mathrm{dR}}$ of the Hodge filtration occurs at the tensor power r — has been actually dictated to Kato by its "compatibility" with the Weil conjectures: as shown by Kato, the Weil conjectures imply that the height defined as the Arakelov degree (1.1.2) satisfies an invariance property that generalizes the invariance by isogenies of the height of Abelian varieties to motives of arbitrary weight.

After Kato's seminal paper [**Kat14**], Kato's heights have been generalized and investigated by Kato himself [**Kat18**], by Koshikawa [**Kos15**] who studied their possible applications to classical boundedness and semisimplicity conjectures concerning motives over number fields, and by Venkatesh [**Ven18**], who started to explore their relationship to automorphic forms.

1.1.2. The Griffiths line bundle and the Griffiths heights of a VHS over a projective curve. As pointed out in [**Kat18**], a forerunner of Kato's definitions (1.1.1) and (1.1.2) already appears in Griffiths' famous work concerning the Hodge structures on the cohomology of complex projective varieties and the period mapping associated to a family of such varieties.

Recall that an integral variation of Hodge structures (VHS) of weight $n \in \mathbb{N}$ over a complex analytic manifold S is a pair:

$$\mathbb{V} := (V_{\mathbb{Z}}, \mathcal{F}^\bullet),$$

where $V_{\mathbb{Z}}$ is a local system of finitely generated free $\mathbb{Z}$-modules over S, and

$$\mathcal{F}^\bullet : \mathcal{F}^0 := \mathcal{V} \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^n \supseteq \mathcal{F}^{n+1} = 0$$

is a decreasing filtration by subbundles of the holomorphic vector bundle

$$\mathcal{V} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S^{\mathrm{an}},$$

attached to the local system $V_{\mathbb{Z}}$. These data must satisfy the Hodge decomposition condition — so that for every $s \in S$, the fiber $\mathbb{V}_s := (V_{\mathbb{Z},s}, \mathcal{F}_s^\bullet)$ is a Hodge structure of weight n — and the Griffiths horizontality condition (see for instance [**Voi02**, III] or [**PS08**, Chapter 10]).

To a holomorphic vector bundle $\mathcal{V}$ over an complex analytic manifold S, equipped with a finite decreasing filtration

$$F^\bullet := F^0 := \mathcal{V} \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0$$

by subbundles, we may attach the holomorphic line bundle on S:

$$\mathcal{GK}_S(\mathcal{V}, F^\bullet) := \bigotimes_{i=1}^{n} \det F^i \simeq \bigotimes_{r=0}^{n} (\det F^r/F^{r+1})^{\otimes r},$$

that we will call the *Griffiths line bundle* attached to $(\mathcal{V}, F^\bullet)$.

In particular, to a variation of Hodge structures $\mathbb{V} = (V_{\mathbb{Z}}, \mathcal{F}^\bullet)$ over S as above is attached its Griffiths line bundle:

$$\mathcal{GK}_S(\mathbb{V}) := \mathcal{GK}_S(\mathcal{V}, \mathcal{F}^\bullet) = \bigotimes_{r=0}^{n} (\det \mathcal{F}^r/\mathcal{F}^{r+1})^{\otimes r}. \tag{1.1.3}$$

This definition is introduced by Griffiths in [**Gri70**, Section 7 b)], under the terminology of *canonical line bundle* of the variation of Hodge structures. The specific tensor product of powers of line bundles that appears in the right hand side of (1.1.3) is the same as in the definition of the "Kato line" in (1.1.1). However it arises in Griffiths' work for completely different reasons. As shown in [**Gri70**], a polarization on the VHS $\mathbb{V}$ induces canonical Hermitian metrics on the vector bundles $\mathcal{F}^r/\mathcal{F}^{r+1}$, and therefore on the line bundle $\mathcal{GK}_S(\mathbb{V})$. The rationale behind the introduction of the line bundle (1.1.3) is that the curvature [1] of $\mathcal{GK}_S(\mathbb{V})$ equipped with this canonical Hermitian structure is always a *non-negative* real $(1,1)$-form on S.

In particular, when S is a connected smooth projective complex curve C, the degree of this line bundle:

$$\mathrm{ht}_{GK}(\mathbb{V}) := \deg_C \mathcal{GK}_S(\mathbb{V})$$

— which we will call the *Griffiths-Kato height*, or simply the *Griffiths height*, of the VHS $\mathbb{V}$ over C — is a non-negative integer when $\mathbb{V}$ admits a polarization.

1.1.3. The construction of Peters and the Griffiths height of the cohomology of a pencil of projective varieties. The Griffiths height $\mathrm{ht}_{GK}(\mathbb{V})$ may be seen as an analog of the Kato height $\mathrm{ht}_K(M)$, where the number field K is replaced by the function field $\mathbb{C}(C)$, and where the motive M is replaced by the VHS $\mathbb{V}$ over C. As shown by Peters [**Pet84**], it is actually possible to extend this definition to the situation where $\mathbb{V} := (V_{\mathbb{Z}}, \mathcal{F}^\bullet)$ is a variation of Hodge structures over a Zariski dense open subset $\mathring{C}$ of C.

Peters' construction relies on Deligne's work ([**Del70**]) on extensions of analytic local systems defined on a Zariski dense open subset of a smooth projective complex variety and on Schmid's results ([**Sch73**]) on variations of Hodge structures.

1. or more precisely the first Chern form.

It naturally arises in two variants, involving a choice of sign and the so-called upper and lower extensions, and leads to the introduction of some canonical extensions over C, denoted by $\mathcal{GK}_{C,+}(\mathbb{V})$ and $\mathcal{GK}_{C,-}(\mathbb{V})$, of the analytic line bundle $\mathcal{GK}_{\mathring{C}}(\mathbb{V})$ over $\mathring{C}$.[2]

The associated Griffiths heights will be denoted by:

$$\mathrm{ht}_{GK,+}(\mathbb{V}) := \deg_C \mathcal{GK}_{C,+}(\mathbb{V}) \quad \text{and} \quad \mathrm{ht}_{GK,-}(\mathbb{V}) := \deg_C \mathcal{GK}_{C,-}(\mathbb{V}).$$

The line bundles $\mathcal{GK}_{C,+}(\mathbb{V})$ and $\mathcal{GK}_{C,-}(\mathbb{V})$, and therefore the heights $\mathrm{ht}_{GK,+}(\mathbb{V})$ and $\mathrm{ht}_{GK,-}(\mathbb{V})$ actually coincide when the local monodromy of $V_{\mathbb{Z}}$ at every point of the complement $C - \mathring{C}$ (which is automatically quasi-unipotent) is unipotent.

The setting considered by Peters covers the variations of Hodge structures that "arise from geometry", namely those associated to a pencil of projective varieties parametrized by C and to the cohomology of its fibers.

Indeed consider a smooth projective complex variety X and

$$f : X \longrightarrow C$$

a surjective morphism. There exists $\mathring{C}$ as above such that the morphism:

$$f_{|X_{\mathring{C}}} : X_{\mathring{C}} := f^{-1}(\mathring{C}) \longrightarrow \mathring{C}$$

is smooth. For every $n \in \mathbb{N}$, we may consider the VHS $\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})$ over $\mathring{C}$, defined by the relative Betti cohomology in degree n, and the filtration on the relative de Rham cohomology induced by the relative Hodge-de Rham spectral sequence, then the line bundles $\mathcal{GK}_{C,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ and $\mathcal{GK}_{C,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ over C, and the corresponding Griffiths heights:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})) := \deg_C \mathcal{GK}_{C,+}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$$

and:

$$\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})) := \deg_C \mathcal{GK}_{C,-}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C})).$$

These numbers depend only on the generic fiber X_η of f over the generic point η of C, and will also be denoted by $\mathrm{ht}_{GK,+}(\mathbb{H}^n(X_\eta/C_\eta))$ and $\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_\eta/C_\eta))$.

1.1.4. Concerning the content of this paper. The starting point of this work has been the observation that, in spite of the significance of Kato's heights for the understanding of pure motives over number fields, basically nothing was known concerning them beyond motives of Abelian varieties. In particular, no explicit computation of these heights was known concerning motives of weight greater than one[3] Somewhat surprisingly, the situation is not better concerning the Griffiths heights $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$ when $n \geq 2$, which constitute their geometric counterparts.

Our aim has been to compute these Griffiths heights $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$, when $n \geq 2$, in a number of significant situations, with a view towards the derivation of similar formulas concerning Kato's heights. These formulas for Kato's heights would constitute some material for exploring various finiteness conjectures of Kato and Koshikawa ([**Kat18, Kos15**]).

In this paper, we establish formulas for these Griffiths heights, when n is the relative dimension of X/C, and when X/C is a semistable pencil of surfaces, a pencil of hypersurfaces in a projective space, or a Lefschetz pencil.

Our approach has been inspired by the classical computation by Hirzebruch of the Hodge numbers of complete intersections in projective space ([**Hir56**] and [**Hir95**, Appendix I]), and by the

2. See section 1.2 below for more details. Peters actually considers only the extension $\mathcal{GK}_{C,+}(\mathbb{V})$, for which the positivity of the Griffiths height in the polarized case still holds. Steenbrink's theory ([**Ste76, Ste77, PS08**]) is naturally related to the extension $\mathcal{GK}_{C,-}(\mathbb{V})$. This discrepancy leads to some mistakes in the published literature, which are clarified notably in [**Kol86**], [**Mor87**], and [**EFiMM21**].

3. apart from motives constructed by tensor operations from the motive of weight 1 associated with an Abelian variety.

recent results of Eriksson, Freixas, and Mourougane concerning BCOV invariants of Calabi-Yau manifolds of arbitrary dimension ([**EFiMM18, EFiMM21, EFiMM22**]).[4]

In the special case of Calabi-Yau manifolds, the alternating tensor product of Griffiths line bundles, which plays a key role in our computation, coincides with the BCOV line bundle that is investigated in the work of Eriksson, Freixas, and Mourougane. Our general formulas shed some light on the remarkable properties of this alternating tensor product when dealing with families of Calabi-Yau manifolds.

The next sections of this introduction are devoted to the statement of our results. Section 1.2 provides some details on Peters' construction alluded to above, and could be skipped at first reading. In the remaining sections, we state our results as equalities of rational numbers, providing closed formulas for suitable Griffiths heights $\mathrm{ht}_{GK,\pm}(\mathbb{H}^n(X_\eta/C_\eta))$. We shall actually establish these formulas in a more refined form, concerning the class of the line bundles $\mathcal{GK}_{C,\pm}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ in the rational Picard group $\mathrm{CH}^1(C)_\mathbb{Q}$ of the base curve C.

The next chapters of the paper are organized as follows.

Chapter 2 is devoted to preliminary results, most of them classical, concerning Steenbrink's theory. They will allow us to express the line bundle $\mathcal{GK}_{C,\pm}(\mathbb{H}^n(X_{\mathring{C}}/\mathring{C}))$ in terms of the higher direct images of relative logarithmic differentials when the singular fibers of X/C are divisors with strict normal crossings.

In Chapter 3, we establish various identities concerning the characteristic classes of relative differentials and relative logarithmic differentials in this situation. We believe that several of these formulas are of independent interest.

Chapter 4 contains the proof of the first of our main results, the computation of the alternating sum:

$$\sum_{n\in\mathbb{N}}(-1)^{n-1}\,\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_\eta/C_\eta)). \tag{1.1.4}$$

Besides the results of Steenbrink's recalled in Chapter 2, this computation relies on the Grothendieck-Riemann-Roch theorem and on the identities on characteristic classes established in Chapter 3. It turns out that the specific numerology in the definitions of the Kato line (1.1.1) and of the Griffiths line bundle (1.1.3), already discussed in 1.1.1 and 1.1.2, leads to remarkable cancellations when applying the Grothendieck-Riemann-Roch theorem, and allows one to derive some (relatively) simple formulas for this alternating sum.

In Chapter 5, we give an expression for the alternating sum (1.1.4) in the situation, better suited to explicit computations, where the singular fibers of X/C have only ordinary double points as singularities.

Finally, in Chapter 6, we combine our previous expressions for the sum (1.1.4) with the weak Lefschetz theorem to derive, in various significant situations, a formula for its middle-dimensional term $\mathrm{ht}_{GK,-}(\mathbb{H}^n(X_\eta/C_\eta))$, where n denotes the relative dimension of X/C.

4. The BCOV invariants of Calabi-Yau manifolds equipped with a Kähler metric are introduced in [**BCOV94**]. The paper [**FLY08**] introduces a normalized version of the BCOV invariant attached to Kählerian Calabi-Yau threefolds which does not depend on the choice of a specific Kähler metric, and investigates its asymptotic behaviour in degenerations; see also [**Yos15**] and [**LX19**]. The BCOV invariants of Calabi-Yau manifolds of arbitrary dimension are introduced and investigated in [**EFiMM18**] and [**EFiMM21**], and their relation with genus one mirror symmetry is studied in [**EFiMM22**].

1.2. Peters' construction and the Griffiths height of a variation of Hodge structures with possible degenerations over a curve

Let C be a connected smooth projective complex curve, let Δ be a finite subset of C, and let us denote by:

$$\mathring{C} := C - \Delta$$

the connected smooth algebraic curve defined by its complement.

Let $D \subset \mathbb{C}$ be a subset such that the projection $\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Z}$ induces a bijection $D \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}$, and let:

$$\log_{2i\pi D} : \mathbb{C}^* \longrightarrow 2i\pi D \subset \mathbb{C}$$

be the inverse of the map $\exp_{|2i\pi D}$.

For example, we can take as D the subset of $\mathbb{C}$:

$$D_+ := \{z \in \mathbb{C} | 0 \leq \Re(z) < 1\},$$

or:

$$D_- := \{z \in \mathbb{C} | -1 < \Re(z) \leq 0\}.$$

We shall note $\log_+$ for $\log_{2i\pi D_+}$, and $\log_-$ for $\log_{2i\pi D_-}$.

Let $(\mathcal{V}, \nabla)$ be an analytic vector bundle on $\mathring{C}$ with a flat connection.

Definition and Proposition 1.2.1 ([**Del70**, II, Prop. 5.4], see also [**Kat76**, "Key Lemma" p. 547], and [**EFiMM21**, section 2.1]). *The bundle with connection $(\mathcal{V}, \nabla)$ admits an unique analytic*[5] *extension with logarithmic connection $(\overline{\mathcal{V}}, \overline{\nabla}_D)$ on C, such that for all x in Δ, the eigenvalues of the residue endomorphism $\mathrm{Res}_x \overline{\nabla}_D \in \mathrm{End}(\overline{\mathcal{V}}_{Dx})$ belong to $-D$.*

Following the convention of [**EFiMM21**], actually introduced in [**Kol86**, Section 2] and expanded upon in [**Mor87**, Section 2], we shall call this extension the *upper* (resp. *lower*) *Deligne extension*, if $D = D_+$ (resp. D_-). Note that the eigenvalues of the residue of the connexion defining the *lower* extension are *non-negative*.

Consider now a VHS on $\mathring{C}$:

$$\mathbb{V} := (V_{\mathbb{Z}}, \nabla, \mathcal{F}^\bullet),$$

where as before $\mathcal{F}^\bullet$ denotes a filtration of the vector bundle $\mathcal{V} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}_{\mathring{C}}} \mathcal{O}^{an}_{\mathring{C}}$. According to a result of Griffiths ([**Sch73**, Theorem (4.13), a)]), the subbundles $(\mathcal{F}^p)_{p\geq 0}$ are algebraic vector subbundles of $\mathcal{V}_{|\mathring{C}}$ equipped with the algebraic structure defined by its Deligne extension.[6]

Let us denote by $(\overline{\mathcal{V}}_+, \overline{\nabla}_+)$ and $(\overline{\mathcal{V}}_-, \overline{\nabla}_-)$ the upper and lower Deligne extensions of the bundle with connection $(\mathcal{V}, \nabla)$, and for every integer $p \geq 0$, by $\overline{\mathcal{F}}^p_+$ and $\overline{\mathcal{F}}^p_-$ the vector subbundles (that is, the locally direct summands) of $\overline{\mathcal{V}}_+$ and $\overline{\mathcal{V}}_-$ over C that extend the vector subbundle $\mathcal{F}^p$ of:

$$\mathcal{V} \simeq \overline{\mathcal{V}}_{+|\mathring{C}} \simeq \overline{\mathcal{V}}_{-|\mathring{C}}.$$

The *upper* and *lower Griffiths line bundles* of the VHS $\mathbb{V}$ on $\mathring{C}$ are defined as the line bundles over C:

$$\mathcal{GK}_{C,+}(\mathbb{V}_\eta) := \mathcal{GK}_C(\overline{\mathcal{V}}_+, \overline{\mathcal{F}}^\bullet_+) = \bigotimes_{p\geq 0} (\det \overline{\mathcal{F}}^p_+ / \overline{\mathcal{F}}^{p+1}_+)^{\otimes p},$$

and:

$$\mathcal{GK}_{C,-}(\mathbb{V}_\eta) := \mathcal{GK}_C(\overline{\mathcal{V}}_-, \overline{\mathcal{F}}^\bullet_-) = \bigotimes_{p\geq 0} (\det \overline{\mathcal{F}}^p_- / \overline{\mathcal{F}}^{p+1}_-)^{\otimes p}.$$

5. Since C is projective, the analytic vector bundle with connection $(\mathcal{V}, \nabla)$ is actually algebraizable.

6. The algebraic structure of the restriction $\mathcal{V}_{\mathring{C}}$, defined by means of Definition and Proposition 1.2.1 is easily seen not to depend on the choice of the subset D.

Finally the *upper* and *lower Griffiths-Kato heights* of $\mathbb{V}$ are defined as the degree of these line bundles:

$$\mathrm{ht}_{GK,+}(\mathbb{V}_\eta) := \deg_C(\mathcal{GK}_{C,+}(\mathbb{V}_\eta)),$$

and:

$$\mathrm{ht}_{GK,-}(\mathbb{V}_\eta) := \deg_C(\mathcal{GK}_{C,-}(\mathbb{V}_\eta)).$$

Observe that, by construction, the line bundles $\mathcal{GK}_{C,\pm}(\mathbb{V}_\eta)$ and the heights $\mathrm{ht}_{GK,\pm}(\mathbb{V}_\eta)$ are unchanged when Δ is replaced by a larger finite subset of C. They depend only on the restriction of the VHS $\mathbb{V}$ to some arbitrary small Zariski open neighborhood $\mathring{C}$ of the generic point η of C, as indicated by our notation.

Observe also that, when the local monodromy of $V_\mathbb{Z}$ at every point of Δ (which is automatically quasi-unipotent according to a result of Borel [**Sch73**, Lemma (4.5)]) is unipotent, the upper and lower Deligne extensions $(\overline{\mathcal{V}}_+, \overline{\nabla}_+)$ and $(\overline{\mathcal{V}}_-, \overline{\nabla}_-)$ coincide, and therefore the Griffiths line bundles $\mathcal{GK}_{C,+}(\mathbb{V}_\eta)$ and $\mathcal{GK}_{C,-}(\mathbb{V}_\eta)$ also, and their degree $\mathrm{ht}_{GK,+}(\mathbb{V}_\eta)$ and $\mathrm{ht}_{GK,-}(\mathbb{V}_\eta)$ as well. In this case, we shall denote them by $\mathcal{GK}_C(\mathbb{V}_\eta)$ and $\mathrm{ht}_{GK}(\mathbb{V}_\eta)$.

These notions are motivated by the following theorem of Peters, which extends an earlier positivity result of Griffiths [**Gri70**]:

THEOREM 1.2.2 ([**Pet84**, Th. 4.1]). *If the VHS $\mathbb{V} := (V_\mathbb{Z}, \mathcal{F}^\bullet)$ over $\mathring{C}$ is polarized. then the upper Griffiths-Kato height is always non-negative:*

$$\mathrm{ht}_{GK,+}(\mathbb{V}_\eta) \geq 0. \tag{1.2.1}$$

When furthermore the weight of the VHS $\mathbb{V}$ is positive, equality holds in (1.2.1) *if and only if all the subbundles $\mathcal{F}^p$ are flat for ∇ and the local monodromy of $V_\mathbb{Z}$ at every point of Δ is unipotent.*

It is possible to introduce a variant of the heights $\mathrm{ht}_{GK,+}(\mathbb{V}_\eta)$ and $\mathrm{ht}_{GK,-}(\mathbb{V}_\eta)$, namely the *Griffiths-Kato stable height* $\mathrm{ht}_{GK,stab}(\mathbb{V}_\eta)$, which is defined as follows (see [**Kat18**]; this construction is inspired by the definition of the stable Faltings height of Abelian varieties over number fields.)

For every $x \in \Delta$, using the result of Borel cited above ([**Sch73**, Lemma (4.5)]), the eigenvalues of the local monodromy of $V_\mathbb{Z}$ at x are roots of unity. We shall denote by r_x the lcm of the order of these roots of unity.

Let C' be a connected smooth projective curve with generic point η', and let:

$$\sigma : C' \longrightarrow C$$

be a finite morphism, such that for all $x' \in \sigma^{-1}(\Delta)$, the index of ramification of σ in x' is a multiple of $r_{\sigma(x')}$. Such pairs (C', σ) are easily seen to exist. They may be constructed for instance as cyclic coverings of C.

Pulling back $\mathbb{V}$ by σ, we get a VHS:

$$\mathbb{V}' := (\sigma^* V_\mathbb{Z}, \sigma^* \mathcal{F}^\bullet)$$

over the complement $\mathring{C}' := C' - \Delta'$ of $\Delta' := \sigma^{-1}(\Delta)$. By construction, the local monodromy of $V'_\mathbb{Z} := \sigma^* V_\mathbb{Z}$ at every point of Δ' is unipotent. Therefore we may define:

$$\mathrm{ht}_{GK,stab}(\mathbb{V}_\eta) := \frac{1}{\deg(\sigma)} \deg_{C'}(\mathcal{GK}_{C'}(\sigma^* \mathbb{V}_\eta)).$$

This rational number is easily seen not to depend on the choice of the ramified covering (C', σ) and to satisfy the following inequalities:

$$\mathrm{ht}_{GK,-}(\mathbb{V}_\eta) \leq \mathrm{ht}_{GK,stab}(\mathbb{V}_\eta) \leq \mathrm{ht}_{GK,+}(\mathbb{V}_\eta).$$

These inequalities are simple consequences of the definition of the upper and lower Deligne extensions and do not require the VHS $\mathbb{V}$ to be polarized.

1.3. Alternating sums of Griffiths heights and characteristic classes

1.3.1. The main formulas for alternating sums of Griffiths heights. Using the Grothendieck-Riemann-Roch formula and Steenbrink's theory, we shall establish expressions in terms of characteristic classes for the alternating sum of Griffiths heights associated to a pencil of projective varieties whose singular fibers are divisors with strict normal crossings.

THEOREM 1.3.1. *Let C be a connected smooth projective complex curve with generic point η, let Y be a connected smooth projective complex variety of dimension N, and let*

$$g : Y \longrightarrow C$$

be a surjective morphism of complex varieties. Let us assume that there exists a finite subset Δ in C such that g is smooth over $C - \Delta$, and such that the divisor Y_Δ is a divisor with strict normal crossings in Y.

Then the following equality holds:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \int_Y \rho_{N-1}(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}, \tag{1.3.1}$$

where $\omega^1_{Y/C}$ denotes the vector bundle over Y of relative logarithmic 1-forms[7]*:*

$$\omega^1_{Y/C} := \Omega^1_{Y/C}(\log Y_\Delta),$$

where $[T_g]$ is the relative tangent class in K-theory:

$$[T_g] := [T_{Y/\mathbb{C}}] - g^*[T_{C/\mathbb{C}}] \in K^0(Y),$$

and where Td *denotes the Todd class and ρ_{N-1} the characteristic class defined in terms of the Chern classes by:*

$$\rho_{N-1} := c_{N-2} - \frac{N-1}{2} c_{N-1} + \frac{1}{12} c_1 c_{N-1}. \tag{1.3.2}$$

The occurence of the height $\mathrm{ht}_{GK,-}$, and not of $\mathrm{ht}_{GK,+}$ or $\mathrm{ht}_{GK,stab}$, in the left-hand side of (1.3.1) is due to the specific role played by the *lower* extensions in Steenbrink's theory.

The class ρ_{N-1}, defined by the right-hand side of (1.3.2), already appears in a computation of anomaly in [**BCOV94**, 5.8].[8]

In the right-hand side of (1.3.1), we denote by:

$$\int_Y : \mathrm{CH}^*(Y)_\mathbb{Q} \longrightarrow \mathbb{Q}$$

the map that sends a class α in $\mathrm{CH}^*(Y)_\mathbb{Q}$ to the degree of its homogeneous component $\alpha^{[N]}$ in $\mathrm{CH}^N(Y)_\mathbb{Q} \simeq \mathrm{CH}_0(Y)_\mathbb{Q}$.

In particular, when the morphism g is smooth, the class in $K^0(Y)$ of $\omega^{1\vee}_{Y/C} = \Omega^{1\vee}_{Y/C}$ and $[T_g]$ coincide, and (1.3.1) becomes the simpler formula:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12} \int_Y c_1(\Omega^{1\vee}_{Y/C}) c_{N-1}(\Omega^{1\vee}_{Y/C}). \tag{1.3.3}$$

---

7. We use the same notation as [**Ill94**]. Observe that this vector bundle does not depend on the choice of the divisor Δ in C satisfying the above conditions.

8. We were not aware of this reference when completing the first version of this memoir. We are grateful to Christophe Mourougane for pointing out this reference.

When the morphism g is not necessarily smooth, the restrictions in $K^0(Y - Y_\Delta)$ of $[T_g]$ and $[\omega^{1\vee}_{Y/C}]$ still coincide, and there exists a cycle with rational coefficients V supported by Y_Δ such that:

$$\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} = 1 + [V],$$

and formula (1.3.1) may be written:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\omega^{1\vee}_{Y/C}) c_{N-1}(\omega^{1\vee}_{Y/C}) + \int_Y \rho_{N-1}(\omega^{1\vee}_{Y/C})[V]. \tag{1.3.4}$$

Formula (1.3.4) is similar to (1.3.3), but its right-hand side contains a second term that depends on the geometry of the morphism g in the infinitesimal neighborhood of the "bad fibers" Y_Δ.

We are going to give a closed formula for this second term, that will be derived by means of an actual computation of the cycle V. To achieve this, we need to introduce some more notation.

With the notation of Theorem 1.3.1, let us write the divisor Y_Δ (defined as the inverse image by g of Δ seen as a reduced subscheme of C) as follows:

$$Y_\Delta = \sum_{i \in I} m_i D_i,$$

where I is a finite set, and where the m_i are positive integers and the D_i are pairwise distinct connected non-singular divisors in Y.

The set I may be written as the disjoint union:

$$I = \bigcup_{x \in \Delta} I_x,$$

where, for every $x \in \Delta$, I_x denotes the non-empty subset of I defined by:

$$I_x := \big\{ i \in I \mid g(D_i) = \{x\} \big\}.$$

Let $\prec$ be a total order on the set I.

For every integer $r \geq 1$, let D^r be the subscheme of codimension r of Y defined as the union of all the intersections of r distinct components (D_i):

$$D^r := \bigcup_{J \subset I, |J| = r} \bigcap_{i \in J} D_i.$$

For every i in I, let:

$$i_{D_i} : D_i \longrightarrow Y$$

be the inclusion map, and

$$\mathcal{N}_i := \mathcal{N}_{D_i} Y$$

be the normal line bundle to D_i in Y.

Observe also that, for every $i \in I$, the scheme:

$$D_i \cap D^2 = \bigcup_{j \in I, j \neq i} D_{ij}$$

is a divisor with strict normal crossings in D_i. Similarly, for every $(i, j) \in I^2$ such that $i \prec j$, the scheme:

$$D_{ij} \cap D^3 = \bigcup_{k \in I - \{i,j\}} D_{ij} \cap D_k$$

is a divisor with strict normal crossings in D_{ij}. We shall also use the notation:

$$\mathring{D}_i :=: D_i - D_i \cap D^2 \quad \text{and} \quad \mathring{D}_{ij} :=: D_{ij} - D_{ij} \cap D^3.$$

Theorem 1.3.2. *Under the assumptions of Theorem 1.3.1, using the above notation, the following equality holds:*

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\omega^{1\vee}_{Y/C}) c_{N-1}(\omega^{1\vee}_{Y/C}) + \sum_{x\in\Delta} \alpha_x, \tag{1.3.5}$$

where for every x in Δ, α_x is the rational number given by the following two expressions[9]*:*

$$\alpha_x = \frac{N-1}{4}\sum_{i\in I_x}(m_i-1)\chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12}\sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} (3 - m_i/m_j - m_j/m_i)\chi_{\mathrm{top}}(\mathring{D}_{ij}) \tag{1.3.6}$$

$$= \frac{1}{12}\sum_{i\in I_x}\Big[3(N-1)(m_i-1)\chi_{\mathrm{top}}(\mathring{D}_i) + \int_{D_i} c_1(\mathcal{N}_i) c_{N-2}(\Omega^{1\vee}_{D_i}(\log D_i\cap D^2))\Big] + \frac{1}{4}\sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} \chi_{\mathrm{top}}(\mathring{D}_{ij}). \tag{1.3.7}$$

The expression (1.3.6) for α_x involves only the topology of the open strata $\mathring{D}_i$ and $\mathring{D}_{ij}$ of the fiber $g^{-1}(x)$ and the multiplicities m_i of its components D_i. The expression (1.3.7) makes clear that α_x belongs to $(1/12)\mathbb{Z}$.

Observe that, when the divisor Y_Δ is reduced, that is when the degenerations of the fibers of g are semistable with smooth components, the variations of Hodge structures $\mathbb{H}^n(Y_\eta/C_\eta)$ have unipotent local monodromy, and formula (1.3.5) takes the simpler form:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\omega^{1\vee}_{Y/C}) c_{N-1}(\omega^{1\vee}_{Y/C}) + \frac{1}{12}\chi_{\mathrm{top}}(D^2 - D^3). \tag{1.3.8}$$

In this semistable situation, using the identification of the lower Deligne extension of the cohomology and the logarithmic de Rham cohomology (see Proposition 2.1.2 below), we also have an isomorphism of line bundles over C:

$$\mathcal{GK}_C(\mathbb{H}^1(Y_\eta/C_\eta)) \xrightarrow{\sim} \det g_*\omega^1_{Y/C}.$$

Moreover, using the unipotence of the local monodromy, Poincaré duality and the existence of an integral structure on the de Rham cohomology vector bundles as in Proposition 2.5.2, we have an isomorphism of line bundles over C, up to 2-torsion:

$$\mathcal{GK}_C(\mathbb{H}^1(Y_\eta/C_\eta)) \simeq \mathcal{GK}_C(\mathbb{H}^{2N-3}(Y_\eta/C_\eta)).$$

Under this assumption of semistable reduction, when $N=3$, we also have the relations:

$$\mathrm{ht}_{GK}(\mathbb{H}^1(Y_\eta/C_\eta)) = \mathrm{ht}_{GK}(\mathbb{H}^3(Y_\eta/C_\eta)) = \deg g_*\omega^1_{Y/C},$$

and therefore the equality (1.3.8) becomes an expression for the Griffiths height of the variation of Hodge structures $\mathbb{H}^2(Y_\eta/C_\eta)$ associated to a family of surfaces with semistable degenerations:

Corollary 1.3.3. *In the situation of Theorem 1.3.1, when $N=3$ and when the degenerations of the fibers of g are semistable with smooth components, the following equality holds:*

$$\mathrm{ht}_{GK}(\mathbb{H}^2(Y_\eta/C_\eta)) = 2\deg g_*\omega^1_{Y/C} + \frac{1}{12}\int_Y c_1(\omega^1_{Y/C})c_2(\omega^1_{Y/C}) - \frac{1}{12}\chi_{\mathrm{top}}(D^2-D^3). \tag{1.3.9}$$

9. By χ_{top}, we denote the topological Euler characteristic.

1.3.2. Application to pencils of projective varieties whose singular fibers have ordinary double points. We shall notably apply Theorem 1.3.2 in the case where Y is the blow-up of a finite number of non-degenerate critical points of a morphism

$$f : H \longrightarrow C.$$

In this case, using a result of Eriksson, Freixas and Mourougane's concerning the comparison of Deligne extensions and the so-called "elementary exponents" of Hodge bundles in the case of singular fibers with ordinary double points ([**EFiMM21**, Prop. 3.10]), and computations of characteristic classes, we shall obtain the following result.

THEOREM 1.3.4. *Let C be a connected smooth projective complex curve with generic point η, H be a smooth projective N-dimensional complex scheme, and let:*

$$f : H \longrightarrow C$$

be a morphism of complex schemes. Let us assume that there exists a finite subset Σ in H such that f is smooth on $H - \Sigma$ and admits a non-degenerate critical point at every point of Σ.[10]

Then the following equalities hold:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(H_\eta/C_\eta)) = \frac{1}{12}\int_H c_1([\Omega^1_{H/C}]^\vee) c_{N-1}([\Omega^1_{H/C}]^\vee) + u_N^- \, |\Sigma|, \tag{1.3.10}$$

and

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,+}(\mathbb{H}^n(H_\eta/C_\eta)) = \frac{1}{12}\int_H c_1([\Omega^1_{H/C}]^\vee) c_{N-1}([\Omega^1_{H/C}]^\vee) + u_N^+ \, |\Sigma|, \tag{1.3.11}$$

where u_N^- and u_N^+ are the rational numbers defined by:

$$u_N^- := \begin{cases} (5N-3)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even,} \end{cases}$$

and:

$$u_N^+ := \begin{cases} -(7N-9)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even.} \end{cases}$$

1.4. The Griffiths height of the middle-dimensional cohomology of a pencil of hypersurfaces

1.4.1. Families of ample hypersurfaces in a smooth pencil. Applying Theorem 1.3.4 to the situation where H is an hypersurface in some smooth C-scheme X, and using the weak Lefschetz theorem (see for instance [**Voi03**, Theorem 1.29]), we obtain the following somewhat technical result.

PROPOSITION 1.4.1. *Let C be a connected smooth projective complex curve with generic point η, X be a smooth projective complex scheme of pure dimension $N+1$, and let*

$$\pi : X \longrightarrow C$$

be a smooth surjective morphism of complex schemes. Let H be a non-singular hypersurface in X such that the morphism

$$\pi_{|H} : H \longrightarrow C$$

is flat[11] *and has a finite set Σ of critical points, all of which are non-degenerate.*[12]

10. Namely the differential of f vanishes and its Hessian is non-degenerate at every point of Σ; in other words, the only singularities of the fibers of f are ordinary double points.

11. or equivalently, is non-constant on every connected component of H, or has fibers of pure dimension $N-1$.

12. Equivalently, the only possible singularities of the fibers $(H_x)_{x\in C}$ are ordinary double points.

If we denote by L the line bundle $\mathcal{O}_X(H)$ on X, then the following equality holds:

$$|\Sigma| = \int_X (1 - c_1(L))^{-1} c(\Omega^1_{X/C}), \tag{1.4.1}$$

If moreover the line bundle L is ample relatively to π, then the following equalities hold:

$$\begin{aligned} \mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) &= \mathrm{ht}_{GK}(\mathbb{H}^{N-1}(X/C)) + \mathrm{ht}_{GK}(\mathbb{H}^{N+1}(X/C)) - \mathrm{ht}_{GK}(\mathbb{H}^{N}(X/C)) \\ &+ \frac{1}{12}\int_X \left[(1-c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C}) - c_1(L) c_N(\Omega^1_{X/C})\right] + v_N^+ \, |\Sigma|, \end{aligned} \tag{1.4.2}$$

and:

$$\begin{aligned} \mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) &= \mathrm{ht}_{GK}(\mathbb{H}^{N-1}(X/C)) + \mathrm{ht}_{GK}(\mathbb{H}^{N+1}(X/C)) - \mathrm{ht}_{GK}(\mathbb{H}^{N}(X/C)) \\ &+ \frac{1}{12}\int_X \left[(1-c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C}) - c_1(L) c_N(\Omega^1_{X/C})\right] + v_N^- \, |\Sigma|, \end{aligned} \tag{1.4.3}$$

where:

$$v_N^+ := \begin{cases} 7(N-1)/24 & \text{if } N \text{ is odd} \\ (N+2)/24 & \text{if } N \text{ is even,} \end{cases}$$

and:

$$v_N^- := \begin{cases} -5(N-1)/24 & \text{if } N \text{ is odd} \\ (N+2)/24 & \text{if } N \text{ is even.} \end{cases}$$

In the expressions in the right-hand side of (1.4.1), (1.4.2), and (1.4.3), we denote by $c(\Omega^1_{X/C})$ the total Chern class of the vector bundle $\Omega^1_{X/C}$ of rank N:

$$c(\Omega^1_{X/C}) := 1 + c_1(\Omega^1_{X/C}) + \cdots + c_N(\Omega^1_{X/C}).$$

These expressions may be expanded in terms of $c_1(L)$, $c_1(\Omega^1_{X/C})$,..., $c_N(\Omega^1_{X/C})$. For instance, (1.4.2) may also be written:

$$\begin{aligned} &\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = \mathrm{ht}_{GK}(\mathbb{H}^{N-1}(X/C)) + \mathrm{ht}_{GK}(\mathbb{H}^{N+1}(X/C)) - \mathrm{ht}_{GK}(\mathbb{H}^{N}(X/C)) \\ &+ \frac{1}{12}\sum_{j=0}^{N}\int_X c_1(L)^j c_1(\Omega^1_{X/C}) c_{N-j}(\Omega^1_{X/C}) - \frac{1}{12}\int_X c_1(L) c_N(\Omega^1_{X/C}) + v_N^+ \sum_{j=1}^{N+1}\int_X c_1(L)^j c_{N+1-j}(\Omega^1_{X/C}). \end{aligned}$$

In Annexe A, we show that the hypotheses in Proposition 1.4.1 on the hypersurface H and the morphism f — namely, H is non-singular, $\pi_{|H}$ is flat, and the critical points of $\pi_{|H}$ are non-degenerate — holds for H the hypersurface defined by a generic section of some line bundle L on X, when L is "positive enough", namely when the global sections of L on X generate L and the relative jet bundle $J_1(L)_{X/C}$. If moreover these global sections separate pairs of points in the same fiber of π, then the hypersurface H defined by a generic section of L satisfies a slightly stronger condition: the morphism $\pi_{|H}$ admits at most one critical point in each fiber of π.

1.4.2. Pencils of hypersurfaces in the projective space. As a first application of Proposition 1.4.1, we may study pencils of projective hypersurfaces.

Let E be a vector bundle of rank $N+1$ over a connected smooth projective complex curve C, and let

$$\pi : \mathbb{P}(E) := \operatorname{Proj} S^\bullet E^\vee \longrightarrow C$$

be the associated projective bundle. We shall denote by $\mathcal{O}_E(-1)$ the tautological rank one subbundle of $\pi^* E$, and by $\mathcal{O}_E(1)$ its dual. An horizontal hypersurface in the projective bundle $\mathbb{P}(E)$ is an effective Cartier divisor H in $\mathbb{P}(E)$ such that the morphism:

$$\pi_{|H} : H \longrightarrow C$$

is flat. Then its fibers:

$$H_x := \pi_{|H}^{-1}(x), \quad x \in C$$

are hypersurfaces in the projective spaces $\mathbb{P}(E_x) \simeq \mathbb{P}^N(\mathbb{C})$. Their degree d is independent of $x \in C$, and defines the *relative degree* of the horizontal hypersurface.

We introduce the *intersection-theoretic height* of an horizontal hypersurface H. It is defined as the rational number:

$$\mathrm{ht}_{int}(H/C) := \int_{\mathbb{P}(E)} c_1(\mathcal{O}_E(1))^N \cap [H] + dN\mu(E), \tag{1.4.4}$$

where

$$\mu(E) := \deg_C E/\mathrm{rk}E = \deg(c_1(E) \cap [C])/(N+1)$$

denotes the slope of the vector bundle E over C.[13]

We will show the following theorem.

THEOREM 1.4.2. *Let C be a connected smooth projective complex curve with generic point η, E a vector bundle of rank $N+1$ over C, and $H \subset \mathbb{P}(E)$ an horizontal hypersurface of relative degree d, smooth over $\mathbb{C}$. If $\pi_{|H}$ has only a finite number of critical points, all of which are non-degenerate, then the cardinality of the set Σ of critical points satisfies:*

$$|\Sigma| = (N+1)(d-1)^N \,\mathrm{ht}_{int}(H/C). \tag{1.4.5}$$

Moreover, under the same hypothesis, the following equalities hold:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_+(d,N)\,\mathrm{ht}_{int}(H/C),$$
$$\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_-(d,N)\,\mathrm{ht}_{int}(H/C),$$

and:

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_{stab}(d,N)\,\mathrm{ht}_{int}(H/C),$$

where $F_+(d,N)$, $F_-(d,N)$ and $F_{stab}(d,N)$ are the elements of $(1/12)\mathbb{Z}$ given when N is odd by:

$$F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N - d^2 - 2dN - 2) + 2(d^2-1)\right],$$

$$F_+(d,N) := F_{stab}(d,N) + \frac{(N+1)(N-1)(d-1)^N}{4},$$

and:

$$F_-(d,N) := F_{stab}(d,N) - \frac{(N+1)(N-1)(d-1)^N}{4},$$

and when N is even by:

$$F_+(d,N) = F_-(d,N) = F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N + 2d^2 - 2dN - 2) - 2(d^2-1)\right].$$

The expression (1.4.5) does not seem to appear explicitly in the literature. However it is a simple consequence of various known results concerning the discriminant of homogeneous polynomials in $N+1$ variables; see 6.2.3 below.

From Theorem 1.4.2, we can also deduce similar formulas for the Griffiths heights of the *primitive* middle-dimensional cohomology.

Indeed, it follows from Lefschetz's weak theorem that for every integer $n < N-1$, the VHS $\mathbb{H}^n(H_{C-\Delta}/C-\Delta)$ on $C-\Delta$ (where Δ denotes the set of critical values of the morphism $\pi_{|H}$) is trivial. Namely its underlying flat vector bundle is trivial, and its Hodge filtration also. Consequently its Griffiths line bundle is trivial.

13. The additive normalization by $dN\mu(E)$ in the right-hand side of (1.4.4) ensures that $\mathrm{ht}_{int}(H/C)$ is unchanged when the vector bundle E is replaced by $E \otimes L$ for some line bundle L over C: it depends only on H as a subscheme of the projective bundle $\mathbb{P} := \mathbb{P}(E)$, and not on the actual choice of a vector bundle E such that $\mathbb{P} \simeq \mathbb{P}(E)$. See Remark 6.2.3 below.

Using the Lefschetz decomposition of the VHS $\mathbb{H}^{N-1}(H_{C-\Delta}/C-\Delta)$ and the easily checked fact that the Griffiths line bundle of a direct sum of VHS is isomorphic to the tensor product of their Griffiths line bundles, we obtain the equalities of rational numbers:

$$\mathrm{ht}_{GK,\pm}(\mathbb{H}^{N-1}_{prim}(H_\eta/C_\eta)) = \mathrm{ht}_{GK,\pm}(\mathbb{H}^{N-1}(H_\eta/C_\eta)),$$

and:

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}_{prim}(H_\eta/C_\eta)) = \mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}(H_\eta/C_\eta)),$$

hence, under the hypothesis of Theorem 1.4.2:

$$\mathrm{ht}_{GK,\pm}(\mathbb{H}^{N-1}_{prim}(H_\eta/C_\eta)) = F_\pm(d,N)\,\mathrm{ht}_{int}(H/C),$$

and:

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}_{prim}(H_\eta/C_\eta)) = F_{stab}(d,N)\,\mathrm{ht}_{int}(H/C).$$

Let us point out that the intersection-theoretic height $\mathrm{ht}_{int}(H/C)$ is non-negative when $d \geq 2$. This follows from equality (1.4.5). The constants $F_{stab}(d,N)$ and $F_+(d,N)$ are easily checked to be non-negative, as predicted by Peters' inequality (1.2.1).

However, for any given value of $d \geq 2$, the constant $F_-(d,N)$ is negative when N is large and odd, and the height $\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}_{prim}(H_\eta/C_\eta))$ may be negative.

1.4.3. Linear pencils of hypersurfaces. Another case in which we can apply Proposition 1.4.1 is the case of linear pencils of hypersurfaces.

Let V be a connected smooth projective complex scheme of pure dimension $N \geq 1$, and let H be a non-singular hypersurface in $V \times \mathbb{P}^1$ such that the morphism

$$\mathrm{pr}_{1|H} : H \longrightarrow V$$

is dominant, or equivalently surjective, and let δ be its degree. The line bundle $\mathcal{O}_{V\times\mathbb{P}^1}(H)$ on $V\times\mathbb{P}^1$ is isomorphic to the line bundle

$$\mathrm{pr}_1^* M \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(\delta),$$

where M denotes some line bundle over V, which is unique up to isomorphism.

Proposition 1.4.3. *With the above notation, let us assume that the morphism*

$$\mathrm{pr}_{2|H} : H \longrightarrow \mathbb{P}^1$$

is surjective, and has a finite set Σ of critical points, all of which are non-degenerate.

Then the cardinality of Σ satisfies:

$$|\Sigma| = \delta \int_V (1 - c_1(M))^{-2} c(\Omega^1_V).$$

Furthermore, if the line bundle M on V is ample, the following equalities of integers hold:[14]

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta)) = \frac{\delta}{12}\int_V (1-c_1(M))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}\delta}{12}\chi_{\mathrm{top}}(V) + v_N^+|\Sigma|$$

and:

$$\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta)) = \frac{\delta}{12}\int_V (1-c_1(M))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}\delta}{12}\chi_{\mathrm{top}}(V) + v_N^-|\Sigma|,$$

where v_N^+ and v_N^- are the rational numbers defined in Proposition 1.4.1.

14. We use the notation introduced at the end of 1.1.3 with $C := \mathbb{P}^1$.

1.4.4. Lefschetz pencils. Proposition 1.4.3 applies notably to Lefschetz pencils.

Let V be a connected smooth projective complex scheme of pure dimension $N \geq 1$, embedded into some projective space $\mathbb{P}^r$, of dimension $r \geq \max(N, 2)$.

Let Λ be a projective subspace of dimension $r-2$ in $\mathbb{P}^r$ that intersects V transversally, and let $P \subset \mathbb{P}^{r\vee}$ be the projective line in the dual projective space $\mathbb{P}^{r\vee}$ corresponding to Λ by projective duality.

Let us denote by:

$$\nu : \widetilde{\mathbb{P}}^r_\Lambda \longrightarrow \mathbb{P}^r$$

the blowing-up of Λ in $\mathbb{P}^r$. If $\widetilde{V}$ denote the proper transform of V by ν, the restriction:

$$\nu_{|\widetilde{V}} : \widetilde{V} \longrightarrow V$$

may be identified with the blowing-up in V of $\Lambda \cap V$, which is smooth of dimension $r-2$.

The projection of center Λ:

$$\mathbb{P}^r - \Lambda \longrightarrow P$$

extends to a smooth morphism of complex schemes:

$$p : \widetilde{\mathbb{P}}^r_\Lambda \longrightarrow P.$$

Recall that the pencil of hyperplanes in $\mathbb{P}^r$ containing Λ — in other words the pencil of hyperplanes defined by P — is said to be a Lefschetz pencil with respect to the subvariety V of $\mathbb{P}^r$ when the morphism:

$$p_{|\widetilde{V}} : \widetilde{V} \longrightarrow P$$

has a finite set Σ of critical points, all of which are non-degenerate, and when the restriction

$$p_{|\Sigma} : \Sigma \longrightarrow P$$

is an injective map.

As a simple instance of Proposition 1.4.3, we shall establish the following result, which notably applies to Lefschetz pencils:

COROLLARY 1.4.4. *With the above notation, let us assume that the morphism:*

$$p_{|\widetilde{V}} : \widetilde{V} \longrightarrow P$$

has a finite set Σ of critical points, all of which are non-degenerate.

Then the cardinality of Σ satisfies:

$$|\Sigma| = \int_V (1 - c_1(\mathcal{O}_V(1)))^{-2} c(\Omega^1_V). \tag{1.4.6}$$

and the following equalities hold:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)) = \frac{1}{12}\int_V (1-c_1(\mathcal{O}_V(1)))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}}{12}\chi_{\mathrm{top}}(V) + v^+_N|\Sigma|, \tag{1.4.7}$$

and:

$$\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)) = \frac{1}{12}\int_V (1-c_1(\mathcal{O}_V(1)))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}}{12}\chi_{\mathrm{top}}(V) + v^-_N|\Sigma|. \tag{1.4.8}$$

The expression (1.4.6) for the number of critical points in a Lefschetz pencil is established by Katz in [**SGA73**, Exposé XVII, cor. 5.6].[15]

15. In *loc. cit.*, the right-hand side of (1.4.6) is shown to be the product of the degree of the Gauss map and of the degree of the dual hypersurface associated to V.

1.5. Griffiths heights and Calabi-Yau manifolds

We conclude this introduction by some observations about the special forms taken by our formulas concerning Griffiths heights when applied to Calabi-Yau manifolds. This sheds some light on the significance of the assumption of being Calabi-Yau in the investigation of BCOV invariants, which have already been mentioned in 1.1.4 above.

1.5.1. Pencils of Calabi-Yau manifolds. Let us consider a pencil of projective varieties:

$$g : Y \longrightarrow C$$

as in Theorems 1.3.1 and 1.3.2, and assume that the smooth fibers of the morphism g are Calabi-Yau manifolds. In other terms, for every point x in $C - \Delta$, we assume that the canonical line bundle:

$$\det \Omega^1_{Y_x} \simeq (\det \omega^1_{Y/C})_{|Y_x}$$

of the fiber $Y_x := g^{-1}(x)$ is trivial.

The coherent sheaf $g_* \det \omega^1_{Y/C}$ is torsion free on C, hence locally free. Over $C - \Delta$, it coincides with the relative Hodge bundle $H^{N-1,0}(Y_{C-\Delta}/C - \Delta)$, and therefore it defines a line bundle over C that we shall denote by:

$$L := g_* \det \omega^1_{Y/C}.$$

The tautological "evaluation morphism" between line bundles over Y:

$$g^* L \longrightarrow \det \omega^1_{Y/C}$$

is an isomorphism on $Y - Y_\Delta$. If we denote by V the divisor in Y defined by the vanishing of this morphism, it defines an isomorphism:

$$g^* L \xrightarrow{\sim} (\det \omega^1_{Y/C})(-V). \tag{1.5.1}$$

From the isomorphism (1.5.1), we deduce an equality in $\mathrm{CH}^1(Y)$:

$$c_1(\omega^1_{Y/C}) = g^* c_1(L) + [V]. \tag{1.5.2}$$

Moreover the support of the divisor V is contained in the support of Y_Δ, and the divisor V can be written:

$$V = \sum_{i \in I} v_i D_i, \tag{1.5.3}$$

where for every i in I, v_i is an integer.

Using (1.5.2) and (1.5.3), we obtain the equality of integers:

$$\begin{aligned}
\int_Y c_1(\omega^1_{Y/C})\, c_{N-1}(\omega^1_{Y/C}) &= \int_Y (g^* c_1(L) + [V])\, c_{N-1}(\omega^1_{Y/C}), \\
&= \int_Y g^* c_1(L)\, c_{N-1}(\omega^1_{Y/C}) + \sum_{i \in I} v_i \int_{D_i} c_{N-1}(\omega^1_{Y/C|D_i}), \\
&= (\deg L) \int_{Y_\eta} c_{N-1}(\Omega^1_{Y_\eta}) + \sum_{i \in I} v_i \int_{D_i} c_{N-1}(\Omega^1_{D_i}(\log D_i \cap D^2)), \qquad (1.5.4) \\
&= (-1)^{N-1} (\deg L)\, \chi_{\mathrm{top}}(Y_\eta) + (-1)^{N-1} \sum_{i \in I} v_i\, \chi_{\mathrm{top}}(\mathring{D}_i), \qquad (1.5.5)
\end{aligned}$$

where in (1.5.4), we have used Proposition 3.1.5 below, and in (1.5.5), $\chi_{\mathrm{top}}(Y_\eta)$ and $\chi_{\mathrm{top}}(\mathring{D}_i)$ denote respectively the Euler characteristic of a general fiber of the morphism g and of $\mathring{D}_i$.[16]

16. See for instance Theorem 4.2.2 below for the equality: $\chi_{\mathrm{top}}(\mathring{D}_i) = (-1)^{N-1} \int_{D_i} c_{N-1}(\Omega^1_{D_i}(\log D_i \cap D^2))$.

Consequently, inserting (1.5.5) in the right-hand side of the equality (1.3.5) in Theorem 1.3.2, we obtain the following expression for the alternating sum of the Griffiths heights of a pencil of Calabi-Yau manifolds, which may be seen as a geometric version of the BCOV invariant:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = -\frac{1}{12}\deg(g_*\det\omega^1_{Y/C})\,\chi_{\mathrm{top}}(Y_\eta) + \sum_{x\in\Delta}\beta_x, \tag{1.5.6}$$

where, for every x in Δ, β_x is the rational number given by:

$$\beta_x = \frac{1}{12}\sum_{i\in I_x}[3(N-1)(m_i-1)-v_i]\,\chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12}\sum_{(i,j)\in I_x^2, i\prec j}(3-m_i/m_j-m_j/m_i)\,\chi_{\mathrm{top}}(\mathring{D}_{ij}).$$

Observe that, in the right-hand side of (1.5.6), the first term only depends on the general fiber of Y and on the line bundle "of modular forms" $L := g_*\det\omega^1_{Y/C}$, and the second term only depends on the geometry of an infinitesimal neighborhood of the singular fibers.

When $N=3$ and when the smooth fibers of g are K3 surfaces, and when the singular fibers are semistable [17] with smooth components, formula (1.5.6) becomes:

$$\mathrm{ht}_{GK}(\mathbb{H}^2(Y_\eta/C_\eta)) = 2\deg g_*\omega_{Y/C} - \frac{1}{12}\chi_{\mathrm{top}}(D^2-D^3) + \frac{1}{12}\sum_{i\in I} v_i\,\chi_{\mathrm{top}}(\mathring{D}_i), \tag{1.5.7}$$

where:

$$\omega_{Y/C} \simeq \det\omega^1_{Y/C}$$

denotes the relative dualizing line bundle of the morphism g.

Indeed, in this situation, the Griffiths heights in the left-hand side of (1.5.6) vanish if $n \neq 2$, and:

$$\chi_{\mathrm{top}}(Y_\eta) = 24;$$

see for instance [**Huy16**]. One may actually expect to obtain a more explicit expression of the last two terms in the right-hand side of (1.5.7) when the degenerations of Y/C are Kulikov models; see [**Huy16**, Chapter 6, §5].

1.5.2. Lefschetz pencils on a Calabi-Yau manifold embedded in a projective space. Let us adopt the notation of Corollary 1.4.4, and let us assume that V is a Calabi-Yau manifold, namely that the line bundle $\det\Omega^1_V$ on V is trivial.

Under this assumption, we have the equality in $\mathrm{CH}^1(V)$:

$$c_1(\Omega^1_V) = 0,$$

and using it in equalities (1.4.7) and (1.4.8) of Corollary 1.4.4 yields the following expressions for the Griffiths heights of the middle-dimensional cohomology of a Lefschetz pencil on a Calabi-Yau manifold:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)) = \frac{(-1)^{N-1}}{12}\chi_{\mathrm{top}}(V) + v_N^+|\Sigma|,$$

and:

$$\mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)) = \frac{(-1)^{N-1}}{12}\chi_{\mathrm{top}}(V) + v_N^-|\Sigma|.$$

In particular, these Griffiths heights only depend on some basic invariants of V and its Lefschetz pencil, namely the Euler characteristic of V and the number of critical points of the pencil.

17. namely, when all the multiplicities m_i equal 1.

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1.7. Conventions and notation

If X is a scheme of finite type over some field k, Z is a closed subscheme of X, locally complete intersection in X, and if $i : Z \to X$ is the inclusion map and α is a class in $\mathrm{CH}^*(X)$, we shall denote by:

$$\alpha_{|Z} := i^*\alpha$$

its image by the Gysin morphism:

$$i^* : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(Z).$$

We shall use this notation only when both X and Z are smooth over k.

If X is a smooth k-scheme of pure dimension N, where N is a non-negative integer, we denote by:

$$\int_X : \mathrm{CH}^*(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

the map that sends a class α in $\mathrm{CH}^*(X)_{\mathbb{Q}}$ to the degree of its homogeneous component $\alpha^{[N]}$ in $\mathrm{CH}^N(X)_{\mathbb{Q}} \simeq \mathrm{CH}_0(X)_{\mathbb{Q}}$.

If X is a complex scheme (separated and of finite type), we denote by $\chi_{\mathrm{top}}(X)$ the topological Euler characteristic of the complex analytic space $X(\mathbb{C})$.

CHAPTER 2

Variations of Hodge structures associated to pencils of complete varieties with DNC degenerations: Steenbrink's theory and elementary exponents

2.1. Steenbrink's theory and logarithmic Hodge bundles

Let C be a connected smooth projective complex curve, let $\Delta \subset C$ be a finite subset, and let:

$$\mathring{C} := C - \Delta,$$

be its complement in C. Let also Y be a smooth complex manifold, and let:

$$g : Y \longrightarrow C$$

be a proper complex analytic map, that is relatively projective, locally (in the analytic topology) over C, and such that the complex analytic map

$$g_{|Y_{\mathring{C}}} : Y_{\mathring{C}} := g^{-1}(\mathring{C}) \longrightarrow \mathring{C}$$

is smooth (that is, a complex analytic submersion).

Furthermore, let us assume that the divisor

$$Y_\Delta := g^{-1}(\Delta)$$

is a (not necessarily reduced) divisor with strict normal crossings, i.e. it can be written $\sum_i a_i D_i$, where the $(D_i)_i$ are smooth divisors that intersect transversally.

Definition 2.1.1 (see for instance [**Kat71**]). *Let $\Omega^1_Y(\log Y_\Delta)$ be the (analytic) sheaf of differentials with logarithmic singularities. The sheaf on Y:*

$$\Omega^1_{Y/C}(\log Y_\Delta) := \Omega^1_Y(\log Y_\Delta)/\mathrm{Im}(g^*\Omega^1_C(\log \Delta) \longrightarrow \Omega^1_Y(\log Y_\Delta))$$

is called the relative sheaf of differentials with logarithmic singularities. *By taking local coordinates near a point of Y_Δ, one can check that this is a locally free sheaf.*

We also define the relative logarithmic de Rham complex*:*

$$\Omega^\bullet_{Y/C}(\log Y_\Delta) := \bigwedge\nolimits^\bullet \Omega^1_{Y/C}(\log Y_\Delta),$$

with the differential induced by the one on the complex $j_*\Omega^\bullet_{Y_{\mathring{C}}}$ by restriction and quotient, where

$$j : Y_{\mathring{C}} = Y - Y_\Delta \longrightarrow Y$$

denotes the inclusion morphism.

The relative logarithmic de Rham cohomology sheaf *in degree n is defined as the analytic coherent sheaf on C:*

$$\mathcal{H}^n_{\log}(Y/C) := R^n g_* \Omega^\bullet_{Y/C}(\log Y_\Delta).$$

For simplicity's sake, we will write $\omega^\bullet_{Y/C}$ for $\Omega^\bullet_{Y/C}(\log Y_\Delta)$ as in [**Ill94**]. Note that this sheaf does not depend on the choice of Δ provided that g is smooth over $C - \Delta$; non-singular fibers in Y_Δ would not contribute to it.

Also note that we have a relative logarithmic Hodge-de Rham spectral sequence:

$$E_2^{p,q} := R^q g_* \omega^p_{Y/C} \Rightarrow R^{p+q} g_* \omega^\bullet_{Y/C}. \tag{2.1.1}$$

The restriction to $\mathring{C}$ of this spectral sequence is the classical Hodge-de Rham spectral sequence of the smooth morphism:

$$g_{|Y_{\mathring{C}}} : Y_{\mathring{C}} \longrightarrow \mathring{C}.$$

This sequence degenerates at the first page (see for instance [**Del68**, Th. (5.5)]).

Consequently, for every integer n, there is a filtration $(\mathcal{F}^p)_{0\leq p\leq n}$ of the vector bundle:

$$\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C}) := R^n g_{|Y_{\mathring{C}}*} \Omega^\bullet_{Y_{\mathring{C}}/\mathring{C}} \simeq R^n g_{|Y_{\mathring{C}}*} \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{\mathring{C}}$$

on $\mathring{C}$, such that for every integer $0 \leq p \leq n$, the subquotient $\mathcal{F}^p/\mathcal{F}^{p+1}$ is isomorphic to

$$R^{n-p} g_{|Y_{\mathring{C}}} \Omega^p_{Y_{\mathring{C}}/\mathring{C}},$$

and is locally free.

Finally, applying the functor g_* to the short exact sequence of complexes of sheaves on Y:

$$0 \longrightarrow g^*\Omega^1_C(\log \Delta) \otimes \omega^\bullet_{Y/C}[-1] \longrightarrow \Omega^\bullet_Y(\log Y_\Delta) \longrightarrow \omega^\bullet_{Y/C} \longrightarrow 0,$$

we obtain a long exact sequence whose connecting homomorphism defines a logarithmic connection:

$$\nabla_{\log} : R^n g_* \omega^\bullet_{Y/C} \longrightarrow R^n g_* \omega^\bullet_{Y/C} \otimes_{\mathcal{O}_C} \Omega^1_C(\log \Delta),$$

whose restriction to $\mathring{C}$ is the Gauss-Manin connection.

The following results are simple consequences of Steenbrink's theory.

Proposition 2.1.2. *For all $n \geq 0$, the lower Deligne extension to C of the relative cohomology bundle $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ with the Gauss-Manin connection is precisely the relative logarithmic de Rham cohomology $\mathcal{H}^n_{\log}(Y/C) := R^n g_* \omega^\bullet_{Y/C}$ equipped with the logarithmic connection $\nabla_{\log}$. In particular $\mathcal{H}^n_{\log}(Y/C)$ is locally free over C.*

Also, denoting by $(a_i)_{i\in I}$ the multiplicities of the components of Y_Δ, the eigenvalues of the residue of the Gauss-Manin connection are in the set $\{\frac{j_i}{a_i} \mid i \in I,\, 0 \leq j_i \leq a_i - 1\}$, and the monodromy endomorphism at every point of Δ is quasi-unipotent.

Proof. By [**Ste76**, Th. (2.18)], for all $n \geq 0$, the sheaf $R^n g_* \omega^\bullet_{Y/C}$ is locally free on C.

By [**Kat71**, VII] (applied to the bundle $M := \mathcal{O}_{Y_{\mathring{C}}}$ equipped with the connection given by the differential d, and to the trivial extension $\overline{M} = \mathcal{O}_Y$ equipped with the differential d), we obtain that the only possible eigenvalues of the residue in Δ of the logarithmic connection $\nabla_{\log}$ are in the set $\{\frac{j_i}{a_i} \mid i \in I,\, 0 \leq j_i \leq a_i - 1\}$. In particular, all the eigenvalues have their real part in $[0, 1[$.

Using the definition of the lower Deligne extension and the fact that the monodromy in a point $x \in \Delta$ is given by:

$$T_x = \exp(-2\pi\sqrt{-1}\mathrm{Res}_x \nabla_{\log}),$$

this gives the result. □

Proposition 2.1.3. *The relative logarithmic Hodge-de Rham spectral sequence* (2.1.1) *degenerates, and for all $p, q \geq 0$, the sheaves $R^q g_* \omega^p_{Y/C}$ are locally free on C. Consequently, for all (p, n) such that $0 \leq p \leq n$, the p-th subquotient of the Hodge filtration $\mathcal{F}^\bullet_{\log}$ on the relative cohomology bundle $\mathcal{H}^n_{\log}(Y/C)$ given by the spectral sequence* (2.1.1)*satisfies:*

$$\mathrm{gr}^p_{\mathcal{F}^\bullet_{\log}}(R^n g_* \omega^\bullet_{Y/C}) = R^{n-p} g_* \omega^p_{Y/C}.$$

In particular, the vector bundles $\mathcal{F}^p$ on $\mathring{C}$ are algebraic vector subbundles of $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ endowed with the algebraic structure defined by means of its Deligne extension.[1]

PROOF. By [**Ste77**, Th. (2.11)], for all $p, q \geq 0$, the sheaves $E_2^{p,q} := R^q g_* \omega^p_{Y/C}$ are locally free.

The restriction of the spectral sequence (2.1.1) to $\mathring{C}$ is the classical relative Hodge-de Rham sequence, which degenerates at the first page. This means that the differentials of the sequence (2.1.1) $\mathrm{d}_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ vanish on $\mathring{C}$, hence everywhere by local freeness of the sheaves $E_2^{p,q}$. So the relative logarithmic Hodge-de Rham spectral sequence degenerates.

The second statement follows by definition of the spectral sequence.

For the third one, from the GAGA theorem applied on the projective manifold C, we have that the analytic vector bundles $\mathcal{F}^p_{\log}$ are algebraic subbundles of the Deligne extension $R^n g_* \omega^\bullet_{Y/C}$ of $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$, hence their restrictions to $\mathring{C}$ are algebraic with respect to the same algebraic structure. □

PROPOSITION 2.1.4. *For every integer $n \geq 0$, letting η be the generic point of C, the lower Griffiths line bundle of the relative cohomology VHS $\mathbb{H}^n(Y_\eta/C_\eta)$ exists and is given by:*

$$\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) \simeq \bigotimes_{0 \leq p \leq n} (\det R^{n-p} g_* \omega^p_{Y/C})^{\otimes p}.$$

PROOF. Let n be a non-negative integer. Using Proposition 2.1.2, the lower Deligne extension of the vector bundle $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ on $\mathring{C}$ equipped with the Gauss-Manin connection coincides with the relative logarithmic cohomology $\mathcal{H}^n_{\log}(Y/C) \simeq R^n g_* \omega^\bullet_{Y/C}$ equipped with the logarithmic connection $\nabla_{\log}$.

Using Proposition 2.1.3, for every integer $p \geq 0$, the vector bundle $\mathcal{F}^p$ on $\mathring{C}$ is an algebraic subbundle of the vector bundle $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ endowed with the algebraic structure defined by $R^n g_* \omega^\bullet_{Y/C}$, and its extension in the vector bundle $R^n g_* \omega^\bullet_{Y/C}$ is the vector bundle $\mathcal{F}^p_{\log}$ given by the logarithmic Hodge-de Rham exact sequence.

Consequently, the lower Griffiths line bundle is given by:

$$\begin{aligned} \mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) &\simeq \bigotimes_{p \geq 0} (\det \mathcal{F}^p_{\log}/\mathcal{F}^{p+1}_{\log})^{\otimes p}, \\ &= \bigotimes_{0 \leq p \leq n} (\det R^{n-p} g_* \omega^p_{Y/C})^{\otimes p} \text{ using Proposition 2.1.3.} \end{aligned}$$ □

2.2. Elementary exponents of a degeneration

In this section, we recall the formalism of the elementary exponents of a normal crossing degeneration, as developed in [**EFiMM21**, Sections 2.2 and 2.3].

Let $\mathbb{D}$ be the unit complex disk with coordinate t, let Y be a smooth complex manifold, and let

$$g : Y \longrightarrow \mathbb{D}$$

be a projective morphism such that the fiber Y_0 is a (not necessarily reduced) divisor with strict normal crossings.

1. This also follows from [**Sch73**, Theorem (4.13), a)]. Observe that the algebraic structures defined by the upper and lower Deligne extensions of a vector bundle with connection on $\mathring{C}$ actually coincide.

Let us consider a *semistable reduction diagram*, i.e. a commutative diagram of complex manifolds

$$\begin{array}{ccc} Y' & \xrightarrow{\rho} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathbb{D}' & \xrightarrow{\sigma} & \mathbb{D} \end{array} \tag{2.2.1}$$

where $\mathbb{D}'$ is a copy of the unit open disk with coordinate t', σ is the map sending t' to t'^l where l is some integer, Y' is a smooth N-dimensional complex manifold, g' is a morphism such that the open subscheme $Y' - Y'_0$ of Y' is isomorphic to the fiber product $(Y - Y_0) \times_{\mathbb{D}^*} \mathbb{D}'^*$, and such that Y'_0 is a *reduced* divisor with strict normal crossings.

Such a diagram exists by the "semistable reduction theorem" ([**KKMSD73**, Chapter II]).

For every pair of integers (p, q), we can consider the coherent sheaf $R^q g_* \omega^p_{Y/\mathbb{D}}$ (resp. $R^q g'_* \omega^p_{Y'/\mathbb{D}'}$) on $\mathbb{D}$ (resp. $\mathbb{D}'$), which is locally free using Proposition 2.1.3. Both locally free sheaves have the same rank, given by the Hodge number of a general fiber:

$$h^{p,q} := h^{p,q}(Y_\infty).$$

DEFINITION 2.2.1 ([**EFiMM21**, Def. 2.5]). *Let $p, q \geq 0$ be integers. With the above notation, the map of locally free sheaves on $\mathbb{D}'$ given by the pullback of forms:*

$$\sigma^* R^q g_* \omega^p_{Y/\mathbb{D}} \longrightarrow R^q g'_* \omega^p_{Y'/\mathbb{D}'}$$

induces an isomorphism on $\mathbb{D}'^*$, so it is injective and its cokernel is an $\mathcal{O}_{\mathbb{D}'}$-module of the form

$$\bigoplus_{1 \leq j \leq h^{p,q}} \mathcal{O}_{\mathbb{D}'}/(t'^{b_j^{p,q}} \mathcal{O}_{\mathbb{D}'}),$$

where $(b_j^{p,q})_{1 \leq j \leq h^{p,q}}$ is a family of integers, unique up to order.

The elementary exponents of the (p, q)-Hodge bundle *are the rational numbers:*

$$(\alpha_j^{p,q})_{1 \leq j \leq h^{p,q}} := \left(\frac{b_j^{p,q}}{l} \right)_{1 \leq j \leq h^{p,q}} \in \left(\frac{1}{l} \mathbb{Z} \right)^{h^{p,q}}.$$

We also define rational numbers by:

$$b^{p,q} := \sum_{1 \leq j \leq h^{p,q}} b_j^{p,q},$$

$$\alpha^{p,q} := \sum_{1 \leq j \leq h^{p,q}} \alpha_j^{p,q} = \frac{b^{p,q}}{l}.$$

Note that these numbers do not depend on the choice of the coordinates t',t.

PROPOSITION 2.2.2 ([**EFiMM21**, Lemma 2.4]). *For all integers p, q, the integers $(b_j^{p,q})_j$ are all in $\{0, ..., l-1\}$. Consequently, the elementary exponents $(\alpha_j^{p,q})_j$ are all in $\{0, \frac{1}{l}, ..., \frac{l-1}{l}\}$.*

Moreover, the elementary exponents do not depend on the choice of the semistable reduction diagram.

The elementary exponents are related to the eigenvalues of the monodromy.

Let ∞ be a point in $\mathbb{D}^*$, and $n \geq 0$ be an integer. As in [**Ste77**, (2.8)], we can equip the fiber

$$(R^n g_* \mathbb{C})_\infty \simeq H^n(Y_\infty, \mathbb{C})$$

with a mixed Hodge structure. Let us denote by $F^\bullet_\infty$ the associated Hodge filtration, not to be confused with the classical Hodge filtration $(\mathcal{F}^\bullet)_\infty$ on the same space.

Using [**Ste77**, Th. (2.13)], the semisimple part of the monodromy:

$$T_s : H^n(Y_\infty) \xrightarrow{\sim} H^n(Y_\infty)$$

conserves the mixed Hodge structure, hence the filtration $F^\bullet_\infty$.

PROPOSITION 2.2.3 ([**EFiMM21**, Cor. 2.8]). *For every pair of integers (p,q) with $p+q=n$, letting $(\alpha_j^{p,q})_j$ be the elementary exponents of the (p,q)-Hodge bundle, the complex numbers*

$$(\exp(-2i\pi\alpha_j^{p,q}))_j$$

are precisely the eigenvalues (counted with multiplicities) of the endomorphism T_s acting on the subquotient $F^p_\infty/F^{p+1}_\infty$.

This proposition yields a new proof that the elementary exponents do not depend on the choice of the semistable reduction diagram.

We can also use them to compare the determinant line bundles of the Hodge bundles.

PROPOSITION 2.2.4. *With the same notation, the canonical isomorphism of line bundles on $\mathbb{D}'^*$:*

$$\sigma^* \det R^q g_* \Omega^p_{Y-Y_0/\mathbb{D}^*} \longrightarrow \det R^q g'_* \Omega^p_{Y'-Y'_0/\mathbb{D}'^*}$$

can be extended into an isomorphism of line bundles on $\mathbb{D}'$:

$$(\sigma^* \det R^q g_* \omega^p_{Y/\mathbb{D}}) \otimes \mathcal{O}_{\mathbb{D}'}(\{0\})^{\otimes l\alpha^{p,q}} \xrightarrow{\sim} \det R^q g'_* \omega^p_{Y'/\mathbb{D}'}.$$

PROOF. This follows from the multiplicativity of the determinant line bundle applied to the exact sequence of coherent sheaves on $\mathbb{D}'$:

$$0 \longrightarrow \sigma^* R^q g_* \omega^p_{Y/\mathbb{D}} \longrightarrow R^q g'_* \omega^p_{Y'/\mathbb{D}'} \longrightarrow \bigoplus_{1\leq j\leq h^{p,q}} \mathcal{O}_{\mathbb{D}'}/(t'^{l\alpha_j^{p,q}} \mathcal{O}_{\mathbb{D}'}) \longrightarrow 0. \quad \square$$

2.3. Comparison of Griffiths line bundles and elementary exponents

Let C be a connected smooth projective complex curve with generic point η, Δ be a finite subset of C, which we also see as a reduced divisor, Y be a smooth N-dimensional complex scheme, and

$$g : Y \longrightarrow C$$

be a projective morphism which is smooth over $\mathring{C} := C - \Delta$, and such that Y_Δ is a divisor with strict normal crossings. As before, we shall denote:

$$Y_{\mathring{C}} := g^{-1}(\mathring{C}) = Y - Y_\Delta.$$

For every point x in Δ, let l_x be an integer such that there exists an open disk $\mathbb{D}$ centered in x with a coordinate t that identifies it with the unit open disk, such that the restriction

$$g_{\mathbb{D}} : Y_{\mathbb{D}} := g^{-1}(\mathbb{D}) \longrightarrow \mathbb{D}$$

and the morphism

$$\sigma : \mathbb{D}' \longrightarrow \mathbb{D}, \quad t' \longrightarrow t'^{l_x}$$

where $\mathbb{D}'$ is a copy of the unit open disk with coordinate t', can be inserted into a semistable reduction diagram of the form (2.2.1).

Let C' be a connected smooth projective complex curve with generic point η', and

$$\sigma : C' \longrightarrow C$$

be a finite morphism satisfying the following condition:

(2.3.1)
For every x' in $\sigma^{-1}(\Delta)$, the ramification index $r_{x'}$ of σ in x' is a multiple of the integer $l_{\sigma(x')}$.

In particular, for every integer n, the monodromy of the bundle $\sigma^*\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ with the connection given by the pullback of the Gauss-Manin connection, is unipotent at every point of $\sigma^{-1}(\Delta)$.

For n an integer, let $\mathcal{V}^n$ be the vector bundle with connection $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ on $\mathring{C}$, and $F^\bullet\mathcal{V}^n$ be its Hodge filtration.

Let $\overline{\mathcal{V}^n}_+$ (resp. $\overline{\mathcal{V}^n}_-$) be its upper (resp. lower) Deligne extension to C.

We know that the vector bundle with connection $\sigma^*\mathcal{V}^n$ on:

$$\mathring{C}' := C' - \sigma^{-1}(\Delta)$$

has unipotent local monodromy at the points of $\Delta' := \sigma^{-1}(\Delta)$, so we can consider its Deligne extension $\overline{\sigma^*\mathcal{V}^n}$ to C'.

Using Proposition 2.1.3, for every integer p, the vector subbundle $F^p\mathcal{V}^n$ on $\mathring{C}$ is an algebraic subbundle of the vector bundle $\mathcal{V}^n$ endowed with the algebraic structure defined by means of its lower Deligne extension $\overline{\mathcal{V}^n}_-$. As the algebraic structures defined by the upper and the lower Deligne extensions actually coincide, the vector subbundle $F^p\mathcal{V}^n$ on $\mathring{C}$ is also an algebraic subbundle of the vector bundle $\mathcal{V}^n$ endowed with the algebraic structure defined by its upper Deligne extension $\overline{\mathcal{V}^n}_+$. Consequently, we can consider the extension $\overline{F^p\mathcal{V}^n}_+$ (resp. $\overline{F^p\mathcal{V}^n}_-$) in the vector bundle $\overline{\mathcal{V}^n}_+$ (resp. $\overline{\mathcal{V}^n}_-$) of the vector subbundle $F^p\mathcal{V}^n$ in $\mathcal{V}^n$.

Similarly, we can consider the extension $\overline{\sigma^*F^p\mathcal{V}^n}$ in $\overline{\sigma^*\mathcal{V}^n}$, over C', of the subbundle $\sigma^*F^p\mathcal{V}^n$ in $\sigma^*\mathcal{V}^n$ over $\mathring{C}'$.

For all integers p, q, and for every x in Δ, let $(\alpha^{p,q}_{j,x})_{1\leq j\leq \mathrm{rg}(H^{p,q}(Y_\eta))}$ be the elementary exponents of the (p,q)-Hodge bundle at point x, and let $\alpha^{p,q}_x$ be their sum.

Let us define a divisor on C with rational coefficients by:

$$A^{p,q} := \sum_{x\in\Delta} \alpha^{p,q}_x\{x\}.$$

PROPOSITION 2.3.1. *With the above notation, for all integers p, n, the identity map of line bundles on $\mathring{C}'$:*

$$\sigma^*\det(F^p\mathcal{V}^n/F^{p+1}\mathcal{V}^n) \longrightarrow \sigma^*\det(F^p\mathcal{V}^n/F^{p+1}\mathcal{V}^n)$$

can be extended into an isomorphism of line bundles on C':

$$\sigma^*\det(\overline{F^p\mathcal{V}^n}_-/\overline{F^{p+1}\mathcal{V}^n}_-)\otimes\mathcal{O}_{C'}(\sigma^*A^{p,n-p}) \simeq \det(\overline{\sigma^*F^p\mathcal{V}^n}/\overline{\sigma^*F^{p+1}\mathcal{V}^n}).$$

Note that even though $A^{p,n-p}$ is a divisor with only rational coefficients, its pullback by σ necessarily has integral coefficients.

PROOF. It is enough to show this locally around a point x' in $\sigma^{-1}(\Delta)$. Let $r_{x'}$ be the ramification index of σ at x', and x be the image of x' by σ.

Let $\mathbb{D}$ be an open disk centered in x with a coordinate t that identifies $\mathbb{D}$ with the unit open disk, and let $\mathbb{D}'$ be an open disk around x' with a coordinate t' that identifies $\mathbb{D}'$ with the unit open disk, such that $\sigma(\mathbb{D}')\subseteq\mathbb{D}$ and that we have the equality of functions on $\mathbb{D}'$:

$$\sigma^*t = t'^{r_{x'}}.$$

Using condition (2.3.1), the integer $r_{x'}$ can be written $r'l_x$, where $r'\geq 1$ and l_x is an integer such that we have a semistable reduction diagram

$$\begin{array}{ccc} (Y_{\mathbb{D}})'' & \longrightarrow & Y_{\mathbb{D}} \\ \downarrow{\scriptstyle (g_{\mathbb{D}})''} & & \downarrow{\scriptstyle g_{\mathbb{D}}} \\ \mathbb{D}'' & \xrightarrow{\sigma'} & \mathbb{D} \end{array}$$

where σ' is the elevation to the power l_x.

Let

$$\tau : \mathbb{D}' \longrightarrow \mathbb{D}''$$

be the elevation to the power r', so that we have the equality of morphisms:

$$\sigma' \circ \tau = \sigma_{|\mathbb{D}'}.$$

Using Proposition 2.1.2, and the fact that $(g_{\mathbb{D}})''$ is semistable, i.e. that the fiber $(Y_{\mathbb{D}})''_0$ is a reduced divisor with strict normal crossings, we obtain that for every $n \geq 0$, the monodromy of the vector bundle $R^n(g_{\mathbb{D}})''_{|(Y_{\mathbb{D}})''-(Y_{\mathbb{D}})''_0 *}\Omega^{\bullet}_{(Y_{\mathbb{D}})''-(Y_{\mathbb{D}})''_0/\mathbb{D}''-\{0\}}$, with the Gauss-Manin connection, is unipotent. Consequently, the upper and lower Deligne extensions on $\mathbb{D}''$ coincide and are compatible with base change by ramified morphisms.

Consequently, the extensions of the subquotients in the lower Deligne extensions are also compatible with base change. So using Proposition 2.1.3, we obtain, for all integers p, n, an isomorphism of vector bundles on $\mathbb{D}'$:

$$\tau^* R^{n-p}(g_{\mathbb{D}})''_* \omega^p_{(Y_{\mathbb{D}})''/\mathbb{D}''} \simeq \overline{\sigma^* F^p \mathcal{V}^n} / \overline{\sigma^* F^{p+1} \mathcal{V}^n} \tag{2.3.2}$$

that extends the canonical isomorphism on $\mathbb{D}' - \{x'\}$.

Using Proposition 2.2.4, we have an isomorphism of line bundles on $\mathbb{D}''$:

$$(\sigma'^* \det R^{n-p} g_{\mathbb{D}*} \omega^p_{Y_{\mathbb{D}}/\mathbb{D}}) \otimes \mathcal{O}_{\mathbb{D}''}(\{0\})^{\otimes l_x \alpha_x^{p,n-p}} \simeq \det R^{n-p}(g_{\mathbb{D}})''_* \omega^p_{(Y_{\mathbb{D}})''/\mathbb{D}''},$$

that extends the canonical isomorphism on $\mathbb{D}'' - \{0\}$. Pulling it back by τ and using the isomorphism (2.3.2), we get an isomorphism of line bundles on $\mathbb{D}'$:

$$\begin{aligned}(\sigma^* \det R^{n-p} g_{\mathbb{D}*} \omega^p_{Y_{\mathbb{D}}/\mathbb{D}}) \otimes \mathcal{O}_{\mathbb{D}'}(\{x'\})^{\otimes r_{x'} \alpha_x^{p,n-p}} &\simeq \tau'^* \det R^{n-p}(g_{\mathbb{D}})''_* \omega^p_{(Y_{\mathbb{D}})''/\mathbb{D}''} \\ &\simeq \det(\overline{\sigma^* F^p \mathcal{V}^n} / \overline{\sigma^* F^{p+1} \mathcal{V}^n}).\end{aligned}$$

Using again Proposition 2.1.3, this can be rewritten as an isomorphism of line bundles on $\mathbb{D}'$:

$$\sigma^* \det(\overline{F^p \mathcal{V}^n}_- / \overline{F^{p+1} \mathcal{V}^n}_-) \otimes \mathcal{O}_{\mathbb{D}'}(\{x'\})^{\otimes r_{x'} \alpha_x^{p,n-p}} \simeq \det(\overline{\sigma^* F^p \mathcal{V}^n} / \overline{\sigma^* F^{p+1} \mathcal{V}^n})$$

that extends the identity map on $\mathbb{D}' - \{x'\}$. Using this extension for every x' in $\sigma^{-1}(\Delta)$ shows that there is an isomorphism of line bundles on C':

$$(\sigma^* \det \overline{F^p \mathcal{V}^n}_- / \overline{F^{p+1} \mathcal{V}^n}_-) \otimes \mathcal{O}_{C'}\left(\sum_{x' \in \sigma^{-1}(\Delta)} r_{x'} \alpha^{p,n-p}_{\sigma(x')} \{x'\} \right) \simeq \det(\overline{\sigma^* F^p \mathcal{V}^n} / \overline{\sigma^* F^{p+1} \mathcal{V}^n}).$$

that extends the canonical isomorphism on $\mathring{C}'$.

The wanted isomorphism follows, using the fact that for every x' in $\sigma^{-1}(\Delta)$, the multiplicity of the divisor $\sigma^* A^{p,n-p}$ at x' is exactly $r_{x'} \alpha^{p,n-p}_{\sigma(x')}$ (which is an integer because the rational number $\alpha^{p,n-p}_{\sigma(x')}$ is in $\frac{1}{l_{\sigma(x')}}\mathbb{Z}$ by definition, and because the integer $r_{x'}$ is a multiple of $l_{\sigma(x')}$ by condition (2.3.1)). □

Proposition 2.3.2. *For all pair (p, n) of integers such that $0 \leq p \leq n$, the identity map of line bundles on $\mathring{C}'$:*

$$\sigma^* \det(F^p \mathcal{V}^n / F^{p+1} \mathcal{V}^n) \longrightarrow \sigma^* \det(F^p \mathcal{V}^n / F^{p+1} \mathcal{V}^n)$$

can be extended into an isomorphism of line bundles on C':

$$\sigma^* \det(\overline{F^p \mathcal{V}^n}_+ / \overline{F^{p+1} \mathcal{V}^n}_+) \otimes \mathcal{O}_{C'}(-\sigma^* A^{(N-1)-p,(N-1)-(n-p)}) \longrightarrow \det(\overline{\sigma^* F^p \mathcal{V}^n} / \overline{\sigma^* F^{p+1} \mathcal{V}^n}).$$

Proof. Using Poincaré and Serre duality, for every integer n, the vector bundle $\mathcal{V}^n$ on $\mathring{C}$ is isomorphic to the dual of the vector bundle $\mathcal{V}^{2(N-1)-n}$, and for every integer p, the subquotient:

$$F^p \mathcal{V}^n / F^{p+1} \mathcal{V}^n \simeq R^{n-p} g_* \Omega^p_{Y_{\mathring{C}}}$$

is identified to the dual of the subquotient

$$F^{(N-1)-p} \mathcal{V}^{2(N-1)-n} / F^{(N-1)-p+1} \mathcal{V}^{2(N-1)-n} \simeq R^{(N-1)-(n-p)} g_* \Omega^{(N-1)-p}_{Y_{\mathring{C}}}.$$

As the duality exchanges the upper and lower Deligne extensions, we obtain an isomorphism of vector bundles on C:

$$\overline{F^p\mathcal{V}^n}_+/\overline{F^{p+1}\mathcal{V}^n}_+ \simeq (\overline{F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}_-/\overline{F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}}_-)^\vee$$

that extends the canonical isomorphism on $\mathring{C}$; hence, pulling back by σ and taking the determinant line bundle, an isomorphism of line bundles on C':

$$\sigma^*\det(\overline{F^p\mathcal{V}^n}_+/\overline{F^{p+1}\mathcal{V}^n}_+) \simeq \sigma^*\det(\overline{F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}_-/\overline{F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}}_-)^\vee$$

that extends the canonical isomorphism on $\mathring{C}'$.

Applying Proposition 2.3.1 to the integers $p' := (N-1)-p$, $n' := 2(N-1)-n$, we obtain an isomorphism:

$$\begin{aligned}&\sigma^*\det(\overline{F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}_-/\overline{F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}}_-)\otimes\mathcal{O}(\sigma^*A^{(N-1)-p,(N-1)-(n-p)})\\ &\simeq \det(\overline{\sigma^*F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}/\overline{\sigma^*F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-p}})\end{aligned}$$

that extends the canonical isomorphism on $\mathring{C}'$; and consequently, an isomorphism:

$$\begin{aligned}&\sigma^*\det(\overline{F^p\mathcal{V}^n}_+/\overline{F^{p+1}\mathcal{V}^n}_+)\otimes\mathcal{O}(-\sigma^*A^{(N-1)-p,(N-1)-(n-p)})\\ &\simeq \det(\overline{\sigma^*F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}/\overline{\sigma^*F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}})^\vee\end{aligned}$$

that extends the canonical isomorphism on $\mathring{C}'$.

Applying Poincaré and Serre duality again to the vector bundle $\sigma^*\mathcal{V}^{2(N-1)-n}$ on $\mathring{C}'$ and its subquotient $\sigma^*F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}/\sigma^*F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}$, and using that duality sends the (upper or lower) Deligne extension on C' of this subquotient to the (upper or lower) Deligne extension of its dual, we obtain an isomorphism:

$$(\overline{\sigma^*F^{(N-1)-p}\mathcal{V}^{2(N-1)-n}}/\overline{\sigma^*F^{(N-1)-p+1}\mathcal{V}^{2(N-1)-n}})^\vee \simeq \overline{\sigma^*F^p\mathcal{V}^n}/\overline{\sigma^*F^{p+1}\mathcal{V}^n},$$

that extends the canonical isomorphism on $\mathring{C}'$.

This shows the result. □

PROPOSITION 2.3.3. *With the same notation, for all integers p, n, the identity map of line bundles on $\mathring{C}$:*

$$\det(F^p\mathcal{V}^n/F^{p+1}\mathcal{V}^n) \longrightarrow \det(F^p\mathcal{V}^n/F^{p+1}\mathcal{V}^n)$$

can be extended into an isomorphism of line bundles on C:

$$\det(\overline{F^p\mathcal{V}^n}_-/\overline{F^{p+1}\mathcal{V}^n}_-)\otimes\mathcal{O}_C(A^{p,n-p}+A^{(N-1)-p,(N-1)-(n-p)}) \longrightarrow \det(\overline{F^p\mathcal{V}^n}_+/\overline{F^{p+1}\mathcal{V}^n}_+).$$

In particular, the coefficients of the divisor $A^{p,n-p}+A^{(N-1)-p,(N-1)-(n-p)}$ are integers.

PROOF. Combining the isomorphisms of Propositions 2.3.1 and 2.3.2 yields an isomorphism of line bundles on C':

$$\sigma^*\det(\overline{F^p\mathcal{V}^n}_-/\overline{F^{p+1}\mathcal{V}^n}_-)\otimes\mathcal{O}_{C'}(\sigma^*(A^{p,n-p}+A^{(N-1)-p,(N-1)-(n-p)})) \simeq \sigma^*\det(\overline{F^p\mathcal{V}^n}_+/\overline{F^{p+1}\mathcal{V}^n}_+)$$

that extends the canonical isomorphism on $\mathring{C}'$.

This canonical isomorphism is exactly the pullback by σ of the canonical isomorphism on $\mathring{C}$, so that this isomorphism has an extension to C as wanted. □

Now, we can apply these results to compute Griffiths bundles.

PROPOSITION 2.3.4. *For every integer $n \geq 0$, we have isomorphisms of line bundles on C':*

$$\sigma^*\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))\otimes\mathcal{O}_{C'}\left(\sum_{0\leq p\leq n} p\,\sigma^*A^{p,n-p}\right) \simeq \mathcal{GK}_{C'}(\sigma^*\mathbb{H}^n(Y_\eta/C_\eta)), \tag{2.3.3}$$

(2.3.4)

$$\sigma^*\mathcal{GK}_{C,+}(\mathbb{H}^n(Y_\eta/C_\eta)) \otimes \mathcal{O}_{C'}\left(-\sum_{0\leq p\leq n} p\,\sigma^* A^{(N-1)-p,(N-1)-(n-p)}\right) \simeq \mathcal{GK}_{C'}(\sigma^*\mathbb{H}^n(Y_\eta/C_\eta)),$$

and an isomorphism of line bundles on C:

(2.3.5)

$$\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) \otimes \mathcal{O}_{C}\left(\sum_{0\leq p\leq n} p(A^{p,n-p} + A^{(N-1)-p,(N-1)-(n-p)})\right) \simeq \mathcal{GK}_{C,+}(\mathbb{H}^n(Y_\eta/C_\eta)).$$

PROOF. This follows from Propositions 2.3.1, 2.3.2 and 2.3.3 and the definition of Griffiths bundles. ☐

COROLLARY 2.3.5. *For every $n \geq 0$, we have the equalities in $\mathrm{CH}_0(C)$:*

$$\begin{aligned} c_1(\mathcal{GK}_{C,+}(\mathbb{H}^n(Y_\eta/C_\eta))) &= c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) + \sum_{0\leq p\leq n} p([A^{p,n-p}] + [A^{(N-1)-p,(N-1)-(n-p)}]), \\ &= \sum_{0\leq p\leq n} pc_1(R^{n-p}g_*\omega^p_{Y/C}) + \sum_{0\leq p\leq n} p([A^{p,n-p}] + [A^{(N-1)-p,(N-1)-(n-p)}]), \end{aligned}$$

hence the equalities of integers:

$$\begin{aligned} \mathrm{ht}_{GK,+}(\mathbb{H}^n(Y_\eta/C_\eta))) &= \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta))) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p} + A^{(N-1)-p,(N-1)-(n-p)}), \\ &= \sum_{0\leq p\leq n} p\deg_C(R^{n-p}g_*\omega^p_{Y/C}) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p} + A^{(N-1)-p,(N-1)-(n-p)}); \end{aligned}$$

and we also have the equalities of rational numbers:

$$\begin{aligned} \mathrm{ht}_{GK,stab}(\mathbb{H}^n(Y_\eta/C_\eta)) &= \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p}), \\ &= \sum_{0\leq p\leq n} p\deg_C(R^{n-p}g_*\omega^p_{Y/C}) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p}). \end{aligned}$$

PROOF. The first four equalities follow from the isomorphism (2.3.5) in Proposition 2.3.4 and Proposition 2.1.4.

For the last two equalities, by definition of the stable height, using the isomorphism (2.3.3) in Proposition 2.3.4 and Proposition 2.1.4, we have the equalities of rational numbers:

$$\begin{aligned} \mathrm{ht}_{GK,stab}(\mathbb{H}^n(Y_\eta/C_\eta)) &= \frac{1}{\deg(\sigma)}\deg_{C'}(\mathcal{GK}_{C'}(\sigma^*\mathbb{H}^n(Y_\eta/C_\eta)), \\ &= \frac{1}{\deg(\sigma)}\deg_{C'}\left(\sigma^*\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) \otimes \mathcal{O}_{C'}\left(\sum_{0\leq p\leq n} p\,\sigma^* A^{p,n-p}\right)\right), \\ &= \deg_C(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p}), \\ &= \sum_{0\leq p\leq n} p\deg_C(R^{n-p}g_*\omega^p_{Y/C}) + \sum_{0\leq p\leq n} p\deg_C(A^{p,n-p}). \end{aligned}$$

☐

2.4. Comparison of Griffiths line bundles in the case of non-degenerate critical points

2.4.1. Non-degenerate critical points and elementary exponents. Let C be a connected smooth projective complex curve, let H be an N-dimensional smooth projective complex manifold, and let:

$$f : H \longrightarrow C$$

be a complex analytic morphism whose only critical points are non-degenerate. Let Σ be the set of these critical points and $\Delta := f(\Sigma)$ be its set-theoretical image by f.

We can rephrase the hypothesis as the divisor in H:

$$H_\Delta := f^*\Delta$$

having only finitely many singularities, all of which are ordinary double points.

Let

$$\nu : Y := \tilde{H} \longrightarrow H$$

be the blow-up of these points and

$$g := f \circ \nu : Y \longrightarrow C$$

be the composition. For every point P in Σ, let us define a subscheme in Y by:

$$E_P := \nu^{-1}(\{P\}),$$

so that the $(E_P)_{P\in\Sigma}$ are the connected components of the exceptional divisor E. In particular, they are disjoint divisors isomorphic to the complex projective space $\mathbb{P}^{N-1}$.

Let us define a divisor in Y by:

$$Y_\Delta := g^*\Delta = \nu^* H_\Delta.$$

Since the divisor H_Δ has its only singularities at points of Σ, and since all of these singularities are ordinary double points, its pullback divisor Y_Δ can be written as follows:

$$Y_\Delta = 2 \sum_{P\in\Sigma} E_P + W \tag{2.4.1}$$

where W is the proper transform in Y of the divisor H_Δ in H: it is a non-singular divisor intersecting transversally the components $(E_P)_{P\in\Sigma}$ of the exceptional divisor, and for every point P, the intersection $E_P \cap W$ is a smooth quadric in the projective space E_P.

In particular, the divisor Y_Δ is a divisor with strict normal crossings.

This section is devoted to the proof of the following result.

Proposition 2.4.1. *With the above notation, for every x in Δ and p, q integers, all the elementary exponents $(\alpha^{p,q}_{j,x})_j$ of the degeneration g at point x vanish unless $N-1$ is even and $p = q = \frac{N-1}{2}$.*

If $N-1$ is even and $p = q = \frac{N-1}{2}$, precisely $|\Sigma_x|$ of the $(\alpha^{p,q}_{j,x})_j$ are equal to $\frac{1}{2}$, and the other ones vanish.

In particular, the rational number $\alpha^{p,q}_x$ vanishes unless $N-1$ is even and $p = q = \frac{N-1}{2}$, in which case it is equal to $\frac{1}{2}|\Sigma_x|$.

In other terms, the divisor $A^{p,q}$ with rational coefficients in C vanishes unless $N-1$ is even and $p = q = \frac{N-1}{2}$, in which case it is given by:

$$A^{\frac{N-1}{2},\frac{N-1}{2}} = \frac{1}{2} f_*[\Sigma].$$

This proposition is stated in [**EFiMM21**, Prop. 3.10], but only proved there in the case where each fiber of the morphism f contains only one critical point. In Subsection 2.4.2 below, we present an argument, which was explained to us by the authors of [**EFiMM21**], that reduces the proposition to this special case.

Corollary 2.4.2. *For every integer n, we have the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:*

$$c_1(\mathcal{GK}_{C,+}(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta)) + \delta^{n,N-1}\eta_N \frac{N-1}{2} f_*[\Sigma], \tag{2.4.2}$$

$$= \sum_{0\leq p\leq n} p c_1(R^{n-p}g_*\omega^p_{Y/C}) + \delta^{n,N-1}\eta_N \frac{N-1}{2} f_*[\Sigma], \tag{2.4.3}$$

hence the equality of rational numbers:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^n(H_\eta/C_\eta))) = \mathrm{ht}_{GK,-}(\mathbb{H}^n(H_\eta/C_\eta))) + \delta^{n,N-1}\eta_N \frac{N-1}{2}|\Sigma| \tag{2.4.4}$$

$$= \sum_{0\leq p\leq n} p \deg_C(R^{n-p}g_*\omega^p_{Y/C}) + \delta^{n,N-1}\eta_N \frac{N-1}{2}|\Sigma|; \tag{2.4.5}$$

and we also have the equality of rational numbers:

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^n(H_\eta/C_\eta)) = \mathrm{ht}_{GK,-}(\mathbb{H}^n(H_\eta/C_\eta))) + \delta^{n,N-1}\eta_N \frac{N-1}{4}|\Sigma| \tag{2.4.6}$$

$$= \sum_{0\leq p\leq n} p \deg_C(R^{n-p}g_*\omega^p_{Y/C}) + \delta^{n,N-1}\eta_N \frac{N-1}{4}|\Sigma|, \tag{2.4.7}$$

where η_N is 1 if N is odd and 0 if N is even.

Proof. For n an integer, using Proposition 2.4.1, the sum in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\sum_{0\leq p\leq n} p([A^{p,n-p}] + [A^{(N-1)-p,(N-1)-(n-p)}])$$

vanishes unless $n = N-1$ and $N-1$ is even, and in this case, it is equal to

$$\frac{N-1}{2}([A^{\frac{N-1}{2},\frac{N-1}{2}}] + [A^{\frac{N-1}{2},\frac{N-1}{2}}]) = \frac{N-1}{2} f_*[\Sigma].$$

So for every integer n, we obtain the equality:

$$\sum_{0\leq p\leq n} p([A^{p,n-p}] + [A^{(N-1)-p,(N-1)-(n-p)}]) = \delta^{n,N-1}\eta_N \frac{N-1}{2} f_*[\Sigma].$$

Similarly, for every integer n, we obtain the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\sum_{0\leq p\leq n} p[A^{p,n-p}] = \delta^{n,N-1}\eta_N \frac{N-1}{4} f_*[\Sigma].$$

Replacing these equalities in Corollary 2.3.5 yields the result. □

2.4.2. Proof of Proposition 2.4.1. Let x be a point in Δ. As the proposition is local around x, we can replace the curve C by the unit open disk $\mathbb{D}$ with coordinate t, the point x by 0, and assume that f is smooth over $\mathbb{D}^* := \mathbb{D} - \{0\}$ and that the fiber H_0 admits only ordinary double points as singularities. In particular, the only critical points of the morphism f are the points in Σ_x, we shall simply call this subset Σ.

Let us introduce a copy $\mathbb{D}'$ of the unit open disk with coordinate t', let us define an analytic morphism by:

$$\sigma : \mathbb{D}' \longrightarrow \mathbb{D}, \quad t' \longmapsto t'^2,$$

and let us define the fiber product:

$$\begin{array}{ccc} \tilde{H}' := Y \times_{\mathbb{D}} \mathbb{D}' & \xrightarrow{\rho} & Y \\ \downarrow g' & & \downarrow g \\ \mathbb{D}' & \xrightarrow{\sigma} & \mathbb{D} \end{array} \tag{2.4.8}$$

Let us define the following divisors in the singular analytic space $\tilde{H}'$:

$$E'_P := \rho^* E_P, P \in \Sigma,$$

$$E' := \bigsqcup_{P \in \Sigma} E'_P,$$

and:

$$W' := \rho^* W.$$

Using (2.4.1), we get the equality of Cartier divisors in $\tilde{H}'$:

$$2\tilde{H}'_0 = \rho^* Y_0 = 2E' + W'. \tag{2.4.9}$$

Let us consider the normalization of the singular analytic space $\tilde{H}'$:

$$\pi : Y' \longrightarrow \tilde{H}',$$

and the composition:

$$h := g' \circ \pi : Y' \longrightarrow \mathbb{D}'.$$

Since the morphism g is smooth on the open subset $Y - E \subset Y$, the morphism g' is smooth on the open subset $\tilde{H}' - E' \subset \tilde{H}'$, in particular the analytic space $\tilde{H}' - E'$ is non-singular, and the morphism π induces an isomorphism between $Y' - \pi^{-1}(E')$ and $\tilde{H}' - E'$.

The cokernel of the canonical injective morphism of $\mathcal{O}_{\tilde{H}'}$-modules

$$\varphi : \mathcal{O}_{\tilde{H}'} \longrightarrow \pi_* \mathcal{O}_{Y'}$$

is supported by the closed subset E', so we can write it as follows:

$$\mathrm{Coker}(\varphi) = \bigoplus_{P \in \Sigma} \mathcal{F}_P,$$

where for every point P in Σ, $\mathcal{F}_P$ is a coherent sheaf supported by the closed subset E'_P.

The proof of Proposition 2.4.1 shall rely on the following three lemmas.

LEMMA 2.4.3. *The three following statements hold.*

(i) *The morphism*

$$h : Y' \longrightarrow \mathbb{D}'$$

defines a semistable reduction of the morphism g; namely the analytic space Y' is non-singular and the fiber $Y'_0 := h^{-1}(0)$ is a reduced divisor with strict normal crossings in Y'.

(ii) *For every integer p, the canonical isomorphism of vector bundles on $Y'_{\mathbb{D}'^*}$:*

$$(\rho \circ \pi)^* \Omega^p_{Y_{\mathbb{D}^*}/\mathbb{D}^*} \longrightarrow \Omega^p_{Y'_{\mathbb{D}'^*}/\mathbb{D}'^*}$$

extends into an isomorphism of vector bundles on Y':

$$(\rho \circ \pi)^* \omega^p_{Y/\mathbb{D}} \longrightarrow \omega^p_{Y'/\mathbb{D}'}.$$

(iii) *For every point P in Σ, the $\mathcal{O}_{\tilde{H}'}$-module $(g'^* t')\mathcal{F}_P$ vanishes.*

PROOF. The three statements can be proved in local models of $\tilde{H}'$ and of Y'. We shall only show them in a local model of $\tilde{H}'$ near a point of $E'_P \cap W'$ and a local model of Y' near a point of $\pi^{-1}(E'_P \cap W')$ (where P is a point in Σ) when $N \geq 2$, which is the most complicated case.

Let P be a point in Σ. Near a point of $E_P \cap W$, we can choose a local coordinate system $(y_1, ..., y_N)$ of Y, such that the divisor W is defined by $(y_1 = 0)$, the divisor E_P is defined by $(y_2 = 0)$, and satisfying the equality of functions:

$$g^* t = y_1 y_2^2.$$

Consequently, near a point of $E'_P \cap W'$, the analytic space $\tilde{H}'$ admits coordinates $(g'^* t', \rho^* y_1, ..., \rho^* y_N)$, subjected to the relation:

$$g'^* t'^2 = (\rho^* y_1)\,(\rho^* y_2)^2.$$

The local model is that of the product of a "Whitney umbrella" singularity by $\mathbb{C}^{N-2}$.

Finally, near a point of $\pi^{-1}(E'_P \cap W')$, the analytic space Y' admits a local coordinate system $(z, (\rho \circ \pi)^* y_2, ..., (\rho \circ \pi)^* y_N)$ satisfying the relations:

$$h^* t' = z\,(\rho \circ \pi)^* y_2$$

and:

$$(\rho \circ \pi)^* y_1 = z^2.$$

In particular, the analytic space Y' is smooth, and the divisor $\pi^* W'$ in Y' is of multiplicity 2, which allows us to define a divisor by $(1/2)\pi^* W'$ (defined by $(z = 0)$ in the above local model). We also have that the morphism h is smooth over $\mathbb{D}'^*$, and that the divisor Y'_0 in Y' can be written:

$$Y'_0 = \sum_{P \in \Sigma} \pi^* E'_P + (1/2)\pi^* W', \tag{2.4.10}$$

where the divisors $(\pi^* E'_P)_{P \in \Sigma}$ are disjoint and non-singular, where the divisor $(1/2)\pi^* W'$ is non-singular, and intersects transversally the divisors $(\pi^* E'_P)_{P \in \Sigma}$.

Consequently, the degeneration h is semistable, which shows (i).

It clearly suffices to prove (ii) when the integer p is 1. In the above local models, the locally free sheaf $\omega^1_{Y/\mathbb{D}}$ on Y is generated by $([\mathrm{d}y_1/y_1], [\mathrm{d}y_2/y_2], [\mathrm{d}y_3], ..., [\mathrm{d}y_N])$ modulo the relation $[\mathrm{d}y_1/y_1] + 2[\mathrm{d}y_2/y_2] = 0$, hence it admits a local frame $([\mathrm{d}y_2/y_2], [\mathrm{d}y_3], ..., [\mathrm{d}y_N])$.

On the other hand, the locally free sheaf $\omega^1_{Y'/\mathbb{D}'}$ on Y' is generated by $([\mathrm{d}z/z], (\rho \circ \pi)^*[\mathrm{d}y_2/y_2], (\rho \circ \pi)^*[\mathrm{d}y_3], ...)$ modulo the relation $[\mathrm{d}z/z] + (\rho \circ \pi)^*[\mathrm{d}y_2/y_2] = 0$, hence it admits a local frame $((\rho \circ \pi)^*[\mathrm{d}y_2/y_2], (\rho \circ \pi)^*[\mathrm{d}y_3], ...)$.

The canonical morphism of sheaves on $Y'_{\mathbb{D}'^*}$:

$$(\rho \circ \pi)^* \Omega^1_{Y_{\mathbb{D}^*}/\mathbb{D}^*} \simeq (\rho \circ \pi)^* \omega^1_{Y/\mathbb{D}|Y_{\mathbb{D}^*}} \longrightarrow \Omega^1_{Y'_{\mathbb{D}'^*}/\mathbb{D}'^*} \simeq \omega^1_{Y'/\mathbb{D}'|Y'_{\mathbb{D}'^*}}$$

is compatible with these local frames, hence extends into an isomorphism on Y', which shows (ii).

For (iii), in the above local models, the $\mathcal{O}_{\tilde{H}'}$-algebra $\pi_* \mathcal{O}_{Y'}$ is generated by z, subjected to the relations:

$$g'^* t' = z\,(\rho^* y_2)$$

and:

$$\rho^* y_1 = z^2.$$

Consequently, the $\mathcal{O}_{\tilde{H}'}$-module $\pi_* \mathcal{O}_{Y'}$ is generated by the powers of z. For every even integer r, the element z^r is contained in $\mathcal{O}_{\tilde{H}'}$, so this module is generated by the odd powers of z.

Let r be an odd integer, we have the equality of elements of the algebra $\pi_* \mathcal{O}_{Y'}$:

$$(g'^* t')\, z^r = z^{r+1}\,(\rho^* y_2) \in \mathcal{O}_{\tilde{H}'},$$

Consequently, the $\mathcal{O}_{\tilde{H}'}$-submodule $(g'^* t')\, \pi_* \mathcal{O}_{Y'}$ of $\pi_* \mathcal{O}_{Y'}$ is contained in $\mathcal{O}_{\tilde{H}'}$. By definition of the morphism φ, the element $(g'^* t')$ is in the annihilator of its cokernel $\mathcal{F}_P$, which shows (iii). □

LEMMA 2.4.4. *For every pair of non-negative integers (p, q), the cokernel of the natural injective map of coherent sheaves on $\mathbb{D}'$:*

$$u^{p,q} : \sigma^* R^q g_* \omega^p_{Y/\mathbb{D}} \longrightarrow R^q h_* \omega^p_{Y'/\mathbb{D}'}$$

is isomorphic to the coherent sheaf:

$$\bigoplus_{P \in \Sigma} R^q g'_* (\mathcal{F}_P \otimes \rho^* \omega^p_{Y/\mathbb{D}}).$$

PROOF. Tensoring the following exact sequence of coherent sheaves on $\tilde{H}'$:

$$0 \longrightarrow \mathcal{O}_{\tilde{H}} \xrightarrow{\varphi} \pi_*\mathcal{O}_{Y'} \longrightarrow \bigoplus_{P\in\Sigma} \mathcal{F}_P \longrightarrow 0,$$

by the locally free sheaf $\rho^*\omega^p_{Y/\mathbb{D}}$, and using the projection formula, yields an exact sequence of coherent sheaves on $\tilde{H}'$:

$$0 \longrightarrow \rho^*\omega^p_{Y/\mathbb{D}} \longrightarrow \pi_*(\rho\circ\pi)^*\omega^p_{Y/\mathbb{D}} \longrightarrow \bigoplus_{P\in\Sigma} \mathcal{F}_P \otimes (\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow 0. \tag{2.4.11}$$

Using Lemma 2.4.3, (ii), we have a canonical isomorphism of vector bundles on Y':

$$(\rho\circ\pi)^*\omega^p_{Y/\mathbb{D}} \longrightarrow \omega^p_{Y'/\mathbb{D}'}.$$

Hence replacing in (2.4.11), we obtain an exact sequence of coherent sheaves on $\tilde{H}'$:

$$0 \longrightarrow \rho^*\omega^p_{Y/\mathbb{D}} \longrightarrow \pi_*\omega^p_{Y'/\mathbb{D}'} \longrightarrow \bigoplus_{P\in\Sigma} \mathcal{F}_P \otimes (\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow 0. \tag{2.4.12}$$

The higher direct images of (2.4.12) by the morphism g' fit into the following long exact sequence of coherent analytic sheaves over $\mathbb{D}'$ (with the convention that $R^{-1}g'_* = 0$):

$$\begin{aligned} \ldots \longrightarrow \bigoplus_{P\in\Sigma} R^{q-1}g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow R^q g'_*(\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow R^q g'_*(\pi_*\omega^p_{Y'/\mathbb{D}'}) \\ \longrightarrow \bigoplus_{P\in\Sigma} R^q g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow R^{q+1}g'_*(\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow \ldots \end{aligned} \tag{2.4.13}$$

Since the morphism σ is flat, by base change using the commutative diagram (2.4.8), we obtain the following isomorphisms of coherent sheaves on $\mathbb{D}'$:

$$R^q g'_*(\rho^*\omega^p_{Y/\mathbb{D}}) \simeq \sigma^* R^q g_*\omega^p_{Y/\mathbb{D}},$$

and:

$$R^{q+1} g'_*(\rho^*\omega^p_{Y/\mathbb{D}}) \simeq \sigma^* R^{q+1} g_*\omega^p_{Y/\mathbb{D}}.$$

These sheaves are locally free on $\mathbb{D}'$ by Proposition 2.1.3.

On the other hand, since for every point P in Σ, the coherent sheaf $\mathcal{F}_P$ on $\tilde{H}'$ is supported on E'_P, the coherent sheaves $\bigoplus_{P\in\Sigma} R^q g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}})$ and $\bigoplus_{P\in\Sigma} R^{q-1} g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}})$ on $\mathbb{D}'$ are clearly supported by the point 0.

Consequently, the connecting morphisms in (2.4.13) vanish, and (2.4.13) induces a short exact sequence of coherent sheaves on $\mathbb{D}'$:

(2.4.14)

$$0 \longrightarrow R^q g'_*(\rho^*\omega^p_{Y/\mathbb{D}}) \simeq \sigma^* R^q g_*\omega^p_{Y/\mathbb{D}} \longrightarrow R^q g'_*(\pi_*\omega^p_{Y'/\mathbb{D}'}) \longrightarrow \bigoplus_{P\in\Sigma} R^q g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}}) \longrightarrow 0.$$

Furthermore, the normalization morphism π is finite, so its higher direct image functors on coherent sheaves vanish. Using the spectral sequence of composed functors, we can rewrite the second sheaf in (2.4.14):

$$R^q g'_*(\pi_*\omega^p_{Y'/\mathbb{D}'}) \simeq R^q h_*\omega^p_{Y'/\mathbb{D}'}.$$

Under this identification, the second arrow in the diagram (2.4.14) is precisely $u^{p,q}$. So we obtain that its cokernel is the coherent sheaf on $\mathbb{D}'$ given by:

$$\operatorname{Coker} u^{p,q} \simeq \bigoplus_{P\in\Sigma} R^q g'_*(\mathcal{F}_P\otimes\rho^*\omega^p_{Y/\mathbb{D}}),$$

which concludes the proof. □

LEMMA 2.4.5. *The isomorphism class of the coherent sheaf on $\mathbb{D}'$, supported by $\{0\}$:*

$$R^q g'_*(\mathcal{F}_P \otimes \rho^* \omega^p_{Y/\mathbb{D}}),$$

where (p, q) is a pair of non-negative integers and P is a point in Σ, only depends on the integers N, p, and q.

PROOF. For every point P in Σ, we can construct local models of neighborhoods of E'_P in $\tilde{H}'$ and of $\pi^{-1}(E'_P)$ in Y', that only depend on the integer N, as follows.

Since the morphism f admits a non-degenerate critical point at P, we can choose an analytic coordinate system $(x_{1,P}, ..., x_{N,P})$ on a neighborhood U_P of P in H, defining a morphism inserted in a commutative diagram:

$$\begin{array}{ccc} U_P & \xrightarrow{\sim} & U \subset \mathbb{C}^N \\ {\scriptstyle f_{|U_P}}\downarrow & \swarrow {\scriptstyle f_{loc}} & \\ \mathbb{D} & & \end{array}$$

where U is some fixed neighborhood of 0 in $\mathbb{C}^N$ (for instance the open polydisk of radius $\sqrt{1/N}$ centered in 0), and f_{loc} is the morphism sending $(x_1, ..., x_N)$ to $x_1^2 + ... + x_N^2$.

Taking the blow-ups of U_P at P and of U at the origin, we obtain a commutative diagram:

$$\begin{array}{ccc} V_P & \xrightarrow{\sim} & V \\ {\scriptstyle g_{|V_P}}\downarrow & \swarrow {\scriptstyle g_{loc}} & \\ \mathbb{D} & & \end{array}$$

where V_P is a neighborhood of E_P in $\tilde{H}$, V is an analytic manifold that only depends on N, containing a divisor E_{loc} identified with E_P, and g_{loc} is an analytic morphism that only depends on N.

Taking the fiber product of the morphisms $g_{|V_P}$ and g_{loc} with the morphism σ, then taking the normalizations of the resulting analytic spaces, we obtain a commutative diagram:

$$\begin{array}{ccccc} W_P & \xrightarrow{\sim} & W & & \\ {\scriptstyle \pi_{|W_P}}\downarrow & & \downarrow{\scriptstyle \pi_{loc}} & & \\ V'_P & \xrightarrow{\sim} & V' & \xrightarrow{\rho_{loc}} & V \\ & {\scriptstyle g'_{|V'_P}}\searrow & \downarrow{\scriptstyle g'_{loc}} & & \downarrow{\scriptstyle g_{loc}} \\ & & \mathbb{D}' & \xrightarrow{\sigma} & \mathbb{D} \end{array}$$

where V'_P (resp. W_P) is a neighborhood of E'_P (resp. $\pi^{-1}(E'_P)$) in $\tilde{H}'$ (resp. Y'), V' (resp. W) is an analytic space (resp. manifold) that only depends on N, containing a divisor E'_{loc} (resp. $\pi^{-1}_{loc}(E'_{loc})$) identified with E'_P (resp. $\pi^{-1}(E'_P)$) via the isomorphisms, and π_{loc}, g'_{loc}, and ρ_{loc} are analytic morphisms that only depend on N.

The morphism of coherent sheaves $\varphi_{|V'_P}$ on V'_P is identified via these isomorphisms with a morphism of coherent sheaves on V':

$$\varphi_{loc} : \mathcal{O}_{V'} \longrightarrow \pi_{loc*}\mathcal{O}_W.$$

Let $\mathcal{F}_{loc}$ be its cokernel: it is a coherent sheaf on V' supported on E'_{loc}, that only depends on N, identified via the isomorphisms with $\mathcal{F}_P$.

Similarly, for every integer p, the locally free sheaf $(\rho^* \omega^p_{Y/\mathbb{D}})_{|V'_P}$ on V'_P is identified with the locally free sheaf $\rho^*_{loc}\omega^p_{V/\mathbb{D}}$ on V', which only depends on N.

Consequently, for every pair of integers (p, q), it follows from the commutativity of the diagram, and from the fact that the coherent sheaf $\mathcal{F}_P \otimes (\rho^* \omega^p_{Y/\mathbb{D}})$ is supported on E'_P, hence on V'_P, that we have an isomorphism of coherent sheaves on $\mathbb{D}'$:

$$\begin{aligned} R^q g'_*(\mathcal{F}_P \otimes \rho^* \omega^p_{Y/\mathbb{D}}) &\simeq R^q g'_*(\mathcal{F}_P \otimes \rho^* \omega^p_{Y/\mathbb{D}})_{|V'_P}, \\ &\simeq R^q g'_{loc*}(\mathcal{F}_{loc} \otimes \rho^*_{loc} \omega^p_{V/\mathbb{D}}), \end{aligned}$$

so that this coherent sheaf only depends on the integers N, p, q. $\square$

Proof of Proposition 2.4.1. Using Lemma 2.4.3, (i), and the fact that π is an isomorphism over $\mathbb{D}'^*$, the commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{\rho\circ\pi} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ \mathbb{D}' & \xrightarrow{\sigma} & \mathbb{D} \end{array}$$

is a semistable reduction diagram. Therefore, according to the definition of elementary exponents, for every pair of integers (p, q), the elementary exponents $(\alpha^{p,q}_{j,x})_j$ can be described in terms of the cokernel of the injective map $u^{p,q}$ from Lemma 2.4.4. By this Lemma, this cokernel is isomorphic to the coherent sheaf:

$$\bigoplus_{P \in \Sigma} R^q g'_*(\mathcal{F}_P \otimes \rho^* \omega^p_{Y/\mathbb{D}}).$$

Using Lemma 2.4.3, (iii), for every point P in Σ, the coherent sheaf $\mathcal{F}_P$ on $\tilde{H}'$ is annihilated by the function g^*t', hence can be written $i_{\tilde{H}'_0 *}\mathcal{F}'_P$, where $\mathcal{F}'_P$ is some coherent sheaf on $\tilde{H}'_0$, and $i_{\tilde{H}'_0}$ is the inclusion map.

This implies that the cokernel of $u^{p,q}$ is isomorphic to the coherent sheaf:

$$\bigoplus_{P \in \Sigma} H^q(\tilde{H}'_0, \mathcal{F}'_P \otimes (\rho^* \omega^p_{Y/\mathbb{D}})_{|\tilde{H}'_0})_0,$$

where $(.)_0$ denotes the skyscraper sheaf on 0 in $\mathbb{D}'$ associated with a vector space.

We can rewrite this coherent sheaf as:

$$\bigoplus_{P \in \Sigma} (\mathcal{O}_{\mathbb{D}'}/(t'\mathcal{O}_{\mathbb{D}'}))^{\oplus h^q(\tilde{H}'_0, \mathcal{F}'_P \otimes (\rho^* \omega^p_{Y/\mathbb{D}})_{|\tilde{H}'_0})}.$$

Hence by definition of the elementary exponents $(\alpha^{p,q}_{j,x})_j$, there are precisely

$$\sum_{P \in \Sigma} h^q(\tilde{H}'_0, \mathcal{F}'_P \otimes (\rho^* \omega^p_{Y/\mathbb{D}})_{|\tilde{H}'_0})$$

of them that are nonzero, and all of these are equal to $1/2$.

Using Lemma 2.4.5, the number:

$$h^q(\tilde{H}'_0, \mathcal{F}'_P \otimes (\rho^* \omega^p_{Y/\mathbb{D}})_{|\tilde{H}'_0})$$

only depends on the integers N, p, q, and we shall denote it $c^{p,q}_N$. Consequently, the number of nonzero elementary exponents is $|\Sigma|.c^{p,q}_N$.

If there is only one critical point in the fiber,[2] the argument in [**EFiMM21**] shows that the number of nonzero elementary exponents is 0 unless $N-1$ is even and $p = q = (N-1)/2$, in which case it is 1. Consequently, the integer $c^{p,q}_N$ is 0 unless $N-1$ is even and $p = q = (N-1)/2$, in which case it is 1, and we obtain in general that all the elementary exponents vanish unless $N-1$ is even and $p = q = (N-1)/2$, in which case precisely $|\Sigma|$ of them do not vanish, and they are equal to $1/2$. $\square$

2. An instance of this situation actually exists for every choice of $N \geq 1$. Consider for instance the hypersurface $X_1^2 + \cdots + X_N^2 = tX_0^2$ in $\mathbb{P}^N \times \mathbb{D}$.

2.5. Poincaré duality and Griffiths line bundles of cohomology in complementary degrees

We return to the notation of Section 2.3. Namely, let Y be a smooth projective complex scheme of pure dimension N, C be a connected smooth projective curve, and

$$g : Y \longrightarrow C$$

be a surjective morphism that is smooth over $\mathring{C} := C - \Delta$, where Δ is a finite subset of C, and such that the divisor Y_Δ is a divisor with strict normal crossings. Finally we denote:

$$Y_{\mathring{C}} := g^{-1}(\mathring{C}) = Y - Y_\Delta.$$

Let η be the generic point of C.

Let n be an integer such that $0 \leq n \leq 2(N-1)$.

For every point x in Δ, let $(\alpha^n_{j,x})_j$ be the reunion over the pairs (p,q) such that $p+q=n$, of the elementary exponents $(\alpha^{p,q}_{j,x})_{j,p,q}$ of the (p,q)-Hodge bundle of the degeneration g at the point x. Let α^n_x be their sum, and let us define a divisor in C with rational coefficients by:

$$A^n := \sum_{x\in\Delta} \alpha^n_x \{x\}.$$

Observe that with the notation of the previous sections, this divisor is also given by:

$$A^n = \sum_{p,q\geq 0, p+q=n} A^{p,q}.$$

Let r be the least common multiple of the denominators of the rational numbers $(\alpha^n_x)_{x\in\Delta}$, so that rA^n is a divisor with integral coefficients.

LEMMA 2.5.1. *With the above notation, the line bundle on C:*

$$\big(\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-\big)^{\otimes r} \otimes \mathcal{O}_C(rA^n)$$

is of 2-torsion.

PROOF. Let us denote simply $\mathcal{V}$ the complex analytic vector bundle $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ on $\mathring{C}$. It has an integral structure, associated to Betti cohomology with $\mathbb{Z}$-coefficients, so its determinant line bundle is of 2-torsion, and we have an isomorphism of line bundles on $\mathring{C}$:

$$\psi : \mathcal{O}_{\mathring{C}} \longrightarrow (\det \mathcal{V})^{\otimes 2},$$

that sends 1 to $(v_1 \wedge ... \wedge v_s)^2$, where $(v_1, ..., v_s)$ is any local integral frame of the vector bundle $\mathcal{V}$.

Let x be a point of Δ, let U be a neighborhood of x with a local coordinate t centered in x, let ∞ be a point in $U - \{x\}$, and let $v_1, ..., v_s$ be an integral frame of the vector space $\mathcal{V}_\infty$.

Let T be the monodromy automorphism on $\mathcal{V}_\infty$ around the point x. As the divisor Y_x in Y is a divisor with strict normal crossings, using Proposition 2.1.2, this automorphism is quasi-unipotent.

As in the construction of the lower Deligne extension (see [**Del70**, II, Prop. 5.4], [**Kat76**, "Key Lemma" p. 547], and [**EFiMM21**, section 2.1]), let B_- be the only automorphism on $\mathcal{V}_\infty$ whose eigenvalues have their real parts in $]-1,0]$, and satisfying the equality,

$$e^{2i\pi B_-} = T.$$

As the automorphism T is quasi-unipotent, these eigenvalues are rational numbers.

By Proposition 2.2.2, the elementary exponents $(\alpha^n_{j,x})_j$ are all in $[0,1[$, and by Proposition 2.2.3, they satisfy that the $(e^{-2i\pi\alpha^n_{j,x}})_j$ are the eigenvalues of the monodromy T.

Consequently, the eigenvalues of the automorphism B_- are exactly the $(-\alpha^n_{j,x})_j$.

By definition of the lower Deligne extension, we have an isomorphism of vector bundles on U:

$$\chi : \mathcal{V}_\infty \otimes_{\mathbb{C}} \mathcal{O}_U \longrightarrow (\overline{\mathcal{V}}_-)_{|U},$$

such that on every point x' in U with coordinate $t(x')$, an integral frame of $\mathcal{V}_{x'}$ is given by:

$$(\chi_{x'}(e^{B_-(\log(t(x'))-\log(t(\infty)))}.v_j))_j,$$

depending on a choice of logarithm of $t(x')$ and of $t(\infty)$.

Observe that adding a multiple of $2i\pi$ to $\log(t(x'))-\log(t(\infty))$ multiplies the basis by $e^{2i\pi B_-}=T$, which conserves the integral structure.

Let x' be a point in U with coordinate $t(x')$, and let us choose a logarithm $\log(t(x'))$ and a logarithm $\log(t(\infty))$. By definition of the morphisms ψ and χ, the image of 1 by the morphism of vector spaces

$$(\det \chi_{x'}^{-1})^{\otimes 2r} \circ \psi_{x'}^{\otimes r} : \mathbb{C} \longrightarrow (\det \mathcal{V}_\infty)^{\otimes 2r},$$

is given by:

$$\begin{aligned}
((\det \chi_{x'}^{-1})^{\otimes 2r} \circ \psi_{x'}^{\otimes r})(1) &= (e^{B_-(\log(t(x'))-\log(t(\infty)))}v_1 \wedge \dots \wedge e^{B_-(\log(t(x'))-\log(t(\infty)))}v_s)^{2r} \\
&= \det(e^{B_-(\log(t(x'))-\log(t(\infty)))})^{2r}.(v_1\wedge\dots\wedge v_s)^{2r} \\
&= e^{2r\mathrm{tr}(B_-(\log(t(x'))-\log(t(\infty))))}.(v_1\wedge\dots\wedge v_s)^{2r} \\
&= e^{2r(\log(t(x'))-\log(t(\infty)))\mathrm{tr}(B_-)}.(v_1\wedge\dots\wedge v_s)^{2r} \\
&= e^{2r(\log(t(x'))-\log(t(\infty)))\sum_j(-\alpha^n_{j,x})}.(v_1\wedge\dots\wedge v_s)^{2r} \qquad (2.5.1)\\
&= e^{-2r(\log(t(x'))-\log(t(\infty)))\alpha^n_x}.(v_1\wedge\dots\wedge v_s)^{2r} \\
&= (t(x')/t(\infty))^{-2r\alpha^n_x}.(v_1\wedge\dots\wedge v_s)^{2r}. \qquad (2.5.2)
\end{aligned}$$

The equality (2.5.1) holds because the eigenvalues of the automorphism B_- are exactly the $(-\alpha^n_{j,x})_j$; (2.5.2) holds because $r\alpha^n_x$ is an integer. Observe that this expression does not depend on the choices of the logarithms $\log(t(x'))$ and $\log(t(\infty))$.

Consequently, the section $((\det\chi^{-1})^{\otimes 2r}\circ\psi^{\otimes r}_{|U-\{x\}})(1)$ over $U-\{x\}$ of the line bundle $(\det \mathcal{V}_\infty)^{\otimes 2r}\otimes \mathcal{O}_U$ admits a pole of order $2r\alpha^n_x$ in x.

Since χ is an isomorphism of vector bundles, the section $\psi^{\otimes r}_{|U-\{x\}}(1)$ of the line bundle

$$(\det \mathcal{V}_{|U-\{x\}})^{\otimes 2r} \simeq (\det \overline{\mathcal{V}}_-)^{\otimes 2r}_{|U-\{x\}}$$

also admits a pole of order $2r\alpha^n_x$ in x.

Consequently, the isomorphism of line bundles $\psi^{\otimes r}$ on $\mathring{C}$ can be extended into an isomorphism of line bundles on C:

$$\overline{\psi^{\otimes r}}_- : \mathcal{O}_C(-2r\sum_{x\in\Delta}\alpha^n_x\{x\}) = \mathcal{O}_C(-2rA^n) \longrightarrow (\det \overline{\mathcal{V}}_-)^{\otimes 2r},$$

so that the line bundle on C:

$$\big(\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-\big)^{\otimes r}\otimes \mathcal{O}_C(rA^n)$$

is of 2-torsion, as wanted. □

Now, let us show the main result of this section.

PROPOSITION 2.5.2. *With the same notation as in Lemma 2.5.1, the line bundle on C:*

$$\mathcal{GK}_{C,+}(\mathbb{H}^{2(N-1)-n}(Y_\eta/C_\eta))^{\otimes r}\otimes \mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))^{\otimes -r}\otimes \mathcal{O}_C(-r(N-1)A^n)$$

is of 2-torsion.

Moreover, if the local monodromy at every point of Δ of the VHS $\mathbb{H}^n(Y_{\mathring{C}}/\mathring{C})$ on $\mathring{C}$ is unipotent, then so is the local monodromy of the VHS $\mathbb{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C})$, and the line bundle on C:

$$\mathcal{GK}_C(\mathbb{H}^{2(N-1)-n}(Y_\eta/C_\eta)) \otimes \mathcal{GK}_C(\mathbb{H}^n(Y_\eta/C_\eta))^\vee$$

is of 2-torsion.

Proof. Let us denote by $F^\bullet\mathcal{H}^n$ (resp. $F^\bullet\mathcal{H}^{2(N-1)-n}$) the Hodge filtration by locally direct summands on the vector bundle $\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})$ (resp. $\mathcal{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C})$).

Using Poincaré and Serre duality, we have an isomorphism of vector bundles on $\mathring{C}$:

$$\varphi_n : \mathcal{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C}) \longrightarrow \mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})^\vee,$$

that sends, for every integer p, the subbundle $F^p\mathcal{H}^{2(N-1)-n}$ into the orthogonal subbundle:

$$(F^{N-p}\mathcal{H}^n)^\perp \subset \mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})^\vee.$$

Consequently, for every integer p, this isomorphism induces by restriction and quotient an isomorphism:

$$\varphi_n^p : \mathrm{gr}_F^p\mathcal{H}^{2(N-1)-n} \longrightarrow (F^{N-p}\mathcal{H}^n)^\perp/(F^{N-1-p}\mathcal{H}^n)^\perp \simeq (\mathrm{gr}_F^{N-1-p}\mathcal{H}^n)^\vee.$$

Recall that the duality of vector bundles exchanges the upper and lower Deligne extensions, therefore, the isomorphism φ_n can be extended into an isomorphism of vector bundles on C:

$$\overline{\varphi_n} : \overline{\mathcal{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C})}_+ \longrightarrow (\overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-)^\vee,$$

that sends for every integer p, the subbundle $\overline{F^p\mathcal{H}^{2(N-1)-n}}_+$ into the subbundle $(\overline{F^{N-p}\mathcal{H}^n}_-)^\perp$, hence induces an isomorphism by restriction and quotient:

$$\overline{\varphi_n^p} : \mathrm{gr}_{\overline{F}_+}^p\overline{\mathcal{H}^{2(N-1)-n}}_+ \longrightarrow (\mathrm{gr}_{\overline{F}_-}^{N-1-p}\overline{\mathcal{H}^n}_-)^\vee.$$

By definition, the upper Griffiths line bundle on C of the VHS $\mathbb{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C})$ on $\mathring{C}$ is given by:

$$\begin{aligned}
\mathcal{GK}_{C,+}(\mathbb{H}^{2(N-1)-n}(Y_\eta/C_\eta)) &= \bigotimes_{p\in\mathbb{Z}}(\det \mathrm{gr}_{\overline{F}_+}^p\overline{\mathcal{H}^{2(N-1)-n}}_+)^{\otimes p} \\
&\simeq \bigotimes_{p\in\mathbb{Z}}(\det \mathrm{gr}_{\overline{F}_-}^{N-1-p}\overline{\mathcal{H}^n}_-)^{\otimes -p} \text{ using the isomorphisms } (\overline{\varphi_n^p})_p \\
&\simeq \bigotimes_{p'\in\mathbb{Z}}(\det \mathrm{gr}_{\overline{F}_-}^{p'}\overline{\mathcal{H}^n}_-)^{\otimes p'-(N-1)} \\
&\simeq \bigotimes_{p'\in\mathbb{Z}}(\det \mathrm{gr}_{\overline{F}_-}^{p'}\overline{\mathcal{H}^n}_-)^{\otimes p'} \otimes \Big(\bigotimes_{p'\in\mathbb{Z}}(\det \mathrm{gr}_{\overline{F}_-}^{p'}\overline{\mathcal{H}^n}_-)\Big)^{\otimes -(N-1)} \\
&\simeq \mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) \otimes (\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-)^{\otimes -(N-1)}.
\end{aligned}$$

Consequently, there is an isomorphism of line bundles on C:

$$\mathcal{GK}_{C,+}(\mathbb{H}^{2(N-1)-n}(Y_\eta/C_\eta)) \otimes \mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))^\vee \simeq \big(\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-\big)^{\otimes -(N-1)}.$$

Using Lemma 2.5.1, the line bundle on C:

$$\big(\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-\big)^{\otimes r} \otimes \mathcal{O}_C(rA^n)$$

is of 2-torsion. Consequently, the line bundle on C:

$$\begin{aligned}
&\mathcal{GK}_{C,+}(\mathbb{H}^{2(N-1)-n}(Y_\eta/C_\eta))^{\otimes r} \otimes \mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))^{\otimes(-r)} \otimes \mathcal{O}_C(-r(N-1)A^n) \\
&\quad\simeq (\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-)^{\otimes -r(N-1)} \otimes \mathcal{O}_C(-r(N-1)A^n), \\
&\quad\simeq \big[(\det \overline{\mathcal{H}^n(Y_{\mathring{C}}/\mathring{C})}_-)^{\otimes r} \otimes \mathcal{O}_C(rA^n)\big]^{\otimes -(N-1)},
\end{aligned}$$

is of 2-torsion, as wanted.

For proving the second assertion in Proposition 2.5.2, we use that the isomorphism φ_n is also compatible with the connections of both VHS, hence with their local monodromy. If the local monodromy at every point of Δ of the VHS $\mathbb{H}^n(Y_{\mathring{C}}/\mathring{C})$ is unipotent, so is the local monodromy of the dual VHS $\mathbb{H}^n(Y_{\mathring{C}}/\mathring{C})^\vee$, and so is the monodromy of the VHS $\mathbb{H}^{2(N-1)-n}(Y_{\mathring{C}}/\mathring{C})$.

Finally, applying the first part of the proof with $A^n = 0$ and $r = 1$, we obtain the last assertion in Proposition 2.5.2. $\square$

CHAPTER 3

Characteristic classes of relative differentials and of logarithmic relative differentials

In this chapter, we establish various results relating the Chern classes of sheaves of differentials, and vector bundles of logarithmic differentials. These results will be used in Chapter 4 to derive Theorems 1.3.2 and 4.2.3 from Theorems 1.3.1 and 4.2.1, and are also of independent interest.

The statements of our results are gathered in Section 3.1. The following sections, which contain their rather technical proofs and will not be referred to in the remainder of this paper, could be skipped at first reading.

3.1. Comparing Chern classes and Todd classes of differentials and logarithmic differentials

3.1.1. Chern classes of differentials and of logarithmic differentials: the absolute case. Although we are ultimately interested in characteristic classes of sheaves of relative differentials, we shall first prove the following results in the absolute case.

Let X be a smooth n-dimensional complex scheme and let E be a reduced divisor with strict normal crossings in X. Let us denote by $(E_i)_{i\in I}$ the irreducible components of E, and for any subset $J\subseteq I$, let us define:[1]

$$E_J := \bigcap_{i\in J} E_i.$$

It is a smooth subscheme of codimension $|J|$ in X, and we shall denote by:

$$i_{E_J} : E_J \hookrightarrow X$$

the inclusion morphism.

As in [**Del70**, II, 3.4], for every non-negative integer r, we denote by E^r the subscheme of codimension r of X defined as the union of the intersections of r distinct components of E:

$$E^r := \bigcup_{J\subset I, |J|=r} E_J.$$

Observe that for every subset J in I, the subscheme:

$$E_J \cap E^{|J|+1} = \bigcup_{i\in I-J} E_{J\cup\{i\}}$$

is a divisor with normal crossings in E_J.

PROPOSITION 3.1.1. *With the previous notation, for every subset J of I, the following relation holds between total Chern classes in* $\mathrm{CH}^*(E_J)$*:*

$$c\big(\Omega^1_X(\log E)_{|E_J}\big) = c\big(\Omega^1_{E_J}(\log E_J \cap E^{|J|+1})\big). \tag{3.1.1}$$

1. In particular, $E_\emptyset = X$.

Moreover the following equality holds in $\mathrm{CH}^*(X)$*:*

$$c(\Omega^1_X(\log E)) = \sum_{J \subseteq I} i_{E_J *} c(\Omega^1_{E_J}), \tag{3.1.2}$$

and the following equality holds in $\mathrm{CH}^*(X)_{\mathbb{Q}}$*:*

$$\mathrm{Td}(\Omega^1_X(\log E)) = \mathrm{Td}(\Omega^1_X) \prod_{i \in I} \mathrm{td}(-[E_i])^{-1}, \tag{3.1.3}$$

where $\mathrm{td}(x)$ *denotes the formal series* $x/(1-e^{-x})$*.*

The relation (3.1.2) may be reformulated as the following equalities, where r denotes an integer such that $0 \leq r \leq n$:

$$c_r(\Omega^1_X(\log E)) = \sum_{J \subseteq I, |J| \leq r} i_{E_J *} c_{r-|J|}(\Omega^1_{E_J}). \tag{3.1.4}$$

3.1.2. Notation. For the rest of the statements, let us adopt the following notation.

Let Y be a connected smooth projective complex scheme of pure dimension N, let C be a connected smooth projective complex curve, and let

$$g : Y \longrightarrow C$$

be a surjective morphism that is smooth outside of Y_Δ, where Δ is a reduced divisor in C.

Let us assume that the divisor Y_Δ is a divisor with strict normal crossings, and let us write it as:

$$D = \sum_{i \in I} m_i D_i,$$

where I is a finite set and for every i in I, m_i is a positive integer, and where D_i is a smooth connected divisor such that the $(D_i)_{i \in I}$ intersect each other transversally.

As before, for every subset J of I, we denote:

$$D_J := \bigcap_{i \in J} D_i,$$

and for every integer $r \geq 1$, we define the subscheme:

$$D^r := \bigcup_{J \subset I, |J| = r} D_J.$$

The subscheme:

$$D_J \cap D^{|J|+1} = \bigcup_{i \in I - J} D_{J \cup \{i\}}$$

is a divisor with normal crossings in D_J.

For every element i in I, we denote by

$$\mathcal{N}_i := \mathcal{N}_{D_i} Y \simeq \mathcal{O}_Y(D_i)_{|D_i}$$

the normal vector bundle of the smooth divisor D_i in Y.

Let $\preceq$ be a total order on I.

3.1.3. Normal bundles to the strata D_J and Chern classes.

PROPOSITION 3.1.2. *For every non-empty subset J in I, we have the equality of total Chern classes in* $\mathrm{CH}^*(D_J)$*:*

$$c\big(\omega^1_{Y/C|D_J}\big) = c\big(\Omega^1_{D_J}(\log D_J \cap D^{|J|+1})\big). \tag{3.1.5}$$

PROPOSITION 3.1.3. *Let i be an element in I, we have the equality in* $\mathrm{CH}^1(D_i)$*:*

$$m_i c_1(\mathcal{N}_i) = -\sum_{j\in I-\{i\}} m_j [D_{ij}],$$

where $D_{ij} := D_{\{i,j\}}$.

COROLLARY 3.1.4. *Let i be an element in I and let β be a class in* $\mathrm{CH}^*(D_i)$*, we have the equality in* $\mathrm{CH}^*(Y)$*:*

$$m_i i_{D_i *}(c_1(\mathcal{N}_i) \cap \beta) = -\sum_{j\in I-\{i\}} m_j i_{D_{ij}*}\beta_{|D_{ij}}.$$

COROLLARY 3.1.5. *Let (i,j) be a pair in I^2 such that $i \prec j$.*

We have the equality in $\mathrm{CH}^1(D_{ij})$*:*

$$m_i c_1(\mathcal{N}_{i|D_{ij}}) + m_j c_1(\mathcal{N}_{j|D_{ij}}) = -\sum_{k\in I-\{i,j\}} m_k [D_{ijk}], \tag{3.1.6}$$

where $D_{ijk} := D_{\{i,j,k\}}$.

3.1.4. Comparing the characteristic classes of $[T_g]$ and $\omega^{1\vee}_{Y/C}$. Finally, we shall show the following results of comparison of the Chern classes of $\omega^1_{Y/C}$ and of the relative tangent class in K-theory:

$$[T_g] := [T_Y] - g^*[T_C].$$

The duality operation on vector bundles on Y extends to an involution $.^\vee$ of the abelian group $K^0(Y)$, which sends $[T_g]$ to the class $[\Omega^1_Y] - g^*[\Omega^1_C]$. Observe that we have a sequence of coherent sheaves on Y:

$$0 \longrightarrow g^*\Omega^1_C \xrightarrow{Dg} \Omega^1_Y \longrightarrow \Omega^1_{Y/C} \longrightarrow 0.$$

This sequence is exact: everything but the injectivity of the differential morphism Dg is a standard property of the sheaf of Kähler differentials, and since the morphism g is smooth on a dense open subset of Y, the morphism Dg does not vanish on a dense open subset, hence is injective.

Consequently, the following equality holds in $K^0(Y)$:

$$[T_g] = [\Omega^1_{Y/C}]^\vee.$$

PROPOSITION 3.1.6. *We have the equality in* $\mathrm{CH}^*(Y)_{\mathbb{Q}}$*:*

$$\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} = \prod_{i\in I} \mathrm{td}([D_i]) - 1/2 \sum_{i\in I} m_i [D_i], \tag{3.1.7}$$

where $\mathrm{td}(x)$ *denotes the formal series with rational coefficients* $x/(1-e^{-x})$.

In particular, we have the equality in $\mathrm{CH}^1(Y)_{\mathbb{Q}}$*:*

$$\left[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\right]^{(1)} = -1/2 \sum_{i\in I} (m_i - 1)[D_i], \tag{3.1.8}$$

and the equalities in $\mathrm{CH}^2(Y)_{\mathbb{Q}}$*:*

$$\left[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega_{Y/C}^{1\vee})}\right]^{(2)} = 1/12\sum_{i\in I} i_{D_i*}c_1(\mathcal{N}_i) + 1/4 \sum_{(i,j)\in I^2, i\prec j}[D_{ij}], \tag{3.1.9}$$

$$= 1/12 \sum_{(i,j)\in I^2, i\prec j}(3 - m_i/m_j - m_j/m_i)[D_{ij}]. \tag{3.1.10}$$

Proposition 3.1.7. *For every non-negative integer* r*, we have the equality in* $\mathrm{CH}^r(Y)$*:*

(3.1.11)

$$c_r(\omega_{Y/C}^{1\vee}) = c_r([T_g]) + \sum_{i\in I} m_i i_{D_i*}[c_{r-1}(T_{D_i}) + c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})] + \sum_{\emptyset\neq J\subset I}(-1)^{|J|} i_{D_J*}c_{r-|J|}(T_{D_J}).$$

Furthermore, if the subscheme D^3 *is empty, then for every non-negative integer* r*, we have the equality in* $\mathrm{CH}^r(Y)$*:*

$$c_r(\omega_{Y/C}^{1\vee}) = c_r([T_g]) + \sum_{i\in I}(m_i - 1)i_{D_i*}c_{r-1}(T_{D_i}) - \sum_{(i,j)\in I^2, i\prec j}(m_i + m_j - 1)i_{D_{ij}*}c_{r-2}(T_{D_{ij}}). \tag{3.1.12}$$

The remainder of this chapter shall be devoted to the proofs of these results.

More specifically, Section 3.2 shall be devoted to the proof of Proposition 3.1.1; Section 3.3 shall be devoted to the proofs of Propositions 3.1.2, 3.1.3 and Corollaries 3.1.4 and 3.1.5; and Section 3.4 shall be devoted to the proofs of Propositions 3.1.6 and 3.1.7.

3.2. Characteristic classes of differentials and logarithmic differentials: the absolute case

Let us adopt the notation of Proposition 3.1.1.

3.2.1. Proof of (3.1.1)**.** Let J be a subset of I. Let

$$\mathcal{N}_J := \mathcal{N}_{E_J}X$$

be the normal vector bundle of the subscheme E_J in X.

One checks in local coordinates that the composition of morphisms of vector bundles on E_J:

$$\mathcal{N}_J^\vee \longrightarrow \Omega^1_{X|E_J} \longrightarrow \Omega^1_X(\log E)_{|E_J}$$

vanishes, so using the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}_J^\vee & \longrightarrow & \Omega^1_{X|E_J} & \longrightarrow & \Omega^1_{E_J} & \longrightarrow & 0 \\
 & & & \searrow^{0} & \downarrow & & & & \\
 & & & & \Omega^1_X(\log E)_{|E_J} & & & &
\end{array}$$

and the exactness of the horizontal exact sequence of vector bundles, we obtain a map of vector bundles on E_J:

$$\alpha_J : \Omega^1_{E_J} \longrightarrow \Omega^1_X(\log E)_{|E_J}.$$

One checks in local coordinates that the map α_J factors through a map of vector bundles:

$$\tilde{\alpha_J} : \Omega^1_{E_J}(\log E_J \cap E^{|J|+1}) \longrightarrow \Omega^1_X(\log E)_{|E_J}.$$

On the other hand, let us define a map of vector bundles on E_J by:

$$\beta_J := (\mathrm{Res}_{E_i})_{i\in J} : \quad \Omega^1_X(\log E)_{|E_J} \longrightarrow \mathcal{O}^J_{E_J} \simeq \mathcal{O}^{\oplus|J|}_{E_J},$$

where, for every i in J, Res_{E_i} denotes the residue morphism relative to the divisor E_i.

Equality (3.1.1) follows immediately from the following proposition.

PROPOSITION 3.2.1. *The morphisms $\tilde{\alpha_J}$ and β_J fit together in an exact sequence of vector bundles on E_J:*

$$0 \longrightarrow \Omega^1_{E_J}(\log E_J \cap E^{|J|+1}) \xrightarrow{\tilde{\alpha_J}} \Omega^1_X(\log E)_{|E_J} \xrightarrow{\beta_J} \mathcal{O}^J_{E_J} \longrightarrow 0.$$

PROOF. We can check this on a neighborhood of a point P of E_J. Let $i_1, ..., i_s$ be the elements of J, with $s := |J|$.

Let r be the largest integer such that P is in E^r, in particular, $s \leq r \leq n$. Let E_{i_1},..., E_{i_s}, $E_{i_{s+1}}$,...,E_{i_r} be the components of E that meet in P, with $\{i_{s+1}, ..., i_r\}$ a subset of $I - J$.

Let us consider local coordinates $(x_1, ..., x_n)$ of X centered in P, such that for every integer $1 \leq l \leq r$, E_{i_l} is locally defined by $(x_l = 0)$.

The vector bundle $\Omega^1_{E_J}$ admits a local frame $(\mathrm{d}x_{s+1}, ..., \mathrm{d}x_n)$, the vector bundle $\Omega^1_{X|E_J}$ admits a local frame $(\mathrm{d}x_1$,...,$\mathrm{d}x_n)$, the vector bundle $\Omega^1_X(\log E)_{|E_J}$ admits a local frame $(\mathrm{d}x_1/x_1$,...,$\mathrm{d}x_r/x_r$, $\mathrm{d}x_{r+1}$,...,$\mathrm{d}x_n)$, and the canonical map from $\Omega^1_{X|E_J}$ to $\Omega^1_X(\log E)_{|E_J}$ sends, for every integer l, $\mathrm{d}x_l$ to $x_l(\mathrm{d}x_l/x_l)$ if $l \leq r$, else to $\mathrm{d}x_l$.

Consequently, the map

$$\alpha_J : \Omega^1_{E_J} \longrightarrow \Omega^1_X(\log E)_{|E_J}$$

sends, for every integer $l \geq s+1$, $\mathrm{d}x_l$ to $x_l(\mathrm{d}x_l/x_l)$ if $l \leq r$, else to $\mathrm{d}x_l$.

The divisor $E_J \cap E^{|J|+1}$ in E_J is locally defined by the equation $(x_{s+1}...x_r = 0)$, so the vector bundle $\Omega^1_{E_J}(\log E_J \cap E^{|J|+1})$ admits a local frame $(\mathrm{d}x_{s+1}/x_{s+1}, ..., \mathrm{d}x_r/x_r, \mathrm{d}x_{r+1}, ..., \mathrm{d}x_n)$.

Consequently, the map

$$\tilde{\alpha_J} : \Omega^1_{E_J}(\log E_J \cap E^{|J|+1}) \longrightarrow \Omega^1_X(\log E)_{|E_J}$$

sends, for every integer $s+1 \leq l \leq r$, $\mathrm{d}x_l/x_l$ to itself, and for every integer $l > r$, $\mathrm{d}x_l$ to itself. It is an injective morphism of vector bundles, whose image is the vector subbundle of $\Omega^1_X(\log E)_{|E_J}$ generated by $(\mathrm{d}x_l/x_l)_{s+1 \leq l \leq r}$ and by $(\mathrm{d}x_l)_{l>r}$.

On the other hand, the morphism of vector bundles:

$$\beta_J : \Omega^1_X(\log E)_{|E_J} \longrightarrow \mathcal{O}^J_{E_J}$$

sends, for every integer l such that $1 \leq l \leq s$, $\mathrm{d}x_l/x_l$ to $(0, ..., 1, ..., 0)$ with the 1 in i_l-th position; for every integer l such that $s+1 \leq l \leq r$, $\mathrm{d}x_l/x_l$ to 0, and for every integer l such that $l > r$, $\mathrm{d}x_l$ to 0.

Consequently, it is surjective, and its kernel is the vector subbundle of $\Omega^1_X(\log E)_{|E_J}$ generated by $(\mathrm{d}x_l/x_l)_{s+1 \leq l \leq r}$ and by $(\mathrm{d}x_l)_{l>r}$, which is exactly the image of $\tilde{\alpha_J}$.

So the sequence is exact, as wanted. □

3.2.2. Proof of (3.1.2)**.** We shall reason by induction over the dimension n of X.

If $n = 0$, then X is a finite scheme and the divisor E is empty, so that equality (3.1.2) holds.

Let $n \geq 1$ be an integer, let us assume that equality (3.1.2) holds for schemes of dimension at most $n-1$, and let us prove it for a scheme X of dimension n.

One easily checks in local coordinates that we have an exact sequence of coherent sheaves on X:

$$0 \longrightarrow \Omega^1_X \xrightarrow{u} \Omega^1_X(\log E) \xrightarrow{(\mathrm{Res}_{E_i})_{i \in I}} \bigoplus_{i \in I} i_{E_i *}\mathcal{O}_{E_i} \longrightarrow 0, \tag{3.2.1}$$

where u is the canonical inclusion morphism, and for every i in I, Res_{E_i} is the residue morphism.

Consequently, we have the equality in $\mathrm{CH}^*(X)$:

$$c(\Omega^1_X) = c(\Omega^1_X(\log E)) \prod_{i \in I} c([i_{E_i *}\mathcal{O}_{E_i}])^{-1}. \tag{3.2.2}$$

For every element i in I, we have an exact sequence of coherent sheaves on X:

$$0 \longrightarrow \mathcal{O}_X(-E_i) \longrightarrow \mathcal{O}_X \longrightarrow i_{E_i *}\mathcal{O}_{E_i} \longrightarrow 0, \tag{3.2.3}$$

hence the equality in $\mathrm{CH}^*(X)$:

$$c([i_{E_i *}\mathcal{O}_{E_i}])^{-1} = c(\mathcal{O}_X(-E_i)) = 1 - [E_i].$$

Hence replacing in (3.2.2):

$$\begin{aligned}
c(\Omega^1_X) &= c(\Omega^1_X(\log E)) \prod_{i\in I}(1-[E_i]),\\
&= c(\Omega^1_X(\log E)) \sum_{K\subset I}\prod_{i\in K}(-[E_i]),\\
&= c(\Omega^1_X(\log E)) \sum_{K\subset I}(-1)^{|K|}[E_K] \text{ because the } (E_i)_{i\in I} \text{ intersect each other transversally},\\
&= \sum_{K\subset I}(-1)^{|K|} i_{E_K *} c(\Omega^1_X(\log E)_{|E_K}),\\
&= \sum_{K\subset I}(-1)^{|K|} i_{E_K *} c(\Omega^1_{E_K}(\log E_K \cap E^{|K|+1})) \text{ using equality (3.1.1)},\\
&= c(\Omega^1_X(\log E)) + \sum_{\emptyset\neq K\subset I}(-1)^{|K|} i_{E_K *} c(\Omega^1_{E_K}(\log E_K \cap E^{|K|+1})). \qquad (3.2.4)
\end{aligned}$$

Let K be a non-empty subset of I. Applying the induction hypothesis to the scheme E_K of dimension at most $n-1$ and to the divisor with strict normal crossings $E_K \cap E^{|K|+1}$, whose components are the $(E_{K\cup\{i\}})_{i\in I-K}$, yields the equality in $\mathrm{CH}^*(E_K)$:

$$c(\Omega^1_{E_K}(\log E_K \cap E^{|K|+1})) = \sum_{K\subset J\subset I} i_{E_J, E_K *} c(\Omega^1_{E_J}).$$

Hence replacing in (3.2.4):

$$\begin{aligned}
c(\Omega^1_X) &= c(\Omega^1_X(\log E)) + \sum_{\emptyset\neq K\subset J\subset I}(-1)^{|K|} i_{E_J *} c(\Omega^1_{E_J}),\\
&= c(\Omega^1_X(\log E)) + \sum_{\emptyset\neq J\subset I}\Big[\sum_{\emptyset\neq K\subset J}(-1)^{|K|}\Big] i_{E_J *} c(\Omega^1_{E_J}),\\
&= c(\Omega^1_X(\log E)) + \sum_{\emptyset\neq J\subset I}\Big[\prod_{i\in J}(1-1) - 1\Big] i_{E_J *} c(\Omega^1_{E_J}),\\
&= c(\Omega^1_X(\log E)) - \sum_{\emptyset\neq J\subset I} i_{E_J *} c(\Omega^1_{E_J}),
\end{aligned}$$

hence we have the equality in $\mathrm{CH}^*(X)$:

$$\begin{aligned}
c(\Omega^1_X(\log E)) &= c(\Omega^1_X) + \sum_{\emptyset\neq J\subset I} i_{E_J *} c(\Omega^1_{E_J}),\\
&= \sum_{J\subset I} i_{E_J *} c(\Omega^1_{E_J}),
\end{aligned}$$

which concludes the induction and the proof of equality (3.1.2).

3.2.3. Proof of (3.1.3)**.** Similarly to the proof of (3.1.2), using exact sequences (3.2.1), (3.2.3) and the multiplicativity of the Todd class Td, we obtain the equality in $\mathrm{CH}^*(X)_{\mathbb{Q}}$:

$$\mathrm{Td}(\Omega^1_X(\log E)) = \mathrm{Td}(\Omega^1_X) \prod_{i\in I} \mathrm{Td}([i_{E_i *}\mathcal{O}_{E_i}]) = \mathrm{Td}(\Omega^1_X) \prod_{i\in I} \mathrm{td}(-[E_i])^{-1},$$

as wanted.

3.3. Normal bundles to the strata of the singular fibers and Chern classes

3.3.1. Proof of Proposition 3.1.2. Let J be a non-empty subset of I. Since the line bundle $\Omega^1_C(\log\Delta)_{|\Delta}$ can be trivialised on Δ, the line bundle $(g^*\Omega^1_C(\log\Delta))_{|D_J}$ can be trivialised on D_J, and we obtain the equality in $\mathrm{CH}^*(D_J)$:

$$c(g^*\Omega^1_C(\log\Delta))_{|D_J} = 1,$$

hence by definition of the relative logarithmic vector bundle $\omega^1_{Y/C}$ on Y, the equality in $\mathrm{CH}^*(D_J)$:

$$c(\omega^1_{Y/C})_{|D_J} = c(\Omega^1_Y(\log Y_\Delta))_{|D_J},$$

hence applying equality (3.1.1) from Proposition 3.1.1 to the scheme $X := Y$ and the reduced divisor with strict normal crossings $E := |D| = \sum_{i\in I} D_i$, and to the subset J in I:

$$c(\omega^1_{Y/C})_{|D_J} = c(\Omega^1_{D_J}(\log D_J \cap D^{|J|+1})),$$

as wanted.

3.3.2. Proofs of Proposition 3.1.3 and Corollaries 3.1.4 and 3.1.5. Let i be an element of I. Since Δ is a scheme of dimension 0, the line bundle $\mathcal{N}_\Delta C$ on Δ can be trivialized. Consequently, we can trivialize the following line bundle on D_i:

$$\begin{aligned}(g^*\mathcal{N}_\Delta C)_{|D_i} &\simeq \mathcal{O}_Y(Y_\Delta)_{|D_i}\\ &\simeq \mathcal{O}_Y(m_iD_i)_{|D_i} \otimes \bigotimes_{j\in I-\{i\}} \mathcal{O}_Y(m_jD_j)_{|D_i}\\ &\simeq \mathcal{N}_i^{\otimes m_i} \otimes \bigotimes_{j\in I-\{i\}} \mathcal{O}_{D_i}(m_jD_{ij}).\end{aligned}$$

Taking Chern classes, we obtain the following equality in $\mathrm{CH}^1(D_i)$:

$$m_ic_1(\mathcal{N}_i) = -\sum_{j\in I-\{i\}} m_j[D_{ij}], \tag{3.3.1}$$

which shows Proposition 3.1.3.

If β is a class in $\mathrm{CH}^*(D_i)$, intersecting (3.3.1) with β then pushing forward by the inclusion i_{D_i} yields Corollary 3.1.4.

Now, let (i,j) be a pair in I^2 such that $i \prec j$. Applying equality (3.3.1) with the element i then restricting to D_{ij} yields the equality in $\mathrm{CH}^1(D_{ij})$:

$$m_ic_1(\mathcal{N}_{i|D_{ij}}) = -\sum_{k\in I-\{i\}} m_k[D_{ik}]_{|D_{ij}},$$

where $[D_{ik}]$ is seen as a divisor in D_i.

If $k \neq j$, this divisor intersects transversally the divisor D_{ij}, so that we have the equality in $\mathrm{CH}^1(D_{ij})$:

$$[D_{ik}]_{|D_{ij}} = [D_{ik}\cap D_{ij}] = [D_{ijk}].$$

If $k = j$, the self-intersection of the divisor D_{ik} is given by:

$$[D_{ik}]_{|D_{ij}} = c_1(\mathcal{N}_{D_{ij}}D_i) = c_1(\mathcal{N}_{j|D_{ij}}).$$

We have used the canonical isomorphism of line bundles on D_{ij}:

$$\mathcal{N}_{D_{ij}}D_i \simeq \mathcal{N}_{j|D_{ij}}.$$

Hence replacing, we have the equality in $\mathrm{CH}^1(D_{ij})$:

$$m_i c_1(\mathcal{N}_{i|D_{ij}}) = -m_j c_1(\mathcal{N}_{j|D_{ij}}) - \sum_{k\in I-\{i,j\}} m_k [D_{ijk}],$$

which shows Corollary 3.1.5.

3.4. Characteristic classes of $[T_g]$ and $\omega^{1\vee}_{Y/C}$

3.4.1. Proof of Proposition 3.1.6. Using equality (3.1.3) from Proposition 3.1.1 applied to the scheme $X := Y$ and the reduced divisor with strict normal crossings $E := |D| = \sum_{i\in I} D_i$, we have the equality in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$\mathrm{Td}(\Omega^1_Y(\log D)) = \mathrm{Td}(\Omega^1_Y)\prod_{i\in I}\mathrm{td}(-[D_i])^{-1},$$

hence the equality:

$$\mathrm{Td}(\Omega^1_Y) = \mathrm{Td}(\Omega^1_Y(\log D))\prod_{i\in I}\mathrm{td}(-[D_i]),$$

hence, using the definition of the relative logarithmic bundle $\omega^1_{Y/C}$ and the multiplicativity of Td:

$$\mathrm{Td}(\Omega^1_Y) = \mathrm{Td}(\omega^1_{Y/C})\, g^*\mathrm{Td}(\Omega^1_C(\log\Delta))\prod_{i\in I}\mathrm{td}(-[D_i]).$$

Since C is a curve, we have a canonical isomorphism of line bundles on C:

$$\Omega^1_C(\log\Delta) \simeq \Omega^1_C(\Delta),$$

hence replacing:

$$\mathrm{Td}(\Omega^1_Y) = \mathrm{Td}(\omega^1_{Y/C})\, g^*\mathrm{td}(c_1(\Omega^1_C) + [\Delta])\prod_{i\in I}\mathrm{td}(-[D_i]).$$

Taking duals, we obtain the equality in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$\mathrm{Td}(T_Y) = \mathrm{Td}(\omega^{1\vee}_{Y/C})\, g^*\mathrm{td}(c_1(T_C) - [\Delta])\prod_{i\in I}\mathrm{td}([D_i]). \tag{3.4.1}$$

By definition of the relative tangent class $[T_g]$ in $K^0(Y)$, we have the identity in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$\begin{aligned}\mathrm{Td}([T_g]) &= \mathrm{Td}(T_Y)g^*\mathrm{Td}(T_C)^{-1},\\ &= \mathrm{Td}(\omega^{1\vee}_{Y/C})\, g^*[\mathrm{td}(c_1(T_C)-[\Delta])\mathrm{td}(c_1(T_C))^{-1}]\prod_{i\in I}\mathrm{td}([D_i]) \text{ using (3.4.1)},\end{aligned}$$

hence the equality:

$$\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} = g^*[\mathrm{td}(c_1(T_C)-[\Delta])\mathrm{td}(c_1(T_C))^{-1}]\prod_{i\in I}\mathrm{td}([D_i]). \tag{3.4.2}$$

Since C is a curve, all the classes in $\mathrm{CH}^2(C)_{\mathbb{Q}}$ vanish. Consequently, we have the equality in $\mathrm{CH}^*(C)_{\mathbb{Q}}$:

$$\begin{aligned}\mathrm{td}(c_1(T_C)-[\Delta])\mathrm{td}(c_1(T_C))^{-1} &= [1 + 1/2\,(c_1(T_C)-[\Delta])][1+1/2\,c_1(T_C)]^{-1},\\ &= [1+1/2(c_1(T_C)-[\Delta])][1-1/2\; c_1(T_C)],\\ &= 1 - 1/2\,[\Delta].\end{aligned}$$

Hence replacing in (3.4.2), we obtain the equality in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$\begin{aligned}\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} &= g^*[1-1/2\,[\Delta]]\prod_{i\in I}\mathrm{td}([D_i]),\\ &= \prod_{i\in I}\mathrm{td}([D_i]) - 1/2\,(g^*[\Delta])\prod_{i\in I}\mathrm{td}([D_i]).\end{aligned}\tag{3.4.3}$$

The line bundle $\mathcal{O}_C(\Delta)_{|\Delta}$ on Δ can be trivialized, so for every element i in I, the line bundle $(g^*\mathcal{O}_C(\Delta))_{|D_i}$ on D_i can be trivialized. Consequently, for every element i in I, we have the equality in $\mathrm{CH}^2(Y)$:

$$(g^*[\Delta])[D_i] = 0,$$

hence we have the equality in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$(g^*[\Delta])\prod_{i\in I}\mathrm{td}([D_i]) = g^*[\Delta] = \sum_{i\in I} m_i[D_i].$$

Hence replacing in (3.4.3), we have the equality in $\mathrm{CH}^*(Y)_{\mathbb{Q}}$:

$$\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} = \prod_{i\in I}\mathrm{td}([D_i]) - 1/2\sum_{i\in I} m_i[D_i],$$

which shows equality (3.1.7).

Taking the terms of codimension 1, we obtain the equality in $\mathrm{CH}^1(Y)_{\mathbb{Q}}$:

$$\begin{aligned}\Big[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big]^{(1)} &= \sum_{i\in I}\mathrm{td}([D_i])^{(1)} - 1/2\sum_{i\in I} m_i[D_i],\\ &= \sum_{i\in I}(1/2[D_i]) - 1/2\sum_{i\in I} m_i[D_i],\\ &= -1/2\sum_{i\in I}(m_i-1)[D_i],\end{aligned}$$

which shows equality (3.1.8).

Taking instead the terms of codimension 2, we obtain the equality in $\mathrm{CH}^2(Y)_{\mathbb{Q}}$:

$$\begin{aligned}\Big[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big]^{(2)} &= \big[\prod_{i\in I}\mathrm{td}([D_i])\big]^{(2)},\\ &= \sum_{i\in I}\mathrm{td}([D_i])^{(2)} + \sum_{(i,j)\in I^2, i\prec j}\mathrm{td}([D_i])^{(1)}.\mathrm{td}([D_j])^{(1)},\\ &= \sum_{i\in I}(1/12\,[D_i]^2) + \sum_{(i,j)\in I^2, i\prec j}(1/2\,[D_i]).(1/2[D_j]),\\ &= 1/12\sum_{i\in I}[D_i]^2 + 1/4\sum_{(i,j)\in I^2, i\prec j}[D_i].[D_j].\end{aligned}\tag{3.4.4}$$

For every element i in I, the self-intersection of the divisor D_i is given in $\mathrm{CH}^2(Y)$ by:

$$[D_i]^2 = i_{D_i *}c_1(\mathcal{N}_i).$$

On the other hand, for every pair (i,j) in I^2 such that $i\prec j$, the divisors D_i and D_j intersect transversally, so that we have the equality in $\mathrm{CH}^2(Y)$:

$$[D_i].[D_j] = [D_{ij}].$$

Hence replacing in (3.4.4):

$$\Big[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big]^{(2)} = 1/12\sum_{i\in I} i_{D_i*}c_1(\mathcal{N}_i) + 1/4\sum_{(i,j)\in I^2, i\prec j}[D_{ij}],$$

which shows equality (3.1.9).

Finally, for every element i in I, applying Proposition 3.1.3 and dividing by m_i, we have the equality in $\mathrm{CH}^1(D_i)_{\mathbb{Q}}$:

$$c_1(\mathcal{N}_i) = -\sum_{j\in I-\{i\}} m_j/m_i[D_{ij}],$$

hence pushing forward by the inclusion of D_i in Y, then summing over i, we have the equality in $\mathrm{CH}^2(Y)_{\mathbb{Q}}$:

$$\sum_{i\in I} i_{D_i*}c_1(\mathcal{N}_i) = -\sum_{(i,j)\in I^2, i\prec j}(m_i/m_j + m_j/m_i)[D_{ij}],$$

hence replacing, we obtain the equality in $\mathrm{CH}^2(Y)_{\mathbb{Q}}$:

$$\begin{aligned}\Big[\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big]^{(2)} &= -1/12\sum_{(i,j)\in I^2, i\prec j}(m_i/m_j + m_j/m_i)[D_{ij}] + 1/4\sum_{(i,j)\in I^2, i\prec j}[D_{ij}],\\ &= 1/12\sum_{(i,j)\in I^2, i\prec j}(3 - m_i/m_j - m_j/m_i)[D_{ij}],\end{aligned}$$

which shows equality (3.1.10).

This concludes the proof of Proposition 3.1.6.

3.4.2. Proof of Proposition 3.1.7. Using equality (3.1.2) from Proposition 3.1.1 applied to the scheme $X := Y$ and the reduced divisor with strict normal crossings $E := |D| = \sum_{i\in I} D_i$, we have the equality in $\mathrm{CH}^*(Y)$:

$$c(\Omega^1_Y(\log D)) = c(\Omega^1_Y) + \sum_{\emptyset\neq J\subset I} i_{D_J*}c(\Omega^1_{D_J}),$$

hence, by definition of the relative logarithmic bundle $\omega^1_{Y/C}$, the equality in $\mathrm{CH}^*(Y)$:

$$c(\omega^1_{Y/C}) = \big[c(\Omega^1_Y) + \sum_{\emptyset\neq J\subset I} i_{D_J*}c(\Omega^1_{D_J})\big]\, g^*c(\Omega^1_C(\log\Delta))^{-1}.$$

Since C is a curve, we have a canonical isomorphism of line bundles on C:

$$\Omega^1_C(\log\Delta) \simeq \Omega^1_C(\Delta),$$

hence replacing:

$$c(\omega^1_{Y/C}) = \big[c(\Omega^1_Y) + \sum_{\emptyset\neq J\subset I} i_{D_J*}c(\Omega^1_{D_J})\big]\, g^*(1 + c_1(\Omega^1_C) + [\Delta])^{-1},$$

hence taking duals:

$$c(\omega^{1\vee}_{Y/C}) = \big[c(T_Y) + \sum_{\emptyset\neq J\subset I}(-1)^{|J|} i_{D_J*}c(T_{D_J})\big]\, g^*(1 + c_1(T_C) - [\Delta])^{-1}. \tag{3.4.5}$$

The line bundles $T_{C|\Delta}$ and $\mathcal{O}_C(\Delta)_{|\Delta}$ on Δ can be trivialized, so for every non-empty subset J in I, the line bundles $(g^*T_C)_{|D_J}$ and $(g^*\mathcal{O}_C(\Delta))_{|D_J}$ on D_J can be trivialized. Consequently, for every non-empty subset J in I, we have the equalities in $\mathrm{CH}^*(Y)$:

$$[i_{D_J*}c(T_{D_J})](g^*c_1(T_C)) = [i_{D_J*}c(T_{D_J})](g^*[\Delta]) = 0,$$

hence we have the equality in $\mathrm{CH}^*(Y)$:

$$\Big[\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J})\Big]\,g^*(1+c_1(T_C)-[\Delta])^{-1}=\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J}).$$

Hence replacing in (3.4.5), we obtain the equality:

$$c(\omega^{1\vee}_{Y/C})=c(T_Y)g^*(1+c_1(T_C)-[\Delta])^{-1}+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J}). \tag{3.4.6}$$

Since C is a curve, all the classes in $\mathrm{CH}^2(C)$ vanish. Consequently, we have the equality in $\mathrm{CH}^*(C)$:

$$(1+c_1(T_C)-[\Delta])^{-1}=(1+c_1(T_C))^{-1}+[\Delta].$$

Hence replacing in (3.4.6):

$$\begin{aligned}c(\omega^{1\vee}_{Y/C})&=c(T_Y)g^*[(1+c_1(T_C))^{-1}+[\Delta]]+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J}),\\&=c(T_Y)g^*c(T_C)^{-1}+c(T_Y)\big(\sum_{i\in I}m_i[D_i]\big)+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J}),\\&=c([T_g])+\sum_{i\in I}m_i i_{D_i*}c(T_{Y|D_i})+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c(T_{D_J})\text{ by definition of the class }[T_g].\end{aligned}$$

Let r be a non-negative integer, taking the terms of codimension r, we obtain the equality in $\mathrm{CH}^r(Y)$:

$$c_r(\omega^{1\vee}_{Y/C})=c_r([T_g])+\sum_{i\in I}m_i i_{D_i*}c_{r-1}(T_{Y|D_i})+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c_{r-|J|}(T_{D_J}). \tag{3.4.7}$$

For every element i in I, we have an exact sequence of vector bundles on D_i:

$$0\longrightarrow T_{D_i}\longrightarrow T_{Y|D_i}\longrightarrow \mathcal{N}_i\longrightarrow 0,$$

hence the equality in $\mathrm{CH}^*(D_i)$:

$$c(T_{Y|D_i})=c(T_{D_i})(1+c_1(\mathcal{N}_i)),$$

hence the equality in $\mathrm{CH}^{r-1}(D_i)$,

$$c_{r-1}(T_{Y|D_i})=c_{r-1}(T_{D_i})+c_1(\mathcal{N}_i)c_{r-2}(T_{D_i}).$$

Hence replacing in (3.4.7), we obtain the equality in $\mathrm{CH}^r(Y)$:

$$c_r(\omega^{1\vee}_{Y/C})=c_r([T_g])+\sum_{i\in I}m_i i_{D_i*}[c_{r-1}(T_{D_i})+c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})]+\sum_{\emptyset\neq J\subset I}(-1)^{|J|}i_{D_J*}c_{r-|J|}(T_{D_J}),$$

which shows equality (3.1.11).

Now, let us assume that the subscheme D^3 is empty. In this case, equality (3.1.11) can be rewritten:

$$\begin{aligned}c_r(\omega^{1\vee}_{Y/C})&=c_r([T_g])+\sum_{i\in I}m_i i_{D_i*}[c_{r-1}(T_{D_i})+c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})]\\&\quad-\sum_{i\in I}i_{D_i*}c_{r-1}(T_{D_i})+\sum_{(i,j)\in I^2,i\prec j}i_{D_{ij}*}c_{r-2}(T_{D_{ij}}),\\&=c_r([T_g])+\sum_{i\in I}(m_i-1)i_{D_i*}c_{r-1}(T_{D_i})\\&\quad+\sum_{i\in I}m_i i_{D_i*}(c_1(\mathcal{N}_i)c_{r-2}(T_{D_i}))+\sum_{(i,j)\in I^2,i\prec j}i_{D_{ij}*}c_{r-2}(T_{D_{ij}}).\end{aligned} \tag{3.4.8}$$

For every element i in I, applying Corollary 3.1.4 to the class $\beta := c_{r-2}(T_{D_i})$ in $\mathrm{CH}^{r-2}(D_i)$ yields the equality in $\mathrm{CH}^r(Y)$:

$$m_i i_{D_i *}(c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})) = -\sum_{j\in I-\{i\}} m_j i_{D_{ij}*}c_{r-2}(T_{D_i|D_{ij}}), \tag{3.4.9}$$

hence summing over i, we obtain the equality in $\mathrm{CH}^r(Y)$:

$$\sum_{i\in I} m_i i_{D_i *}(c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})) = -\sum_{(i,j)\in I^2, i\prec j} i_{D_{ij}*}(m_j c_{r-2}(T_{D_i|D_{ij}}) + m_i c_{r-2}(T_{D_j|D_{ij}})).$$

For every pair (i,j) in I^2 such that $i \prec j$, we have the following normal exact sequences of vector bundles on D_{ij}:

$$0 \longrightarrow T_{D_{ij}} \longrightarrow T_{D_i|D_{ij}} \longrightarrow \mathcal{N}_{D_{ij}}D_i \longrightarrow 0,$$
$$0 \longrightarrow T_{D_{ij}} \longrightarrow T_{D_j|D_{ij}} \longrightarrow \mathcal{N}_{D_{ij}}D_j \longrightarrow 0,$$

hence the following equalities in $\mathrm{CH}^{r-2}(D_{ij})$:

$$c_{r-2}(T_{D_i|D_{ij}}) = c_{r-2}(T_{D_{ij}}) + c_1(\mathcal{N}_{D_{ij}}D_i)c_{r-3}(T_{D_{ij}}),$$
$$c_{r-2}(T_{D_j|D_{ij}}) = c_{r-2}(T_{D_{ij}}) + c_1(\mathcal{N}_{D_{ij}}D_j)c_{r-3}(T_{D_{ij}}),$$

hence using the canonical isomorphisms of line bundles on D_{ij} between $\mathcal{N}_{D_{ij}}D_j$ and $\mathcal{N}_{i|D_{ij}}$, as well as between $\mathcal{N}_{D_{ij}}D_i$ and $\mathcal{N}_{j|D_{ij}}$:

$$c_{r-2}(T_{D_i|D_{ij}}) = c_{r-2}(T_{D_{ij}}) + c_1(\mathcal{N}_{j|D_{ij}})c_{r-3}(T_{D_{ij}}),$$
$$c_{r-2}(T_{D_j|D_{ij}}) = c_{r-2}(T_{D_{ij}}) + c_1(\mathcal{N}_{i|D_{ij}})c_{r-3}(T_{D_{ij}}),$$

hence replacing in (3.4.9), we obtain the equality in $\mathrm{CH}^r(Y)$:

$$\begin{aligned}\sum_{i\in I} m_i i_{D_i *}(c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})) = &-\sum_{(i,j)\in I^2, i\prec j} (m_j + m_i) i_{D_{ij}*}c_{r-2}(T_{D_{ij}})\\ &-\sum_{(i,j)\in I^2, i\prec j} i_{D_{ij}*}[\{m_i c_1(\mathcal{N}_{i|D_{ij}}) + m_j c_1(\mathcal{N}_{j|D_{ij}})\}c_{r-3}(T_{D_{ij}})];\end{aligned}$$

using Corollary 3.1.5, and using that D^3 is empty, we get:

$$\sum_{i\in I} m_i i_{D_i *}(c_1(\mathcal{N}_i)c_{r-2}(T_{D_i})) = -\sum_{(i,j)\in I^2, i\prec j} (m_i + m_j) i_{D_{ij}*}c_{r-2}(T_{D_{ij}}).$$

Hence replacing in (3.4.8), we obtain the equality in $\mathrm{CH}^r(Y)_{\mathbb{Q}}$:

$$\begin{aligned}c_r(\omega^{1\vee}_{Y/C}) &= c_r([T_g]) + \sum_{i\in I}(m_i - 1) i_{D_i *}c_{r-1}(T_{D_i})\\ &\quad - \sum_{(i,j)\in I^2, i\prec j}(m_i + m_j) i_{D_{ij}*}c_{r-2}(T_{D_{ij}}) + \sum_{(i,j)\in I^2, i\prec j} i_{D_{ij}*}c_{r-2}(T_{D_{ij}}),\\ &= c_r([T_g]) + \sum_{i\in I}(m_i - 1) i_{D_i *}c_{r-1}(T_{D_i}) - \sum_{(i,j)\in I^2, i\prec j}(m_i + m_j - 1) i_{D_{ij}*}c_{r-2}(T_{D_{ij}}).\end{aligned}$$

This proves equality (3.1.12).

CHAPTER 4

The alternating product of Griffiths line bundles associated to a pencil of complete varieties with DNC degenerations

In this chapter, we work over a base field k of characteristic zero, and by the word "scheme", we mean a separated algebraic scheme of finite type over k.

4.1. The characteristic classes ρ and ρ_r and the Grothendieck-Riemann-Roch formula

In this section, we will work on a fixed smooth k-scheme X.

4.1.1. The classes λ_y, φ_y, and ρ. Recall that the *Grothendieck λ-map*:

$$\lambda_y : K^0(X) \longrightarrow 1 + y\, K^0(X)[[y]],$$

where y denotes an indeterminate, is defined as the only map that is multiplicative for exact sequences, and that is given on (the K-theory class of) a vector bundle V by:

$$\lambda_y([V]) := \sum_{i=0}^{\mathrm{rk}(V)} y^i \left[\textstyle\bigwedge^i V\right] \in 1 + y\, K^0(X)[y];$$

see for instance [**SGA71**, app. to Exposé 0].

Definition 4.1.1. *For every vector bundle V on X, we define $\varphi_y(V)$ as the element of* $\mathrm{CH}^*(X)_{\mathbb{Q}}[y]$ *given by:*

$$\varphi_y(V) := \mathrm{ch}\big(\lambda_y(V^\vee)\big)\ \mathrm{Td}(V) = \sum_{i=0}^{\mathrm{rk}(V)} y^i \mathrm{ch}\left(\textstyle\bigwedge^i V^\vee\right)\ \mathrm{Td}(V). \tag{4.1.1}$$

The ring $\mathrm{CH}^*(X)_{\mathbb{Q}}[y]$ has a graduation given by the sum of the cohomological degree in CH^* and the degree in the indeterminate y:

$$(\mathrm{CH}^*(X)_{\mathbb{Q}}[y])^{(d)} := \bigoplus_{e=0}^{d} \mathrm{CH}^e(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}[y]_{d-e}.$$

For every vector bundle V over X, the term of total degree 0 of the element $\varphi_y(V)$ is 1, so this element is in $1 + (\mathrm{CH}^*(X)_{\mathbb{Q}}[y])^{(\geq 1)}$.

Proposition 4.1.2. *The map φ_y is multiplicative for short exact sequences.*

Proof. The λ-map λ_y and the Todd class Td are both multiplicative for short exact sequences, and the Chern character ch is a morphism of rings. □

Remark 4.1.3. We could use this to prolong φ_y to a morphism of groups from $K^0(X)$ to $1 + (\mathrm{CH}^*(X)_{\mathbb{Q}}[[y]])^{(\geq 1)}$, but we would then lose the property that $\varphi_y(V)$ is a polynomial, which is necessary for our purpose.

PROPOSITION 4.1.4. *Let L be a line bundle on X, we have the identity in $\mathrm{CH}^*(X)_{\mathbb{Q}}[y]$:*

$$\varphi_y(L) = \mathrm{td}(c_1(L)) \left[1 + y\, e^{-c_1(L)}\right]. \tag{4.1.2}$$

In particular, replacing y with -1:

$$\varphi_{-1}(L) = c_1(L). \tag{4.1.3}$$

PROOF. The first equality (4.1.2) comes from a direct computation of $\varphi_y(L)$, using the equality in $\mathrm{CH}^*(X)_{\mathbb{Q}}$:

$$\mathrm{ch}(L^\vee) = e^{-c_1(L)}.$$

The second equality (4.1.3) follows by definition of the series td. □

The identity (4.1.3) is also a particular case of [**Ful98**, Ex. 3.2.5]. Indeed, the upcoming Proposition 4.1.8 can be seen as a variant of this example.

DEFINITION 4.1.5. *If V is a vector bundle on X, the class $\rho(V)$ is defined by:*

$$\rho(V) := \dot{\varphi}_{-1}(V) = \left(\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{ch}(\lambda_y(V^\vee))\right)_{|y=-1} \mathrm{Td}(V) \in \mathrm{CH}^*(X)_{\mathbb{Q}}.$$

the derivative of φ applied to $y = -1$, it is well-defined since $\varphi_y(V)$ is a polynomial.

If $r \geq 1$ is an integer, we also define the characteristic class:

$$\rho_r := c_{r-1} - \frac{r}{2}c_r + \frac{1}{12}c_1 c_r : K^0(X) \longrightarrow \mathrm{CH}^*(X)_{\mathbb{Q}}.$$

We are interested in comparing the classes $\rho(V)$ and $\rho_r([V])$ when V is a vector bundle of rank r.

PROPOSITION 4.1.6. *Let L be a line bundle on X, we have the identity in $\mathrm{CH}^*(X)_{\mathbb{Q}}$:*

$$\rho(L) = \mathrm{td}(-c_1(L)). \tag{4.1.4}$$

PROOF. We differentiate the equality (4.1.2). Since $\varphi_y(L)$ is a polynomial of degree at most 1, its derivative is its coefficient of degree 1, which gives the equality:

$$\rho(L) = \mathrm{td}(c_1(L))\, e^{-c_1(L)}.$$

Now, we have an equality of formal series in an indeterminate x:

$$\mathrm{td}(x)\, e^{-x} = \mathrm{td}(-x),$$

that we can show using the definition of td in terms of exponentials, as follows:

$$\begin{aligned} \mathrm{td}(x)\, e^{-x} &= \left(\frac{x}{1-e^{-x}}\right) e^{-x}, \\ &= \frac{x}{e^x - 1}, \\ &= -\frac{x}{1-e^{-(-x)}}, \\ &= \mathrm{td}(-x). \end{aligned}$$

Replacing x with $c_1(L)$, this becomes the result. □

COROLLARY 4.1.7. *Let V be a vector bundle of rank r on X, and let*

$$V = V_r \supset \ldots \supset V_0 = 0$$

be a filtration such that $L_i := V_i/V_{i-1}$ is a line bundle for every integer $i \in \{1, \ldots, r\}$.

We have the equality in $\mathrm{CH}^*(X)_{\mathbb{Q}}$*:*

$$\rho(V) = \sum_{i=1}^{r} \mathrm{td}(-c_1(L_i)) \prod_{\substack{j=1 \\ j\neq i}}^{r} c_1(L_j).$$

PROOF. Using Proposition 4.1.2, the map φ_y is multiplicative for exact sequences, so we have the equality:

$$\varphi_y(V) = \prod_{i=1}^{r} \varphi_y(L_i).$$

Differentiating and applying Leibniz's rule:

$$\dot{\varphi}_{-1}(V) = \sum_{i=1}^{r} \dot{\varphi}_{-1}(L_i) \prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{-1}(L_j).$$

Using (4.1.3) and (4.1.4), we obtain the result. □

This allows us to compare ρ and ρ_r:

PROPOSITION 4.1.8 (compare [**BCOV94**, p. 374]). *Let V be a vector bundle of rank r on X, we have the identity in* $\mathrm{CH}^*(X)_{\mathbb{Q}}$*:*

(4.1.5) $$\rho(V) = \rho_r([V]) + (\textit{terms of codimension} \geq r+2).$$

PROOF. First, suppose that V has a filtration by subbundles $V = V_r \supset ... \supset V_0 = 0$, such that for $1 \leq i \leq r$, $L_i := V_i/V_{i-1}$ is a line bundle. Using Corollary 4.1.7, we have the equality:

$$\rho(V) = \sum_{i=1}^{r} \mathrm{td}(-c_1(L_i)) \prod_{\substack{j=1 \\ j\neq i}}^{r} c_1(L_j).$$

Recall the first terms of the development of td:

$$\mathrm{td}(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + ... \quad .$$

Accordingly we have:

$$\begin{aligned}
\rho(V) &= \sum_{i=1}^{r} \left[\left(1 - \frac{c_1(L_i)}{2} + \frac{c_1(L_i)^2}{12} + (\text{terms of codimension} \geq 3)\right) \prod_{\substack{j=1 \\ j\neq i}}^{r} c_1(L_j) \right], \\
&= \sum_{i=1}^{r} \prod_{\substack{j=1 \\ j\neq i}}^{r} c_1(L_j) - \frac{1}{2} \sum_{i=1}^{r} \prod_{j=1}^{r} c_1(L_j) + \frac{1}{12} \sum_{i=1}^{r} \Big(c_1(L_i) \prod_{j=1}^{r} c_1(L_j) \Big) + (\text{codim} \geq r+2)), \\
&= \sum_{i=1}^{r} \prod_{\substack{j=1 \\ j\neq i}}^{r} c_1(L_j) - \frac{r}{2} \prod_{j=1}^{r} c_1(L_j) + \frac{1}{12} \Big(\sum_{i=1}^{r} c_1(L_i) \Big) \prod_{j=1}^{r} c_1(L_j) + (\text{codim} \geq r+2),
\end{aligned}$$

where by $(\text{codim} \geq r+2)$ we mean an element in $\mathrm{CH}^{\geq r+2}(X)_{\mathbb{Q}}$.

By multiplicativity of the total Chern class, the elementary symmetric polynomials of $(c_1(L_i))_i$ are simply the Chern classes of V, which gives the equality:

$$\begin{aligned}
\rho(V) &= c_{r-1}(V) - \frac{r}{2} c_r(V) + \frac{1}{12} c_1(V) c_r(V) + (\text{codim} \geq r+2), \\
&= \rho_r([V]) + (\text{codim} \geq r+2)).
\end{aligned}$$

This shows the desired formula when V has such a filtration.

In the general case, we are reduced to this situation by the "splitting principle" ([**Ful98**, Remark 3.2.3]). □

4.1.2. The class ρ and the Grothendieck-Riemann-Roch formula. The Riemann-Roch formula implies the following result.

PROPOSITION 4.1.9. *Let X, S be two smooth k-schemes of pure dimension, let:*

$$f : X \longrightarrow S$$

be a proper morphism of relative dimension $m \geq 0$, and let V be a vector bundle on X. Let us denote by

$$[T_f] \in K^0(X)$$

the relative tangent class associated with the l.c.i. morphism f.[1]

We have the equality in $\mathrm{CH}^*(S)_{\mathbb{Q}}$:

$$f_*\left(\rho(V)\frac{\mathrm{Td}([T_f])}{\mathrm{Td}(V)}\right) = \sum_{\substack{0\leq p\leq \mathrm{rk}(V),\\ 0\leq q\leq m}} p(-1)^{p+q-1}\mathrm{ch}(R^q f_* \textstyle\bigwedge^p V^\vee).$$

In particular, taking the terms of codimension 1, we have the equality in $\mathrm{CH}^1(S)_{\mathbb{Q}}$:

$$f_*\left[\left(\rho(V)\frac{\mathrm{Td}([T_f])}{\mathrm{Td}(V)}\right)^{(m+1)}\right] = \sum_{\substack{0\leq p\leq \mathrm{rk}(V),\\ 0\leq q\leq m}} p(-1)^{p+q-1}c_1(R^q f_* \textstyle\bigwedge^p V^\vee).$$

PROOF. We have the equality:

$$\begin{aligned}
&(4.1.6) \quad & f_*\left(\rho(V)\frac{\mathrm{Td}([T_f])}{\mathrm{Td}(V)}\right) &= f_*\left(\left(\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{ch}(\lambda_y(V^\vee))\right)_{|y=-1}\mathrm{Td}(V)\frac{\mathrm{Td}[T_f]}{\mathrm{Td}(V)}\right),\\
&&&= \frac{\mathrm{d}}{\mathrm{d}y} f_*\left(\mathrm{ch}(\lambda_y(V^\vee))\mathrm{Td}([T_f])\right)_{|y=-1},\\
&(4.1.7) &&= \frac{\mathrm{d}}{\mathrm{d}y} f_*\left(\sum_{0\leq p\leq \mathrm{rk}(V)} y^p \mathrm{ch}(\textstyle\bigwedge^p V^\vee)\mathrm{Td}([T_f])\right)_{|y=-1},\\
&(4.1.8) &&= \frac{\mathrm{d}}{\mathrm{d}y}\left(\sum_{0\leq p\leq \mathrm{rk}(V)} y^p \mathrm{ch}(R^\bullet f_* \textstyle\bigwedge^p V^\vee)\right)_{|y=-1},\\
&&&= \sum_{0\leq p\leq \mathrm{rk}(V)} p(-1)^{p-1}\mathrm{ch}(R^\bullet f_* \textstyle\bigwedge^p V^\vee),\\
&(4.1.9) &&= \sum_{\substack{0\leq p\leq \mathrm{rk}(V)\\ 0\leq q\leq m}} p(-1)^{p-1}(-1)^q\mathrm{ch}(R^q f_* \textstyle\bigwedge^p V^\vee),
\end{aligned}$$

where in (4.1.6), we have used the definition of the class ρ; in (4.1.7), we have used the definition of the class λ_y; in (4.1.8), we have used the Grothendieck-Riemann-Roch formula; and in (4.1.9), we have used the definition of the morphism of abelian groups $R^\bullet f_*$. □

1. see for instance [**Ful98**, B.7.6].

4.2. Application to Griffiths line bundles of fibrations over curves with DNC fibers

4.2.1. An application of Steenbrink's theory. We are now able to compute a certain combination of the Griffiths line bundles in terms of the function ρ_r.

THEOREM 4.2.1. *Let C be a connected smooth projective complex curve with generic point η, Y be a connected smooth projective N-dimensional complex scheme, and let*

$$g : Y \longrightarrow C$$

be a surjective morphism of complex schemes. Let us assume that there exists a finite subset Δ in C such that g is smooth over $C - \Delta$, and such that the divisor Y_Δ is a divisor with strict normal crossings in Y.

We have the equality in $\mathrm{CH}^1(C)_\mathbb{Q}$*:*

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = g_* \left[\left(\rho_{N-1}(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} \right)^{(N)} \right].$$

PROOF. Using Proposition 2.1.4, for every integer n, we have an isomorphism of line bundles over C:

$$\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)) \simeq \bigotimes_{0 \leq p \leq n} \left(\det R^{n-p} g_* \omega^p_{Y/C} \right)^{\otimes p}.$$

Consequently, taking Chern classes and taking the alternate sum, we obtain the equality in $\mathrm{CH}^1(C)_\mathbb{Q}$:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = \sum_{n=0}^{2(N-1)} (-1)^{n-1} \sum_{0 \leq p \leq n} p c_1(R^{n-p} g_* \omega^p_{Y/C}).$$

We reindex this sum on p and $q := n - p$, and we get:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = \sum_{\substack{p,q \geq 0, \\ p+q \leq 2(N-1)}} p(-1)^{p+q-1} c_1(R^q g_* \omega^p_{Y/C}).$$

Since the morphism g is of relative dimension $N-1$, the term in the sum vanishes unless $p, q \leq N-1$, so we can rewrite:

$$\begin{aligned} \sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) &= \sum_{0 \leq p,q \leq N-1} p(-1)^{p+q-1} c_1(R^q g_* \omega^p_{Y/C}), \\ &= \sum_{0 \leq p,q \leq N-1} p(-1)^{p+q-1} c_1(R^q g_* {\textstyle\bigwedge}^p (\omega^{1\vee}_{Y/C})^\vee). \end{aligned}$$

Now, we apply Proposition 4.1.9 to the morphism g of relative dimension $N-1$ and the locally free sheaf

$$V := \omega^{1\vee}_{Y/C}$$

of rank $N-1$. This yields the equality:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = g_* \left[\left(\rho(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} \right)^{(N)} \right],$$

hence applying Proposition 4.1.8 to the vector bundle V of rank $N-1$:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)))$$
$$= g_* \left[\left(\{\rho_{N-1}(\omega^{1\vee}_{Y/C}) + (\text{terms of codimension} \geq N+1)\} \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} \right)^{(N)} \right],$$
$$= g_* \left[\left(\rho_{N-1}(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})} \right)^{(N)} \right]. \qquad \square$$

4.2.2. The alternating product of Griffiths line bundles and the geometry of singular fibers. We adopt the notation of the previous subsection, and assume, without loss of generality, that the divisor Δ in C is reduced.

Let us adopt the notation of Subsection 3.1.2. Namely, we write the divisor with strict normal crossings Y_Δ as:

$$D = \sum_{i \in I} m_i D_i,$$

where I is a finite set, for every i in I, $m_i \geq 1$ is an integer, and D_i is a smooth connected divisor, such that the $(D_i)_{i\in I}$ intersect each other transversally. We denote by $\mathcal{N}_i$ the normal bundle $N_{D_i}Y$ of D_i in Y.

The set I may be written as the disjoint union:

$$I = \bigcup_{x \in \Delta} I_x,$$

where, for every $x \in \Delta$, I_x denotes the non-empty subset of I defined by:

$$I_x := \big\{ i \in I \mid g(D_i) = \{x\} \big\}.$$

For every subset J of I, let us denote:

$$D_J := \bigcap_{i \in J} D_i,$$

it is a smooth subscheme of codimension $|J|$.

We adopt the notation of [**Del70**, II, 3.4]. Namely, we denote, for every integer $r \geq 1$, D^r the subscheme of codimension r of Y defined by the union of all the intersections of r different components (D_i):

$$D^r := \bigcup_{J \subset I, |J| = r} D_J.$$

Let us choose a total order $\preceq$ on I.

For every element i in I, let us define an open subset $\mathring{D}_i$ of the divisor D_i by:

$$\mathring{D}_i := D_i - D_i \cap D^2.$$

Similarly, for every pair (i,j) in I^2 such that $i \prec j$, let us define an open subset $\mathring{D}_{ij}$ of the subscheme

$$D_{ij} := D_{\{i,j\}}$$

by:

$$\mathring{D}_{ij} := D_{ij} - D_{ij} \cap D^3.$$

Let χ_{top} denote the topological Euler characteristic.

First recall the following classical theorem:

THEOREM 4.2.2. *Let X be a proper smooth n-dimensional complex scheme and E be a reduced divisor with strict normal crossings in X. We have the equality of integers:*

$$\chi_{\mathrm{top}}(X-E) := \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X-E,\mathbb{C}) = \int_X c_n(\Omega_X^{1\vee}(\log E)).$$

PROOF. We can give a proof of this theorem using the results of this paper. Let us adopt the notation of Proposition 3.1.1, we have the equality of integers:

$$\begin{aligned}\int_X c_n(\Omega_X^{1\vee}(\log E)) &= (-1)^n \int_X c_n(\Omega_X^1(\log E)),\\ &= (-1)^n \sum_{J\subset I} \int_{E_J} c_{n-|J|}(\Omega^1_{E_J}) \text{ using equality (3.1.2) from Proposition 3.1.1,}\\ &= \sum_{J\subset I} (-1)^{|J|} \int_{E_J} c_{n-|J|}(T_{E_J}),\\ &= \sum_{J\subset I} (-1)^{|J|} \chi_{\mathrm{top}}(E_J),\\ &= \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(E) \text{ using the inclusion-exclusion formula,}\\ &= \chi_{\mathrm{top}}(X-E).\end{aligned}$$

□

THEOREM 4.2.3. *With the notation of this subsection, we have the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:*

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = \frac{1}{12} g_*(c_1 c_{N-1})(\omega^{1\vee}_{Y/C}) + \sum_{x\in\Delta} \alpha_x [x], \tag{4.2.1}$$

where for every x in Δ, α_x is the rational number given by:

(4.2.2)
$$\alpha_x = \frac{N-1}{4} \sum_{i\in I_x} (m_i - 1)\chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12} \sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} (3 - m_i/m_j - m_j/m_i)\chi_{\mathrm{top}}(\mathring{D}_{ij}),$$

(4.2.3)
$$= \frac{1}{12} \sum_{i\in I_x} \left[3(N-1)(m_i-1)\chi_{\mathrm{top}}(\mathring{D}_i) + \int_{D_i} c_1(\mathcal{N}_i) c_{N-2}(\Omega^{1\vee}_{D_i}(\log D_i \cap D^2))\right] + \frac{1}{4} \sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} \chi_{\mathrm{top}}(\mathring{D}_{ij}).$$

The expression (4.2.2) for α_x involves only the topology of the open strata $\mathring{D}_i$ and $\mathring{D}_{ij}$ of the singular fibers of g and the multiplicities m_i of their components D_i. The expression (4.2.3) makes clear that α_x belongs to $(1/12)\mathbb{Z}$.

PROOF. By Theorem 4.2.1 and by definition of the class ρ_{N-1}, the following equalities hold in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}
&\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta)))\\
&\quad = g_*\Big[\Big(\rho_{N-1}(\omega^{1\vee}_{Y/C}).\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(N)}\Big],\\
&\quad = g_*\Big[\Big((c_{N-2}(\omega^{1\vee}_{Y/C}) - \frac{N-1}{2}c_{N-1}(\omega^{1\vee}_{Y/C}) + \frac{1}{12}(c_1c_{N-1})(\omega^{1\vee}_{Y/C}))\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(N)}\Big],\\
&\quad = \frac{1}{12}g_*(c_1c_{N-1})(\omega^{1\vee}_{Y/C}) - \frac{N-1}{2}g_*\Big[c_{N-1}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(1)}\Big]\\
&\quad + g_*\Big[c_{N-2}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(2)}\Big]. \qquad (4.2.4)
\end{aligned}$$

Using successively equality (3.1.8) from Proposition 3.1.6 and Proposition 3.1.2, we get the following equality in $\mathrm{CH}^N(Y)_{\mathbb{Q}}$:

$$\begin{aligned}
c_{N-1}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(1)} &= -\frac{1}{2}\sum_{i\in I}(m_i-1)i_{D_i*}c_{N-1}(\omega^{1\vee}_{Y/C|D_i}),\\
&= -\frac{1}{2}\sum_{i\in I}(m_i-1)i_{D_i*}c_{N-1}(\Omega^{1\vee}_{D_i}(\log D_i\cap D^2))
\end{aligned}$$

hence pushing forward by g, the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}
g_*\Big[c_{N-1}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(1)}\Big] &= -\frac{1}{2}\sum_{i\in I}(m_i-1)g_*i_{D_i*}c_{N-1}(\Omega^{1\vee}_{D_i}(\log D_i\cap D^2)),\\
&= \sum_{x\in\Delta}\alpha_{1,x}[x],
\end{aligned}$$

where for every point x in Δ, $\alpha_{1,x}$ is the rational number given by:

$$\begin{aligned}
\alpha_{1,x} &= -\frac{1}{2}\sum_{i\in I_x}(m_i-1)\int_{D_i} c_{N-1}(\Omega^{1\vee}_{D_i}(\log D_i\cap D^2)),\\
&= -\frac{1}{2}\sum_{i\in I_x}(m_i-1)\,\chi_{\mathrm{top}}(\mathring{D}_i) \qquad \text{using Theorem 4.2.2.}
\end{aligned}$$

Similarly, using equality (3.1.10) from Proposition 3.1.6 and Proposition 3.1.2, the following equalities hold in $\mathrm{CH}^N(Y)_{\mathbb{Q}}$:

$$\begin{aligned}
c_{N-2}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(2)} &= \frac{1}{12}\sum_{\substack{(i,j)\in I^2,\\ i\prec j}}(3-m_i/m_j-m_j/m_i)i_{D_{ij}*}c_{N-2}(\omega^{1\vee}_{Y/C|D_{ij}}),\\
&= \frac{1}{12}\sum_{\substack{(i,j)\in I^2,\\ i\prec j}}(3-m_i/m_j-m_j/m_i)i_{D_{ij}*}c_{N-2}(\Omega^{1\vee}_{D_{ij}}(\log D_{ij}\cap D^3)),
\end{aligned}$$

hence pushing forward by g, the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$g_*\Big[c_{N-2}(\omega^{1\vee}_{Y/C})\Big(\frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}\Big)^{(2)}\Big] = \sum_{x\in\Delta}\alpha_{2,x}[x],$$

where, for every point x in Δ, $\alpha_{2,x}$ is the rational number given by:

$$\begin{aligned}\alpha_{2,x} &= \frac{1}{12} \sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} (3 - m_i/m_j - m_j/m_i) \int_{D_{ij}} c_{N-2}(\Omega^{1\vee}_{D_{ij}}(\log D_{ij} \cap D^3)),\\ &= \frac{1}{12} \sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} (3 - m_i/m_j - m_j/m_i) \chi_{\mathrm{top}}(\mathring{D}_{ij}) \text{ using again Theorem 4.2.2.}\end{aligned}$$

Hence replacing in (4.2.4), we obtain the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(Y_\eta/C_\eta))) = \frac{1}{12} g_*(c_1 c_{N-1})(\omega^{1\vee}_{Y/C}) + \sum_{x\in\Delta} \alpha_x[x],$$

where for every point x in Δ, the rational number α_x is given by:

$$\begin{aligned}\alpha_x &= -\frac{N-1}{2}\alpha_{1,x} + \alpha_{2,x},\\ &= \frac{N-1}{4} \sum_{i\in I_x} (m_i - 1)\chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12} \sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} (3 - m_i/m_j - m_j/m_i)\chi_{\mathrm{top}}(\mathring{D}_{ij}).\end{aligned}$$

For the other expression of the rational number α_x, one reasons similarly using the expression from (3.1.9) from Proposition 3.1.6 instead of (3.1.10). □

CHAPTER 5

The alternating product of Griffiths line bundles associated to a proper fibration with non-degenerate critical points

The objective of this chapter is to prove the following theorem.

THEOREM 5.0.1. *Let C be a connected smooth projective complex curve with generic point η, H be a smooth projective N-dimensional complex scheme, and let*

$$f : H \longrightarrow C$$

be a morphism of complex schemes. Let us assume that there exists a finite subset Σ in H such that f is smooth on $H - \Sigma$ and admits a non-degenerate critical point at any point of Σ. [1]

Let $[\Sigma]$ be the reduced 0-cycle in H associated with the finite subset Σ.

We have the equalities in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1\big(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta))\big) = \frac{1}{12} f_*\big((c_1 c_{N-1})([\Omega^1_{H/C}]^\vee)\big) + u_N^- f_*[\Sigma], \tag{5.0.1}$$

and

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1\big(\mathcal{GK}_{C,+}(\mathbb{H}^n(H_\eta/C_\eta))\big) = \frac{1}{12} f_*\big((c_1 c_{N-1})([\Omega^1_{H/C}]^\vee)\big) + u_N^+ f_*[\Sigma], \tag{5.0.2}$$

where u_N^- and u_N^+ are the rational numbers defined by:

$$u_N^- := \begin{cases} (5N-3)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even,} \end{cases}$$

and:

$$u_N^+ := \begin{cases} -(7N-9)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even.} \end{cases}$$

Observe that, as the morphism of non-singular complex schemes $f : H \to C$ is smooth on an open dense subset of H, we have the following equality in $K^0(H) \simeq K_0(H)$:

$$[T_f] = [\Omega^1_{H/C}]^\vee,$$

where $.^\vee$ denotes the duality involution on $K^0(H)$, by the same reasoning as in Subsection 3.1.4.

5.1. Reduction of the proof of Theorem 5.0.1 to a computation in $\mathrm{CH}_0(H)$

As in Theorem 5.0.1, let C be a connected smooth projective complex curve, let H be a smooth projective complex scheme of pure dimension $N \geq 1$, and let

$$f : H \longrightarrow C$$

1. Namely the differential of f vanishes and its Hessian is non-degenerate at any point of Σ; in other words, the only singularities of the fibers of f are ordinary double points.

be a proper surjective morphism with a finite subset Σ of critical points, all of which are non-degenerate.

Let $\Delta := f(\Sigma)$ be the set-theoretic image of Σ, that we see as a reduced divisor in C. We can rephrase the hypothesis as the divisor in H:

$$H_\Delta := f^*\Delta$$

having only finitely many singularities, all of which are ordinary double points.

Let

$$\nu : \tilde{H} \longrightarrow H$$

be the blow-up of H at Σ, and let E be the exceptional divisor. Let

$$g := f \circ \nu : \tilde{H} \longrightarrow C$$

be the composition. Over the generic point η of C, this morphism can be identified with the morphism f.

For every point P in Σ, let us define a subscheme in $\tilde{H}$ by:

$$E_P := \nu^{-1}(\{P\}),$$

so that the $(E_P)_{P\in\Sigma}$ are the connected components of the exceptional divisor E. In particular, they are disjoint divisors, isomorphic to the complex projective space $\mathbb{P}^{N-1}$; see for instance Proposition 5.2.1 below.

Let us define a divisor in $\tilde{H}$ by:

$$\tilde{H}_\Delta := g^*\Delta = \nu^* H_\Delta.$$

Since the divisor H_Δ has its only singularities at points of Σ, and since all of these singularities are ordinary double points, its pullback divisor $\tilde{H}_\Delta$ can be written as follows:

$$\tilde{H}_\Delta = 2\sum_{P\in\Sigma} E_P + W \tag{5.1.1}$$

where W is the proper transform in $\tilde{H}$ of the divisor H_Δ in H: it is a non-singular divisor intersecting transversally the components $(E_P)_{P\in\Sigma}$ of the exceptional divisor, and for every point P, the intersection $E_P \cap W$ is a smooth quadric in the projective space E_P.

In particular, the divisor $\tilde{H}_\Delta$ is a divisor with strict normal crossings, which allows us to define the relative logarithmic bundle $\omega^1_{\tilde{H}/C}$.

We are going to apply Theorem 4.2.3 to the morphism g and to the variations of Hodge structures of relative cohomology:

$$\mathbb{H}^n(H_\eta/C_\eta) \xrightarrow{\sim} \mathbb{H}^n(\widetilde{H}_\eta/C_\eta).$$

For this, we are going to prove the following lemma using the results of Chapter 3.

Lemma 5.1.1. *With the above notation, the following formula holds in* $\mathrm{CH}^N(H)$*:*

$$\nu_*\big((c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C})\big) = (c_1c_{N-1})([T_f]) - (N-2)\frac{1-(-1)^N}{2}[\Sigma]. \tag{5.1.2}$$

Let us prove Theorem 5.0.1 using this lemma.

For every point P in Σ, the complex scheme E_P is isomorphic to the projective space $\mathbb{P}^{N-1}$, hence its topological Euler characteristic is given by:

$$\chi_{\mathrm{top}}(E_P) = N.$$

Furthermore, the intersection $E_P \cap W$ is isomorphic to a smooth quadric in $\mathbb{P}^{N-1}$, hence its topological Euler characteristic is given by:

$$\chi_{\mathrm{top}}(E_P \cap W) = \frac{1}{2}[(-1)^N + 2N - 1].$$

This follows for instance from equalities (5.2.5) and (5.2.3), from Proposition 5.2.1 below, applied to the integers $m_P := 2$ and $r := N-2$ and the hypersurface $Q_P := E_P \cap W$, and from equality (5.3.1) in Proposition 5.3.3 below, applied to the integers $n := N$ and $r := 2$ and the complex number $a := 2$.

By additivity of the topological Euler characteristic, the characteristic of the complement $E_P - E_P \cap W$ is given by:

$$\chi_{\mathrm{top}}(E_P - E_P \cap W) = N - \frac{1}{2}[(-1)^N + 2N - 1] = \frac{1}{2}[1 - (-1)^N].$$

Consequently, for every point x in Δ, the expression (4.2.2) for the rational number α_x introduced in Theorem 4.2.3 reduces to:

$$\begin{aligned}\alpha_x &= \frac{N-1}{4}\sum_{P\in\Sigma_x}\chi_{\mathrm{top}}(E_P - E_P\cap W) + \frac{1}{12}\sum_{P\in\Sigma_x}\frac{1}{2}\chi_{\mathrm{top}}(E_P\cap W),\\ &= \frac{N-1}{8}[1-(-1)^N]|\Sigma_x| + \frac{1}{48}[(-1)^N + 2N - 1]|\Sigma_x|,\\ &= \frac{1}{48}[6(N-1)(1-(-1)^N) + (-1)^N + 2N - 1]|\Sigma_x|,\\ &= \frac{1}{48}[(6N-7)(1-(-1)^N) + 2N]|\Sigma_x|.\end{aligned}$$

Hence replacing in Theorem 4.2.3 and using Lemma 5.1.1, we have the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}\sum_{n=0}^{2(N-1)}(-1)^{n-1}c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta))) &= \frac{1}{12}g_*(c_1c_{N-1})(\omega^{1\vee}_{Y/C}) + \sum_{x\in\Delta}\alpha_x[x],\\ &= \frac{1}{12}f_*(c_1c_{N-1})([T_f]) - \frac{1}{12}(N-2)\frac{1-(-1)^N}{2}f_*[\Sigma]\\ &+ \frac{1}{48}[(6N-7)(1-(-1)^N)+2N]\sum_{x\in\Delta}|\Sigma_x|[x],\\ &= \frac{1}{12}f_*(c_1c_{N-1})([T_f]) + u_N^- f_*[\Sigma],\end{aligned}$$

where u_N^- is the rational number given by:

$$\begin{aligned}u_N^- &= -\frac{1}{24}(N-2)(1-(-1)^N) + \frac{1}{48}[(6N-7)(1-(-1)^N)+2N],\\ &= \frac{1}{48}[-2(N-2)(1-(-1)^N) + (6N-7)(1-(-1)^N) + 2N],\\ &= \frac{1}{48}[(4N-3)(1-(-1)^N)+2N],\\ &= \begin{cases}(5N-3)/24 & \text{if } N \text{ is odd}\\ N/24 & \text{if } N \text{ is even,}\end{cases}\end{aligned}$$

which shows equality (5.0.1).

Furthermore, for every integer n, equality (2.4.2) from Corollary 2.4.2 gives the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$c_1(\mathcal{GK}_{C,+}(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta)) + \delta^{n,N-1}\eta_N\frac{N-1}{2}f_*[\Sigma],$$

where η_N is 1 if N is odd and 0 if N is even.

Taking the alternating sum over n yields the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,+}(\mathbb{H}^n(H_\eta/C_\eta))) &= \sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta))) \\ &+ (-1)^N \eta_N \frac{N-1}{2} f_*[\Sigma], \\ &= \frac{1}{12} f_*(c_1 c_{N-1})([T_f]) + u_N^- f_*[\Sigma] - \eta_N \frac{N-1}{2} f_*[\Sigma], \\ &= \frac{1}{12} f_*(c_1 c_{N-1})([T_f]) + u_N^+ f_*[\Sigma],\end{aligned}$$

where u_N^+ is the rational number given by:

$$u_N^+ = u_N^- - \eta_N \frac{N-1}{2} = \begin{cases} -(7N-9)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even,} \end{cases}$$

which shows equality (5.0.2) and completes the proof of Theorem 5.0.1.

The rest of this chapter shall be devoted to proving Lemma 5.1.1.

5.2. Blowing up points in smooth schemes and Chern classes

We adopt the following notation. Let k be a field, H be a smooth k-scheme of pure dimension $N \geq 1$, Σ be a finite subset of H, let

$$\nu : \tilde{H} \longrightarrow H$$

be the blow-up of H at Σ, and let E be the exceptional divisor.

For every point P in Σ, let us define a subscheme in $\tilde{H}$ by:

$$E_P := \nu^{-1}(\{P\}),$$

so that the $(E_P)_{P\in\Sigma}$ are the connected components of the exceptional divisor E. In particular, they are disjoint divisors.

For every point P in Σ, let us denote the class of the divisor E_P by:

$$\eta_P := [E_P] = c_1(\mathcal{O}_{\tilde{H}}(E_P)) \in \mathrm{CH}^1(\tilde{H}).$$

Observe that for every point P in Σ, the restriction to P of the vector bundle T_H can be trivialized, so we obtain, for every cycle α in H of positive codimension, the equality in $\mathrm{CH}^*(\tilde{H})$:

$$\eta_P \cdot \nu^* \alpha = 0. \tag{5.2.1}$$

Furthermore, for every couple (P, P') in Σ^2 such that $P \neq P'$, the divisors E_P and $E_{P'}$ in $\tilde{H}$ are disjoint, so we obtain the equality in $\mathrm{CH}^2(\tilde{H})$:

$$\eta_P \cdot \eta_{P'} = 0. \tag{5.2.2}$$

Proposition 5.2.1. *Let P be a point in Σ. The scheme E_P is isomorphic to a projective space of dimension $N-1$.*

We have the equality in $\mathrm{CH}^N(H)$:

$$\nu_* \eta_P^N = (-1)^{N-1}[P]. \tag{5.2.3}$$

Moreover, for every non-negative integer r, we have the equality in $\mathrm{CH}^r(E_P)$:

$$c_r(T_{E_P}) = (-1)^r \binom{N}{r} \eta^r_{P|E_P}. \tag{5.2.4}$$

Finally, let m_P be a positive integer, and let Q_P be a smooth hypersurface of degree m_P in E_P and let $i_{Q_P} : Q_P \to E_P$ be the inclusion morphism.

For every non-negative integer r, we have the equality in $\mathrm{CH}^{r+1}(E_P)$:

$$i_{Q_P *} c_r(T_{Q_P}) = (-1)^{r+1} m_P \Big[\frac{(1+y)^N}{1+m_P y} \Big]^{[r]} \eta_{P|E_P}^{r+1}, \tag{5.2.5}$$

where $f(y)^{[r]}$ denotes the coefficient of y^r in some formal series $f(y) \in \mathbb{C}[[y]]$.

Proof. Since E_P is a connected component of the exceptional divisor E, there is a canonical isomorphism of k-schemes:

$$\varphi : E_P \longrightarrow \mathbb{P}_k(T_P H),$$

and a canonical isomorphism of line bundles on E_P:

$$\varphi^* \mathcal{O}_{T_P H}(-1) \simeq \mathcal{O}_{\tilde{H}}(E_P)_{|E_P}.$$

Consequently, we have the equality in $\mathrm{CH}^1(E_P)$:

$$\eta_{P|E_P} = \varphi^* c_1(\mathcal{O}_{T_P H}(-1)). \tag{5.2.6}$$

Now, let us compute in $\mathrm{CH}^N(\tilde{H})$:

$$\begin{aligned} \eta_P^N &= \eta_P^{N-1}[E_P], \\ &= i_{E_P *} \eta_{P|E_P}^{N-1} \text{ by the projection formula}, \\ &= (-1)^{N-1} i_{E_P *} \varphi^* c_1(\mathcal{O}_{T_P H}(1))^{N-1}. \end{aligned}$$

Hence pushing forward, we have the equality in $\mathrm{CH}^N(H)$:

$$\begin{aligned} \nu_* \eta_P^N &= (-1)^{N-1} \nu_* i_{E_P *} \varphi^* c_1(\mathcal{O}_{T_P H}(1))^{N-1}, \\ &= (-1)^{N-1} \Big[\int_{\mathbb{P}(T_P H)} c_1(\mathcal{O}_{T_P H}(1))^{N-1} \Big] [P], \\ &= (-1)^{N-1} [P], \end{aligned}$$

which shows equality (5.2.3).

Let r be a non-negative integer. Since φ is an isomorphism, we have the equality in $\mathrm{CH}^r(E_P)$:

$$c_r(T_{E_P}) = \varphi^* c_r(T_{\mathbb{P}(T_P H)}),$$

using the exact sequence of vector bundles on $\mathbb{P}(T_P H)$ from [**Ful98**, B.5.8]:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(T_P H)} \longrightarrow T_P H \otimes_k \mathcal{O}_{T_P H}(1) \longrightarrow T_{\mathbb{P}(T_P H)} \longrightarrow 0,$$

we can replace:

$$c_r(T_{E_P}) = \varphi^* [(1 + c_1(\mathcal{O}_{T_P H}(1)))^N]^{(r)} = \binom{N}{r} \varphi^* c_1(\mathcal{O}_{T_P H}(1))^r = (-1)^r \binom{N}{r} \eta_{P|E_P}^r \text{ using (5.2.6)},$$

which shows equality (5.2.4).

Now, let m_P be a positive integer and Q_P be an hypersurface of degree m_P in E_P.

We have the following exact sequence of vector bundles on Q_P:

$$0 \longrightarrow T_{Q_P} \longrightarrow T_{E_P|Q_P} \longrightarrow \mathcal{N}_{Q_P} E_P \longrightarrow 0,$$

hence the following equality in $\mathrm{CH}^*(Q_P)$:

$$\begin{aligned} c(T_{Q_P}) &= \frac{c(T_{E_P|Q_P})}{c(\mathcal{N}_{Q_P}E_P)}, \\ &= \frac{\varphi^*_{|Q_P}(1+c_1(\mathcal{O}_{T_PH}(1)))^N}{1+c_1(\mathcal{O}_{E_P}(Q_P)_{|Q_P})}, \\ &= \frac{\varphi^*_{|Q_P}(1+c_1(\mathcal{O}_{T_PH}(1)))^N}{1+m_P\varphi^*_{|Q_P}c_1(\mathcal{O}_{T_PH}(1))} \text{ because } Q_P \text{ is of degree } m_P, \\ &= \varphi^*_{|Q_P}\Big[\frac{(1+c_1(\mathcal{O}_{T_PH}(1)))^N}{1+m_Pc_1(\mathcal{O}_{T_PH}(1))}\Big]. \end{aligned}$$

Hence, for every integer r, we have the equality in $\mathrm{CH}^r(Q_P)$:

$$c_r(T_{Q_P}) = \Big[\frac{(1+y)^N}{1+m_Py}\Big]^{[r]}\varphi^*_{|Q_P}c_1(\mathcal{O}_{T_PH}(1))^r = (-1)^r\Big[\frac{(1+y)^N}{1+m_Py}\Big]^{[r]}\eta^r_{P|Q_P} \text{ using (5.2.6).} \tag{5.2.7}$$

Pushing forward (5.2.7) by the inclusion map of Q_P in E_P, and using the projection formula and the equality in $\mathrm{CH}^1(E_P)$:

$$[Q_P] = \varphi^*\mathcal{O}_{T_PH}(m_P) = -m_P\eta_{P|E_P},$$

yields equality (5.2.5). □

Recall the following consequence of the "Grothendieck-Riemann-Roch without denominators" formula:

PROPOSITION 5.2.2 ([**Ful98**, Example 15.4.2, (c)]). *We have the equality in* $\mathrm{CH}^*(\tilde{H})$*:*

$$c(T_{\tilde{H}}) = \nu^*c(T_H) + \sum_{P\in\Sigma}[(1+\eta_P)(1-\eta_P)^N - 1]. \tag{5.2.8}$$

The following proposition is a reformulation of Proposition 5.2.2, and we leave its derivation from Proposition 5.2.2 to the reader.

PROPOSITION 5.2.3. *For every integer* $r \geq 1$*, we have the equality in* $\mathrm{CH}^r(\tilde{H})$*:*

$$c_r(T_{\tilde{H}}) = \nu^*c_r(T_H) + (-1)^r\left[\binom{N}{r} - \binom{N}{r-1}\right]\sum_{P\in\Sigma}\eta_P^r.$$

COROLLARY 5.2.4. *We have the equality in* $\mathrm{CH}^N(\tilde{H})$*:*

$$(c_1c_{N-1})(T_{\tilde{H}}) = \nu^*(c_1c_{N-1})(T_H) + (-1)^{N+1}N(N-1)(N-3)/2\sum_{P\in\Sigma}\eta_P^N \tag{5.2.9}$$

and the equality in $\mathrm{CH}^{N-1}(\tilde{H})$*:*

$$(c_{N-1}+c_1c_{N-2})(T_{\tilde{H}}) = \nu^*(c_{N-1}+c_1c_{N-2})(T_H) + a_N\sum_{P\in\Sigma}\eta_P^{N-1}, \tag{5.2.10}$$

where a_N *is the integer defined by:*

$$a_N := \begin{cases} 0 & \text{if } N=1, \\ (-1)^NN(N-3)/2 + (-1)^NN(N-1)^2(N-5)/6 & \text{if } N\geq 2. \end{cases}$$

PROOF. Applying Proposition 5.2.3 to the integer $r := 1$ yields the equality in $\mathrm{CH}^1(\tilde{H})$:

$$c_1(T_{\tilde{H}}) = \nu^*c_1(T_H) - \left[\binom{N}{1} - \binom{N}{0}\right]\sum_{P\in\Sigma}\eta_P = \nu^*c_1(T_H) - (N-1)\sum_{P\in\Sigma}\eta_P. \tag{5.2.11}$$

If the integer N is at least 3, applying the same proposition to $r := N-2$ yields the equality in $\mathrm{CH}^{N-2}(\tilde{H})$:

$$\begin{aligned} c_{N-2}(T_{\tilde{H}}) &= \nu^* c_{N-2}(T_H) + (-1)^{N-2}\left[\binom{N}{N-2} - \binom{N}{N-3}\right] \sum_{P\in\Sigma} \eta_P^{N-2}, \\ &= \nu^* c_{N-2}(T_H) + (-1)^N [N(N-1)/2 - N(N-1)(N-2)/6] \sum_{P\in\Sigma} \eta_P^{N-2}, \\ &= \nu^* c_{N-2}(T_H) + (-1)^{N+1} N(N-1)(N-5)/6 \sum_{P\in\Sigma} \eta_P^{N-2}. \end{aligned} \tag{5.2.12}$$

If the integer N is at least 2, applying the same proposition to $r := N-1$ yields the equality in $\mathrm{CH}^{N-1}(\tilde{H})$:

$$\begin{aligned} c_{N-1}(T_{\tilde{H}}) &= \nu^* c_{N-1}(T_H) + (-1)^{N-1}\left[\binom{N}{N-1} - \binom{N}{N-2}\right] \sum_{P\in\Sigma} \eta_P^{N-1}, \\ &= \nu^* c_{N-1}(T_H) - (-1)^N [N - N(N-1)/2] \sum_{P\in\Sigma} \eta_P^{N-1}, \\ &= \nu^* c_{N-1}(T_H) + (-1)^N N(N-3)/2 \sum_{P\in\Sigma} \eta_P^{N-1}. \end{aligned} \tag{5.2.13}$$

Let us show equality (5.2.9). If the integer N is 1, then it follows immediately from equality (5.2.11) (observe that the coefficient before $\sum_P \eta_P$ vanishes in that case).

Let us assume that the integer N is at least 2.

Multiplying (5.2.11) and (5.2.13) yields the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\begin{aligned} (c_1 c_{N-1})(T_{\tilde{H}}) &= (\nu^* c_1(T_H) - (N-1)\sum_{P\in\Sigma} \eta_P)(\nu^* c_{N-1}(T_H) + (-1)^N N(N-3)/2 \sum_{P\in\Sigma} \eta_P^{N-1}), \\ &= \nu^*(c_1 c_{N-1})(T_H) + (-1)^N N(N-3)/2 \sum_{P\in\Sigma} \eta_P^{N-1}\, \nu^* c_1(T_H) \\ &\quad - (N-1)\sum_{P\in\Sigma} \eta_P\, \nu^* c_{N-1}(T_H) + (-1)^{N+1} N(N-1)(N-3)/2 \sum_{P,P'\in\Sigma} \eta_P\, \eta_{P'}^{N-1}. \end{aligned} \tag{5.2.14}$$

Since we have assumed that N was at least 2, we can apply equality (5.2.1) to the cycle $\alpha := c_{N-1}(T_H)$ of positive codimension to obtain, for every point P in Σ, the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\eta_P\, \nu^* c_{N-1}(T_H) = 0.$$

Furthermore, we can apply this same equality to the cycle $\alpha := c_1(T_H)$ to obtain, for every point P in Σ, the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\eta_P^{N-1}\, \nu^* c_1(T_H) = \eta_P^{N-2}\big(\eta_P\, \nu^* c_1(T_H)\big) = 0.$$

Finally, applying equality (5.2.2), we obtain the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\sum_{P,P'\in\Sigma} \eta_P\, \eta_{P'}^{N-1} = \sum_{P\in\Sigma} \eta_P^N.$$

Hence replacing in (5.2.14):

$$(c_1 c_{N-1})(T_{\tilde{H}}) = \nu^*(c_1 c_{N-1})(T_H) + (-1)^{N+1} N(N-1)(N-3)/2 \sum_{P\in\Sigma} \eta_P^N,$$

which shows equality (5.2.9) if the integer N is at least 2. Consequently, this equality holds for every positive integer N.

Now let us show equality (5.2.10). If the integer N is 1, it is trivial. If the integer N is 2, the coefficient a_2 is equal to -2, so it follows from multiplying equality (5.2.11) by 2.

Let us assume that the integer N is at least 3.

Multiplying (5.2.11) and (5.2.12), we obtain the equality in $\mathrm{CH}^{N-1}(\tilde{H})$:

$$(c_1c_{N-2})(T_{\tilde{H}}) = (\nu^*c_1(T_H) - (N-1)\sum_{P\in\Sigma}\eta_P)(\nu^*c_{N-2}(T_H) + (-1)^{N+1}\frac{N(N-1)(N-5)}{6}\sum_{P\in\Sigma}\eta_P^{N-2});$$

developing and using equality (5.2.2), we get:

$$\begin{aligned}(c_1c_{N-2})(T_{\tilde{H}}) &= \nu^*(c_1c_{N-2})(T_H) + (-1)^{N+1}N(N-1)(N-5)/6\sum_{P\in\Sigma}\eta_P^{N-2}\nu^*c_1(T_H)\\ &- (N-1)\sum_{P\in\Sigma}\eta_P\nu^*c_{N-2}(T_H) + (-1)^N N(N-1)^2(N-5)/6\sum_{P\in\Sigma}\eta_P^{N-1}.\end{aligned} \tag{5.2.15}$$

Since we have assumed that the integer N was at least 3, then we can apply equality (5.2.1) to the cycle $\alpha := c_{N-2}(T_H)$ of positive dimension to obtain, for every point P in Σ, the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\eta_P\,\nu^*c_{N-2}(T_H) = 0.$$

Furthermore, we can apply this same equality to the cycle $\alpha := c_1(T_H)$ to obtain, for every point P in Σ, the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\eta_P^{N-2}\,\nu^*c_1(T_H) = \eta_P^{N-3}\big(\eta_P\,\nu^*c_1(T_H)\big) = 0.$$

Hence replacing in (5.2.15), we obtain:

$$(c_1c_{N-2})(T_{\tilde{H}}) = \nu^*(c_1c_{N-2})(T_H) + (-1)^N N(N-1)^2(N-5)/6\sum_{P\in\Sigma}\eta_P^{N-1},$$

hence, adding equality (5.2.13), we obtain:

$$\begin{aligned}(c_{N-1} + c_1c_{N-2})(T_{\tilde{H}}) &= \nu^*(c_{N-1} + c_1c_{N-2})(T_H)\\ &+ [(-1)^N N(N-3)/2 + (-1)^N N(N-1)^2(N-5)/6]\sum_{P\in\Sigma}\eta_P^{N-1},\end{aligned}$$

which shows equality (5.2.10) if the integer N is at least 3. Consequently, this equality holds for every positive integer N. □

5.3. The cycle classes $\nu_*(c_1c_{N-1})([T_g])$ and $\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C})$ in terms of $(c_1c_{N-1})([T_f])$

In order to prove Lemma 5.1.1, we want to compare the cycle classes $\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C})$ and $(c_1c_{N-1})([T_f])$ on H.

Using Corollary 5.2.4, we can compare the cycle classes $(c_1c_{N-1})(T_{\tilde{H}})$ and $\nu^*(c_1c_{N-1})(T_H)$ on $\tilde{H}$. Then in Subsection 5.3.1, we shall combine Corollary 5.2.4 and the definition of the relative tangent classes to compare the cycle classes $\nu_*(c_1c_{N-1})([T_g])$ and $(c_1c_{N-1})([T_f])$ on H: this will be the object of Proposition 5.3.2.

In Subsection 5.3.2, we shall apply equality (3.1.12) from Proposition 3.1.7 to compare the cycle classes $(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C})$ and $(c_1c_{N-1})([T_g])$ on $\tilde{H}$ (this will be the object of Proposition 5.3.5), then we shall push forward by ν and apply Proposition 5.3.2 to prove Lemma 5.1.1. This will complete the proof of Theorem 5.0.1.

5.3.1. Computation of $\nu_*(c_1c_{N-1})([T_g])$.

PROPOSITION 5.3.1. *Let k be a field, V be a smooth projective k-scheme of pure dimension N, C be a smooth projective k-curve, and*

$$h : V \longrightarrow C$$

be a flat morphism.

Recall that the relative tangent class in K-theory of the morphism h is defined by:

$$[T_h] := [T_V] - h^*[T_C] \in K^0(V).$$

We have the equality in $\mathrm{CH}^N(V)$:

$$(c_1c_{N-1})([T_h]) = (c_1c_{N-1})(T_V) - (c_{N-1} + c_1c_{N-2})(T_V)\, h^*c_1(T_C).$$

PROOF. By definition of the class $[T_h]$, we have the equality in $\mathrm{CH}^*(V)$:

$$\begin{aligned} c([T_h]) &= c(T_V)\, h^*c(T_C)^{-1}, \\ &= c(T_V)\, h^*(1 + c_1(T_C))^{-1}, \end{aligned}$$

since C is a curve, the class $c_1(T_C)^2$ vanishes, so that:

$$c([T_h]) = c(T_V)h^*(1 - c_1(T_C)).$$

Consequently, we have the equality in $\mathrm{CH}^1(V)$:

$$c_1([T_h]) = c_1(T_V) - h^*c_1(T_C),$$

and the equality in $\mathrm{CH}^{N-1}(V)$:

$$c_{N-1}([T_h]) = c_{N-1}(T_V) - c_{N-2}(T_V)\, h^*c_1(T_C).$$

Hence multiplying, we have the equality in $\mathrm{CH}^N(V)$:

$$(c_1c_{N-1})([T_h]) = (c_1(T_V) - h^*c_1(T_C))(c_{N-1}(T_V) - c_{N-2}(T_V)\, h^*c_1(T_C)),$$

and developing, and using that the class $c_1(T_C)^2$ vanishes:

$$(c_1c_{N-1})([T_h]) = (c_1c_{N-1})(T_V) - (c_{N-1} + c_1c_{N-2})(T_V)\, h^*c_1(T_C),$$

as wanted. □

Now let us show the main proposition of this subsection.

We shall use the notation of Section 5.2, in a relative setting, over a smooth projective k-curve C. Namely, let H be a smooth k-scheme of pure dimension $N \geq 1$, let:

$$f : H \longrightarrow C$$

be a flat morphism, let Σ be a finite subset of H, let:

$$\nu : \tilde{H} \longrightarrow H$$

be the blow-up of H at Σ, and let E be the exceptional divisor. Let:

$$g := f \circ \nu : \tilde{H} \longrightarrow C$$

be the composition. Let $(E_P)_{P \in \Sigma}$ be the connected components of the exceptional divisor E.

For every point P in Σ, let us denote the class of the divisor E_P by:

$$\eta_P := [E_P] = c_1(\mathcal{O}_{\tilde{H}}(E_P)) \in \mathrm{CH}^1(\tilde{H}).$$

PROPOSITION 5.3.2. *With the above notation, we have the equality in $\mathrm{CH}^N(H)$:*

$$\nu_*(c_1c_{N-1})([T_g]) = (c_1c_{N-1})([T_f]) + N(N-1)(N-3)/2[\Sigma].$$

Proof. Let us compute the class $(c_1c_{N-1})([T_g])$ in $\mathrm{CH}^N(\tilde{H})$. Applying Proposition 5.3.1 to the scheme $V := \tilde{H}$ and the morphism $h := g$ yields the equality:

$$(c_1c_{N-1})([T_g]) = (c_1c_{N-1})(T_{\tilde{H}}) - (c_{N-1} + c_1c_{N-2})(T_{\tilde{H}})g^*c_1(T_C),$$

using (5.2.9) and (5.2.10):

$$= \nu^*(c_1c_{N-1})(T_H) + (-1)^{N+1}N(N-1)(N-3)/2 \sum_{P\in\Sigma} \eta_P^N$$

$$- [\nu^*(c_{N-1} + c_1c_{N-2})(T_H) + a_N \sum_{P\in\Sigma} \eta_P^{N-1}]g^*c_1(T_C),$$

applying equality (5.2.1) to the cycle $\alpha := f^*c_1(T_C)$, we obtain that for every point P in Σ, the class $\eta_P\, g^*c_1(T_C)$ vanishes, hence replacing:

$$= \nu^*((c_1c_{N-1})(T_H)) + (-1)^{N+1}N(N-1)(N-3)/2 \sum_{P\in\Sigma} \eta_P^N - \nu^*((c_{N-1} + c_1c_{N-2})(T_H))\, g^*c_1(T_C),$$

$$= \nu^*[(c_1c_{N-1})(T_H)) - (c_{N-1} + c_1c_{N-2})(T_H)f^*c_1(T_C)] + (-1)^{N+1}N(N-1)(N-3)/2 \sum_{P\in\Sigma} \eta_P^N,$$

applying Proposition 5.3.1 to the scheme $V := H$ and the morphism $h := f$:

$$= \nu^*((c_1c_{N-1}([T_f])) + (-1)^{N+1}N(N-1)(N-3)/2 \sum_{P\in\Sigma} \eta_P^N.$$

Hence pushing forward and applying equality (5.2.3) from Proposition 5.2.1, we obtain the equality in $\mathrm{CH}^N(H)$:

$$\nu_*((c_1c_{N-1})([T_g])) = \nu_*\nu^*((c_1c_{N-1})([T_f])) + N(N-1)(N-3)/2 \sum_{P\in\Sigma} [P];$$

using the projection formula and the fact that the morphism ν is of generic degree 1, we obtain:

$$\nu_*\big((c_1c_{N-1})([T_g])\big) = (c_1c_{N-1})([T_f]) + N(N-1)(N-3)/2\, [\Sigma].$$

This concludes the proof. □

5.3.2. Computation of $\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C})$. We begin with a purely combinatorial result.

If y denotes an indeterminate and $f(y)$ a formal series in $\mathbb{C}[[y]]$, for every integer p, we shall denote by $f(y)^{[p]}$ its coefficient of degree p. It vanishes if p is negative.

Proposition 5.3.3. *For every a in $\mathbb{C}^*$, and every n and r in $\mathbb{N}$, we have the formulas:*

(5.3.1)
$$\left[\frac{(1+y)^n}{1+ay}\right]^{[n-r]} = (-1)^r \sum_{r\le k\le n} \binom{n}{k}(-1)^k a^{k-r} = \frac{(-1)^{n+r}}{a^r}\Big[(a-1)^n - \sum_{0\le k\le r-1} \binom{n}{k}(-1)^{n-k}a^k\Big],$$

and:

(5.3.2)
$$\left[\frac{(1+y)^{n+1}}{(1+ay)^2}\right]^{[n-r]} = (-1)^{r+1} \sum_{r+1\le k\le n+1} (k-r)\binom{n+1}{k}(-1)^k a^{k-r-1}$$

(5.3.3)
$$= \frac{(-1)^{n+r}}{a^{r+1}}\Big[(r+(n+1-r)a)(a-1)^n - \sum_{0\le k\le r} (k-r)\binom{n+1}{k}(-1)^{n+1-k}a^k\Big].$$

PROOF. The equality between the left-hand side and the middle side in (5.3.1) follows from the computation:

$$\begin{aligned}\left[\frac{(1+y)^n}{1+ay}\right]^{[n-r]} &= \sum_{0\leq i\leq n-r} [(1+y)^n]^{[i]}\left[\frac{1}{1+ay}\right]^{[n-r-i]}\\ &= \sum_{0\leq i\leq n-r}\binom{n}{i}(-a)^{n-r-i}\\ &= \sum_{r\leq k\leq n}\binom{n}{n-k}(-a)^{k-r} \qquad (5.3.4)\\ &= (-1)^r\sum_{r\leq k\leq n}\binom{n}{k}(-1)^k a^{k-r},\end{aligned}$$

where in (5.3.4) we have introduced $k := n-i$.

The equality between the middle side and the right-hand side in (5.3.1) follows from the binomial formula applied to $(a-1)^n$.

Equality (5.3.2) follows from the equality of formal series in $\mathbb{C}[[a,y]]$:

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[\frac{(1+y)^{n+1}}{1+ay}\right] = -y\frac{(1+y)^{n+1}}{(1+ay)^2},$$

which implies the equality of complex numbers:

$$\begin{aligned}\left[\frac{(1+y)^{n+1}}{(1+ay)^2}\right]^{[n-r]} &= -\left[y^{-1}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{(1+y)^{n+1}}{1+ay}\right)\right]^{[n-r]}\\ &= -\frac{\mathrm{d}}{\mathrm{d}a}\left[\frac{(1+y)^{n+1}}{1+ay}\right]^{[n+1-r]}\\ &= -\frac{\mathrm{d}}{\mathrm{d}a}\Big[(-1)^r\sum_{r\leq k\leq n+1}\binom{n+1}{k}(-1)^k a^{k-r}\Big] \qquad (5.3.5)\\ &= (-1)^{r+1}\sum_{r+1\leq k\leq n+1}(k-r)\binom{n+1}{k}(-1)^k a^{k-r-1},\end{aligned}$$

where in (5.3.5), we have used (5.3.1) applied to $n' := n+1$.

Equality (5.3.3) follows from the following development:

$$\begin{aligned}(r+(n+1-r)a)(a-1)^n &= (n+1)a(a-1)^n - r(a-1)^{n+1}\\ &= (n+1)\sum_{1\leq k\leq n+1}\binom{n}{k-1}(-1)^{n-k+1}a^k\\ &\quad - r\sum_{0\leq k\leq n+1}\binom{n+1}{k}(-1)^{n+1-k}a^k\\ &= \sum_{0\leq k\leq n+1}\left((n+1)\binom{n}{k-1} - r\binom{n+1}{k}\right)(-1)^{n+1-k}a^k\\ &= \sum_{0\leq k\leq n+1}(k-r)\binom{n+1}{k}(-1)^{n+1-k}a^k, \qquad (5.3.6)\end{aligned}$$

where in (5.3.6), we have used the classical identity, for every integer k:

$$(n+1)\binom{n}{k-1} = k\binom{n+1}{k}. \quad \square$$

Let us adopt the notation of Section 5.1.

For every point P in Σ, let us denote the class of the divisor E_P by:

$$\eta_P := [E_P] = c_1(\mathcal{O}_{\tilde{H}}(E_P)) \in \mathrm{CH}^1(\tilde{H}).$$

Observe that equalities (5.2.1) and (5.2.2) still hold in this setting.

Proposition 5.3.4. *With the above notation, for every non-negative integer r, we have the equality in $\mathrm{CH}^r(\tilde{H})$:*

$$c_r(\omega^{1\vee}_{\tilde{H}/C}) = c_r([T_g]) + \alpha(N,r) \sum_{P\in\Sigma} \eta_P^r,$$

where $\alpha(N,r)$ is the integer given by:

$$\alpha(N,r) := (-1)^{r-1}\left(\binom{N}{r-1} - 4\Big[\frac{(1+y)^N}{1+2y}\Big]^{[r-2]}\right).$$

Observe that the integer $\alpha(N,1)$ is simply 1.

Proof. Let us apply equality (3.1.12) from Proposition 3.1.7 to the scheme

$$Y := \tilde{H},$$

and the morphism g, whose divisor of singular fibers $\tilde{H}_\Delta$ is of the form given by (5.1.1). Observe that with the notation of that proposition, the subscheme D^3 is empty.

We obtain, for every integer r, the equality in $\mathrm{CH}^r(\tilde{H})$:

$$c_r(\omega^{1\vee}_{\tilde{H}/C}) = c_r([T_g]) + \sum_{P\in\Sigma} i_{E_P*}c_{r-1}(T_{E_P}) - 2\sum_{P\in\Sigma} i_{E_P\cap W*}c_{r-2}(T_{E_P\cap W}). \tag{5.3.7}$$

Using (5.2.4) from Proposition 5.2.1, we have the equality in $\mathrm{CH}^{r-1}(E_P)$:

$$c_{r-1}(T_{E_P}) = (-1)^{r-1}\binom{N}{r-1}\eta^{r-1}_{P|E_P},$$

hence pushing forward by the inclusion in $\tilde{H}$ and using the projection formula, we have the equality in $\mathrm{CH}^r(\tilde{H})$:

$$i_{E_P*}c_{r-1}(T_{E_P}) = (-1)^{r-1}\binom{N}{r-1}\eta_P^r.$$

On the other hand, the intersection $Q_P := E_P \cap W$ is a smooth hypersurface of degree $m_P := 2$ in the projective space E_P, hence using equality (5.2.5) from Proposition 5.2.1, we obtain the equality in $\mathrm{CH}^{r-1}(E_P)$:

$$i_{E_P\cap W*}c_{r-2}(T_{E_P\cap W}) = 2(-1)^{r-1}\Big[\frac{(1+y)^N}{1+2y}\Big]^{[r-2]}\eta^{r-1}_{P|E_P},$$

hence pushing forward by the inclusion in $\tilde{H}$ and using the projection formula, we have the equality in $\mathrm{CH}^r(\tilde{H})$:

$$i_{E_P\cap W*}c_{r-2}(T_{E_P\cap W}) = 2(-1)^{r-1}\Big[\frac{(1+y)^N}{1+2y}\Big]^{[r-2]}\eta_P^r.$$

Replacing in (5.3.7) yields the result. □

Proposition 5.3.5. *We have the equality in $\mathrm{CH}^N(\tilde{H})$:*

$$(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) = (c_1c_{N-1})([T_g]) + \beta(N)\sum_{P\in\Sigma}\eta_P^N,$$

where $\beta(N)$ is an integer given by:

$$\beta(N) = \frac{(-1)^N}{2}[N^3 - 4N^2 + 4N - 2 + (-1)^{N+1}(N-2)].$$

Proof. If the integer N is 1, this follows directly from Proposition 5.3.4 applied to the integer $r := 1$ (observe that the combinatorial coefficient is 1).

Let us assume that the integer N is at least 2.

Applying Proposition 5.3.4 to the integer $r := 1$, we have the equality in $\mathrm{CH}^1(\tilde{H})$:

$$\begin{aligned} c_1(\omega^{1\vee}_{\tilde{H}/C}) &= c_1([T_g]) + \alpha(N,1)\sum_{P\in\Sigma}\eta_P, \\ &= c_1([T_g]) + \sum_{P\in\Sigma}\eta_P, \end{aligned}$$

and applying it to the integer $r := N-1$, we have the equality in $\mathrm{CH}^{N-1}(\tilde{H})$:

$$c_{N-1}(\omega^{1\vee}_{\tilde{H}/C}) = c_{N-1}([T_g]) + \alpha(N,N-1)\sum_{P\in\Sigma}\eta_P^{N-1}.$$

Hence multiplying, we have the equality in $\mathrm{CH}^N(\tilde{H})$:

$$\begin{aligned} (c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) = (c_1c_{N-1})([T_g]) &+ \alpha(N,N-1)c_1([T_g]).\sum_{P\in\Sigma}\eta_P^{N-1} \\ &+ c_{N-1}([T_g])\sum_{P\in\Sigma}\eta_P + \alpha(N,N-1)\Big(\sum_{P\in\Sigma}\eta_P\Big)\Big(\sum_{P\in\Sigma}\eta_P^{N-1}\Big); \end{aligned}$$

using the projection formula, equality (5.2.2) and the hypothesis that N is at least 2, we get:

$$\begin{aligned} (c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) = (c_1c_{N-1})([T_g]) &+ \alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}(c_1([T_g]_{|E_P})\eta^{N-2}_{P|E_P}) \\ &+ \sum_{P\in\Sigma} i_{E_P*}c_{N-1}([T_g]_{|E_P}) + \alpha(N,N-1)\sum_{P\in\Sigma}\eta_P^N. \end{aligned} \tag{5.3.8}$$

Using the definition of the relative tangent class $[T_g]$ and the fact that for every P in Σ, the line bundle $(g^*T_C)_{|E_P}$ can be trivialized on E_P, the middle two sums in (5.3.8) can be rewritten:

$$\begin{aligned} \alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}(c_1([T_g]_{|E_P})\eta^{N-2}_{P|E_P}) &+ \sum_{P\in\Sigma} i_{E_P*}c_{N-1}([T_g]_{|E_P}) \\ &= \alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}(c_1(T_{\tilde{H}|E_P})\eta^{N-2}_{P|E_P}) + \sum_{P\in\Sigma} i_{E_P*}c_{N-1}(T_{\tilde{H}|E_P}). \end{aligned}$$

For every point P in Σ, we have an exact sequence of vector bundles on E_P:

$$0 \longrightarrow T_{E_P} \longrightarrow T_{\tilde{H}|E_P} \longrightarrow \mathcal{N}_{E_P}\tilde{H} \longrightarrow 0,$$

hence for every integer r, we have the equality in $\mathrm{CH}^r(E_P)$:

$$c_r(T_{\tilde{H}|E_P}) = c_r(T_{E_P}) + c_1(\mathcal{N}_{E_P}\tilde{H})c_{r-1}(T_{E_P}) = c_r(T_{E_P}) + \eta_{P|E_P}c_{r-1}(T_{E_P}).$$

The middle two sums in (5.3.8) can be rewritten as follows, using notably the equality (5.2.4) from Proposition 5.2.1:

$$\begin{aligned}
&\alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}(c_1([T_g]_{|E_P})\eta_{P|E_P}^{N-2}) + \sum_{P\in\Sigma} i_{E_P*}c_{N-1}([T_g]_{|E_P}) \\
&\quad = \alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}((c_1(T_{E_P}) + \eta_{P|E_P})\eta_{P|E_P}^{N-2}) \\
&\quad + \sum_{P\in\Sigma} i_{E_P*}(c_{N-1}(T_{E_P}) + \eta_{P|E_P}c_{N-2}(T_{E_P})), \\
&\quad = \alpha(N,N-1)\sum_{P\in\Sigma} i_{E_P*}\Big[\Big(-\binom{N}{1}\eta_{P|E_P} + \eta_{P|E_P}\Big)\eta_{P|E_P}^{N-2}\Big] \\
&\quad + \sum_{P\in\Sigma} i_{E_P*}\Big[(-1)^{N-1}\binom{N}{N-1}\eta_{P|E_P}^{N-1} + (-1)^N\binom{N}{N-2}\eta_{P|E_P}^{N-1}\Big], \\
&\quad = \Big[\alpha(N,N-1)\Big(1-\binom{N}{1}\Big) + (-1)^{N-1}\binom{N}{N-1} + (-1)^N\binom{N}{N-2}\Big]\sum_{P\in\Sigma} i_{E_P*}\eta_{P|E_P}^{N-1}, \\
&\quad = [-(N-1)\alpha(N,N-1) + (-1)^N(N(N-1)/2 - N)]\sum_{P\in\Sigma}\eta_{P|E_P}^{N}, \\
&\quad = [-(N-1)\alpha(N,N-1) + (-1)^N N(N-3)/2]\sum_{P\in\Sigma}\eta_{P|E_P}^{N}.
\end{aligned}$$

By replacing in (5.3.8), we obtain the equality in $\mathrm{CH}^N(\tilde{H})$:

$$(c_1c_{N-1})(\omega_{\tilde{H}/C}^{1\vee}) = (c_1c_{N-1})([T_g]) + \beta(N)\sum_{P\in\Sigma}\eta_P^N,$$

where $\beta(N)$ is the integer given by:

$$\begin{aligned}
\beta(N) &= -(N-1)\alpha(N,N-1) + (-1)^N N(N-3)/2 + \alpha(N,N-1), \\
&= -(N-2)\alpha(N,N-1) + (-1)^N N(N-3)/2.
\end{aligned}$$

Now let us compute further the combinatorial coefficient $\beta(N)$.

By definition, the coefficient $\alpha(N,N-1)$ is given by:

$$\alpha(N,N-1) = (-1)^N\Big[\binom{N}{N-2} - 4\Big[\frac{(1+y)^N}{1+2y}\Big]^{[N-3]}\Big].$$

By using equality (5.3.1) of Proposition 5.3.3 applied to the integers $n := N$ and $r := 3$, and to the complex number $a := 2$, we obtain:

$$\begin{aligned}
\alpha(N,N-1) &= (-1)^N[N(N-1)/2 - 1/2((-1)^{N+1} + 2N(N-1) - 2N + 1)], \\
&= \frac{(-1)^N}{2}[N(N-1) - (-1)^{N+1} - 2N(N-1) + 2N - 1], \\
&= \frac{(-1)^N}{2}[-N^2 + 3N - 1 - (-1)^{N+1}],
\end{aligned}$$

hence the coefficient $\beta(N)$ is given by:

$$\begin{aligned}
\beta(N) &= -(N-2)\frac{(-1)^N}{2}[-N^2+3N-1-(-1)^{N+1}] + (-1)^N N(N-3)/2,\\
&= \frac{(-1)^N}{2}[-(N-2)(-N^2+3N-1-(-1)^{N+1}) + N(N-3)],\\
&= \frac{(-1)^N}{3}[-(-N^3+2N^2+3N^2-6N-N+2) + (-1)^{N+1}(N-2) + N^2-3N],\\
&= \frac{(-1)^N}{2}[N^3-4N^2+4N-2+(-1)^{N+1}(N-2)],
\end{aligned}$$

as wanted. □

Proof of Lemma 5.1.1. By Proposition 5.3.5, we have the equality in $\mathrm{CH}^N(\tilde{H})$:

$$(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) = (c_1c_{N-1})([T_g]) + \frac{(-1)^N}{2}[N^3-4N^2+4N-2+(-1)^{N+1}(N-2)]\sum_{P\in\Sigma}\eta_P^N,$$

hence pushing forward by the morphism ν, whose generic degree is one, and using equality (5.2.3) from Proposition 5.2.1, we have the equality in $\mathrm{CH}^N(H)$:

$$\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) = \nu_*(c_1c_{N-1})([T_g]) - \frac{1}{2}[N^3-4N^2+4N-2+(-1)^{N+1}(N-2)][\Sigma].$$

Combining this with Proposition 5.3.2 yields the equality in $\mathrm{CH}^N(H)$:

$$\begin{aligned}
\nu_*(c_1c_{N-1})(\omega^{1\vee}_{\tilde{H}/C}) &= (c_1c_{N-1})([T_f])\\
&\quad - \frac{1}{2}[N^3-4N^2+4N-2+(-1)^{N+1}(N-2) - N(N-1)(N-3)][\Sigma],\\
&= (c_1c_{N-1})([T_f]) - \frac{1}{2}[N-2+(-1)^{N+1}(N-2)][\Sigma],\\
&= (c_1c_{N-1})([T_f]) - (N-2)\frac{1-(-1)^N}{2}[\Sigma],
\end{aligned}$$

as wanted. □

CHAPTER 6

The Griffiths height of the middle-dimensional cohomology of a pencil of hypersurfaces

6.1. Families of ample divisors in a smooth pencil

The objective of this section is to prove the following proposition.

PROPOSITION 6.1.1. *Let C be a connected smooth projective complex curve with generic point η, X be a smooth projective complex scheme of pure dimension $N+1$, and let*

$$\pi : X \longrightarrow C$$

be a smooth surjective morphism of complex schemes. Let H be a non-singular hypersurface in X such that the morphism

$$\pi_{|H} : H \longrightarrow C$$

is flat[1] *and has a finite set Σ of critical points, all of which are non-degenerate.*[2]

If we denote by L the line bundle $\mathcal{O}_X(H)$ on X, then the following equality holds in $\mathrm{CH}_0(X)$:

$$[\Sigma] = \big((1 - c_1(L))^{-1} c(\Omega^1_{X/C})\big)^{(N+1)}, \tag{6.1.1}$$

If moreover the line bundle L is ample relatively to π, then the following equalities hold in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned} &c_1(\mathcal{GK}_{C,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta))) \\ &\quad = c_1(\mathcal{GK}_C(\mathbb{H}^{N-1}(X/C))) + c_1(\mathcal{GK}_C(\mathbb{H}^{N+1}(X/C))) - c_1(\mathcal{GK}_C(\mathbb{H}^{N}(X/C))) \\ &\quad + \frac{1}{12}\pi_*\Big[\big((1 - c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C})\big)^{(N+1)}\Big] - \frac{1}{12}\pi_*\big(c_1(L) c_N(\Omega^1_{X/C})\big) + v_N^+\, \pi_*[\Sigma], \end{aligned} \tag{6.1.2}$$

and:

$$\begin{aligned} &c_1(\mathcal{GK}_{C,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta))) \\ &\quad = c_1(\mathcal{GK}_C(\mathbb{H}^{N-1}(X/C))) + c_1(\mathcal{GK}_C(\mathbb{H}^{N+1}(X/C))) - c_1(\mathcal{GK}_C(\mathbb{H}^{N}(X/C))) \\ &\quad + \frac{1}{12}\pi_*\Big[\big((1 - c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C})\big)^{(N+1)}\Big] - \frac{1}{12}\pi_*\big(c_1(L) c_N(\Omega^1_{X/C})\big) + v_N^-\, \pi_*[\Sigma], \end{aligned} \tag{6.1.3}$$

where:

$$v_N^+ := \begin{cases} 7(N-1)/24 & \text{if } N \text{ is odd} \\ (N+2)/24 & \text{if } N \text{ is even.} \end{cases}$$

and:

$$v_N^- := \begin{cases} -5(N-1)/24 & \text{if } N \text{ is odd} \\ (N+2)/24 & \text{if } N \text{ is even.} \end{cases}$$

1. or equivalently, is non-constant on every connected component of H, or has fibers of pure dimension $N-1$.
2. Equivalently, the only possible singularities of the fibers $(H_x)_{x \in C}$ are ordinary double points.

6.1.1. Computation of $i_*[\Sigma]$ and of $i_*(c_1 c_{N-1})([T_f])$. Let C be a connected smooth projective complex curve with generic point η, let X be a smooth projective complex scheme of pure dimension $N+1$, and let

$$\pi : X \longrightarrow C$$

be a smooth surjective morphism of complex schemes. Let H be a non-singular hypersurface in X such that the morphism

$$f := \pi_{|H} : H \longrightarrow C$$

is flat and has a finite set Σ of critical points, all of which are non-degenerate.

Let

$$i : H \longrightarrow X$$

be the inclusion map. Let us also denote by L the line bundle $\mathcal{O}_X(H)$ on X.

Observe that the line bundle $L^\vee_{|H}$ on H can be identified with the conormal line bundle $\mathcal{N}_H X^\vee$ of the hypersurface H in X.

Let us define a map s of vector bundles on H by the composition:

$$s : L^\vee_{|H} \simeq \mathcal{N}_H X^\vee \longrightarrow \Omega^1_{X|H} \longrightarrow \Omega^1_{X/C|H}.$$

We will also see s as a section of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$.

Some of the content of the upcoming results, Lemmas 6.1.2 and 6.1.3, can be summarized in the commutative diagram below, where all the horizontal and vertical sequences are exact sequences of coherent sheaves on H:

(6.1.4)
$$\begin{array}{ccccccccc}
 & & & & f^*\Omega^1_C & = & f^*\Omega^1_C & & \\
 & & & & \downarrow {\scriptstyle {}^t(D\pi)} & & \downarrow {\scriptstyle {}^t(Df)} & & \\
0 & \longrightarrow & \mathcal{N}_H X^\vee & \longrightarrow & \Omega^1_{X|H} & \xrightarrow{{}^t(Di)} & \Omega^1_H & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_H X^\vee & \xrightarrow{s} & \Omega^1_{X/C|H} & \xrightarrow{{}^t(Di)} & \Omega^1_{H/C} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In (6.1.4), ${}^t(Df)$ (resp. ${}^t(D\pi)$, ${}^t(Di)$) denotes the morphism between coherent sheaves of Kähler differentials induced by the morphism of schemes f (resp. π, i).

LEMMA 6.1.2. *The section s does not vanish on the open dense subset $H - \Sigma$ in H, and for every point P in Σ, there exists a system $(z_1, ..., z_N)$ of local analytic coordinates on H near P such that, in some local frame of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$, the section s is given by:*

$$s = (2z_i)_{1 \leq i \leq N}.$$

PROOF. By definition of the subset Σ, the morphism of non-singular complex schemes f is smooth on the open dense subset $H - \Sigma$, so that by elementary properties of the sheaf of Kähler differentials, the section s does not vanish on $H - \Sigma$.

Now, let P be a point in Σ, and let y in C be its image by f. The hypersurfaces H and X_y in X are both non-singular and meet in P. By definition of Σ, the morphism f has a critical point in P, so we have the equality of hyperplanes in $T_{X,P}$:

$$T_{H,P} = T_{X_y,P} \tag{6.1.5}$$

Let t be a local coordinate of C in some analytic neighborhood of y. Since the morphism π is smooth, we can extend the function π^*t into a local coordinate system $(\pi^*t, z_1, ..., z_N)$ in some analytic neighborhood V of P.

Using (6.1.5) and the implicit function theorem, after possibly shrinking V, there exists a diffeomorphism:

$$\varphi : X_y \cap V \longrightarrow H \cap V$$

which sends P to P and whose differential at P is the identity map

$$\mathrm{Id} : T_{X_y,P} \longrightarrow T_{H,P}.$$

In other terms, there exists an analytic neighborhood U of 0 in $\mathbb{C}^N$, and a function:

$$u : U \longrightarrow \mathbb{C},$$

vanishing in the origin, and whose derivative vanishes in the origin, and such that $H \cap V$ is defined by the equation:

$$\pi^*t = u(z_1, ..., z_N). \tag{6.1.6}$$

The morphism φ and the function u are related by the equality of morphisms from X_y to $\mathbb{C}^{N+1}$:

$$(\pi^*t, z_1, ..., z_N) \circ \varphi = (u(z_1, ..., z_N), z_1, ..., z_N). \tag{6.1.7}$$

Observe that H admits a local coordinate system given by $(z_{1|H}, ..., z_{N|H})$.

By hypothesis, y is a non-degenerate critical point of the morphism $f = \pi_{|H}$ on H, hence it is also a non-degenerate critical point of the morphism:

$$\pi_{|H\cap V} \circ \varphi : X_y \cap V \longrightarrow C.$$

Using (6.1.7), this morphism is given in local coordinates by:

$$(\pi^*t)_{|H\cap V} \circ \varphi = u(z_1, ..., z_N),$$

so we obtain that the Hessian at 0 of the function u is non-degenerate. Consequently, after possibly shrinking the neighborhood V and changing the coordinates $z_1, ..., z_N$, we can assume that the function u is given by:

$$u(z_1, ..., z_N) = z_1^2 + ... + z_N^2. \tag{6.1.8}$$

The vector bundle Ω^1_X admits a local frame given by $(\pi^*\mathrm{d}t, \mathrm{d}z_1, ..., \mathrm{d}z_N)$. Using that the hypersurface H is defined by the equation (6.1.6), the conormal line bundle $L^\vee_{|H} \simeq \mathcal{N}_H X^\vee$, as embedded in the vector bundle $\Omega^1_{X|H}$, is generated by the non-vanishing section:

$$s_0 := \frac{\partial u}{\partial z_1}\mathrm{d}z_1 + ... + \frac{\partial u}{\partial z_N}\mathrm{d}z_N - \pi^*\mathrm{d}t = 2z_{1|H}\mathrm{d}z_1 + ... + 2z_{N|H}\mathrm{d}z_N - \pi^*\mathrm{d}t \text{ using (6.1.8)}.$$

The vector bundle $\Omega^1_{X/C}$ admits a local frame given by $([\mathrm{d}z_1], ..., [\mathrm{d}z_N])$. In this frame, the image by the projection

$$\Omega^1_{X|H} \longrightarrow \Omega^1_{X/C|H}$$

of the section s_0 is given by:

$$2z_{1|H}[\mathrm{d}z_1] + ... + 2z_{N|H}[\mathrm{d}z_N].$$

This image is precisely the section s, which proves the lemma. □

LEMMA 6.1.3. *We have an exact sequence of coherent sheaves on H:*

$$0 \longrightarrow L^\vee_{|H} \xrightarrow{s} \Omega^1_{X/C|H} \xrightarrow{{}^t(Di)} \Omega^1_{H/C} \longrightarrow 0, \tag{6.1.9}$$

where ${}^t(Di)$ is the map between sheaves of Kähler differentials associated with the morphism of C-schemes i.

Furthermore, the subscheme of H defined by the vanishing of s is precisely the reduced subscheme defined by the finite subset Σ.

Proof. For the exactness of (6.1.9), everything but the injectivity of the morphism s is a standard property of sheaves of relative Kähler differentials. By Lemma 6.1.2, the morphism s does not vanish on the open dense subset $H - \Sigma$ in H. Since both coherent sheaves $L^\vee_{|H}$ and $\Omega^1_{X/C|H}$ are locally free on H, the morphism of coherent sheaves s is therefore injective, which completes the proof of the exactness of (6.1.9).

Furthermore, by Lemma 6.1.2, the morphism s vanishes with order 1 in every point of Σ. Consequently, the subscheme defined by the vanishing of the morphism s is precisely the reduced subscheme defined by Σ, as wanted. □

Proposition 6.1.4. *We have the following equalities in* $\mathrm{CH}_0(X)$*:*

$$i_*[\Sigma] = [(1 - c_1(L))^{-1} c(\Omega^1_{X/C})]^{(N+1)}, \tag{6.1.10}$$

and:

$$\begin{aligned} i_*(c_1 c_{N-1})([T_f]) = (c_1 c_N)([T_\pi]) \\ + (-1)^N \big([(1 - c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C})]^{(N+1)} + i_*[\Sigma] - c_1(L) c_N(\Omega^1_{X/C})\big). \end{aligned} \tag{6.1.11}$$

Proof. Using Lemma 6.1.3, the subscheme defined by the vanishing of the section s of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$ on H is exactly the reduced subscheme defined by Σ, in particular, it is of dimension 0, hence of codimension N in H. The rank of the vector bundle $\Omega^1_{X/C|H} \otimes L_{|H}$ on H is exactly N, so that s is a regular section in the sense of [**Ful98**, B.3.4].

Consequently, using [**Ful98**, Example 3.2.16, (ii)], we have the equality in $\mathrm{CH}_0(H)$:

$$\begin{aligned} [\Sigma] &= c_N(\Omega^1_{X/C|H} \otimes L_{|H}), \\ &= i^* c_N(\Omega^1_{X/C} \otimes L), \\ &= i^* \sum_{0 \leq k \leq N} c_1(L)^{N-k} c_k(\Omega^1_{X/C}), \end{aligned}$$

where the last expression follows from the formula for the top Chern class of the tensor product of a vector bundle by a line bundle ([**Ful98**, Remark 3.2.3, (b)]).

Pushing forward by the inclusion i and applying the projection formula, we obtain the equality in $\mathrm{CH}_0(X)$:

$$\begin{aligned} i_*[\Sigma] &= [H] \cdot \sum_{0 \leq k \leq N} c_1(L)^{N-k} c_k(\Omega^1_{X/C}), \\ &= c_1(L) \sum_{0 \leq k \leq N} c_1(L)^{N-k} c_k(\Omega^1_{X/C}), \\ &= \sum_{0 \leq k \leq N} c_1(L)^{N+1-k} c_k(\Omega^1_{X/C}), \qquad (6.1.12) \\ &= \sum_{0 \leq k \leq N+1} c_1(L)^{N+1-k} c_k(\Omega^1_{X/C}) \text{ because the vector bundle } \Omega^1_{X/C} \text{ is of rank } N, \\ &= [(1 - c_1(L))^{-1} c(\Omega^1_{X/C})]^{(N+1)}, \end{aligned}$$

which shows equality (6.1.10).

Now let us show equality (6.1.11). Using exact sequence (6.1.9) and the multiplicativity of total Chern classes, we have the equality in $\mathrm{CH}^*(H)$:

$$c([\Omega^1_{H/C}]) = c(\Omega^1_{X/C|H}) c(L^\vee_{|H})^{-1} = i^*[(1 - c_1(L))^{-1} c(\Omega^1_{X/C})].$$

Taking the terms of codimension 1, we obtain the equality in $\mathrm{CH}^1(H)$:

$$c_1([\Omega^1_{H/C}]) = i^*[c_1(\Omega^1_{X/C}) + c_1(L)], \tag{6.1.13}$$

and taking the terms of codimension $N-1$, we obtain the equality in $\mathrm{CH}^{N-1}(H)$:

$$c_{N-1}([\Omega^1_{H/C}]) = i^* \sum_{0\le k\le N-1} c_1(L)^{N-1-k} c_k(\Omega^1_{X/C}). \tag{6.1.14}$$

Multiplying (6.1.13) and (6.1.14), we obtain the equality in $\mathrm{CH}^N(H) \simeq \mathrm{CH}_0(H)$:

$$(c_1c_{N-1})([\Omega^1_{H/C}]) = i^*\Big[\sum_{0\le k\le N-1} c_1(L)^{N-k}c_k(\Omega^1_{X/C}) + \sum_{0\le k\le N-1} c_1(L)^{N-1-k}(c_1c_k)(\Omega^1_{X/C})\Big].$$

Pushing forward by i and applying the projection formula, we obtain the equality in $\mathrm{CH}_0(X)$:

$$\begin{aligned} i_*(c_1c_{N-1})([\Omega^1_{H/C}]) &= c_1(L).\Big[\sum_{0\le k\le N-1} c_1(L)^{N-k}c_k(\Omega^1_{X/C}) + \sum_{0\le k\le N-1} c_1(L)^{N-1-k}(c_1c_k)(\Omega^1_{X/C})\Big],\\ &= \sum_{0\le k\le N-1} c_1(L)^{N+1-k}c_k(\Omega^1_{X/C}) + \sum_{0\le k\le N-1} c_1(L)^{N-k}(c_1c_k)(\Omega^1_{X/C}). \end{aligned} \tag{6.1.15}$$

The first sum in (6.1.15) is given by:

$$\begin{aligned} \sum_{0\le k\le N-1} c_1(L)^{N+1-k}c_k(\Omega^1_{X/C}) &= \sum_{0\le k\le N} c_1(L)^{N+1-k}c_k(\Omega^1_{X/C}) - c_1(L)c_N(\Omega^1_{X/C}),\\ &= i_*[\Sigma] - c_1(L)c_N(\Omega^1_{X/C}) \text{ using equality (6.1.12)} . \end{aligned}$$

Similarly, the second sum in (6.1.15) is given by:

$$\begin{aligned} \sum_{0\le k\le N-1} c_1(L)^{N-k}(c_1c_k)(\Omega^1_{X/C}) &= \sum_{0\le k\le N} c_1(L)^{N-k}(c_1c_k)(\Omega^1_{X/C}) - (c_1c_N)(\Omega^1_{X/C}),\\ &= [(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})]^{(N+1)} - (c_1c_N)(\Omega^1_{X/C}). \end{aligned}$$

Hence replacing in (6.1.15), we obtain the equality in $\mathrm{CH}_0(X)$:

$$i_*(c_1c_{N-1})([\Omega^1_{H/C}]) = i_*[\Sigma] - c_1(L)c_N(\Omega^1_{X/C}) + [(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})]^{(N+1)} - (c_1c_N)(\Omega^1_{X/C}).$$

Since the morphism of non-singular complex schemes f is smooth on an open dense subset of H, similarly to Subsection 3.1.4, we have the following equality in $K^0(H) \simeq K_0(H)$:

$$[T_f] = [\Omega^1_{H/C}]^\vee,$$

where $.^\vee$ denotes the duality involution on $K^0(H)$.

Consequently, we have the equality in $\mathrm{CH}_0(X)$:

$$\begin{aligned} i_*(c_1c_{N-1})([T_f]) &= (-1)^N i_*(c_1c_{N-1})([\Omega^1_{H/C}]),\\ &= (c_1c_N)([T_\pi])\\ &\quad + (-1)^N\big([(1-c_1(L))^{-1}c_1(\Omega^1_{X/C})c(\Omega^1_{X/C})]^{(N+1)} + i_*[\Sigma] - c_1(L)c_N(\Omega^1_{X/C})\big), \end{aligned}$$

which shows equality (6.1.11). □

6.1.2. An application of Lefschetz's weak theorem. We can also prove the following consequence of Lefschetz's weak theorem and of Poincaré duality.

PROPOSITION 6.1.5. *If the line bundle L is ample relatively to the morphism π, then for every integer n such that $n < N-1$, the local monodromy of $\mathbb{H}^n(H_\eta/C_\eta)$ is unipotent and we have the equality in* $\mathrm{CH}_0(C)$*:*

$$c_1(\mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_C(\mathbb{H}^n(X/C))), \tag{6.1.16}$$

and for every integer n such that $n > N-1$, the local monodromy of $\mathbb{H}^n(H_\eta/C_\eta)$ is unipotent and we have the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$*:*

$$c_1(\mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_C(\mathbb{H}^{n+2}(X/C))). \tag{6.1.17}$$

A minor variant of the proof below would actually show that the local monodromy of $\mathbb{H}^n(H_\eta/C_\eta)$ is trivial when $n \neq N-1$.

Proof. Let us assume that the line bundle L is ample relatively to the morphism π. Let Δ be the set-theoretic image:

$$\Delta := f(\Sigma),$$

and let:

$$\mathring{C} := C - \Delta$$

be its complement in C.

Let n be an integer such that $n < N-1$.

Since the line bundle $L = \mathcal{O}_X(H)$ is ample relatively to the morphism π, using Lefschetz's weak theorem (see for instance [**Voi03**, Theorem 1.29]), we obtain that the pullback morphism of variations of Hodge structures on $\mathring{C}$:

$$\mathbb{H}^n(i^*) : \mathbb{H}^n(X - X_\Delta/\mathring{C}) \longrightarrow \mathbb{H}^n(H - H_\Delta/\mathring{C})$$

is an isomorphism.

Since X is smooth over C, the VHS $\mathbb{H}^n(X - X_\Delta/\mathring{C})$ on $\mathring{C}$ can be extended into the VHS $\mathbb{H}^n(X/C)$ on C, hence its local monodromy at every point of Δ is trivial.

Consequently, the local monodromy of the VHS $\mathbb{H}^n(H - H_\Delta/\mathring{C})$ at every point of Δ is trivial, in particular unipotent, and the isomorphism of VHS $\mathbb{H}^n(i^*)$ extends into an isomorphism on the (upper or lower) Deligne extension.

Consequently, it induces an isomorphism of line bundles on C:

$$\mathcal{GK}_C(\mathbb{H}^n(i^*)) : \mathcal{GK}_C(\mathbb{H}^n(X/C)) \longrightarrow \mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta)).$$

Taking Chern classes, we obtain immediately equality (6.1.16).

Now, let n be an integer such that $n > N-1$. Applying Proposition 2.5.2 to the blow-up Y of H at every point of Σ, satisfying that the divisor Y_Δ is a divisor with strict normal crossings, and to the integer

$$n' := 2(N-1) - n < N-1,$$

and using that the local monodromy at every point of Δ of the VHS $\mathbb{H}^{n'}(Y - Y_\Delta/\mathring{C})$ is unipotent (which we showed above), we obtain that the local monodromy at every point of Δ of the VHS $\mathbb{H}^n(Y - Y_\Delta/\mathring{C})$, hence of the VHS $\mathbb{H}^n(H_\eta/C_\eta)$, is unipotent and that the line bundle on C:

$$\mathcal{GK}_C(\mathbb{H}^n(Y - Y_\Delta/\mathring{C})) \otimes \mathcal{GK}_C(\mathbb{H}^{2(N-1)-n}(Y - Y_\Delta/\mathring{C}))^\vee \\ \simeq \mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta)) \otimes \mathcal{GK}_C(\mathbb{H}^{2(N-1)-n}(H_\eta/C_\eta))$$

is of 2-torsion. Consequently, we have the equality in $\mathrm{CH}_0(C)_\mathbb{Q}$,

$$c_1(\mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_C(\mathbb{H}^{2(N-1)-n}(H_\eta/C_\eta))).$$

Applying equality (6.1.16) to the integer $n' := 2(N-1) - n < N-1$, we obtain:

$$c_1(\mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_C(\mathbb{H}^{2(N-1)-n}(X/C))).$$

Applying Proposition 2.5.2 to X, with the set of critical values being empty, we obtain:

$$c_1(\mathcal{GK}_C(\mathbb{H}^n(H_\eta/C_\eta))) = c_1(\mathcal{GK}_C(\mathbb{H}^{2N-(2(N-1)-n)}(X/C))) = c_1(\mathcal{GK}_C(\mathbb{H}^{n+2}(X/C)),$$

which shows equality (6.1.17). $\square$

Now we can conclude the proof of Proposition 6.1.1. Let us assume that the line bundle L is ample relatively to the morphism π.

Applying equality (5.0.1) from Theorem 5.0.1 to the C-scheme H, then applying equality (6.1.11) from Proposition 6.1.4, we have the equalities in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}
&\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1\big(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta))\big)\\
&= \frac{1}{12} f_*\big((c_1 c_{N-1})([\Omega^1_{H/C}]^\vee)\big) + u_N^- f_*[\Sigma],\\
&= \frac{1}{12}\pi_*(c_1 c_N)([T_\pi]) + (u_N^- + \frac{(-1)^N}{12}) f_*[\Sigma]\\
&+ \frac{(-1)^N}{12}\pi_*\big([(1-c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C})]^{(N+1)} - c_1(L) c_N(\Omega^1_{X/C})\big),
\end{aligned} \tag{6.1.18}$$

where u_N^- is the rational number defined by:

$$u_N^- := \begin{cases} (5N-3)/24 & \text{if } N \text{ is odd} \\ N/24 & \text{if } N \text{ is even.} \end{cases}$$

Applying Theorem 5.0.1 to the smooth C-scheme X yields the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\sum_{n=0}^{2N} (-1)^{n-1} c_1(\mathcal{GK}_C(\mathbb{H}^n(X/C))) = \frac{1}{12}\pi_*(c_1 c_N)([T_\pi]). \tag{6.1.19}$$

Finally, it follows from Proposition 6.1.5 that we have the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$\begin{aligned}
&\sum_{n=0}^{2(N-1)} (-1)^{n-1} c_1\big(\mathcal{GK}_{C,-}(\mathbb{H}^n(H_\eta/C_\eta))\big) - \sum_{n=0}^{2N} (-1)^{n-1} c_1(\mathcal{GK}_C(\mathbb{H}^n(X/C)))\\
&= (-1)^N [c_1(\mathcal{GK}_{C,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta))) - c_1(\mathcal{GK}_C(\mathbb{H}^{N-1}(X/C)))\\
&\qquad + c_1(\mathcal{GK}_C(\mathbb{H}^N(X/C))) - c_1(\mathcal{GK}_C(\mathbb{H}^{N+1}(X/C)))].
\end{aligned} \tag{6.1.20}$$

Equality (6.1.3) follows by combining equalities (6.1.18), (6.1.19) and (6.1.20). Equality (6.1.2) is proved in the same way, using equality (5.0.2) from Theorem 5.0.1 instead of equality (5.0.1).

This concludes the proof of Proposition 6.1.1.

6.2. Pencils of hypersurfaces in the projective space

In this section, we shall use Proposition 6.1.1 to compute the Griffiths height of the middle -dimensional cohomology of pencils of projective hypersurfaces. Before stating our results, we introduce some notation and recall some basic facts concerning projective bundles over a projective curve, their Chow groups, and their hypersurfaces.

6.2.1. Preliminary: projective bundles over a curve and horizontal hypersurfaces. Let C be a connected smooth projective complex curve with generic point η, let E be a vector bundle of rank $N+1 \geq 1$ over C, and let:

$$\pi : \mathbb{P}(E) := \operatorname{Proj} S^\bullet E^\vee \longrightarrow C,$$

be the associated projective bundle over C.

We denote by $\mathcal{O}_E(1)$ the tautological quotient line bundle over $\mathbb{P}(E)$; it is the dual of the tautological subbundle of rank 1 of $\pi^* E$.

We shall use the fact (see for instance [**Ful98**, B.5.8]) that the relative tangent bundle

$$T_{\mathbb{P}(E)/C} := T_\pi$$

fits into a short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \pi^* E \otimes \mathcal{O}_E(1) \longrightarrow T_{\mathbb{P}(E)/C} \longrightarrow 0. \tag{6.2.1}$$

Let H be an hypersurface, namely an effective Cartier divisor, in $\mathbb{P}(E)$. It is a projective complex scheme of dimension N. Its generic fiber H_η is an hypersurface in the projective space $\mathbb{P}(E)_\eta$ of dimension N over $\mathbb{C}(C)$. We shall denote its degree by d.

We shall also denote by:

$$i : H \longrightarrow \mathbb{P}(E)$$

the inclusion morphism of H in the projective bundle $\mathbb{P}(E)$, and by:

$$f := \pi \circ i : H \longrightarrow C$$

its projection to the curve C.

As is well-known, the hypersurface H_η in the projective space $\mathbb{P}(E)_\eta$ may be defined, as a subscheme, by the vanishing of an homogeneous polynomial of degree d — namely by a non-zero element of $S^d E_\eta^\vee$ — unique up to the multiplication by an element of $\mathbb{C}(C)^*$.

This description of H_η extends to a description of the "relative hypersurface" H over C as follows.

Since $\mathbb{P}(E)$ is a regular scheme, its Picard group is isomorphic to its Chow group of codimension one $\mathrm{CH}^1(\mathbb{P}(E))$. According to the structure of the Chow groups of a projective bundle (see [**Ful98**, Th. 3.3 (b)], with $k = \dim(\mathbb{P}(E)) - 1$), there is an isomorphism:

$$\mathbb{Z} \oplus \mathrm{CH}^1(C) \xrightarrow{\sim} \mathrm{CH}^1(\mathbb{P}(E)), \quad (\delta, \alpha) \longmapsto \delta c_1(\mathcal{O}_E(1)) + \pi^* \alpha.$$

Consequently the line bundle $\mathcal{O}_{\mathbb{P}(E)}(H)$ is isomorphic to $\mathcal{O}_E(\delta) \otimes \pi^* M$ for some integer δ and some line bundle M over C. By considering the restriction of this isomorphism to $\mathbb{P}(E)_\eta$, we see that δ coincides with the generic degree d of H. Moreover the line bundle M is unique up to isomorphism.

We will denote by:

$$\sigma : \mathcal{O}_{\mathbb{P}(E)}(H) \xrightarrow{\sim} \mathcal{O}_E(d) \otimes \pi^* M \tag{6.2.2}$$

the isomorphism obtained by this construction. This isomorphism may be seen as a morphism of line bundles over $\mathbb{P}(E)$:

$$\sigma : \pi^* M^\vee \longrightarrow \mathcal{O}_E(d), \tag{6.2.3}$$

the vanishing of which defines H scheme-theoretically. In turn, the data of the morphism (6.2.3) is equivalent to the data of a morphism of sheaves of $\mathcal{O}_C$-modules:

$$\tau : M^\vee \longrightarrow \pi_* \mathcal{O}_E(d) \simeq S^d E^\vee, \tag{6.2.4}$$

which is clearly injective.

For every point x of C, the image $\tau_x(y)$ of any non-zero element y of $M_x^\vee$ is an element of $S^d E_x^\vee$, which is a (scheme-theoretic) equation for the fiber H_x of H in the projective space $\mathbb{P}(E)_x$ over $\mathbb{C}$. This shows that the hypersurface H is horizontal — namely, that no fiber of π is a component of H, or equivalently that the morphism $f : H \to C$ is flat — if and only if τ_x is non-zero for every point x of C, or equivalently if and only if τ is an isomorphism onto a locally direct summand of $S^d E^\vee$.

In the sequel, we shall use the following notation for the first Chern classes of E and M, and of $\mathcal{O}_E(1)$, in $\mathrm{CH}^1(C)$ and $\mathrm{CH}^1(\mathbb{P}(E))$ respectively:

$$e := c_1(E), \quad m := c_1(M), \quad \text{and} \quad h := c_1(\mathcal{O}_E(1)).$$

Using the isomorphism (6.2.2), we obtain the equality in $\mathrm{CH}^1(\mathbb{P}(E))$:

$$[H] = d\, h + \pi^* m. \tag{6.2.5}$$

Using the definition of Segre classes and their relation to Chern classes ([**Ful98**, 3.1, 3.2]), and the fact that C is one-dimensional, we obtain the following equality, for every $i \in \mathbb{N}$:

$$\pi_* h^i =: s_{i-N}(E) = \begin{cases} 0 & \text{if } i < N, \\ [C] & \text{if } i = N, \\ -e & \text{if } i = N+1, \\ 0 & \text{if } i > N+1. \end{cases} \tag{6.2.6}$$

From the relations above and the projection formula, we deduce:

$$\pi_*(h^{N-1} \cap [H]) = d[C], \tag{6.2.7}$$

and

$$\pi_*(h^N \cap [H]) = -de + m. \tag{6.2.8}$$

In computations, we will use that $\mathrm{CH}^i(C)$ vanishes for $i > 1$; in particular, we have:

$$m \cdot m = e \cdot e = m \cdot e = 0. \tag{6.2.9}$$

REMARK 6.2.1. The classes m and e depend on the choice of the bundle E, and not only on the C-scheme $\mathbb{P}(E)$.

More specifically, if L is a line bundle on C, and E is replaced by $E' := E \otimes L$, then we may identify the C-schemes $\mathbb{P}(E')$ and $\mathbb{P}(E)$, but e is replaced by:

$$e' := c_1(E') = c_1(E \otimes L) = e + (N+1)\ c_1(L).$$

The line bundle $\mathcal{O}_E(1)$ is replaced by:

$$\mathcal{O}_{E'}(1) \simeq \mathcal{O}_E(1) \otimes \pi^* L^\vee,$$

and the class h is replaced by:

$$h' := c_1(\mathcal{O}_{E'}(1)) = h - \pi^* c_1(L).$$

Moreover, the bundle $S^d E'^\vee$ may be identified with $(S^d E^\vee) \otimes L^{-d}$. The morphism of vector bundles:

$$\tau' := \tau \otimes \mathrm{Id}_{L^{-d}} : M^\vee \otimes L^{-d} \hookrightarrow S^d E'^\vee$$

clearly represents the same equation as τ, hence it also defines the hypersurface H in $\mathbb{P}(E')$.

Finally the line bundle M is replaced by $M' := M \otimes L^d$, and consequently the class m is replaced by:

$$m' := c_1(M') = m + d\ c_1(L).$$

Observe that this implies the equality of classes in $\mathrm{CH}^1(C)_\mathbb{Q}$:

$$(N+1)m' - de' = (N+1)m - de.$$

We shall denote the relative canonical sheaf of the smooth morphism

$$\pi : \mathbb{P}(E) \longrightarrow C$$

by:

$$\omega_{\mathbb{P}(E)/C} \simeq \Omega^N_{\mathbb{P}(E)/C} \simeq \det T^\vee_{\mathbb{P}(E)/C}. \tag{6.2.10}$$

DEFINITION AND PROPOSITION 6.2.2. *Let $h^* \in \mathrm{CH}^1(\mathbb{P}(E))_\mathbb{Q}$ be the class defined by:*

$$h^* := -\frac{1}{N+1} c_1(\omega_{\mathbb{P}(E)/C}),$$

and let $\mathrm{ht}_{int}(H/C)$ be the rational number defined by:

$$\mathrm{ht}_{int}(H/C) := \int_{\mathbb{P}(E)} h^{*N} \cap [H].$$

They satisfy the following relations:

$$h^* = h + \frac{1}{N+1}\pi^* e, \tag{6.2.11}$$

$$\mathrm{ht}_{int}(H/C) = \int_{\mathbb{P}(E)} h^N \cap [H] + \frac{dN}{N+1} \deg E, \tag{6.2.12}$$

$$= \deg M - \frac{d}{N+1} \deg E. \tag{6.2.13}$$

In particular, the rational number $(N+1)\mathrm{ht}_{int}(H/C)$ *is an integer.*

Proof. The equality (6.2.11) comes from taking the determinant of the exact sequence (6.2.1). This gives isomorphisms:

$$\det T_{\mathbb{P}(E)/C} \simeq \det\left(\pi^* E \otimes \mathcal{O}_E(1)\right) \simeq \mathcal{O}_E(N+1) \otimes \pi^* \det E.$$

Together with (6.2.10), this establishes (6.2.11).

By using successively (6.2.9), (6.2.5) and (6.2.9) again, we have:

$$\begin{aligned} h^{*N} \cap [H] &= \left(h + \frac{1}{N+1}\pi^* e\right)^N \cap [H], \\ &= \left(h^N + \frac{N}{N+1} h^{N-1}\pi^* e\right) \cap [H], \\ &= h^N \cap [H] + \frac{N}{N+1} h^{N-1}\pi^* e \ . \ (dh + \pi^* m), \end{aligned}$$

$$= h^N \cap [H] + \frac{dN}{N+1} h^N \pi^* e. \tag{6.2.14}$$

After pushing forward by π and taking the degree, using (6.2.6), this becomes equality (6.2.12).

From (6.2.14) and (6.2.5), we obtain:

$$\begin{aligned} h^{*N} \cap [H] &= h^N. \ (dh + \pi^* m) + \frac{dN}{N+1} h^N \pi^* e, \\ &= dh^{N+1} + h^N \pi^* m + \frac{dN}{N+1} h^N \pi^* e, \end{aligned}$$

so pushing it forward by π and using (6.2.6), we get:

$$\pi_*(h^{*N} \cap [H]) = -de + m + \frac{dN}{N+1} e = m - \frac{d}{N+1} e.$$

Consequently, we have:

$$\int_{\mathbb{P}(E)} h^{*N} \cap [H] = \deg(\pi_*(h^{*N} \cap [H])) = \deg M - \frac{d}{N+1} \deg E.$$

This shows equality (6.2.13). □

Remark 6.2.3. The subscript stands for "intersection-theoretic." The height $\mathrm{ht}_{int}(H/C)$ clearly only depends on $\mathbb{P}(E)$ as a scheme over C and on its subscheme H. We can also check this property on the expression (6.2.13): according to Remark 6.2.1, if E is replaced by $E' := E \otimes L$, with L a line bundle, the class $m - \frac{d}{N+1} e$ is unchanged.

We can also express this height in terms of slopes. The slope of the line bundle M is $\mu(M) = \deg M$, the slope of E is:

$$\mu(E) = \frac{\deg E}{N+1},$$

and the one of $S^dE^\vee$ is:

$$\mu(S^dE^\vee) = -d\,\mu(E) = -\frac{d\deg E}{N+1},$$

which gives the following expression:

$$\mathrm{ht}_{int}(H/C) = \mu(S^dE^\vee) - \mu(M^\vee). \tag{6.2.15}$$

The existence of an effective section of the line bundle $M^{\otimes(N+1)(d-1)^N} \otimes (\det E)^{\otimes -d(d-1)^N}$ on C defined by the discriminant of the horizontal hypersurface H (see (6.2.33) below), along with equality (6.2.13), will prove that the rational number $\mathrm{ht}_{int}(H/C)$ is non-negative when $d \geq 2$.

When the vector bundle $S^dE^\vee$ is semistable, the non-negativity of $\mathrm{ht}_{int}(H/C)$ would follow from equation (6.2.15). Indeed, since $M^\vee$ is naturally embedded as a coherent subsheaf of $S^dE^\vee$, using the expression (6.2.15) for $\mathrm{ht}_{int}(H/C)$, if the vector bundle $S^dE^\vee$ is stable (resp. semistable), then $\mathrm{ht}_{int}(H/C)$ is positive (resp. non-negative).

Combining this result with the compatibility of vector bundle stability with tensor operations in characteristic 0 (see for instance [**NS65**]), we immediately obtain that if the vector bundle E is semistable, then $\mathrm{ht}_{int}(H/C)$ is non-negative.

The non-negativity of $\mathrm{ht}_{int}(H/C)$ when the vector bundle E is semistable would also follow from [**Miy87**, Th. 3.1]: the vector bundle E is semistable if and only if the class h^* (which Miyaoka denotes by $\lambda_{\mathcal{E}}$) is nef, and when this holds, the rational number

$$\mathrm{ht}_{int}(H/C) := \int_{\mathbb{P}(E)} h^{*N} \cap [H]$$

is non-negative.

6.2.2. The Griffiths height of the middle-dimensional cohomology of a pencil of hypersurfaces in projective spaces. The objective of this subsection is to show the following theorem, which immediately implies Theorem 1.4.2 using equality (6.2.13). As observed after the statement of Theorem 1.4.2, this theorem implies the validity of the same formulas for the Griffiths heights of the primitive part of the middle-dimensional cohomology $\mathbb{H}^{N-1}(H_\eta/C_\eta)$.

THEOREM 6.2.4. *Let C be a connected smooth projective complex curve with generic point η, E a vector bundle of rank $N+1$ over C, and $H \subset \mathbb{P}(E)$ an horizontal hypersurface of relative degree d, smooth over $\mathbb{C}$. If $\pi_{|H}$ has only a finite number of critical points, all of which are non-degenerate, then the reduced 0-cycle in H defined by the set of critical points Σ satisfies the equality in* $\mathrm{CH}_0(C)$*:*

$$f_*[\Sigma] = (d-1)^N((N+1)m - de). \tag{6.2.16}$$

Moreover, we have the following equalities in $\mathrm{CH}_0(C)_\mathbb{Q}$*:*

$$c_1(\mathcal{GK}_{C,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_+(d,N)(m - \frac{d}{N+1}e), \tag{6.2.17}$$

and:

$$c_1(\mathcal{GK}_{C,-}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_-(d,N)(m - \frac{d}{N+1}e), \tag{6.2.18}$$

and the following equality of rational numbers:

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_{stab}(d,N)\,\mathrm{ht}_{int}(H/C), \tag{6.2.19}$$

where $F_+(d,N)$, $F_-(d,N)$ and $F_{stab}(d,N)$ are the elements of $(1/12)\mathbb{Z}$ given when N is odd by:

$$F_+(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(7d^2N - 7d^2 - 2dN - 2) + 2(d^2-1)\right],$$

$$F_-(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(-5d^2N + 5d^2 - 2dN - 2) + 2(d^2-1)\right],$$

and:

$$F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N-d^2-2dN-2)+2(d^2-1)\right],$$

and, when N is even, by:

$$F_+(d,N) = F_-(d,N) = F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N+2d^2-2dN-2)-2(d^2-1)\right].$$

Observe that the morphism of complex schemes:

$$\pi : X := \mathbb{P}(E) \longrightarrow C$$

is smooth and surjective, and using the isomorphism 6.2.2, the line bundle L is relatively ample. Consequently, we are in the situation of Section 6.1.

Proposition 6.2.5. *With the above notation, and denoting L the line bundle $\mathcal{O}_{\mathbb{P}(E)}(H)$ on $\mathbb{P}(E)$, we have the equalities in $\mathrm{CH}_0(\mathbb{P}(E))$:*

$$i_*[\Sigma] = (d-1)^N h^N[(d-1)h+(N+1)\pi^*m-\pi^*e], \tag{6.2.20}$$

$$c_1(L)c_N(\Omega^1_{\mathbb{P}(E)/C}) = (-1)^N h^N[d(N+1)h+(N+1)\pi^*m+dN\pi^*e], \tag{6.2.21}$$

and:

$$[(1-c_1(L))^{-1}c_1(\Omega^1_{\mathbb{P}(E)/C})c(\Omega^1_{\mathbb{P}(E)/C})]^{(N+1)} = h^N(a_{N,d}h+b_{N,d}\pi^*m+c_{N,d}\pi^*e), \tag{6.2.22}$$

with the rational numbers $a_{N,d}$, $b_{N,d}$ and $c_{N,d}$ given by:

$$\begin{aligned} a_{N,d} &:= \frac{N+1}{d}\big(-(d-1)^{N+1}+(-1)^{N+1}\big),\\ b_{N,d} &:= \frac{N+1}{d^2}\big(-(d-1)^N(dN+1)+(-1)^N\big),\\ c_{N,d} &:= \frac{1}{d}\big(-(d-1)^N(d-N-2)+(-1)^{N+1}(N+2)\big).\end{aligned}$$

Proof. Applying equality (6.1.10) from Proposition 6.1.4 with the above notation yields the equality in $\mathrm{CH}_0(\mathbb{P}(E))$:

$$\begin{aligned} i_*[\Sigma] &= \left[(1-c_1(L))^{-1}c(\Omega^1_{\mathbb{P}(E)/C})\right]^{(N+1)}\\ &= (-1)^{N+1}\left[(1+c_1(L))^{-1}c(T_{\mathbb{P}(E)/C})\right]^{(N+1)}\\ &= (-1)^{N+1}\left[(1+dh+\pi^*m)^{-1}c(\pi^*E\otimes\mathcal{O}(1))\right]^{(N+1)} &(6.2.23)\\ &= (-1)^{N+1}\Big[\big((1+dh)^{-1}-\pi^*m\,(1+dh)^{-2}\big)\big((1+h)^{N+1}+\pi^*e\,(1+h)^N\big)\Big]^{(N+1)} &(6.2.24)\\ &= (-1)^{N+1}[(1+dh)^{-1}(1+h)^{N+1}]^{(N+1)} - (-1)^{N+1}\pi^*m\,[(1+dh)^{-2}(1+h)^{N+1}]^{(N)}\\ &\quad + (-1)^{N+1}\pi^*e\,[(1+dh)^{-1}(1+h)^N]^{(N)} &(6.2.25)\\ &= (-1)^{N+1}[(-1)^{N+1}(d-1)^{N+1}]h^{N+1} - (-1)^{N+1}\pi^*m\,[(-1)^N(N+1)(d-1)^N]h^N\\ &\quad + (-1)^{N+1}\pi^*e\,[(-1)^N(d-1)^N]h^N &(6.2.26)\\ &= (d-1)^N h^N[(d-1)h+(N+1)\pi^*m-\pi^*e],\end{aligned}$$

where in (6.2.23), we used exact sequence (6.2.1) and equality (6.2.5); in (6.2.24), we used (6.2.9) and [**Ful98**, Ex. 3.2.2] ; in (6.2.25), we used again (6.2.9); and in (6.2.26), we used Proposition 5.3.3.

This shows (6.2.20). The other equalities are proved similarly with straightforward computations. □

PROOF OF THEOREM 6.2.4. Pushing equality (6.2.20) forward by the morphism π yields the equality in $\mathrm{CH}^1(C)$:

$$\begin{aligned} f_*[\Sigma] &= (d-1)^N[(d-1)\pi_* h^{N+1} + (N+1)\pi_*(h^N\pi^*m) - \pi_*(h^N\pi^*e)], \\ &= (d-1)^N[-(d-1)e + (N+1)m - e] \text{ using the projection formula and (6.2.6)}, \\ &= (d-1)^N[(N+1)m - de], \end{aligned}$$

which shows equality (6.2.16).

Let us show equality (6.2.17). Since $X = \mathbb{P}(E)$ is a projective bundle over C, all its relative Hodge vector bundles on C are trivial, and for every integer n, the Griffiths line bundle $\mathcal{GK}_C(\mathbb{H}^n(X/C))$ is trivial. Since the line bundle L is ample relatively to the morphism π, we may apply the equality (6.1.2) established in Proposition 6.1.1. With the above notation, this yields the equality in $\mathrm{CH}_0(C)_{\mathbb{Q}}$:

$$(6.2.27)\quad c_1(\mathcal{GK}_{C,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta))) = \frac{1}{12}\pi_*\Big[\big((1-c_1(L))^{-1}c_1(\Omega^1_{\mathbb{P}(E)/C})c(\Omega^1_{\mathbb{P}(E)/C})\big)^{(N+1)}\Big] - \frac{1}{12}\pi_*\big(c_1(L)c_N(\Omega^1_{\mathbb{P}(E)/C})\big) + v_N^+ f_*[\Sigma].$$

Using (6.2.22), the first term in (6.2.27) is given by:

$$\begin{aligned} &\frac{1}{12}\pi_*\Big[\big((1-c_1(L))^{-1}c_1(\Omega^1_{\mathbb{P}(E)/C})c(\Omega^1_{\mathbb{P}(E)/C})\big)^{(N+1)}\Big] \\ &= \frac{1}{12}\pi_*(a_{N,d}h^{N+1} + b_{N,d}h^N\pi^*m + c_{N,d}h^N\pi^*e), \\ (6.2.28)\quad & \\ &= \frac{1}{12}b_{N,d}m + \frac{1}{12}(c_{N,d} - a_{N,d})e, \\ &= \frac{N+1}{12d^2}\big[-(d-1)^N(dN+1) + (-1)^N\big]m \\ &+ \frac{1}{12d}\big[-(d-1)^N(d-N-2) + (-1)^{N+1}(N+2) + (N+1)(d-1)^{N+1} - (N+1)(-1)^{N+1}\big]e, \\ &= \frac{N+1}{12d^2}\big[-(d-1)^N(dN+1) + (-1)^N\big]m - \frac{1}{12d}\big[-(d-1)^N(dN+1) + (-1)^N\big]e, \\ (6.2.29)\quad & \\ &= \frac{N+1}{12d^2}\big(-(d-1)^N(dN+1) + (-1)^N\big)\big(m - \frac{d}{N+1}e\big), \end{aligned}$$

where in (6.2.28), we have used the projection formula and (6.2.6).

Reasoning similarly using (6.2.21), the second term in (6.2.27) is given by:

$$\begin{aligned} \frac{1}{12}\pi_*\big(c_1(L)c_N(\Omega^1_{\mathbb{P}(E)/C})\big) &= \frac{(-1)^N}{12}\pi_*(d(N+1)h^{N+1} + (N+1)h^N\pi^*m + dNh^N\pi^*e), \\ &= \frac{(-1)^N}{12}(-d(N+1)e + (N+1)m + dNe), \\ (6.2.30)\qquad &= \frac{(-1)^N(N+1)}{12}\big(m - \frac{d}{N+1}e\big). \end{aligned}$$

Replacing (6.2.16), (6.2.29) and (6.2.30) in (6.2.27) yields the equality:

$$c_1(\mathcal{GK}_{C,+}(\mathbb{H}^{N-1}(H_\eta/C_\eta))) = F_+(d,N)\big(m - \frac{d}{N+1}e\big),$$

where $F_+(d,N)$ is the rational number given by:

$$F_+(d,N) = \frac{N+1}{12d^2}\big(-(d-1)^N(dN+1) + (-1)^N\big) - \frac{(-1)^N(N+1)}{12} + v_N^+(N+1)(d-1)^N.$$

If the integer N is odd, $F_+(d,N)$ is given by:

$$\begin{aligned} F_+(d,N) &= -\frac{N+1}{12d^2}\big((d-1)^N(dN+1)+1\big) + \frac{N+1}{12} + \frac{7(N-1)}{24}(N+1)(d-1)^N, \\ &= \frac{N+1}{24d^2}\Big[-2\big((d-1)^N(dN+1)+1\big) + 2d^2 + 7d^2(N-1)(d-1)^N\Big], \\ &= \frac{N+1}{24d^2}\Big[(d-1)^N(7d^2N - 7d^2 - 2dN - 2) + 2(d^2-1)\Big], \end{aligned}$$

as wanted, and if the integer N is even, it is given by:

$$\begin{aligned} F_+(d,N) &= \frac{N+1}{12d^2}\big(-(d-1)^N(dN+1)+1\big) - \frac{N+1}{12} + \frac{N+2}{24}(N+1)(d-1)^N, \\ &= \frac{N+1}{24d^2}\Big[2\big(-(d-1)^N(dN+1)+1\big) - 2d^2 + d^2(N+2)(d-1)^N\Big], \\ &= \frac{N+1}{24d^2}\Big[(d-1)^N(d^2N + 2d^2 - 2dN - 2) - 2(d^2-1)\Big], \end{aligned}$$

as wanted. This shows equality (6.2.17).

For equality (6.2.18), we reason similarly using (6.1.3).

For equality (6.2.19), it follows from equality (6.2.16) and the definition of the height $\mathrm{ht}_{int}(H/C)$ that we have the following equality of integers:

$$|\Sigma| = (d-1)^N[(N+1)\deg(m) - d\deg(e)] = (N+1)(d-1)^N \mathrm{ht}_{int}(H/C).$$

Equality (6.2.19) is a simple consequence of this equality combined with equality (2.4.6) from Corollary 2.4.2 and with equality (6.2.18). □

6.2.3. Complements: horizontal hypersurfaces in a projective bundle and discriminant. Equality (6.2.16) from Theorem 6.2.4 can also be deduced from the classical theory of the discriminant, which we shall now recall.

Recall the definition ([**Dem12**]) of the discriminant of a homogeneous polynomial in an arbitrary number of indeterminates.

Let us denote by $n \geq 1$ the number of indeterminates and by d the degree of the homogeneous polynomial. Let:

$$I_{n,d} := \left\{ \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n | \sum_{i=1}^n \alpha_i = d \right\}$$

be the set of indexes, let:

$$U_{n,d} := \mathbb{Z}\left[(T_\alpha)_{\alpha \in I_{n,d}}\right]$$

be the universal polynomial ring, and let:

$$P_{n,d} := \sum_{\alpha \in I_{n,d}} X_1^{\alpha_1} ... X_n^{\alpha_n}\, T_\alpha \in U_{n,d}[X_1, ..., X_n]$$

be the *universal homogeneous polynomial* of degree d in n indeterminates. It satisfies the following tautological property.

For R a ring, let us denote by $R[X_1, ..., X_n]_d$ the submodule of $R[X_1, ..., X_n]$ of homogeneous polynomials of degree d. For every polynomial $F \in R[X_1, ..., X_n]_d$, there is a unique morphism of rings:

$$h_F : U_{n,d} \longrightarrow R,$$

such that the polynomial $h_F(P_{n,d})$ deduced from $P_{n,d}$ by replacing its coefficients in $U_{n,d}$ by their images by h_F — in other terms, its base change by h_F — is F itself.

Definition and Proposition 6.2.6 ([**Dem12**, 5]; see also [**GKZ08**, Chapter 13]). *The* universal discriminant $\mathrm{disc}_{n,d}(P_{n,d})$ *is an element of* $U_{n,d}$ *given by:*

$$\mathrm{disc}_{n,d}(P_{n,d}) := d^{-a(n,d)}\mathrm{Res}\left(\frac{\partial P_{n,d}}{\partial X_1}, \cdots, \frac{\partial P_{n,d}}{\partial_{X_n}}\right),$$

where Res *denotes the resultant of* n *polynomials in* n *indeterminates, and the integer* $a(n,d)$ *is defined by:*

$$a(n,d) := \frac{(d-1)^n - (-1)^n}{d}.$$

For every ring R*, if* F *is a polynomial in* $R[X_1, ..., X_n]_d$*, if* $h_F : U_{n,d} \to R$ *denotes the morphism of rings such that* $h_F(P_{n,d}) = F$*, then:*

$$\mathrm{disc}_{n,d}(F) := h_F(\mathrm{disc}_{n,d}(P_{n,d})) \in R$$

defines the discriminant *of* F*.*

The universal discriminant $\mathrm{disc}_{n,d}(P_{n,d})$ *is characterized (up to a sign) by the following two properties:*

(1) It is a prime element of $U_{n,d}$*.* [3]

(2) If k *is a field, and* F *is a polynomial in* $k[X_1, ..., X_n]_d$*, then* $\mathrm{disc}_{n,d}(F) = 0$ *if and only if the subscheme* $(F = 0)$ *in* $\mathbb{P}_k^{n-1}$ *is not smooth.* [4]

Moreover, for every ring R *and for every homogeneous polynomial* F *in* $R[X_1, ..., X_n]_d$*, the following equalities hold in* $U_{n,d}$ *:* [5]

(3) If γ *is a scalar in* R*, then we have:*

$$\mathrm{disc}_{n,d}(\gamma F) = \gamma^{n(d-1)^{n-1}}\mathrm{disc}_{n,d}(F). \tag{6.2.31}$$

(4) If $A = (a_{i,j})_{1\leq i,j\leq n}$ *is a matrix with entries in* R*, then the discriminant of the polynomial* $F' := F\left(\sum_j a_{1,j}X_1, \sum_j a_{2,j}X_2, ...\right)$ *is given by:*

$$\mathrm{disc}_{n,d}(F') = (\det A)^{d(d-1)^{n-1}}\mathrm{disc}_{n,d}(F). \tag{6.2.32}$$

We can extend this definition to families of polynomials parametrised by a scheme in the following way.

Definition and Proposition 6.2.7. *Let* X *be a scheme,* M *be a line bundle over* X*,* E *be a vector bundle of rank* n *on* X*, and let* τ *be a section over* X *of the vector bundle* $S^d E^\vee \otimes M$*.*

There exists a unique section $\mathrm{disc}_{n,d}(\tau)$ *over* X *of the line bundle:*

$$M^{\otimes n(d-1)^{n-1}} \otimes (\det E)^{\otimes -d(d-1)^{n-1}}$$

satisfying the following property:

(1) If $V = \mathrm{Spec}\, R$ *is an affine open subset of* X*, if*

$$\varphi : \mathcal{O}_V \xrightarrow{\sim} M_V, \quad \psi : \mathcal{O}_V^{\oplus n} \xrightarrow{\sim} E_V$$

are local trivializations of the line bundle M *and the vector bundle* E*, then the scalar*

$$\left((\varphi^{-1})^{\otimes n(d-1)^{n-1}} \otimes (\det \psi^{-1})^{\otimes -d(d-1)^{n-1}}\right)(\mathrm{disc}_{n,d}(\tau)) \in \Gamma(V, \mathcal{O}_V) \simeq R,$$

is precisely the discriminant $\mathrm{disc}_{n,d}(F)$ *of the polynomial*

$$F := \left((S^d\, {}^t\psi) \otimes \varphi^{-1}\right)(\tau_{|V}) \in \Gamma(V, S^d\mathcal{O}_V^{\oplus n}) \simeq R[X_1, ..., X_n]_d.$$

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3. see [**Dem12**, 6, Cor. 1].
4. see [**Dem12**, Prop. 12].
5. see [**Dem12**, Prop. 11, c) and e)].

This section also satisfies the following property:

(2) *If x is a point in X with residue field $\kappa(x)$, then the section* $\mathrm{disc}_{n,d}(\tau)$ *vanishes at x if and only if the hypersurface in $\mathbb{P}(E_{\kappa(x)})$ defined by the vanishing of the polynomial $\tau(x) \in S^d E^\vee_{\kappa(x)} \otimes M_{\kappa(x)}$ is not smooth over $\kappa(x)$.*

The proof is straightforward.

Consider now a connected smooth projective complex curve C with generic point η, a vector bundle E of rank $N+1$ over C, and a horizontal hypersurface $H \subset \mathbb{P}(E)$ such that H_η is smooth over η, and let us adopt the notation of the previous subsections.

In particular, we denote by τ a section of $S^d E^\vee \otimes M$ representing the "equation" of H.

By applying to τ the construction in the previous proposition, we get a section:

$$\mathrm{disc}_{N+1,d}(\tau) \in \Gamma\left(C, M^{\otimes (N+1)(d-1)^N} \otimes (\det E)^{\otimes -d(d-1)^N}\right). \tag{6.2.33}$$

The property 2 of Definition-Proposition 6.2.7 implies that for every scheme point x in C, the section $\mathrm{disc}_{N+1,d}(\tau)$ vanishes at x if and only if the hypersurface $H_{\kappa(x)} \subset \mathbb{P}(E_{\kappa(x)})$ is singular. In particular, $\mathrm{disc}_{N+1,d}(\tau)$ is not the zero section of $M^{\otimes (N+1)(d-1)^N} \otimes (\det E)^{\otimes -d(d-1)^N}$.

Let Δ be the finite subset of critical values in C of the morphism

$$f : H \longrightarrow C.$$

Thanks to the work of Eriksson ([**Eri16**, Th. 1.2]), one may express the order of vanishing of the section $\mathrm{disc}_{N+1,d}(\tau)$ at some point x in Δ in terms of the localized N-th Chern class of the coherent sheaf $\Omega^1_{H/C}$, which is supported on the singular fibers of H.

In the special case where H is smooth and the morphism f has only non-degenerate critical points, his result easily implies the equality, for every x in Δ:

$$\mathrm{ord}_x\big(\mathrm{disc}_{N+1,d}(\tau)\big) = |\Sigma_x|,$$

where Σ denotes the set of critical points of f and:

$$\Sigma_x := \Sigma \cap f^{-1}(x).$$

This implies the equality in $\mathrm{CH}_0(C)$:

$$\begin{aligned} f_*[\Sigma] := \sum_{x\in\Delta} |\Sigma_x|\, x &= \sum_{x\in\Delta} \mathrm{ord}_x(\mathrm{disc}_{N+1,d}(\tau))\, x, \\ &= \mathrm{div}\big(\mathrm{disc}_{N+1,d}(\tau)\big), \\ &= c_1\big(M^{\otimes (N+1)(d-1)^N} \otimes (\det E)^{\otimes -d(d-1)^N}\big), \\ &= (d-1)^N\big((N+1)m - de\big), \end{aligned}$$

and provides an alternative proof of (6.2.16).

6.3. Linear pencils of hypersurfaces and Lefschetz pencils

Another setting where Proposition 6.1.1 applies is provided by linear pencils of hypersurfaces.

Let V be a connected smooth projective complex scheme of pure dimension $N \geq 1$, and let H be a non-singular hypersurface in $V \times \mathbb{P}^1$ such that the morphism

$$\mathrm{pr}_{1|H} : H \longrightarrow V$$

is dominant, or equivalently surjective, and let δ be its degree. The line bundle $\mathcal{O}_{V\times\mathbb{P}^1}(H)$ is isomorphic to the line bundle

$$\mathrm{pr}_1^* M \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(\delta),$$

where M denotes some line bundle over V, which is unique up to isomorphism.

PROPOSITION 6.3.1. *With the above notation, let us assume that the morphism*

$$\mathrm{pr}_{2|H} : H \longrightarrow \mathbb{P}^1$$

is surjective, and has a finite set Σ *of critical points, all of which are non-degenerate.*

The following equality holds in $\mathrm{CH}_0(V \times \mathbb{P}^1)$*:*

$$[\Sigma] = \delta\, \mathrm{pr}_1^*[(1 - c_1(M))^{-2} c(\Omega^1_V)]^{(N)}\, \mathrm{pr}_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(1)). \tag{6.3.1}$$

In particular, the cardinality of Σ *satisfies:*

$$|\Sigma| = \delta \int_V (1 - c_1(M))^{-2} c(\Omega^1_V).$$

Furthermore, if the line bundle M *on* V *is ample, the following equalities of integers hold:*

$$\begin{aligned} &\deg_{\mathbb{P}^1}(\mathcal{GK}_{\mathbb{P}^1,+}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta))) \\ &\qquad = \frac{\delta}{12} \int_V (1 - c_1(M))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}\delta}{12} \chi_{\mathrm{top}}(V) + v_N^+ |\Sigma|, \end{aligned} \tag{6.3.2}$$

and:

$$\begin{aligned} &\deg_{\mathbb{P}^1}(\mathcal{GK}_{\mathbb{P}^1,-}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta))) \\ &\qquad = \frac{\delta}{12} \int_V (1 - c_1(M))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}\delta}{12} \chi_{\mathrm{top}}(V) + v_N^- |\Sigma| \end{aligned} \tag{6.3.3}$$

where:

$$\chi_{\mathrm{top}}(V) = (-1)^N \int_V c_N(\Omega^1_V)$$

denotes the topological Euler characteristic of V*, and where* v_N^+ *and* v_N^- *are the rational numbers defined in Proposition 6.1.1.*

We could have formulated (6.3.2) and (6.3.3) as equalities in the Chow group $\mathrm{CH}^1(\mathbb{P}^1)_{\mathbb{Q}}$ of the base $\mathbb{P}^1$ of the pencil, as in Proposition 6.1.1. However the degree map establishes an isomorphism $\mathrm{CH}^1(\mathbb{P}^1)_{\mathbb{Q}} \simeq \mathbb{Q}$, and the present formulation is actually equivalent to this *a priori* more precise one.

6.3.1. Proof of Proposition 6.3.1. For simplicity, let us denote h the class $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ in $\mathrm{CH}^1(\mathbb{P}^1)$ and m the class $c_1(M)$ in $\mathrm{CH}^1(V)$.

We can apply equality (6.1.1) from Proposition 6.1.1 to the smooth projective complex scheme $X := V \times \mathbb{P}^1$; the smooth surjective morphism of complex schemes

$$\pi := \mathrm{pr}_2 : X \longrightarrow \mathbb{P}^1$$

and the line bundle on X:

$$L := \mathcal{O}_{V\times\mathbb{P}^1}(H) \simeq \mathrm{pr}_1^* M \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(\delta).$$

We obtain the equality in $\mathrm{CH}_0(V \times \mathbb{P}^1)$:

$$\begin{aligned} [\Sigma] &= \big[(1 - c_1(L))^{-1} c(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1})\big]^{(N+1)} \\ &= \big[(1 - \mathrm{pr}_1^* m - \delta\, \mathrm{pr}_2^* h)^{-1} \mathrm{pr}_1^* c(\Omega^1_V)\big]^{(N+1)} \\ &= \mathrm{pr}_1^*\big[(1-m)^{-1} c(\Omega^1_V)\big]^{(N+1)} + \delta\, \mathrm{pr}_2^* h\, \mathrm{pr}_1^*\big[(1-m)^{-2} c(\Omega^1_V)\big]^{(N)} && (6.3.4) \\ &= \delta\, \mathrm{pr}_2^* h\, \mathrm{pr}_1^*\big[(1-m)^{-2} . c(\Omega^1_V)\big]^{(N)}, && (6.3.5) \end{aligned}$$

where in (6.3.4), we have used that $\mathrm{CH}^2(\mathbb{P}^1)$ vanishes, and in (6.3.5), we have used that $\mathrm{CH}^{N+1}(V)$ vanishes.

This shows (6.3.1).

Now, let us assume that the line bundle M on V is ample. This implies that the line bundle L on X is ample relatively to the morphism π, so that the hypothesis of (6.1.2) and (6.1.3) is satisfied.

Since X is the product of $\mathbb{P}^1$ by V, all its relative Hodge vector bundles over $\mathbb{P}^1$ are trivial, and for every integer n, the Griffiths line bundle $\mathcal{GK}_{\mathbb{P}^1}(\mathbb{H}^n(X/\mathbb{P}^1))$ is trivial.

Consequently, taking degrees in equality (6.1.2) yields the following equality of rational numbers:

$$\begin{aligned}\deg_{\mathbb{P}^1}(\mathcal{GK}_{\mathbb{P}^1,+}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta))) = \frac{1}{12}\int_{V\times\mathbb{P}^1}(1-c_1(L))^{-1}c_1(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1})c(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1})\\ -\frac{1}{12}\int_{V\times\mathbb{P}^1}(c_1(L)c_N(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1}))+v_N^+|\Sigma|.\end{aligned} \tag{6.3.6}$$

Reasoning as above, the first term in (6.3.6) can be rewritten:

$$\begin{aligned}&\frac{1}{12}\int_{V\times\mathbb{P}^1}(1-c_1(L))^{-1}c_1(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1})c(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1})\\ &=\frac{1}{12}\int_{V\times\mathbb{P}^1}(1-\mathrm{pr}_1^*m-\delta\mathrm{pr}_2^*h)^{-1}\mathrm{pr}_1^*\big(c_1(\Omega^1_V)c(\Omega^1_V)\big)\\ &=\frac{1}{12}\int_{V\times\mathbb{P}^1}\mathrm{pr}_1^*(1-m)^{-1}\mathrm{pr}_1^*\big(c_1(\Omega^1_V)c(\Omega^1_V)\big)\\ &+\frac{\delta}{12}\int_{V\times\mathbb{P}^1}\mathrm{pr}_2^*h\,\mathrm{pr}_1^*(1-m)^{-2}\mathrm{pr}_1^*(c_1(\Omega^1_V)c(\Omega^1_V)),\\ &=\frac{\delta}{12}\int_V(1-m)^{-2}c_1(\Omega^1_V)c(\Omega^1_V),\end{aligned} \tag{6.3.7}$$

where we have used the fact that the group $\mathrm{CH}^{N+1}(V)$ vanishes, as well as the classical equality, for α a 0-cycle in V:

$$\int_{V\times\mathbb{P}^1}\mathrm{pr}_1^*\alpha\,\mathrm{pr}_2^*h=\int_V\alpha.$$

Similarly, the second term in (6.3.6) can be rewritten:

$$\begin{aligned}\frac{1}{12}\int_{V\times\mathbb{P}^1}c_1(L)c_N(\Omega^1_{V\times\mathbb{P}^1/\mathbb{P}^1}) &= \frac{1}{12}\int_{V\times\mathbb{P}^1}(\mathrm{pr}_1^*m+\delta\mathrm{pr}_2^*h)\mathrm{pr}_1^*c_N(\Omega^1_V),\\ &=\frac{1}{12}\int_{V\times\mathbb{P}^1}\mathrm{pr}_1^*(m\,c_N(\Omega^1_V))+\frac{\delta}{12}\int_{V\times\mathbb{P}^1}\mathrm{pr}_2^*h\,\mathrm{pr}_1^*c_N(\Omega^1_V),\\ &=\frac{\delta}{12}\int_V c_N(\Omega^1_V),\\ &=\frac{(-1)^N\delta}{12}\chi_{\mathrm{top}}(V).\end{aligned} \tag{6.3.8}$$

Replacing (6.3.7) and (6.3.8) in (6.3.6) yields the equality:

$$\deg_{\mathbb{P}^1}(\mathcal{GK}_{\mathbb{P}^1,+}(\mathbb{H}^{N-1}(H_\eta/\mathbb{P}^1_\eta)))=\frac{\delta}{12}\int_V(1-m)^{-2}c_1(\Omega^1_V)c(\Omega^1_V)+\frac{(-1)^{N+1}\delta}{12}\chi_{\mathrm{top}}(V)+v_N^+|\Sigma|,$$

which shows equality (6.3.2). Equality (6.3.3) follows similarly from equality (6.1.3).

6.3.2. Application to Lefschetz pencils. Proposition 6.3.1 applies notably to Lefschetz pencils.

Let V be a connected smooth projective complex scheme of pure dimension $N \geq 1$, embedded into some projective space $\mathbb{P}^r$, of dimension $r \geq \max(N, 2)$.

Let Λ be a projective subspace of dimension $r-2$ in $\mathbb{P}^r$ that intersects V transversally, and let $P \subset \mathbb{P}^{r\vee}$ the projective line in the dual projective space $\mathbb{P}^{r\vee}$ corresponding to Λ by projective duality.

Let us denote by:

$$\nu : \widetilde{\mathbb{P}}^r_\Lambda \longrightarrow \mathbb{P}^r$$

the blowing-up of Λ in $\mathbb{P}^r$. If $\widetilde{V}$ denote the proper transform of V by ν, the restriction:

$$\nu_{|\widetilde{V}} : \widetilde{V} \longrightarrow V$$

may be identified with the blowing-up in V of $\Lambda \cap V$, which is smooth of dimension $r-2$.

Let I be the incidence subscheme in $\mathbb{P}^r \times \mathbb{P}^{r\vee}$. It is an hypersurface in $\mathbb{P}^r \times \mathbb{P}^{r\vee}$, and both projections:

$$\mathrm{pr}_{1|I} : I \longrightarrow \mathbb{P}^r \quad \text{and} \quad \mathrm{pr}_{2|I} : I \longrightarrow \mathbb{P}^{r\vee}$$

are smooth morphisms. Moreover the line bundle $\mathcal{O}(I)$ over $\mathbb{P}^r \times \mathbb{P}^{r\vee}$ is isomorphic to $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{r\vee}}(1)$.

The projection of center Λ:

$$\mathbb{P}^r - \Lambda \longrightarrow P$$

extends to a smooth morphism of complex schemes:

$$p : \widetilde{\mathbb{P}}^r_\Lambda \longrightarrow P,$$

and the pair of morphisms (ν, p) establishes an isomorphism:

$$(\nu, p) : \widetilde{\mathbb{P}}^r_\Lambda \xrightarrow{\sim} I \cap (\mathbb{P}^r \times P)$$

between the scheme $\widetilde{\mathbb{P}}^r_\Lambda$ and the smooth hypersurface $I \cap (\mathbb{P}^r \times P)$ of $\mathbb{P}^r \times P$.

Consequently, by restriction, the pair of morphism $(\nu_{|\widetilde{V}}, p_{|\widetilde{V}})$ defines an isomorphism:

$$(\nu_{|\widetilde{V}}, p_{|\widetilde{V}}) : \widetilde{V} \xrightarrow{\sim} I \cap (V \times P)$$

between $\widetilde{V}$ and the smooth hypersurface $I \cap (V \times P)$ of $V \times P$. Moreover the line bundle $\mathcal{O}\big(I \cap (V \times P)\big)$ over $V \times P$ is isomorphic to $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^r}(1)_{|V} \otimes \mathrm{pr}_2^*\mathcal{O}_P(1)$.

Recall that the pencil of hyperplanes in $\mathbb{P}^r$ containing Λ is said to be a Lefschetz pencil with respect to the subvariety V of $\mathbb{P}^r$ when the morphism:

$$p_{|\widetilde{V}} : \widetilde{V} \longrightarrow P$$

has a finite set Σ of critical points, all of which are non-degenerate, and when the restriction

$$p_{|\Sigma} : \Sigma \longrightarrow P$$

is an injective map.

We may apply Proposition 6.3.1 to the hypersurface $I \cap (V \times P)$ in $V \times P$. With the notation of this proposition in this situation, the line bundle M is (isomorphic to) the restriction $\mathcal{O}_V(1)$ of $\mathcal{O}_{\mathbb{P}^r}(1)$ to V, and $\delta = 1$. Accordingly, we obtain the following result, which notably applies to Lefschetz pencils:

COROLLARY 6.3.2. *With the above notation, let us assume that the morphism:*

$$p_{|\widetilde{V}} : \widetilde{V} \longrightarrow P$$

has a finite set Σ of critical points, all of which are non-degenerate.

The following equality holds in $\mathrm{CH}_0(V \times P)$:

$$(\nu, p)_*[\Sigma] = \mathrm{pr}_1^*[(1 - c_1(\mathcal{O}_V(1)))^{-2} c(\Omega^1_V)]^{(N)}\, \mathrm{pr}_2^* c_1(\mathcal{O}_P(1)).$$

In particular, the cardinality of Σ *satisfies:*

$$|\Sigma| = \int_V (1 - c_1(\mathcal{O}_V(1)))^{-2} c(\Omega^1_V). \tag{6.3.9}$$

Furthermore, the following equalities of integers hold:

$$\deg_P(\mathcal{GK}_{P,+}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)))$$
$$= \frac{1}{12}\int_V (1 - c_1(\mathcal{O}_V(1)))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}}{12}\chi_{\mathrm{top}}(V) + v_N^+|\Sigma|,$$

and:

$$\deg_P(\mathcal{GK}_{P,-}(\mathbb{H}^{N-1}(\widetilde{V}_\eta/P_\eta)))$$
$$= \frac{1}{12}\int_V (1 - c_1(\mathcal{O}_V(1)))^{-2} c_1(\Omega^1_V) c(\Omega^1_V) + \frac{(-1)^{N+1}}{12}\chi_{\mathrm{top}}(V) + v_N^-|\Sigma|.$$

As already mentioned in 1.4.4, the expression (6.3.9) for the number of critical points in a Lefschetz pencil is established by Katz in [**SGA73**, Exposé XVII, cor. 5.6].

Annexe A

Sections génériques et pinceaux d'hypersurfaces

A.1. Introduction

Dans tout ce chapitre, k désigne un corps algébriquement clos de caractéristique 0. Par k-schéma, on entend un k-schéma séparé de type fini. On ne distinguera pas un k-schéma réduit X et l'ensemble de ses k-points $X(k)$.

Nous allons montrer les deux énoncés suivants :

THÉORÈME A.1.1. *Soit X un k-schéma lisse et connexe. Soit E un fibré vectoriel sur X, et soit*

$$W \subset H^0(X, E)$$

un sous-espace vectoriel de k-dimension finie du k-espace vectoriel des sections de E, qu'on identifie au schéma affine sur k

$$\operatorname{Spec}_k(S^\bullet W^\vee).$$

Si le fibré E est engendré sur X par ses sections dans W, alors il existe un ouvert Zariski non vide U de W tel que pour tout s dans U, le fermé de X défini par l'annulation de s soit lisse de codimension $\operatorname{rg} E$.

Quand le fibré E est un fibré en droites, l'espace vectoriel W définit un système linéaire de diviseurs de X, et le théorème A.1.1 est le théorème de Bertini bien connu (voir par exemple [**Har77**, III, Corollary 10.9]).

THÉORÈME A.1.2. *Reprenons les notations du théorème A.1.1, et supposons que E soit un fibré en droites L.*

Pour tout s dans U, notons $\mathcal{H}_s$ l'hypersurface lisse définie par l'annulation de s.

Soit C une courbe connexe lisse sur k, et soit

$$\pi : X \longrightarrow C$$

un k-morphisme lisse.

(1) *Si le fibré L (resp. le fibré des 1-jets verticaux $J_1(L)_{X/C}$) est engendré sur X par les sections de L dans W (resp. par leurs 1-jets verticaux), alors il existe un ouvert Zariski non vide U' de U tel que, pour tout s dans U', tous les points critiques de la restriction*

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \longrightarrow C$$

soient non dégénérés.

(2) *Sous les hypothèses précédentes, si X (et donc C) est projective, et si pour tout (x, y) dans X^2 tel que $x \neq y$ et $\pi(x) = \pi(y)$, le morphisme d'évaluation :*

$$W \longrightarrow L_x \oplus L_y, \quad s \longmapsto (s(x), s(y))$$

est surjectif, alors il existe un ouvert Zariski non vide U'' de U' tel que, pour tout s dans U'', chaque fibre de $\pi_{|\mathcal{H}_s}$ contienne au plus un point critique.

Noter que les ouverts U, U', U'' sont forcément denses dans W.

Lorsque le schéma X est projectif et que L est engendré sur X par ses sections dans W, on peut aussi remarquer que l'ensemble des conditions sur W apparaissant dans le théorème A.1.2 est simultanément vérifié si et seulement si, pour tout t dans C, le morphisme

$$\pi^{-1}(\{t\}) \longrightarrow \mathbb{P}_k(W^\vee) := \operatorname{Proj}_k(S^\bullet W),$$

qui à un point associe l'hyperplan de W des sections s'annulant en ce point, est un plongement fermé. (Voir par exemple [**Har77**, II, Proposition 7.3], appliqué au schéma projectif $\pi^{-1}(\{t\})$.)

Ces deux théorèmes sont des analogues algébriques de résultats de topologie différentielle bien connus. Par exemple, l'assertion (1) du théorème A.1.2 est un analogue algébrique de [**Mil63**, I, Corollary 6.8 p. 37]. Dans leur démonstration, le théorème de lissité générique (Théorème A.2.1 *infra*) joue le rôle du théorème de Sard en topologie différentielle.

La possibilité d'établir des énoncés de ce type qui complètent le classique théorème de Bertini, et leur analogie avec des énoncés de transversalité classiques en topologie différentielle sont certainement bien connus des spécialistes ; voir par exemple [**AGZV85**]. Toutefois la littérature publiée semble ne contenir que peu d'énoncés précis de ce type ; voir toutefois [**Kle74**].

Dans la situation où le k-schéma X est un fibré en projectifs sur la courbe C, le théorème A.1.2 nous permettra d'établir l'énoncé suivant, qui montre que les hypothèses de régularité sur les pinceaux d'hypersurfaces dans l'espace projectif apparaissant dans les théorèmes 1.4.2 et 6.2.4 sont satisfaites génériquement dès que la hauteur du pinceau d'hypersurfaces est suffisamment grande.

THÉORÈME A.1.3. *Soient C une k-courbe projective lisse connexe de genre g, E un fibré vectoriel de rang non nul sur C, M un fibré en droites sur C et $d \geq 1$ un entier. Soit de plus :*

$$\pi : \mathbb{P}(E) := \operatorname{Proj}_C(S^\bullet E^\vee) \longrightarrow C$$

le fibré en projectifs associé à E.

Si le degré de M satisfait à la minoration :

$$\deg_C M > 2g - 1 + d\,\mu_{\max}(E),$$

alors il existe un ouvert Zariski U'' non vide de l'espace vectoriel de sections $H^0(\mathbb{P}(E), \mathcal{O}_E(d) \otimes \pi^* M)$ tel que, pour toute section s dans U'', l'hypersurface H_s de $\mathbb{P}(E)$ définie par s soit lisse sur k, que tous les points critiques de la restriction

$$\pi_{|H_s} : H_s \longrightarrow C$$

soient non dégénérés, et que chaque fibre de $\pi_{|H_s}$ contienne au plus un point critique.

Dans la section A.6, nous montrerons comment l'existence de pinceaux de Lefschetz sur une variété projective lisse connexe apparaît aussi comme un cas particulier des théorèmes A.1.1 et A.1.2.

A.2. Transversalité et sections génériques

A.2.1. Transversalité de morphismes algébriques.

A.2.1.1. *Lissité générique et transversalité.*

THÉORÈME A.2.1 (Théorème de lissité générique, voir [**Har77**, III, Corollary 10.7]). *Soient X et Y deux k-schémas intègres*[1]*, avec X lisse sur k, et soit*

$$f : X \longrightarrow Y$$

un k-morphisme. Il existe un ouvert dense U de Y tel que la restriction

$$f_{|f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

1. En d'autres termes, X et Y sont des ≪ abstract varieties ≫ au sens de [loc. cit., II, 4].

soit un morphisme lisse.

Ce théorème admet la généralisation suivante, valable pour des k-schémas non irréductibles :

Corollaire A.2.2. *Soient X et Y deux k-schémas réduits, avec X lisse sur k, et soit*

$$f : X \longrightarrow Y$$

un k-morphisme. Il existe un ouvert dense U de Y tel que la restriction

$$f_{|f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

soit un morphisme lisse.

Démonstration. Soient $(Y_j)_{j\in J}$ les composantes irréductibles de Y.

Pour tout j dans J, on considère l'ouvert de Y_j défini par

$$\overset{\circ}{Y_j} := Y_j - \bigcup_{j'\in J-\{j\}} Y_{j'}.$$

C'est en fait un ouvert de Y, donc son image réciproque $f^{-1}(\overset{\circ}{Y_j})$ est un ouvert de X. Soit $(X_{i,j})_{i\in I_j}$ la famille finie des composantes irréductibles, c'est-à-dire des composantes connexes, de cette image réciproque.

Pour tout i dans I_j, le schéma $X_{i,j}$ est irréductible et réduit donc intègre. De plus, le schéma $\overset{\circ}{Y_j}$ est réduit, et un ouvert de Y_j irréductible, donc il est intègre.

En appliquant le théorème A.2.1 au morphisme de schémas intègres :

$$f^{|Y_j}_{|X_{i,j}} : X_{i,j} \longrightarrow \overset{\circ}{Y_j}$$

on obtient un ouvert dense $U_{i,j}$ de $\overset{\circ}{Y_j}$ tel que la restriction

$$f^{-1}(U_{i,j}) \cap X_{i,j} \longrightarrow U_{i,j}$$

soit lisse.

On définit un ouvert dense de $\overset{\circ}{Y_j}$ par

$$U_j := \bigcap_{i\in I_j} U_{i,j}.$$

Chacune des composantes connexes $(X_{i,j})_{i\in I_j}$ est alors lisse au-dessus de U_j, donc leur union disjointe, qui est $f^{-1}(\overset{\circ}{Y_j})$, est lisse au-dessus de U_j.

Donc en définissant un ouvert de Y par

$$U := \bigcup_{j\in J} U_j,$$

on obtient que la restriction

$$f^{-1}(U) \longrightarrow U$$

est lisse.

De plus, comme l'intersection de U avec chaque composante irréductible de Y est un ouvert non vide, U est dense dans Y, comme voulu. □

Définition A.2.3. *Soient*

$$f : X \longrightarrow Y$$

un morphisme de k-schémas lisses, et Z un fermé lisse de Y.

Soit x un k-point de X dont l'image $y := f(x)$ est dans Z. On note $T_{X,x}$ (resp. $T_{Y,y}$, resp. $T_{Z,y}$) l'espace tangent à X en x (resp. à Y en y, resp. à Z en y), et on note

$$(\mathrm{d}f)_x : T_{X,x} \longrightarrow T_{Y,y}$$

l'application k-linéaire différentielle de f en x.

On dit que f est transverse à Z en x *si on a l'égalité de sous-espaces vectoriels de $T_{Y,y}$:*

$$T_{Y,y} = (\mathrm{d}f)_x(T_{X,x}) + T_{Z,y}.$$

De plus, on dit que f est transverse à Z *si pour tout x point de $f^{-1}(Z)$, f est transverse à Z en x.*

Cette définition est un analogue algébrique de la définition classique de transversalité en topologie différentielle, voir par exemple [**GP10**, Chapter 1, §5] ou [**AGZV85**, Part I, 2.3].

Proposition A.2.4. *Avec les notations de la définition A.2.3, si f est transverse à Z, alors le sous-schéma fermé $f^{-1}(Z)$ dans X est lisse.*

De plus, si Z est de codimension pure p dans Y, alors f est transverse à Z si et seulement si le sous-schéma fermé $f^{-1}(Z)$ est lisse de codimension pure p dans X.

L'implication directe est bien connue en topologie différentielle, elle est par exemple montrée dans [**GP10**, Chapter 1, §5]. En géométrie algébrique, on peut compléter cette preuve et établir la réciproque, en utilisant le fait que le critère jacobien de lissité est une équivalence.

Démonstration. Soit x un point de $f^{-1}(Z)$ et y son image par f. Soit p_y la codimension de Z en y.

Par le critère jacobien (voir par exemple [**BLR90**, 2.2, Proposition 7]), il existe, au voisinage de y, des fonctions k-régulières $g_1, ..., g_{p_y}$ qui définissent le sous-schéma Z, dont les différentielles $(\mathrm{d}g_1)_y, ..., (\mathrm{d}g_{p_y})_y$ dans $\Omega^1_{Y,y}$ sont linéairement indépendantes.

Par conséquent, comme la fibre $\Omega^1_{Y,y}$ est le dual de l'espace tangent $T_{Y,y}$, l'application k-linéaire

$$((\mathrm{d}g_1)_y, ..., (\mathrm{d}g_{p_y})_y) : T_{Y,y} \longrightarrow k^{\oplus p_y}$$

est surjective, et son noyau est exactement l'espace tangent $T_{Z,y}$.

Au voisinage de x, le sous-schéma fermé $f^{-1}(Z)$ est défini par les équations $(g_1 \circ f, ..., g_{p_y} \circ f)$.

Donc par le critère jacobien, il est lisse de codimension p_y en x si et seulement si les différentielles $(\mathrm{d}(g_1 \circ f))_x, ..., (\mathrm{d}(g_{p_y} \circ f))_x$ sont linéairement indépendantes dans $\Omega^1_{X,x}$, c'est-à-dire si et seulement si l'application k-linéaire

$$(\mathrm{d}(g_1 \circ f)_x, ..., \mathrm{d}(g_{p_y} \circ f)_x) = (\mathrm{d}(g_1)_y, ..., \mathrm{d}(g_{p_y})_y) \circ (\mathrm{d}f)_x : T_{X,x} \longrightarrow k^{\oplus p_y}$$

est surjective, c'est-à-dire si et seulement si la restriction

$$((\mathrm{d}g_1)_y, ..., (\mathrm{d}g_{p_y})_y)_{|(\mathrm{d}f)_x(T_{X,x})} : (\mathrm{d}f)_x(T_{X,x}) \longrightarrow k^{\oplus p_y}$$

est surjective.

Or, on sait que l'application k-linéaire $((\mathrm{d}g_1)_y, ..., (\mathrm{d}g_{p_y})_y)$ est surjective et que son noyau est le sous-espace $T_{Z,y}$, donc elle induit un isomorphisme entre le quotient $T_{Y,y}/T_{Z,y}$ et $k^{\oplus p_y}$.

Donc la restriction ci-dessus est surjective si et seulement si la composition

$$(\mathrm{d}f)_x(T_{X,x}) \hookrightarrow T_{Y,y} \twoheadrightarrow T_{Y,y}/T_{Z,y}$$

est surjective, c'est-à-dire si et seulement si les sous-espaces $T_{Z,y}$ et $(\mathrm{d}f)_x(T_{X,x})$ engendrent $T_{Y,y}$, c'est-à-dire si et seulement si f est transverse à Z en x.

On a ainsi montré que pour tout point x de $f^{-1}(Z)$, si p_y est la codimension de Z en y, le sous-schéma Z est lisse de codimension p_y en x si et seulement si f est transverse à Z en x.

En appliquant cette équivalence sur chaque composante connexe de Z, on obtient que si f est transverse à Z, alors $f^{-1}(Z)$ est une union disjointe de sous-schémas lisses, donc est lisse, ce qui montre la première affirmation.

De plus, si Z est purement de codimension p, alors tous les p_y valent p, donc on obtient la deuxième affirmation. □

Appliquée aux sections d'un fibré vectoriel, la proposition A.2.4 va nous permettre d'établir l'énoncé suivant :

PROPOSITION A.2.5. *Soient X un k-schéma lisse, et E un fibré vectoriel de rang $e \geq 1$ sur X. On note*

$$\mathbb{V}(E) := \operatorname{Spec}_X(S^\bullet E^\vee)$$

l'espace total de E sur X, et

$$0_X \subset \mathbb{V}(E)$$

l'image de la section nulle.

Soit s une section de E sur X, et soit

$$f_s : X \longrightarrow \mathbb{V}(E)$$

le morphisme de k-schémas induit par s.

Les trois assertions suivantes sont équivalentes :

(1) Le sous-schéma fermé défini par l'annulation de s est lisse de codimension e.

(2) Pour tout x dans X tel que $s(x)$ s'annule, la différentielle

$$(\mathrm{d}s)_x : T_{X,x} \longrightarrow E_x$$

est surjective.

(3) L'application f_s est transverse au fermé 0_X.

Lorsque les conditions (1)-(3) sont réalisées, on dit que s est *transverse à la section nulle.*

DÉMONSTRATION. L'équivalence entre (1) et (3) vient de la proposition A.2.4 appliquée au morphisme f_s, et du fait que le fermé de X défini par l'annulation de s est exactement l'image réciproque $f_s^{-1}(0_X)$.

Montrons l'équivalence entre (2) et (3).

Soit x un k-point de X où s s'annule.

L'espace tangent à $\mathbb{V}(E)$ en $f_s(x)$ se décompose canoniquement en somme directe de ses sous-espaces

$$T_{0_X, f_s(x)} \simeq T_{X,x}$$

et

$$T_{\mathbb{V}(E)_x, f_s(x)} \simeq T_{E_x,0} \simeq E_x.$$

Dans cette décomposition, la différentielle de f_s au point x s'écrit

$$(\mathrm{d}f_s)_x = \begin{pmatrix} \mathrm{Id}_{T_{X,x}} \\ (\mathrm{d}s)_x \end{pmatrix}.$$

Donc f_s est transverse à la section nulle en x si et seulement si la composition de $(\mathrm{d}f_s)_x$ avec la projection sur E_x relativement à $T_{0_X,f_s(x)}$ est surjective, c'est-à-dire si et seulement si $(\mathrm{d}s)_x$ est surjective.

Comme cela est vrai pour tout x où s s'annule, on obtient l'équivalence entre (2) et (3). □

A.2.1.2. *Théorème de transversalité générique.* On peut déduire du corollaire A.2.2 des énoncés de transversalité générique.

L'énoncé suivant apparaît en substance dans la preuve de [**GP10**, Chapter 2, §3, « The transversality theorem »].

LEMME A.2.6. *Soient V_1, V_2, V_3 trois k-espaces vectoriels, et V_2' un sous-espace de V_2.*

Soient

$$f : V_1 \longrightarrow V_2 \quad \textit{et} \quad g : V_1 \longrightarrow V_3$$

des applications k-linéaires.

On définit des sous-espaces vectoriels de V_1 par :

$$V_1' := f^{-1}(V_2')$$

et

$$V_1'' := \mathrm{Ker}(g).$$

On suppose qu'on a l'égalité de sous-espaces de V_2 :

$$V_2 = f(V_1) + V_2' \tag{A.2.1}$$

et que la restriction

$$g_{|V_1'} : V_1' \longrightarrow V_3$$

est surjective.

Alors on a l'égalité de sous-espaces de V_2 :

$$V_2 = f(V_1'') + V_2'. \tag{A.2.2}$$

DÉMONSTRATION. Soit v_2 un vecteur de V_2.

En utilisant l'égalité (A.2.1), on obtient une décomposition

$$v_2 = f(v_1) + v_2',$$

où v_1 est un vecteur de V_1 et v_2' est un vecteur de V_2'.

On considère le vecteur

$$v_3 := g(v_1) \in V_3.$$

Comme $g_{|V_1'}$ est surjective, il existe un vecteur v_1' de V_1' que g envoie sur v_3.

On définit un vecteur dans V_1 :

$$v_1'' := v_1 - v_1'.$$

Par définition de v_1', le vecteur $g(v_1'')$ est nul, donc v_1'' est dans V_1''.

D'autre part, on a l'égalité dans V_2 :

$$\begin{aligned} v_2 &= f(v_1) + v_2', \\ &= f(v_1'' + v_1') + v_2', \\ &= f(v_1'') + (f(v_1') + v_2'), \end{aligned}$$

où v_1'' est dans V_1'', et $f(v_1')$ est dans $f(V_1')$, donc dans V_2' par définition.

Donc v_2 est dans le sous-espace $f(V_1'') + V_2'$. □

THÉORÈME A.2.7. *Soient X, Y, S des k-schémas lisses, et Z un fermé lisse de Y. Soient*

$$F : X \longrightarrow Y$$

et

$$\pi : X \longrightarrow S$$

des k-morphismes. On suppose que le morphisme π est lisse.

Si F est transverse à Z, alors il existe un ouvert dense U de S tel que pour tout k-point s de U, la restriction

$$F_{|\pi^{-1}(\{s\})} : \pi^{-1}(\{s\}) \longrightarrow Y$$

soit transverse à Z.

C'est un analogue algébrique du théorème de transversalité en topologie différentielle dans sa version élémentaire donnée dans [**GP10**, Chapter 2, §3, ≪ The transversality theorem ≫]. Il découle du théorème de lissité générique A.2.1 comme le théorème de transversalité en topologie différentielle découle du théorème de Sard.

Quand Z est de codimension pure dans Y, on peut le déduire de [**Kle74**, Lemma 1], avec g l'inclusion de Z dans Y, et de la proposition A.2.4.

Démonstration. Nous allons reprendre la preuve de [**GP10**].

Si s est un point de S, nous noterons plus simplement les fibres de π de la manière suivante :

$$X_s := \pi^{-1}(\{s\}).$$

On définit un fermé

$$W := F^{-1}(Z) \subset X.$$

Comme F est transverse à Z, par la proposition A.2.4, ce fermé est lisse sur k.

On applique le corollaire A.2.2 au morphisme de k-schémas lisses (en particulier réduits) :

$$\pi_{|W} : W \longrightarrow S.$$

On obtient un ouvert dense U de S tel que le morphisme

$$\pi_{|W\cap\pi^{-1}(U)} : W \cap \pi^{-1}(U) \longrightarrow U$$

soit lisse.

Soit s un point de U, et x un point de X_s dont l'image par F, notée y, est dans Z. En particulier, x est dans W.

On va appliquer le lemme A.2.6 aux k-espaces vectoriels :

$$V_1 := T_{X,x},$$
$$V_2 := T_{Y,y},$$
$$V_3 := T_{S,s},$$

au sous-espace vectoriel :

$$V_2' := T_{Z,y} \subset T_{Y,y},$$

et aux applications linéaires :

$$f := (\mathrm{d}F)_x : V_1 \longrightarrow V_2,$$
$$g := (\mathrm{d}\pi)_x : V_1 \longrightarrow V_3.$$

On remarque que les sous-espaces vectoriels V_1' et V_1'' de V_1 sont alors donnés par :

$$V_1' = T_{W,x} \subset T_{X,x}, \quad V_1'' = T_{X_s,x} \subset T_{X,x}.$$

Comme s est dans U, et comme la restriction $\pi_{|W\cap\pi^{-1}(U)}$ est lisse, la différentielle

$$(\mathrm{d}\pi_{|W})_x = g_{|V_1'}$$

est une application linéaire surjective.

D'autre part, comme F est transverse à Z en x, on a l'égalité de sous-espaces vectoriels de $T_{Y,y}$:

$$T_{Y,y} = (\mathrm{d}F)_x(T_{X,x}) + T_{Z,y},$$

donc l'égalité (A.2.1) est vérifiée.

Donc on peut appliquer le lemme A.2.6, et on obtient l'égalité (A.2.2), qui s'écrit comme l'égalité de sous-espaces vectoriels de $T_{Y,y}$:

$$T_{Y,y} = (\mathrm{d}F_{|X_s})_x(T_{X_s,x}) + T_{Z,y}.$$

Autrement dit, le morphisme $F_{|X_s}$ est transverse à Z en x.

Comme cela est vrai pour tout x dans X_s tel que $F(x)$ est dans Z, cela montre que pour tout s dans U, le morphisme $F_{|X_s}$ est transverse à Z. □

A.2.1.3. *Théorème de transversalité relative générique.* On dispose aussi d'une variante relative du théorème A.2.7.

Lemme A.2.8. *On considère un diagramme commutatif :*

(A.2.3)
$$\begin{array}{ccccc} V_1 & \xrightarrow{f} & V_2 & \supset & V_2' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ V_3 & = & V_3 & & \end{array}$$

où V_1, V_2 et V_3 sont des k-espaces vectoriels, V_2' est un sous-espace vectoriel de V_2, et f, g et h sont des applications linéaires.

On définit des sous-espaces vectoriels par :

$$V_1'' = \mathrm{Ker}(g) \subset V_1,$$
$$V_2'' = \mathrm{Ker}(h) \subset V_2,$$

si bien que par commutativité du diagramme (A.2.3)*, f envoie V_1'' dans V_2''.*

Si on a l'égalité de sous-espaces vectoriels de V_2 :

(A.2.4) $$V_2 = f(V_1'') + V_2',$$

alors on a l'égalité de sous-espaces vectoriels de V_2'' :

(A.2.5) $$V_2'' = f(V_1'') + (V_2' \cap V_2'').$$

D'autre part, si g est surjective, et si on a l'égalité (A.2.5)*, alors on a l'égalité de sous-espaces de V_2 :*

(A.2.6) $$V_2 = f(V_1) + V_2'.$$

Démonstration. Supposons qu'on ait (A.2.4).

Soit v_2'' un vecteur de V_2''.

En utilisant l'égalité (A.2.4), on obtient une décomposition

$$v_2'' = f(v_1'') + v_2'$$

où v_1'' est un vecteur de V_1'' et v_2' un vecteur de V_2'.

On a l'égalité dans V_3 :

$$\begin{aligned} h(v_2') &= h(v_2'') - h(f(v_1'')), \\ &= h(v_2'') - g(v_1'') \text{ par commutativité du diagramme (A.2.3)}, \\ &= 0 \text{ par définition de } V_1'' \text{ et } V_2''. \end{aligned}$$

Donc le vecteur v_2' est dans le sous-espace V_2'', donc dans $V_2' \cap V_2''$.

Donc on a une décomposition dans V_2'' :

$$v_2'' = f(v_1'') + v_2',$$

où v_1'' est dans V_1'' et v_2' est dans $V_2' \cap V_2''$.

Donc v_2'' est dans le sous-espace $f(V_1'') + V_2'$ de V_2'', ce qui montre l'égalité (A.2.5), et donc la première affirmation du lemme.

Supposons maintenant qu'on ait l'égalité (A.2.5).

Soit v_2 un vecteur de V_2.

Comme g est surjective, il existe un vecteur v_1 de V_1 tel qu'on ait l'égalité dans V_3 :

$$g(v_1) = h(v_2).$$

En particulier, par commutativité du diagramme, on a l'égalité dans V_3 :

$$h(f(v_1)) = h(v_2),$$

donc le vecteur $v_2 - f(v_1)$ est dans V_2''.

On lui applique l'égalité (A.2.5), pour obtenir une décomposition

$$v_2 - f(v_1) = f(v_1'') + v_2',$$

où v_1'' est dans V_1'' et v_2' est dans V_2'.

On a donc l'égalité dans V_2 :

$$v_2 = f(v_1 + v_1'') + v_2',$$

où $v_1 + v_1''$ est dans V_1 et v_2' dans V_2'.

Donc le vecteur v_2 est dans $f(V_1) + V_2'$.

On obtient ainsi l'égalité (A.2.6), et donc la deuxième affirmation. □

Corollaire A.2.9. *On considère un diagramme commutatif*

$$\begin{array}{ccccc} X & \xrightarrow{F} & Y & \supset & Z \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \\ S & = & S & & \end{array} \tag{A.2.7}$$

où S, X et Y sont des k-schémas lisses, Z est un fermé lisse de Y, et F, π et π' sont des k-morphismes.

On suppose que les morphismes π et π' sont lisses, ainsi que la restriction $\pi'_{|Z}$.

Si F est transverse à Z, alors il existe un ouvert dense U de S tel que pour tout s k-point de U, le morphisme

$$F_{|\pi^{-1}(\{s\})}^{|\pi'^{-1}(\{s\})} : \pi^{-1}(\{s\}) \longrightarrow \pi'^{-1}(\{s\})$$

soit transverse à $Z \cap \pi'^{-1}(\{s\})$.

Réciproquement, si pour tout k-point s de S, le morphisme $F_{|\pi^{-1}(\{s\})}^{|\pi'^{-1}(\{s\})}$ est transverse à $Z \cap \pi'^{-1}(\{s\})$, alors F est transverse à Z.

Démonstration. Si s est un point de S, nous noterons plus simplement ses fibres de la manière suivante :

$$X_s := \pi^{-1}(\{s\}),$$
$$Y_s := \pi'^{-1}(\{s\}).$$

Supposons que F est transverse à Z.

Par le théorème A.2.7, il existe un ouvert dense U de S tel que pour tout s dans U, la restriction

$$F_{|X_s} : X_s \longrightarrow Y$$

soit transverse à Z.

Soit s dans U, et soit x dans X_s un point dont l'image par F, notée y, est dans $Z \cap Y_s$.

On va appliquer le lemme A.2.8 aux k-espaces vectoriels :

$$V_1 := T_{X,x},$$
$$V_2 := T_{Y,y},$$
$$V_3 := T_{S,s},$$

au sous-espace vectoriel

$$V_2' := T_{Z,y} \subset T_{Y,y},$$

et aux applications linéaires

$$f := (\mathrm{d}F)_x : V_1 \longrightarrow V_2,$$
$$g := (\mathrm{d}\pi)_x : V_1 \longrightarrow V_3,$$
$$h := (\mathrm{d}\pi')_x : V_2 \longrightarrow V_3.$$

Comme les morphismes F, π et π' s'insèrent dans le diagramme commutatif (A.2.7), les applications linéaires f, g et h s'insèrent dans le diagramme commutatif (A.2.3).

De plus, on remarque que les sous-espaces vectoriels V_1'' et V_2'' de V_1 et V_2 respectivement, sont donnés par :

$$V_1' = T_{X_s,x} \subset T_{X,x},$$
$$V_2' = T_{Y_s,y} \subset T_{Y,y}.$$

Comme s est dans U, le morphisme $F_{|X_s}$ est transverse à Z en x, donc on a l'égalité de sous-espaces vectoriels :

$$T_{Y,y} = (\mathrm{d}F_{|X_s})_x(T_{X_s,x}) + T_{Z,y},$$

donc on a l'égalité (A.2.4).

Donc en appliquant la première affirmation du lemme A.2.8, on obtient l'égalité (A.2.5), qui s'écrit comme l'égalité de sous-espaces vectoriels de $T_{Y_s,y}$:

$$T_{Y_s,y} = (\mathrm{d}F_{|X_s}^{|Y_s})_x(T_{X_s,x}) + T_{Z\cap Y_s,y},$$

donc le morphisme $F_{|X_s}^{|Y_s}$ est transverse à $Z \cap Y_s$ en x.

Comme cela est vrai en tout x de X_s tel que $F(x)$ est dans Z, cela montre que pour tout s dans U, le morphisme $F_{|X_s}^{|Y_s}$ est transverse à $Z \cap Y_s$.

Réciproquement, supposons que pour tout s dans S, le morphisme $F_{|X_s}^{|Y_s}$ est transverse à $Z \cap Y_s$.

Soit x un point de X dont l'image par F, notée y, est dans Z. Soit s dans S son image par π.

On va appliquer le lemme A.2.8 avec les mêmes notations que ci-dessus.

Comme le morphisme π est lisse, l'application linéaire

$$g = (\mathrm{d}\pi)_x : T_{X,x} \longrightarrow T_{S,s}$$

est surjective.

De plus, comme $F_{|X_s}^{|Y_s}$ est transverse à $Z \cap Y_s$ en x, on a l'égalité de sous-espaces vectoriels de $T_{Y_s,y}$:

$$T_{Y_s,y} = (\mathrm{d}F_{|X_s}^{|Y_s})_x(T_{X_s,x}) + T_{Z\cap Y_s,y},$$

c'est-à-dire qu'on a l'égalité (A.2.5).

Donc en appliquant la deuxième affirmation du lemme A.2.8, on obtient l'égalité (A.2.6), qui s'écrit comme l'égalité de sous-espaces vectoriels de $T_{Y,y}$:

$$T_{Y,y} = (\mathrm{d}F)_x(T_{X,x}) + T_{Z,y},$$

donc le morphisme F est transverse à Z en x.

Cela est vrai pour tout x dans X tel que $F(x)$ est dans Z, donc on obtient que F est transverse à Z en x.

Cela conclut la preuve. □

A.2.2. Sections de fibrés dont le lieu des zéros est lisse : démonstration du théorème A.1.1. Soient X un k-schéma lisse et connexe de dimension n, E un fibré vectoriel de rang e sur X, et

$$W \subset H^0(X, E)$$

un sous-espace vectoriel de dimension finie w.

On considère le sous-schéma fermé $\mathcal{Z}$ de $X \times W$ défini par l'équation :

$$(x, s) \in \mathcal{Z} \Longleftrightarrow s(x) = 0 \quad (\in E_x),$$

pour tout (x, s) dans $X \times W$. En d'autres termes, on considère la section tautologique $\underline{s}$ du fibré $\mathrm{pr}_1^* E$ sur $X \times W$, définie par :

$$\underline{s}(x, s) := s(x) \in E_x,$$

pour tout (x, s) dans $X \times W$. Le sous-schéma fermé $\mathcal{Z}$ est alors le sous-schéma de $X \times W$ défini par l'annulation de cette section $\underline{s}$.

Proposition A.2.10. *Pour tout k-point (x, s) de $\mathcal{Z}$, et pour tout $(\delta x, \delta s)$ dans $T_{(x,s)}(X \times W)$, la différentielle de la section tautologique $\underline{s}$ sur $X \times W$ est donnée par*

$$(\mathrm{d}\underline{s})_{(x,s)}(\delta x, \delta s) = (\delta s)(x) + (\mathrm{d}s)_x(\delta x) \quad \in E_x \simeq (\mathrm{pr}_1^* E)_{(x,s)}, \tag{A.2.8}$$

où les espaces vectoriels W et $T_{W,s}$ sont identifiés, et où $(\mathrm{d}s)_x$ est un élément de $E_x \otimes T_{X,x}^\vee$ bien défini car s s'annule en x.

En particulier, si la section $\underline{s}$ est transverse à la section nulle (et donc le k-schéma $\mathcal{Z}$ est lisse), alors l'espace tangent de $\mathcal{Z}$ en (x, s) est décrit par l'égalité :

$$T_{\mathcal{Z},(x,s)} = \{(\delta x, \delta s) \in T_{X,x} \times T_{W,s} \quad | (\delta s)(x) + (\mathrm{d}s)_x(\delta x) = 0\}. \tag{A.2.9}$$

Démonstration. Par linéarité, il suffit de calculer la différentielle sur les couples de la forme $(0, \delta s)$ ou $(\delta x, 0)$ dans $T_{X,x} \times T_{W,s}$.

Comme la restriction de $\underline{s}$ à $\{x\} \times W$:

$$(x, s') \in \{x\} \times W \longmapsto \underline{s}(x, s') = s'(x) \in E_x$$

est linéaire, elle est sa propre différentielle en s. Donc pour tout couple de la forme $(0, \delta s)$ avec δs dans W, on a l'égalité

$$(\mathrm{d}\underline{s})_{(x,s)}(0, \delta s) = (\delta s)(x) \in E_x.$$

D'autre part, la restriction de $\underline{s}$ à $X \times \{s\}$ s'identifie à s, donc sa différentielle est celle de s. Donc pour tout couple de la forme $(\delta x, 0)$, on a l'égalité :

$$(\mathrm{d}\underline{s})_{(x,s)}(\delta x, 0) = (\mathrm{d}s)_x(\delta x) \in E_x.$$

Cela montre l'égalité voulue. □

Proposition A.2.11. *Les assertions suivantes sont équivalentes :*

(1) Le fibré E est engendré sur X par ses sections dans W.

(2) La projection

$$(\mathrm{pr}_1)_{|\mathcal{Z}} : \mathcal{Z} \longrightarrow X$$

est lisse de dimension relative $w - e$.

(3) Le schéma $\mathcal{Z}$ est lisse de codimension e dans $X \times W$.

(4) La section $\underline{s}$ est transverse à la section nulle. En d'autres termes, pour tout (x, s) point de $\mathcal{Z}$, la différentielle

$$(\mathrm{d}\underline{s})_{(x,s)} : T_{X \times W,(x,s)} \longrightarrow (\mathrm{pr}_1^* E)_{(x,s)}$$

est surjective.

Démonstration. Nous allons montrer les implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

Supposons que (1) est vérifiée. On a un morphisme de fibrés sur X :

$$\mathrm{ev} : W \otimes_k \mathcal{O}_X \longrightarrow E,$$

défini par l'évaluation. Comme E est engendré par ses sections dans W, ce morphisme est surjectif, donc son noyau K est un fibré vectoriel sur X de rang $w - e$.

On peut considérer l'espace total

$$\mathbb{V}(K) := \operatorname{Spec}_X(S^\bullet K^\vee),$$

qui est un sous-schéma fermé du k-schéma

$$\mathbb{V}(W \otimes_k \mathcal{O}_X) \simeq X \times W.$$

Ce sous-schéma fermé est identifié à $\mathcal{Z}$, et la projection de $\mathbb{V}(K)$ sur X est identifiée au morphisme $(\mathrm{pr}_1)_{|\mathcal{Z}}$, qui est donc lisse de dimension relative $w - e$.

Cela montre l'implication (1) ⇒ (2).

L'implication (2) ⇒ (3) est évidente car X est lisse sur k.

Comme le sous-schéma fermé $\mathcal{Z}$ est exactement le lieu d'annulation de la section $\underline{s}$ sur $X \times W$, l'implication (3) ⇒ (4) découle de la proposition A.2.5.

Enfin, supposons que (4) est vérifiée. Soit x un k-point de X. On considère le point $(x, 0)$ dans $X \times W$, il est trivialement dans $\mathcal{Z}$.

Par l'égalité (A.2.8) de la proposition A.2.10, la différentielle de la section $\underline{s}$ au point $(x, 0)$ est donnée par l'égalité, pour tout $(\delta x, \delta s)$ dans $T_{X\times W,(x,0)}$:

$$(\mathrm{d}\underline{s})_{(x,0)}(\delta x, \delta s) = (\delta s)(x) \quad \in E_x \simeq (\mathrm{pr}_1^* E)_{(x,s)}.$$

Comme cette différentielle est surjective, on obtient que le morphisme d'évaluation en x :

$$\delta s \in W \longmapsto (\delta s)(x) \in E_x$$

est surjectif.

Cela est vrai pour tout point x de X, donc le fibré E est engendré par ses sections dans W.

Cela montre l'implication (4) ⇒ (1) et finit la preuve. □

Corollaire A.2.12. *Si les assertions équivalentes de la proposition A.2.11 sont vérifiées, alors il existe un ouvert non vide U de W tel que la restriction*

$$(\mathrm{pr}_2)_{|\mathcal{Z}_U} : \mathcal{Z}_U \longrightarrow U$$

soit lisse de dimension relative $n - e$.

En particulier, pour tout s dans U, le fermé défini par l'annulation de s est lisse de codimension e.

Démonstration. Par hypothèse, le k-schéma $\mathcal{Z}$ est lisse de codimension e, donc en appliquant le corollaire A.2.2 au morphisme de projection

$$(\mathrm{pr}_2)_{|\mathcal{Z}} : \mathcal{Z} \longrightarrow W,$$

on obtient un ouvert dense U de W tel que la restriction $(\mathrm{pr}_2)_{|\mathcal{Z}_U}$ soit lisse.

Comme le schéma $\mathcal{Z}$ est de codimension e dans $X \times W$, il est de dimension $n + w - e$, donc la dimension relative de $(\mathrm{pr}_2)_{|\mathcal{Z}_U}$ est bien $n - e$.

Enfin, pour tout s dans U, par définition de $\mathcal{Z}$, la fibre

$$(\mathrm{pr}_2)^{-1}_{|\mathcal{Z}_U}(\{s\}) = \mathcal{Z}_U \cap (X \times \{s\})$$

s'identifie au sous-schéma fermé de X défini par l'annulation de s, qui est donc lisse de codimension e. □

La dernière assertion du corollaire A.2.12 établit le théorème A.1.1.

Dans les sections suivantes, nous allons nous intéresser au cas où E est un fibré en droites et où X est fibrée sur une courbe.

A.3. Hypersurfaces dans un pinceau de variétés lisses et points critiques non dégénérés

Dans cette section et la suivante, comme dans le théorème A.1.2, on considère X un k-schéma lisse et connexe de dimension n, C une k-courbe lisse et connexe,

$$\pi : X \longrightarrow C$$

un morphisme lisse, L un fibré en droites sur X, et

$$W \subset H^0(X, L)$$

un sous-espace de dimension finie w, qu'on identifie au schéma affine sur k

$$\operatorname{Spec}_k(S^\bullet W^\vee).$$

On suppose que L est engendré sur X par ses sections dans W.

A.3.1. Notations et définitions.

A.3.1.1. *Le diviseur $\mathcal{H}$.* Le sous-schéma $\mathcal{Z}$ de $X \times W$ introduit dans A.2.2 sera noté :

$$\mathcal{H} := \{(x, s) \in X \times W \quad | s(x) = 0\}.$$

D'après la proposition A.2.11, $\mathcal{H}$ est un diviseur lisse dans $X \times W$ et la première projection

$$(\mathrm{pr}_1)_{|\mathcal{H}} : \mathcal{H} \hookrightarrow X \times W \longrightarrow X$$

est lisse de dimension relative $w-1$. En outre, d'après le corollaire A.2.12, il existe un ouvert Zariski non vide U dans W tel que la projection

$$(\mathrm{pr}_2)_{|\mathcal{H}_U} : \mathcal{H}_U \hookrightarrow X \times U \longrightarrow U$$

soit lisse de dimension relative $n-1$.

Pour tout s dans W, notons

$$\mathcal{H}_s := \mathrm{pr}_2^{-1}(\{s\}) \subset X \times \{s\} \simeq X$$

la fibre de s, qu'on identifiera au diviseur de X défini par l'annulation de s. En particulier, si s est dans U, le sous-schéma $\mathcal{H}_s$ est une hypersurface lisse.

A.3.1.2. *Le fibré vectoriel F.* Nous noterons T_{pr_2} le sous-fibré vectoriel du fibré vectoriel $T_{\mathcal{H}_U}$ sur $\mathcal{H}_U$ défini comme l'espace tangent relatif du morphisme $(\mathrm{pr}_2)_{|\mathcal{H}_U}$.

On définit alors un fibré vectoriel sur $\mathcal{H}_U$ par :

$$F := \mathrm{Hom}(T_{\mathrm{pr}_2}, (\pi \circ \mathrm{pr}_1)^* T_C).$$

Proposition A.3.1. *Les fibrés T_{pr_2} et F sont de rang $n-1$.*

Démonstration. On sait que sur U, le morphisme

$$(\mathrm{pr}_2)_{|\mathcal{H}_U} : \mathcal{H}_U \longrightarrow U \subset W$$

est de dimension relative $n-1$.

Donc le fibré T_{pr_2} sur $\mathcal{H}_U$ est de rang $n-1$.

Comme C est une courbe, le fibré T_C est de rang 1, donc le fibré F est de rang $n-1$. □

A.3.1.3. *La section D.* Construisons maintenant une section canonique D du fibré F.

La différentielle du morphisme
$$\pi \circ \mathrm{pr}_1 : \mathcal{H}_U \longrightarrow C,$$
induit une section
$$\mathrm{d}(\pi \circ \mathrm{pr}_1) \in H^0(\mathcal{H}_U, \mathrm{Hom}(T_{\mathcal{H}_U}, (\pi \circ \mathrm{pr}_1)^* T_C)).$$
Par restriction à T_{pr_2}, on en déduit une section :
$$D := \mathrm{d}(\pi \circ \mathrm{pr}_1)_{|T_{\mathrm{pr}_2}} \in H^0(\mathcal{H}_U, F).$$

Pour tout point (x, s) dans $\mathcal{H}_U$, comme s est dans l'ouvert U, l'hypersurface $\mathcal{H}_s$ de X est lisse, et la valeur $D(x, s)$ dans $F_{(x,s)}$ s'interprète comme la différentielle en x du morphisme
$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \hookrightarrow X \longrightarrow C.$$

Proposition A.3.2. *Soit (x, s) un k-point de $\mathcal{H}_U$ tel que $D(x, s)$ s'annule.*

On a alors l'égalité de sous-espaces vectoriels de $T_{X,x}$:
$$((\mathrm{d}s)_x = 0) = ((\mathrm{d}\pi)_x = 0). \tag{A.3.1}$$

De plus, pour tout élément $(\delta x, \delta s)$ de $T_{\mathcal{H}_U,(x,s)}$, avec δs dans W, les trois assertions suivantes sont équivalentes :

(1) On a l'égalité dans L_x :
$$(\delta s)(x) = 0.$$

(2) On a l'égalité dans L_x :
$$(\mathrm{d}s)_x(\delta x) = 0.$$

(3) On a l'égalité dans $T_{C,\pi(x)}$:
$$(\mathrm{d}\pi)_x(\delta x) = 0.$$

Démonstration. Si (x, s) est un point de $\mathcal{H}_U$ tel que $D(x, s)$ s'annule, alors par définition, le morphisme $\pi_{|\mathcal{H}_s}$ a un point critique en x, donc on a l'inclusion de sous-espaces de $T_{X,x}$:
$$((\mathrm{d}s)_x = 0) \subset ((\mathrm{d}\pi)_x = 0). \tag{A.3.2}$$
Comme s est dans U, l'hypersurface $\mathcal{H}_s$ est lisse, donc le sous-espace de gauche est un hyperplan de $T_{X,x}$. De plus, comme le morphisme π est lisse, le sous-espace de droite est aussi un hyperplan de $T_{X,x}$. Donc l'inclusion (A.3.2) est en fait une égalité.

Comme le couple $(\delta x, \delta s)$ est dans $T_{\mathcal{H},(x,s)}$, l'égalité (A.2.9) de la proposition A.2.10 assure l'équivalence entre (1) et (2). D'autre part, on sait que $D(x, s)$ s'annule, et donc d'après (A.3.1), les hyperplans de $T_{X,x}$ définis par l'annulation de $(\mathrm{d}s)_x$ et $(\mathrm{d}\pi)_x$ coïncident. Cela montre l'équivalence entre (2) et (3). □

A.3.1.4. *Points critiques non dégénérés.* Soient Y un k-schéma lisse et
$$p : Y \longrightarrow C$$
un k-morphisme. Considérons la section $\mathrm{d}p$ du fibré vectoriel $\mathrm{Hom}(T_Y, p^* T_C)$ sur Y définie par la différentielle de p.

En notant comme précédemment
$$\mathbb{V}(E) := \mathrm{Spec}_Y(S^\bullet E^\vee)$$
l'espace total d'un fibré vectoriel E sur Y, la section $\mathrm{d}p$ définit un morphisme
$$f_{\mathrm{d}p} : Y \longrightarrow \mathbb{V}(\mathrm{Hom}(T_Y, p^* T_C)).$$

Pour tout k-point y de Y, p a un point critique en y si et seulement si la section $\mathrm{d}p$ s'annule en y, et ce point critique est non dégénéré si et seulement si la différentielle en y de cette section est surjective, c'est-à-dire si et seulement si le morphisme $f_{\mathrm{d}p}$ est en x transverse à l'image de la section nulle.

En particulier, les points critiques de p sont tous non dégénérés si et seulement si la section $\mathrm{d}p$ est transverse à la section nulle de $\mathrm{Hom}(T_Y, p^*T_C)$.

A.3.2. Transversalité de D à la section nulle.

A.3.2.1. *Calcul de la différentielle de D.* On peut calculer explicitement la différentielle de la section D en un point de $\mathcal{H}_U$ où elle s'annule. Soit (x, s) un tel point. Par l'égalité (A.3.1), les noyaux des morphismes non nuls d'espaces vectoriels :

$$(\mathrm{d}s)_x : T_{X,x} \longrightarrow L_x$$

et

$$(\mathrm{d}\pi)_x : T_{X,x} \longrightarrow T_{C,\pi(x)}$$

définissent le même hyperplan V_x, qui s'identifie à $T_{\mathrm{pr}_2,(x,s)}$. Il existe donc un unique isomorphisme d'espaces vectoriels :

$$\varphi_x : L_x \xrightarrow{\sim} T_{C,\pi(x)}$$

satisfaisant l'égalité de morphismes :

$$(\mathrm{d}\pi)_x = \varphi_x \circ (\mathrm{d}s)_x : T_{X,x} \longrightarrow T_{C,\pi(x)}.$$

Soit Ω un voisinage ouvert de x dans X où les fibrés vectoriels T_X, L, et π^*T_C sont triviaux, et soient

$$\psi : T_{X,x} \times_k \Omega \xrightarrow{\sim} T_{X|\Omega},$$

$$\chi : L_x \times_k \Omega \xrightarrow{\sim} L_{|\Omega},$$

et

$$\iota : T_{C,\pi(x)} \times_k \Omega \xrightarrow{\sim} (\pi^*T_C)_{|\Omega}$$

des trivialisations dont les fibres au point x sont les applications $\mathrm{Id}_{T_{X,x}}$, Id_{L_x} et $\mathrm{Id}_{T_{C,\pi(x)}}$.

Pour tout k-point x' de Ω et tout vecteur v dans V_x, on peut considérer l'élément

$$(\mathrm{d}\pi)_{x'}(\psi(v, x')) \in T_{C,\pi(x')},$$

et ainsi définir une section de $(\pi^*T_C)_{|\Omega}$, qui s'annule en x car $(\mathrm{d}\pi)_x(v)$ s'annule. On peut donc prendre sa différentielle en x :

$$\mathrm{d}_{x'}((\mathrm{d}\pi)_{x'}(\psi(v, x')))_x \in T_{C,\pi(x)} \otimes T^\vee_{X,x}.$$

De même, on peut définir une application :

$$\chi^{-1} \circ s : \Omega \longrightarrow L_x, \quad x'' \longmapsto (\chi^{-1}_{x''} \circ s)(x''),$$

dont on peut prendre la dérivée en un point x' dans Ω selon le vecteur $\psi(v, x')$:

$$\mathrm{d}(\chi^{-1} \circ s)_{x'}(\psi(v, x')) \in L_x.$$

Cela définit une application de Ω dans L_x dont on peut considérer la différentielle en x :

$$\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1} \circ s)_{x'}(\psi(v, x')))_x \in L_x \otimes T^\vee_{X,x}.$$

Proposition A.3.3. *Avec les notations ci-dessus, pour tout $(\delta x, \delta s)$ dans $T_{\mathcal{H}_U,(x,s)}$, et pour tout v dans $V_x \simeq T_{\mathrm{pr}_2,(x,s)}$, on a l'égalité dans $T_{C,\pi(x)}$:*

$$(\mathrm{d}D)_{(x,s)}(\delta x, \delta s)(v) = \mathrm{d}_{x'}((\mathrm{d}\pi)_{x'}(\psi(v, x')))_x(\delta x) - \varphi_x(\mathrm{d}(\chi^{-1} \circ \delta s)_x(v)) \\ - \varphi_x\left(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1} \circ s)_{x'}(\psi(v, x')))_x(\delta x)\right).$$

Démonstration. Nous pouvons définir un isomorphisme de fibrés vectoriels :

$$\varphi := \iota \circ (\varphi_x \times \mathrm{Id}_\Omega) \circ \chi^{-1} : L_{|\Omega} \xrightarrow{\sim} (\pi^* T_C)_{|\Omega}$$

dont la fibre en x est φ_x.

Soit u_x un élément de $T_{X,x}$ tel que l'élément

$$(\mathrm{d}s)_x(u_x) \in L_x$$

ne s'annule pas. En particulier, on a que l'élément

$$(\mathrm{d}\pi)_x(u_x) = \varphi_x \circ (\mathrm{d}s)_x(u_x) \in T_{C,\pi(x)}$$

ne s'annule pas, et que l'espace vectoriel $T_{X,x}$ se décompose de la manière suivante :

$$T_{X,x} = V_x \oplus k\, u_x.$$

La trivialisation ψ du fibré vectoriel $T_{X|\Omega}$ permet de prolonger l'espace vectoriel V_x en un sous-fibré vectoriel V de $T_{X|\Omega}$, et l'élément u_x en une section u de $T_{X|\Omega}$, tels que, quitte à restreindre Ω, on ait la décomposition suivante :

$$T_{X|\Omega} = V \oplus \mathcal{O}_\Omega\, u.$$

Quitte à restreindre Ω, on peut prendre un voisinage ouvert $\Omega \times \Omega_W$ de (x,s) dans $X \times U$ tel que pour tout (x', s') dans $\mathcal{H}_U \cap (\Omega \times \Omega_W)$, les éléments

$$(\mathrm{d}s')_{x'}(u_{x'}) \in L_{x'},$$

et

$$(\mathrm{d}\pi)_{x'}(u_{x'}) \in T_{C,\pi(x')}$$

ne s'annulent pas.

Soit (x', s') un k-point de $\mathcal{H}_U \cap (\Omega \times \Omega_W)$.

Comme l'élément $(\mathrm{d}s')_{x'}(u_{x'})$ ne s'annule pas, et comme $T_{\mathcal{H}_{s'},x'}$ est l'hyperplan de $T_{X,x'}$ défini par l'annulation de la forme linéaire $(\mathrm{d}s')_{x'}$, on a un isomorphisme d'espaces vectoriels entre deux hyperplans de $T_{X,x'}$:

$$V_{x'} \xrightarrow{\sim} T_{\mathcal{H}_{s'},x'}, \quad v \longmapsto v - \frac{(\mathrm{d}s')_{x'}(v)}{(\mathrm{d}s')_{x'}(u_{x'})}\, u_{x'},$$

et l'espace d'arrivée s'identifie à $T_{\mathrm{pr}_2,(x',s')}$.

Ces isomorphismes définissent un isomorphisme de fibrés vectoriels sur $\mathcal{H}_U \cap (\Omega \times \Omega_W)$:

$$\Phi : \mathrm{pr}_1^* V \simeq V_x \times_k (\mathcal{H}_U \cap (\Omega \times \Omega_W)) \xrightarrow{\sim} T_{\mathrm{pr}_2|\mathcal{H}_U \cap (\Omega \times \Omega_W)},$$

et la fibre en (x,s) de cet isomorphisme est l'identité. La différentielle en (x,s) de la section D, qui s'annule en (x,s), coïncide donc avec celle de $D \circ \Phi$.

Soit (x', s') dans $\mathcal{H}_U \cap (\Omega \times \Omega_W)$, et soit v dans V_x. On note simplement $v_{x'}$ pour $\psi(v, x')$. On a l'égalité dans $T_{C,\pi(x')}$:

$$\begin{aligned}
(D \circ \Phi)(x', s')(v) &= (\mathrm{d}\pi)_{x'}\Big(v_{x'} - \frac{(\mathrm{d}s')_{x'}(v_{x'})}{(\mathrm{d}s')_{x'}(u_{x'})}\, u_{x'}\Big),\\
&= (\mathrm{d}\pi)_{x'}(v_{x'}) - (\mathrm{d}\pi)_{x'}(u_{x'})\, \frac{(\mathrm{d}s')_{x'}(v_{x'})}{(\mathrm{d}s')_{x'}(u_{x'})},\\
&= (\mathrm{d}\pi)_{x'}(v_{x'}) - (\mathrm{d}\pi)_{x'}(u_{x'})\, \frac{\varphi_{x'}\big((\mathrm{d}s')_{x'}(v_{x'})\big)}{\varphi_{x'}\big((\mathrm{d}s')_{x'}(u_{x'})\big)},
\end{aligned}$$

et donc l'égalité dans $T_{C,\pi(x)}$:

$$\iota_{x'}^{-1}\big((D\circ\Phi)(x',s')(v)\big) = \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(v_{x'})\big) - \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(u_{x'})\big)\,\frac{(\iota_{x'}^{-1}\circ\varphi_{x'})\big((\mathrm{d}s')_{x'}(v_{x'})\big)}{(\iota_{x'}^{-1}\circ\varphi_{x'})\big((\mathrm{d}s')_{x'}(u_{x'})\big)},$$

$$(\text{A.3.3})\qquad = \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(v_{x'})\big) - \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(u_{x'})\big)\,\frac{(\varphi_{x}\circ\chi_{x'}^{-1})\big((\mathrm{d}s')_{x'}(v_{x'})\big)}{(\varphi_{x}\circ\chi_{x'}^{-1})\big((\mathrm{d}s')_{x'}(u_{x'})\big)},$$

$$(\text{A.3.4})\qquad = \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(v_{x'})\big) - \iota_{x'}^{-1}\big((\mathrm{d}\pi)_{x'}(u_{x'})\big)\,\frac{\varphi_{x}\big(\mathrm{d}(\chi^{-1}\circ s')_{x'}(v_{x'})\big)}{\varphi_{x}\big(\mathrm{d}(\chi^{-1}\circ s')_{x'}(u_{x'})\big)},$$

où (A.3.3) découle de la définition du morphisme de fibrés φ, et (A.3.4) du fait qu'on peut calculer la différentielle en x' de la section s' dans n'importe quelle trivialisation.

On peut remarquer que le membre de droite de (A.3.4) a un sens pour tout (x', s') dans $\Omega\times\Omega_W$, même si la section s' ne s'annule pas au point x'.

On peut calculer la différentielle de $(D\circ\Phi)$ en fixant le vecteur v, et en se plaçant dans l'espace vectoriel $T_{C,\pi(x)}$ avec la trivialisation ι^{-1}.

Soit donc v dans V_x un vecteur, et soit $(\delta x,\delta s)$ dans $T_{\mathcal{H}_U,(x,s)}$. En différentiant (A.3.4) selon le vecteur $(\delta x,\delta s)$, au point (x,s), et en utilisant que le morphisme d'espaces vectoriels ι_x est l'identité, on obtient l'égalité dans $T_{C,\pi(x)}$:

$$\begin{aligned}
(\mathrm{d}(D\circ\Phi))_{((x,s),v)}(\delta x,\delta s) &= \mathrm{d}_{(x',s')}\Big(\iota_{x'}^{-1}\big((D\circ\Phi)(x',s')(v)\big)\Big)_{(x,s)}(\delta x,\delta s),\\
&= \mathrm{d}_{x'}(\iota_{x'}^{-1}((\mathrm{d}\pi)_{x'}(v_{x'})))_x(\delta x)\\
&- \mathrm{d}_{x'}(\iota_{x'}^{-1}((\mathrm{d}\pi)_{x'}(u_{x'})))_x(\delta x)\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(v))}{\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(u_x))}\\
&- (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ \delta s)_x(v))}{\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(u_x))}\\
&- (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(v_{x'}))_x(\delta x))}{\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(u_x))}\\
&+ (\mathrm{d}\pi)_x(u_x)\,\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(v))\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ \delta s)_x(u_x))}{(\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(u_x)))^2}\\
&+ (\mathrm{d}\pi)_x(u_x)\,\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(v))\,\frac{\varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(u_{x'}))_x(\delta x))}{(\varphi_x(\mathrm{d}(\chi^{-1}\circ s)_x(u_x)))^2}.
\end{aligned}$$

En utilisant que la section s (resp. $x'\to(\mathrm{d}\pi)_{x'}(v_{x'})$) s'annule au point x, donc que sa différentielle est indépendante du choix de la trivialisation χ^{-1} (resp. ι^{-1}), on obtient :

$$\begin{aligned}
(\mathrm{d}(D\circ\Phi))_{((x,s),v)}(\delta x,\delta s) &= \mathrm{d}_{x'}((\mathrm{d}\pi)_{x'}(v_{x'}))_x(\delta x)\\
&- \mathrm{d}_{x'}(\iota_{x'}^{-1}((\mathrm{d}\pi)_{x'}(u_{x'})))_x(\delta x)\,\frac{\varphi_x((\mathrm{d}s)_x(v))}{\varphi_x((\mathrm{d}s)_x(u_x))}\\
&- (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ \delta s)_x(v))}{\varphi_x((\mathrm{d}s)_x(u_x))}\\
&- (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(v_{x'}))_x(\delta x))}{\varphi_x((\mathrm{d}s)_x(u_x))}\\
&+ (\mathrm{d}\pi)_x(u_x)\,\varphi_x((\mathrm{d}s)_x(v))\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ \delta s)_x(u_x))}{(\varphi_x((\mathrm{d}s)_x(u_x)))^2}\\
&+ (\mathrm{d}\pi)_x(u_x)\,\varphi_x((\mathrm{d}s)_x(v))\,\frac{\varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(u_{x'}))_x(\delta x))}{(\varphi_x((\mathrm{d}s)_x(u_x)))^2}.
\end{aligned}$$

En utilisant que v est dans V_x, donc que $(\mathrm{d}s)_x v$ s'annule, on obtient :

$$\begin{aligned}\mathrm{d}(D\circ\Phi)_{((x,s),v)}(\delta x,\delta s) &= \mathrm{d}_{x'}((\mathrm{d}\pi)_{x'}(v_{x'}))_x(\delta x) - (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}(\chi^{-1}\circ\delta s)_x(v))}{\varphi_x((\mathrm{d}s)_x(u_x))}\\ &\quad - (\mathrm{d}\pi)_x(u_x)\,\frac{\varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(v_{x'}))_x(\delta x))}{\varphi_x((\mathrm{d}s)_x(u_x))}.\end{aligned}$$

En utilisant l'égalité dans $T_{C,\pi(x)}$:

$$(\mathrm{d}\pi)_x(u_x) = (\varphi_x\circ(\mathrm{d}s)_x)(u_x),$$

on obtient :

$$\begin{aligned}\mathrm{d}(D\circ\Phi)_{((x,s),v)}(\delta x,\delta s) &= \mathrm{d}_{x'}((\mathrm{d}\pi)_{x'}(v_{x'}))_x(\delta x) - \varphi_x(\mathrm{d}(\chi^{-1}\circ\delta s)_x(v))\\ &\quad - \varphi_x(\mathrm{d}_{x'}(\mathrm{d}(\chi^{-1}\circ s)_{x'}(v_{x'}))_x(\delta x)).\end{aligned}$$

Comme Φ est un isomorphisme de fibrés qui induit l'identité en (x,s), cela montre l'égalité voulue. □

A.3.2.2. *Transversalité de D à la section nulle et 1-jets verticaux $J_1(L)_{X/C}$.*

Corollaire A.3.4. *Si le fibré des 1-jets verticaux $J_1(L)_{X/C}$ est engendré sur X par les 1-jets des sections de L dans W, alors en tout point (x,s) où D s'annule, la restriction de la différentielle :*

$$\mathrm{d}D_{(x,s)|T_{\mathrm{pr}_1}} : T_{\mathrm{pr}_1,(x,s)} \longrightarrow F_{(x,s)}$$

est surjective, et a fortiori, la différentielle :

$$\mathrm{d}D_{(x,s)} : T_{\mathcal{H}_U,(x,s)} \longrightarrow F_{(x,s)}$$

est surjective.

Démonstration. Soit (x,s) un point de $\mathcal{H}_U$ tel que $D(x,s)$ s'annule.

Considérons un élément :

$$f\in F_{(x,s)} := \mathrm{Hom}(T_{\mathrm{pr}_2,(x,s)}, T_{C,\pi(x)}).$$

On veut trouver un élément de $T_{\mathrm{pr}_1,(x,s)}$ qui est envoyé sur f par la différentielle de D.

Reprenons les notations de la proposition A.3.3. En particulier, considérons un isomorphisme d'espaces vectoriels :

$$\varphi_x : L_x \longrightarrow T_{C,\pi(x)}$$

satisfaisant l'égalité :

$$(\mathrm{d}\pi)_x = \varphi_x\circ(\mathrm{d}s)_x \quad (\in \mathrm{Hom}(T_{X,x}, T_{C,\pi(x)})).$$

On peut donc considérer la composition

$$\varphi_x^{-1}\circ f : T_{\pi,x} = T_{\mathcal{H}_s,x} \longrightarrow L_x.$$

Comme les 1-jets verticaux de L relativement à π sont engendrés par les 1-jets des sections dans W, on peut trouver une section δs dans W qui s'annule en x et satisfait à l'égalité :

$$\mathrm{d}(\delta s)_x = -\varphi_x^{-1}\circ f \quad (\in \mathrm{Hom}(T_{\pi,x}, L_x)).$$

Comme δs s'annule en x, le couple $(0,\delta s)$ est dans $T_{\mathcal{H}_U,(x,s)}$. On lui applique la proposition A.3.3 pour obtenir l'égalité dans $\mathrm{Hom}(T_{\mathrm{pr}_2,(x,s)}, (\pi\circ\mathrm{pr}_1)^*T_{C,\pi(x)})$:

$$\mathrm{d}D_{(x,s)}(0,\delta s) = -\varphi_x\circ\mathrm{d}(\delta s)_x = f.$$

Comme le couple $(0,\delta s)$ est dans $T_{\mathrm{pr}_1,(x,s)}$, cela montre la surjectivité. □

Rappelons que l'on note
$$\mathbb{V}(F) := \operatorname{Spec}_{\mathcal{H}_U}(S^\bullet F^\vee)$$
l'espace total du fibré vectoriel F sur $\mathcal{H}_U$ et
$$0_{\mathcal{H}_U} \subset \mathbb{V}(F)$$
l'image de la section nulle, que la section D induit un morphisme
$$f_D : \mathcal{H}_U \longrightarrow \mathbb{V}(F),$$
et que l'on dit que D est transverse à la section nulle lorsque f_D est transverse à $0_{\mathcal{H}_U}$.

Corollaire A.3.5. *Si le fibré des* 1*-jets verticaux* $J_1(L)_{X/C}$ *est engendré par les* 1*-jets des sections de* L *dans* W*, alors la section* D *de* F *est transverse à la section nulle sur* $\mathcal{H}_U$*, et il existe un ouvert non vide* U' *de* U *tel que, pour tout* s *dans* U'*, la restriction de* D *à* $\mathcal{H}_s$ *soit transverse à la section nulle de* $F_{|\mathcal{H}_s}$*.*

De plus, pour tout point (x,s) *dans* $\mathcal{H}_{U'}$*, la différentielle*
$$\mathrm{d}D_{(x,s)} : T_{\mathrm{pr}_2,(x,s)} \longrightarrow F_{(x,s)}$$
est un isomorphisme de k*-espaces vectoriels.*

Démonstration. Par le corollaire A.3.4, on sait que pour tout point (x,s) de $\mathcal{H}_U$, la différentielle
$$\mathrm{d}D_{(x,s)} : T_{\mathcal{H}_U,(x,s)} \longrightarrow F_{(x,s)}$$
est surjective.

Donc en utilisant la proposition A.2.5, le morphisme f_D est transverse au fermé $0_{\mathcal{H}_U}$, ce qui montre la première affirmation.

Soit
$$p : \mathbb{V}(F) \longrightarrow \mathcal{H}_U$$
la projection.

Considérons le diagramme commutatif
$$\begin{array}{ccccc} \mathcal{H}_U & \xrightarrow{D} & \mathbb{V}(F) & \supset & 0_{\mathcal{H}_U} \\ \Big\downarrow {\scriptstyle \mathrm{pr}_{2|\mathcal{H}_U}} & & \Big\downarrow {\scriptstyle \mathrm{pr}_{2|\mathcal{H}_U}\circ p} & & \\ U & = & U & & \end{array}$$
Par définition de U (voir le corollaire A.2.12), les morphismes verticaux sont lisses, et nous venons de voir que f_D est transverse au fermé $0_{\mathcal{H}_U}$.

Donc en appliquant le corollaire A.2.9, il existe un ouvert non vide U' de U tel que pour tout s dans U', la restriction de f_D à $\mathcal{H}_s$ soit transverse au fermé $0_{\mathcal{H}_U} \cap \mathbb{V}(F_{|\mathcal{H}_s})$.

Cela montre la deuxième affirmation.

Enfin, pour tout (x,s) dans $\mathcal{H}_{U'}$, par la proposition A.2.5, nous venons de montrer que la différentielle
$$\mathrm{d}D_{(x,s)} : T_{\mathrm{pr}_2,(x,s)} \longrightarrow F_{(x,s)}$$
est surjective.

Or, par la proposition A.3.1, les espaces vectoriels $T_{\mathrm{pr}_2,(x,s)}$ et $F_{(x,s)}$ sont de même rang $n-1$, donc cette différentielle est un isomorphisme. □

Corollaire A.3.6. *Si le fibré des* 1*-jets verticaux* $J_1(L)_{X/C}$ *est engendré par les* 1*-jets des sections de* L *dans* W*, alors en prenant* U' *l'ouvert non vide du corollaire A.3.5, pour tout* s *dans* U'*, tous les points critiques de la restriction*
$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \hookrightarrow X \longrightarrow C$$

sont non dégénérés.

Démonstration. Soit s dans U'. La section $(f_D)_{|\mathcal{H}_s}$ s'identifie à la section du fibré $\mathrm{Hom}(T_{\mathcal{H}_s}, \pi^*T_C)$ donnée par la différentielle de la restriction de π à $\mathcal{H}_s$.

Par le corollaire A.3.5, le morphisme associé est transverse à l'image de la section nulle, ce qui veut dire que tous les points critiques du morphisme $\pi_{|\mathcal{H}_s}$ sont non dégénérés. □

Le corollaire A.3.6 établit l'assertion (1) du théorème A.1.2.

On peut noter qu'en tout point (x,s) de $\mathcal{H}_{U'}$ où D s'annule, la différentielle $\mathrm{d}D_{(x,s)}$ restreinte à l'espace tangent $T_{\mathrm{pr}_2,(x,s)}$ s'interprète comme la Hessienne de $\pi_{|\mathcal{H}_s}$ au point critique x.

A.3.3. Le sous-schéma Σ des lieux critiques. Supposons que le fibré des 1-jets verticaux $J_1(L)_{X/C}$ soit engendré par les 1-jets des sections de L dans W.

Soit Σ le sous-schéma fermé de $\mathcal{H}_U$ défini par l'annulation de la section D.

Un k-point (x,s) de $\mathcal{H}_U$ est dans le support de Σ si et seulement si la restriction

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \hookrightarrow X \longrightarrow C$$

a un point critique en x.

De plus, pour tout s dans U, nous pouvons identifier l'intersection

$$\Sigma_s := \Sigma \cap \mathcal{H}_s$$

au schéma des points critiques du morphisme $\pi_{|\mathcal{H}_s}$.

Proposition A.3.7. *Le sous-schéma Σ est lisse de dimension pure w.*

Démonstration. Par la proposition A.3.1, on a que le fibré F sur $\mathcal{H}_U$ est de rang $n-1$.

De plus, on sait par le corollaire A.3.5 que la section D de F est transverse à la section nulle sur $\mathcal{H}_U$. Donc d'après la proposition A.2.5, son lieu des zéros Σ est lisse de dimension pure

$$\dim(\Sigma) = \dim(\mathcal{H}_U) - \mathrm{rg} F = (n+w-1)-(n-1) = w. \quad \square$$

Soit U' l'ouvert du corollaire A.3.5, et soit

$$\Sigma_{U'} := \Sigma \cap (X \times U').$$

Proposition A.3.8. *La restriction*

$$\mathrm{pr}_{2|\Sigma_{U'}} : \Sigma_{U'} \hookrightarrow X \times U' \longrightarrow U'$$

est un morphisme étale.

Démonstration. Soit (x,s) un point de $\Sigma_{U'}$, avec s dans U'.

On s'intéresse au morphisme d'espaces vectoriels

$$\mathrm{d}(\mathrm{pr}_{2|\Sigma_{U'}}) : T_{\Sigma_{U'},(x,s)} \longrightarrow T_{U',s} \simeq W.$$

Soit u un vecteur dans son noyau, on a donc que u est dans l'espace vectoriel

$$T_{\mathrm{pr}_2,(x,s)} \subset T_{\mathcal{H}_{U'},(x,s)}.$$

De plus, par définition de Σ, le vecteur u est annulé par la différentielle de la section D.

Or, par le corollaire A.3.5, comme s est dans U', la différentielle

$$\mathrm{d}D_{(x,s)} : T_{\mathrm{pr}_2,(x,s)} \longrightarrow F_{(x,s)}$$

est un isomorphisme d'espaces vectoriels, donc u s'annule.

Donc le morphisme $\mathrm{d}(\mathrm{pr}_{2|\Sigma_{U'}})$ est injectif.

Par la proposition A.3.7, l'espace de départ et l'espace d'arrivée de $\mathrm{d}(\mathrm{pr}_{2|\Sigma_{U'}})$ ont même dimension, donc ce morphisme est bijectif.

C'est vrai en tout point de $\Sigma_{U'}$, donc le morphisme $\mathrm{pr}_{2|\Sigma_{U'}}$ est étale. □

A.4. Hypersurfaces dans un pinceau projectif avec au plus un point critique dans chaque fibre

Dans cette section, nous complétons la preuve de l'assertion (2) du théorème A.1.2.

On reprend les notations et hypothèses de la section A.3. Notamment on suppose que le fibré des 1-jets verticaux $J^1(L)_{X/C}$ est engendré par les 1-jets des sections de L dans W, et l'on dispose des ouverts U et U' de W construits en A.3.1 et A.3.2. On suppose en outre que le k-schéma X est projectif.

A.4.1. Nombre de points critiques et degré de τ. La projectivité de X implique que la projection

$$\mathrm{pr}_2 : X \times W \longrightarrow W$$

est propre, donc que sa restriction

$$\tau := \mathrm{pr}_{2|\Sigma_{U'}} : \Sigma_{U'} \longrightarrow U'$$

l'est aussi. D'après la proposition A.3.8, τ est donc un morphisme fini étale.

De plus, le schéma U' est irréductible, donc le morphisme τ est surjectif et on peut considérer son degré $\deg(\tau)$, un entier ≥ 1.

En particulier, on obtient :

Proposition A.4.1. *Avec les notations et hypothèses ci-dessus, la fonction qui à un s dans U' associe le nombre de points critiques du morphisme*

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \hookrightarrow X \longrightarrow C,$$

est constante égale au degré $\deg(\tau)$.

Démonstration. Soit s dans U'. Par définition de Σ, les points critiques de $\pi_{|\mathcal{H}_s}$ sont identifiés aux points de

$$\mathrm{pr}_{2|\Sigma}^{-1}(\{s\}) = \tau^{-1}(\{s\}) \subset \Sigma_{U'}.$$

Il y en a donc exactement $\deg(\tau)$. □

On peut calculer explicitement $\deg(\tau)$ en termes de classes caractéristiques.

Proposition A.4.2. *Soit s un élément de U'. On a l'égalité dans* $\mathrm{CH}_0(X)$ *:*

$$[\Sigma_s] = [(1 - c_1(L))^{-1} c(\Omega^1_{X/C})]^{(n)}.$$

où $[\Sigma_s]$ est le 0-cycle réduit de X défini par l'ensemble fini Σ_s.

En particulier, on a l'égalité d'entiers :

$$\deg(\tau) = |\Sigma_s| = \int_X (1 - c_1(L))^{-1} c(\Omega^1_{X/C}).$$

Démonstration. Comme la section s est dans U, l'hypersurface $\mathcal{H}_s$ dans X est lisse, et comme la section s est dans U', par le corollaire A.3.6, la restriction

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \longrightarrow C$$

a un nombre fini de points critiques, qui sont tous non dégénérés. De plus, par la discussion de la sous-section A.3.3, l'ensemble de ces points critiques est exactement Σ_s.

Nous sommes donc dans la situation de la sous-section 6.1.1, avec l'entier $N := n-1$ et l'hypersurface $H := \mathcal{H}_s$ dans X. L'équation (6.1.10) de la proposition 6.1.4 implique immédiatement le résultat [2]. □

A.4.2. Trivialisation après revêtement du morphisme fini étale τ. Rappelons un résultat classique sur les morphismes finis étales.

PROPOSITION A.4.3. *Soient X, Y des k-schémas lisses non vides, avec Y connexe, et*

$$\tau : X \longrightarrow Y$$

un morphisme fini étale.

Il existe un k-schéma lisse $\tilde{Y}$ non vide, un morphisme fini étale

$$\sigma : \tilde{Y} \longrightarrow Y,$$

induisant un diagramme commutatif :

$$\begin{array}{ccccc} \tilde{X} := & \tilde{Y} \times_Y X & \xrightarrow{\tilde{\sigma}} & X \\ & \big\downarrow{\scriptstyle \tilde{\tau}} & & \big\downarrow{\scriptstyle \tau} \\ & \tilde{Y} & \xrightarrow{\sigma} & Y \end{array}$$

et des sections de $\tilde{\tau}$:

$$\iota_1, \ldots, \iota_{\deg(\tau)} : \tilde{Y} \longrightarrow \tilde{X},$$

qui induisent un isomorphisme de $\tilde{Y}$-schémas :

$$\tilde{Y} \sqcup \ldots \sqcup \tilde{Y} \xrightarrow{\sim} \tilde{X}.$$

Nous rappelons la preuve.

DÉMONSTRATION. On raisonne par récurrence sur le degré p du morphisme τ. Si ce degré est 1, τ est un isomorphisme, donc il suffit de prendre $\sigma = \mathrm{Id}_Y$.

Supposons que $p > 1$, et que le résultat soit établi pour les morphismes de degré $p-1$.

Soit X' une composante connexe de X. On forme le diagramme commutatif de morphismes finis étales :

$$\begin{array}{ccc} X' \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \tau} \\ X' & \xrightarrow{\tau_{|X'}} & Y \end{array}$$

On dispose alors de la section de pr_1 donnée par la diagonale :

$$\iota : X' \longrightarrow X' \times_Y X, \quad x \longmapsto (x, x).$$

La diagonale $\iota(X')$ est une composante connexe de $X' \times_Y X$. En notant Z son complémentaire, qui est un ouvert fermé de $X' \times_Y X$, on a l'égalité de sous-schémas de $X' \times_Y X$:

$$X' \times_Y X = \iota(X') \sqcup Z \tag{A.4.1}$$

On applique l'hypothèse de récurrence au morphisme fini étale de degré $p-1$:

$$\mathrm{pr}_{1|Z} : Z \longrightarrow X'.$$

Cela donne un k-schéma lisse $\tilde{Y}$, un morphisme fini étale

$$\sigma' : \tilde{Y} \to X',$$

2. Certains arguments dans la sous-section en question nécessitent que le corps de base soit $\mathbb{C}$, mais comme l'énoncé de la proposition 6.1.4 est de nature purement algébrique, il est aussi valable sur le corps algébriquement clos k de caractéristique 0.

un diagramme commutatif

$$\begin{array}{ccc} \tilde{Y} \times_{X'} Z & \xrightarrow{\tilde{\sigma}'} & Z \\ \Big\downarrow{\scriptstyle \widetilde{\mathrm{pr}}_{1|Z}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{1|Z}} \\ \tilde{Y} & \xrightarrow{\sigma'} & X' \end{array}$$

et des sections de $\widetilde{\mathrm{pr}}_{1|Z}$:

$$\iota_2, ..., \iota_p : \tilde{Y} \longrightarrow \tilde{Y} \times_{X'} Z,$$

induisant un isomorphisme de $\tilde{Y}$-schémas :

$$\tilde{Y} \sqcup ... \sqcup \tilde{Y} \xrightarrow{\sim} \tilde{Y} \times_{X'} Z.$$

On considère le morphisme fini étale

$$\sigma := \tau_{|X'} \circ \sigma' : \tilde{Y} \longrightarrow Y.$$

On a alors l'égalité de schémas lisses :

$$\begin{aligned} \tilde{Y} \times_Y X &= \tilde{Y} \times_{X'} (X' \times_Y X) \qquad \text{par les propriétés du produit fibré} \\ &= \tilde{Y} \times_{X'} (\iota(X') \sqcup Z) \qquad \text{par (A.4.1)} \\ &= (\tilde{Y} \times_{X'} \iota(X')) \sqcup (\tilde{Y} \times_{X'} Z) \\ &= (\tilde{Y} \times_{X'} \iota(X')) \sqcup (\iota_2(\tilde{Y}) \sqcup ... \sqcup \iota_p(\tilde{Y})). \end{aligned}$$

Le schéma $\tilde{Y} \times_{X'} \iota(X')$ est canoniquement isomorphe à $\tilde{Y}$ via le tiré en arrière de la section ι par σ', qu'on note ι_1.

On a donc des sections $\iota_1, ..., \iota_p$ du morphisme de projection

$$\tilde{Y} \times_Y X \longrightarrow \tilde{Y}$$

induisant un isomorphisme de $\tilde{Y}$-schémas :

$$\tilde{Y} \sqcup ... \sqcup \tilde{Y} \xrightarrow{\sim} \tilde{Y} \times_Y X.$$

Cela conclut la récurrence. □

On applique la proposition A.4.3 au morphisme fini étale

$$\tau := \mathrm{pr}_{2|\Sigma_{U'}} : \Sigma_{U'} \longrightarrow U'.$$

Si l'on note p son degré, on obtient ainsi un k-schéma lisse $\tilde{U}'$, un morphisme fini étale

$$\sigma : \tilde{U}' \longrightarrow U',$$

induisant un diagramme commutatif

(A.4.2)
$$\begin{array}{ccccc} \widetilde{\Sigma}_{U'} : = & \tilde{U}' \times_{U'} \Sigma_{U'} & \xrightarrow{\tilde{\tau}} & \tilde{U}' \\ & \Big\downarrow{\scriptstyle \tilde{\sigma}} & & \Big\downarrow{\scriptstyle \sigma} \\ & \Sigma_{U'} & \xrightarrow{\tau} & U' \end{array}$$

et des morphismes

$$\iota_1, ..., \iota_p : \tilde{U}' \longrightarrow \widetilde{\Sigma}_{U'}$$

qui sont des sections de $\tilde{\tau}$, et qui induisent un isomorphisme de $\tilde{U}'$-schémas

$$\tilde{U}' \sqcup ... \sqcup \tilde{U}' \xrightarrow{\sim} \widetilde{\Sigma}_{U'}.$$

Quitte à remplacer $\tilde{U}'$ par une de ses composantes connexes et les sections $\iota_1, ..., \iota_p$ par leurs restrictions, on peut supposer que le k-schéma lisse $\tilde{U}'$ est connexe, donc irréductible.

A.4.3. Le fermé E des hypersurfaces avec plusieurs points critiques dans une même fibre. Avec les notations précédentes, pour chaque entier $1 \leq i \leq p$, on définit une flèche

$$\varphi_i := \pi \circ \mathrm{pr}_1 \circ \tilde{\sigma} \circ \iota_i : \tilde{U}' \longrightarrow C.$$

Notons

$$\Delta_{C \times C} \subset C \times_k C$$

la diagonale de C, et considérons le fermé de $\tilde{U}'$ défini par

$$\tilde{E} := \bigcup_{1 \leq i < j \leq p} (\varphi_i, \varphi_j)^{-1}(\Delta_{C \times C}).$$

Comme le morphisme σ est fini étale, il est propre, et donc on peut considérer le fermé de U' défini par

$$E := \sigma(\tilde{E}).$$

Proposition A.4.4. *Un point s de U' est dans E si et seulement si la restriction*

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \hookrightarrow X \longrightarrow C$$

a au moins deux points critiques dans une même fibre, c'est-à-dire si et seulement si la restriction

$$\pi_{|\Sigma_s} : \Sigma_s \hookrightarrow X \longrightarrow C$$

est non injective.

Démonstration. Soit s un point de U'. Comme le schéma U' est irréductible et $\tilde{U}'$ est non vide, le morphisme

$$\sigma : \tilde{U}' \longrightarrow U'$$

est surjectif. Soit $\tilde{s}$ une des préimages de s par ce morphisme.

Comme le morphisme σ est surjectif, par changement de base, le morphisme $\tilde{\sigma}$ est aussi surjectif. Par commutativité du diagramme (A.4.2), on voit que l'application entre ensembles finis

$$\tilde{\sigma}_{|\tilde{\tau}^{-1}(\{\tilde{s}\})} : \tilde{\tau}^{-1}(\{\tilde{s}\}) \longrightarrow \tau^{-1}(\{s\})$$

est bien définie et surjective, donc bijective car les deux ensembles ont cardinal p.

Par définition des morphismes $(\iota_i)_{1 \leq i \leq p}$, on obtient donc l'égalité de sous-ensembles de $\Sigma_{U'}$:

$$\tau^{-1}(\{s\}) = \{(\tilde{\sigma} \circ \iota_i)(\tilde{s}) \quad | 1 \leq i \leq p\}.$$

Par définition de $\Sigma_{U'}$ et du morphisme τ, on obtient donc que pour tout entier $1 \leq i \leq p$, le point

$$(\tilde{\sigma} \circ \iota_i)(\tilde{s}) \in \Sigma_{U'}$$

est de la forme (x_i, s), pour x_i un point de X, et que les $(x_i)_{1 \leq i \leq p}$ sont exactement les points de Σ_s, c'est-à-dire les points critiques du morphisme :

$$\pi_{|\mathcal{H}_s} : \mathcal{H}_s \longrightarrow C.$$

Pour tout entier $1 \leq i \leq p$, par définition de φ_i, on a l'égalité dans C :

$$\varphi_i(\tilde{s}) = (\pi \circ \mathrm{pr}_1)(x_i, s) = \pi(x_i).$$

Donc le point $\tilde{s}$ est dans $\tilde{E}$ si et seulement si les points $\pi(x_i)$ ne sont pas distincts deux à deux, donc si et seulement si le morphisme $\pi_{|\mathcal{H}_s}$ a au moins deux points critiques dans une même fibre.

En particulier, cette propriété ne dépend pas du choix de la préimage $\tilde{s}$.

Donc on obtient que la section s est dans le fermé

$$E = \sigma(\tilde{E}) \subset U'$$

si et seulement si le morphisme $\pi_{|\mathcal{H}_s}$ a au moins deux points critiques dans une même fibre, comme voulu. □

Pour montrer l'assertion (2) du théorème A.1.2, il suffit donc de montrer que le fermé E est strict si l'hypothèse de cette assertion est satisfaite.

A.4.4. Fin de la preuve du théorème A.1.2.

PROPOSITION A.4.5. *Si pour tout couple (x,y) de X^2 avec $x \neq y$ et $\pi(x) = \pi(y)$, l'espace vectoriel $L_x \oplus L_y$ est engendré par les sections de L dans W, alors pour tout couple d'entiers (i,j) avec $1 \leq i < j \leq p$, le morphisme*

$$(\varphi_i, \varphi_j) : \tilde{U}' \longrightarrow C \times C$$

est transverse à la diagonale $\Delta_{C\times C}$.

DÉMONSTRATION. Soit (i,j) un couple d'entiers avec $1 \leq i < j \leq p$.

Soit $\tilde{s}$ un point de $\tilde{U}'$ envoyé par (φ_i, φ_j) dans la diagonale, c'est-à-dire vérifiant l'égalité dans C :

$$\varphi_i(\tilde{s}) = \varphi_j(\tilde{s}) =: t. \tag{A.4.3}$$

On définit un point

$$s := \sigma(\tilde{s}) \in U'.$$

Par commutativité du diagramme (A.4.2), les points

$$\tilde{\sigma} \circ \iota_i(\tilde{s}) \in \Sigma_{U'}$$

et

$$\tilde{\sigma} \circ \iota_j(\tilde{s}) \in \Sigma_{U'}$$

sont tous deux envoyés sur s par la projection τ, et ils sont distincts car l'application entre ensembles finis

$$\tilde{\sigma}_{|\tilde{\tau}^{-1}(\{\tilde{s}\})} : \tilde{\tau}^{-1}(\{\tilde{s}\}) \longrightarrow \tau^{-1}(\{s\})$$

est bijective. Ces points s'écrivent donc (x,s) et (y,s), avec x et y deux points distincts de X. D'après la définition de φ_i et φ_j et l'hypothèse (A.4.3) sur $\tilde{s}$, on a les égalités dans C :

$$\pi(x) = \varphi_i(\tilde{s}) = t = \varphi_j(\tilde{s}) = \pi(y).$$

Lorsque les hypothèses de la proposition sont satisfaites, il existe une section δs dans W qui s'annule en y et non en x. On la voit comme un élément de $T_{U',s}$.

Comme le morphisme σ est étale, la différentielle

$$\mathrm{d}\sigma_{\tilde{s}} : T_{\tilde{U}',\tilde{s}} \longrightarrow T_{U',s}$$

est un isomorphisme. Donc il existe un unique élément $\widetilde{\delta s}$ dans $T_{\tilde{U}',\tilde{s}}$ que la différentielle $\mathrm{d}\sigma_{\tilde{s}}$ envoie sur δs.

Par commutativité du diagramme (A.4.2), les éléments

$$\mathrm{d}(\tilde{\sigma} \circ \iota_i)_{\tilde{s}}(\widetilde{\delta s}) \in T_{\Sigma_{U'},(x,s)},$$

et

$$\mathrm{d}(\tilde{\sigma} \circ \iota_j)_{\tilde{s}}(\widetilde{\delta s}) \in T_{\Sigma_{U'},(y,s)},$$

sont tous les deux envoyés sur δs par la différentielle $\mathrm{d}\tau$, donc s'écrivent $(\delta x, \delta s)$ et $(\delta y, \delta s)$ avec δx dans $T_{X,x}$ et δy dans $T_{X,y}$.

Le couple (x,s) est dans Σ, le couple $(\delta x, \delta s)$ dans $T_{\mathcal{H}_U,(x,s)}$ et la section δs ne s'annule pas en x. Donc d'après la proposition A.3.2, l'élément

$$(\mathrm{d}\pi)_x(\delta x) \in T_{C,\pi(x)}$$

est non nul.

De même, comme la section δs s'annule en y, l'élément

$$(\mathrm{d}\pi)_y(\delta y) \in T_{C,\pi(x)}$$

est nul.

Par conséquent, on voit que l'élément

$$(\mathrm{d}\varphi_i)_{\tilde{s}}(\widetilde{\delta s}) = \mathrm{d}(\pi \circ \mathrm{pr}_1)_{(x,s)}(\delta x, \delta s) = (\mathrm{d}\pi)_x(\delta x) \in T_{C,t}$$

est non nul, et que l'élément

$$(\mathrm{d}\varphi_j)_{\tilde{s}}(\widetilde{\delta s}) = \mathrm{d}(\pi \circ \mathrm{pr}_1)_{(y,s)}(\delta y, \delta s) = (\mathrm{d}\pi)_y(\delta y) \in T_{C,t}$$

est nul.

On a donc montré que l'image

$$\mathrm{d}(\varphi_i, \varphi_j)_{\tilde{s}}(T_{\tilde{U}',\tilde{s}}) \subset T_{C\times C,(t,t)}$$

contient la droite

$$(T_{C,t}) \times \{0\} \subset T_{C\times C,(t,t)}.$$

Par définition de la diagonale, on a l'égalité de sous-espaces vectoriels de $T_{C\times C,(t,t)}$:

$$T_{C\times C,(t,t)} = (T_{C,t}) \times \{0\} + T_{\Delta_{C\times C},(t,t)},$$

et donc l'égalité :

$$T_{C\times C,(t,t)} = \mathrm{d}(\varphi_i, \varphi_j)_{\tilde{s}}(T_{\tilde{U}',\tilde{s}}) + T_{\Delta_{C\times C},(t,t)}.$$

Ceci est vrai en tout point $\tilde{s}$ que le morphisme (φ_i, φ_j) envoie dans la diagonale, donc ce morphisme est transverse à la diagonale. □

Corollaire A.4.6. *Si pour tout couple (x, y) de X^2 tel que $x \neq y$ et $\pi(x) = \pi(y)$, l'espace vectoriel $L_x \oplus L_y$ est engendré par les sections de L dans W, alors E est un fermé strict de U'.*

Démonstration. Pour tout couple d'entiers (i, j) avec $1 \leq i < j \leq p$, par la proposition A.4.5, le morphisme

$$(\varphi_i, \varphi_j) : \tilde{U}' \longrightarrow C \times C$$

est transverse à la diagonale $\Delta_{C\times C}$, et donc le fermé $(\varphi_i, \varphi_j)^{-1}(\Delta_{C\times C})$ est strict dans $\tilde{U}'$.

Comme $\tilde{U}'$ est irréductible, le fermé

$$\tilde{E} = \bigcup_{1\leq i<j\leq p} (\varphi_i, \varphi_j)^{-1}(\Delta_{C\times C}) \subset \tilde{U}'$$

est strict. En particulier, il est de dimension strictement inférieure à $\dim(\tilde{U}')$, c'est-à-dire à $\dim(U')$.

Donc son image :

$$E = \sigma(\tilde{E}) \subset U'$$

est de dimension strictement inférieure à $\dim(U')$, et est donc un fermé strict. □

Corollaire A.4.7. *Si pour tout couple (x, y) de X^2 tel que $x \neq y$ et $\pi(x) = \pi(y)$, l'espace vectoriel $L_x \oplus L_y$ est engendré par les sections de L dans W, alors l'ouvert*

$$U'' := U' - E \subset U'$$

est non vide, et pour tout s dans U'', la restriction $\pi_{|\mathcal{H}_s}$ a au plus un point critique dans chaque fibre.

Démonstration. Par la proposition A.4.6, le fermé E de U' est strict, donc son complémentaire U'' est non vide. De plus, pour tout s dans U'', par la proposition A.4.4, la restriction $\pi_{|\mathcal{H}_s}$ a au plus un point critique dans chaque fibre. □

Le corollaire A.4.7 complète la preuve du théorème A.1.2.

A.5. Hypersurfaces génériques dans les fibrés en projectifs

Soit C une k-courbe projective lisse connexe de genre g.

A.5.1. Rappels sur les pentes des fibrés vectoriels. Si F est un fibré vectoriel de rang non nul sur C, on notera :

$$\mu(F) := \frac{\deg F}{\mathrm{rg} F} \tag{A.5.1}$$

la pente de F, et $\mu_{\min}(F)$ et $\mu_{\max}(F)$ les pentes minimales et maximales apparaissant dans la filtration d'Harder-Narasimhan de F ; voir par exemple [**HL10**, Section 1.3].

Ces dernières sont aussi définies par les relations :

$$\mu_{\max}(F) := \max\left\{\mu(G),\ G \text{ sous-fibré vectoriel non nul de } F\right\}, \tag{A.5.2}$$

et :

$$\mu_{\min}(F) := -\mu_{\max}(F^{\vee}). \tag{A.5.3}$$

On déduit aisément des relations (A.5.1), (A.5.2) et (A.5.3) que, pour tout fibré en droites L sur C, on a :

(A.5.4)

$$\mu(F\otimes L) = \mu(F) + \deg L, \quad \mu_{\max}(F\otimes L) = \mu_{\max}(F) + \deg L, \quad \text{et} \quad \mu_{\min}(F\otimes L) = \mu_{\min}(F) + \deg L.$$

Dans cette sous-section, nous rappelons quelques propriétés classiques des pentes $\mu_{\min}(F)$ et $\mu_{\max}(F)$.

PROPOSITION A.5.1. *Soit F un fibré vectoriel de rang non nul sur C.*

Si on a l'inégalité de nombres rationnels :

$$\mu_{\max}(F) < 0,$$

alors le k-espace vectoriel $H^0(C, F)$ est nul.

C'est une conséquence directe de l'expression (A.5.2) de la pente maximale. En utilisant la dualité de Serre et (A.5.4), on en déduit :

PROPOSITION A.5.2. *Soit F un fibré vectoriel de rang non nul sur C.*

Si on a l'inégalité de nombres rationnels :

$$\mu_{\min}(F) > 2g - 1,$$

alors, pour tout point x de C, $H^1(F \otimes \mathcal{O}_C(-\{x\}))$ est nul, et donc le fibré vectoriel F est engendré par ses sections globales sur C.

COROLLAIRE A.5.3. *Soient F un fibré vectoriel de rang non nul sur C et M un fibré en droites sur C. Si on a l'inégalité de nombres rationnels :*

$$\deg(M) > 2g - 1 - \mu_{\min}(F),$$

alors le fibré vectoriel $F \otimes M$ est engendré par ses sections globales.

La propriété suivante est de nature plus profonde, et utilise que le corps de base k est de caractéristique nulle. Elle est par exemple une conséquence des résultats de [**NS65**] sur le lien entre fibrés stables sur une surface de Riemann et représentations unitaires, qui impliquent que la stabilité est compatible aux produits tensoriels.

PROPOSITION A.5.4. *Pour tout fibré vectoriel $E \neq 0$ sur C et tout $d \in \mathbb{N}$, on a :*

$$\mu_{\max}(S^d E) = d\, \mu_{\max}(E).$$

Finalement, en combinant le corollaire A.5.3 et la proposition A.5.4, nous obtenons :

Corollaire A.5.5. *Soient E un fibré vectoriel de rang non nul sur C, M un fibré en droites sur C, et $d \geq 0$ un entier.*

Si $\deg_C M > 2g - 1 + d\,\mu_{\max}(E)$, alors le fibré vectoriel $S^d E^\vee \otimes M$ est engendré par ses sections globales sur C.

A.5.2. Preuve du théorème A.1.3. Le théorème A.1.3 découle du théorème A.1.2, du corollaire A.5.5, et de l'énoncé suivant :

Proposition A.5.6. *Soient S un k-schéma, E un fibré vectoriel de rang $N+1$ sur S, M un fibré en droites sur S et $d \geq 1$ un entier. Considérons le fibré en projectifs associé à E :*

$$\pi : \mathbb{P}(E) := \operatorname{Proj}_S(S^\bullet E^\vee) \longrightarrow S,$$

et le fibré en droites sur $\mathbb{P}(E)$:

$$L := \mathcal{O}_E(d) \otimes \pi^* M.$$

Si le fibré vectoriel $S^d E^\vee \otimes M$ sur S est engendré par ses sections globales, alors le fibré vectoriel des 1-jets verticaux $J_1(L)_{\mathbb{P}(E)/S}$ est engendré par les 1-jets des sections globales de L sur $\mathbb{P}(E)$, et pour tout couple (x, y) dans $\mathbb{P}(E)^2$ tel $x \neq y$ et $\pi(x) = \pi(y)$, le morphisme d'évaluation :

$$H^0(\mathbb{P}(E), L) \longrightarrow L_x \oplus L_y, \quad s \longmapsto (s(x), s(y))$$

est surjectif.

Démonstration. Supposons que le fibré vectoriel $S^d E^\vee \otimes M$ sur S soit engendré par ses sections globales.

(1) Commençons par montrer que L est engendré par ses sections globales sur $\mathbb{P}(E)$.

Soit x un point de $\mathbb{P}(E)$. Nous allons construire une section de L sur $\mathbb{P}(E)$ qui ne s'annule pas en ce point.

Soit $s \in S(k)$ l'image de x par π. On prend des trivialisations de k-espaces vectoriels :

$$\varphi : k^{\oplus N+1} \xrightarrow{\sim} E_s,$$
$$\iota : k \xrightarrow{\sim} M_s.$$

Ces isomorphismes induisent un isomorphisme d'espaces vectoriels :

$$S^d({}^t\varphi^{-1}) \otimes \iota : S^d(k^{\oplus N+1})^\vee \xrightarrow{\sim} S^d E_s^\vee \otimes M_s.$$

De plus, l'isomorphisme φ induit un isomorphisme de k-schémas :

$$\mathbb{P}(\varphi) : \mathbb{P}_k^N \xrightarrow{\sim} \mathbb{P}(E)_s.$$

Comme le fibré $\mathcal{O}(d)$ sur $\mathbb{P}_k^N$ est engendré par ses sections globales, il existe une section :

$$F \in H^0(\mathbb{P}_k^N, \mathcal{O}(d)) \simeq S^d(k^{\oplus N+1})^\vee,$$

qui ne s'annule pas au point

$$\mathbb{P}(\varphi)^{-1}(x) \in \mathbb{P}^N(k).$$

On considère le vecteur

$$v := \left(S^d({}^t\varphi^{-1}) \otimes \iota\right)(F) \in S^d E_s^\vee \otimes M_s.$$

Comme le fibré vectoriel $S^d E^\vee \otimes M$ est engendré par ses sections globales, il existe une section globale τ de $S^d E^\vee \otimes M$ dont la valeur en s est v.

Cette section définit une section globale σ de L sur $\mathbb{P}(E)$ dont la restriction sur $\mathbb{P}(E_s)$ est le polynôme homogène donné par v. Par construction de F et v, la section σ ne s'annule pas au point x, comme voulu.

(2) Pour compléter la preuve, rappelons que, comme le fibré en droites $\mathcal{O}(d)$ sur $\mathbb{P}_k^N$ est très ample, les sections du système linéaire

$$H^0(\mathbb{P}^N, \mathcal{O}(d)) \simeq S^d(k^{\oplus N+1})^\vee$$

engendrent les 1-jets de $\mathcal{O}(d)$ et séparent les points.

On peut alors raisonner comme ci-dessus : on prend un 1-jet vertical de L en un point x de $\mathbb{P}(E)$ (respectivement, on prend deux points distincts x, y de $\mathbb{P}(E)$ qui sont dans la même fibre de π), on considère son image s par π (resp. leur image commune s par π), on choisit des trivialisations φ, ι des fibres E_s, M_s.

En appliquant l'isomorphisme $\mathbb{P}(\varphi)^{-1}$, on obtient un 1-jet de $\mathcal{O}(d)$ en un point de $\mathbb{P}^N_k$ (resp. deux points distincts de $\mathbb{P}^N_k$), on peut alors choisir un vecteur F de $S^d(k^{\oplus N+1})^\vee$ qui, quand on le voit comme une section de $\mathcal{O}(d)$, a le 1-jet voulu en $\mathbb{P}(\varphi)^{-1}(x)$ (resp. deux vecteurs F, F' de $S^d(k^{\oplus N+1})^\vee$ qui engendrent l'espace vectoriel $\mathcal{O}(d)_{\mathbb{P}(\varphi)^{-1}(x)} \oplus \mathcal{O}(d)_{\mathbb{P}(\varphi)^{-1}(y)}$).

Comme $S^d E^\vee \otimes M$ est engendré par ses sections globales sur S, on obtient une section globale τ (resp. deux sections globales τ, τ') de ce fibré, dont la valeur en s est $\big(S^d({}^t\varphi^{-1}) \otimes \iota\big)(F)$ (resp. dont les valeurs en s sont $\big(S^d({}^t\varphi^{-1}) \otimes \iota\big)(F)$ et $\big(S^d({}^t\varphi^{-1}) \otimes \iota\big)(F')$). Cette section induit une section σ de L (resp. ces sections induisent deux sections σ, σ' de L) dont le 1-jet en x est le 1-jet voulu (resp. qui engendrent l'espace vectoriel $L_x \oplus L_y$). □

A.6. Hypersurfaces génériques et pinceaux de Lefschetz

Dans cette section, nous expliquons comment utiliser les théorèmes A.1.1 et A.1.2 pour redémontrer l'existence de pinceaux de Lefschetz sur une variété projective lisse connexe sur le corps k, algébriquement clos de caractéristique 0 (voir par exemple [**SGA73**, Exposé XVII, théorème 2.5.2] ou [**Voi03**, Corollary 2.10]).

A.6.1. Les pinceaux d'hyperplans dans $\mathbb{P}^M$ et le système linéaire $|\mathcal{O}_{\mathbb{P}^M}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$. Soit $M \geq 1$ un entier.

Rappelons le dictionnaire entre plongements projectifs de $\mathbb{P}^1$ dans l'espace projectif dual $\mathbb{P}^{M\vee}$ de $\mathbb{P}^M$ et éléments du système linéaire complet $|\mathcal{O}_{\mathbb{P}^M}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$ de diviseurs de $\mathbb{P}^M \times \mathbb{P}^1$, définis par les sections non nulles du fibré en droites sur $\mathbb{P}^M \times \mathbb{P}^1$:

$$L_0 := \mathcal{O}_{\mathbb{P}^M}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) := \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^M}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Parmi les éléments de ce système linéaire

$$\begin{aligned} |L_0| := \mathbb{P}\big(H^0(\mathbb{P}^M \times \mathbb{P}^1, L_0)\big) &\simeq \mathbb{P}\big(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\big) \\ &\simeq \mathbb{P}\big(\operatorname{Hom}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^\vee, H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)))\big) \\ &\simeq \mathbb{P}\big(\operatorname{Hom}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^\vee, H^0(\mathbb{P}^{M\vee}, \mathcal{O}_{\mathbb{P}^M}(1))^\vee)\big), \end{aligned}$$

nous pouvons considérer ceux définis par l'image dans

$$\mathbb{P}\big(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\big)$$

des éléments de rang maximal 2 du produit tensoriel

$$H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)),$$

ou de manière équivalente par l'image dans

$$\mathbb{P}\big(\operatorname{Hom}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^\vee, H^0(\mathbb{P}^{M\vee}, \mathcal{O}_{\mathbb{P}^M}(1))^\vee)\big)$$

des morphismes injectifs dans

$$\operatorname{Hom}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^\vee, H^0(\mathbb{P}^{M\vee}, \mathcal{O}_{\mathbb{P}^M}(1))^\vee).$$

Ces éléments forment un ouvert Zariski dense

$$\Omega \subset |L_0| := \mathbb{P}\big(H^0(\mathbb{P}^M \times \mathbb{P}^1, L_0)\big),$$

et sont en bijection avec les plongements projectifs de $\mathbb{P}^1$ dans $\mathbb{P}^{M\vee}$.[3]

À un plongement projectif p de $\mathbb{P}^1$ dans $\mathbb{P}^{M\vee}$ est ainsi associé un élément de Ω, qui définit un diviseur

$$\mathcal{I}_p \subset \mathbb{P}^M \times \mathbb{P}^1$$

dans le système linéaire $|L_0| := |\mathcal{O}_{\mathbb{P}^M}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$.

Ce diviseur $\mathcal{I}_p$ est lisse ; c'est la variété d'incidence du pinceau d'hyperplans de $\mathbb{P}^M$ défini par l'image $p(\mathbb{P}^1)$ de p dans $\mathbb{P}^{M\vee}$: un point (y, x) de $\mathbb{P}^M \times \mathbb{P}^1$ est dans $\mathcal{I}_p$ si et seulement si le point y est dans l'hyperplan projectif $p(x)$ de $\mathbb{P}^M$.

À un plongement projectif $p : \mathbb{P}^1 \to \mathbb{P}^{M\vee}$ est aussi associé le sous-espace projectif Λ_p de codimension 2 dans $\mathbb{P}^M$, dual de la droite projective $p(\mathbb{P}^1)$ dans $\mathbb{P}^{M\vee}$. C'est aussi le centre du pinceau d'hyperplans $p(\mathbb{P}^1)$. C'est encore l'ensemble des points y dans $\mathbb{P}^M$ tels que $\{y\} \times \mathbb{P}^1$ est contenu dans $\mathcal{I}_p$.

Considérons enfin l'éclatement de $\mathbb{P}^M$ le long de Λ_p :

$$\nu_p : \widetilde{\mathbb{P}}^M_p \longrightarrow \mathbb{P}^M.$$

Le morphisme de projection associé au plongement p :

$$\mathbb{P}^M - \Lambda_p \longrightarrow \mathbb{P}^1,$$

dont le graphe est $\mathcal{I}_p \cap \big((\mathbb{P}^M - \Lambda_p) \times \mathbb{P}^1\big)$, se prolonge en un morphisme :

$$\pi_p : \widetilde{\mathbb{P}}^M_p \longrightarrow \mathbb{P}^1,$$

et le couple de morphismes (ν_p, π_p) établit un isomorphisme :

$$(\nu_p, \pi_p) : \widetilde{\mathbb{P}}^M_p \xrightarrow{\sim} \mathcal{I}_p. \tag{A.6.1}$$

A.6.2. Les pinceaux de Lefschetz sur V comme éléments du système linéaire $|\mathcal{O}_V(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|$. Soit V une k-variété projective lisse connexe de dimension r, plongée dans l'espace projectif $\mathbb{P}^M$ de dimension $M \geq 2$.

On suppose que V n'est contenu dans aucun hyperplan de $\mathbb{P}^M$. Si l'on désigne par $\mathcal{O}_V(1)$ le fibré en droites $\mathcal{O}_{\mathbb{P}^M}(1)_{|V}$ sur V, cette condition est équivalente à l'injectivité du morphisme de restriction :

$$H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \longrightarrow H^0(V, \mathcal{O}_V(1)), \quad s \longmapsto s_{|V}.$$

Nous reprenons les notations de la sous-section précédente et considérons un plongement p de $\mathbb{P}^1$ dans $\mathbb{P}^{M\vee}$.

Comme V n'est pas contenu dans un hyperplan de $\mathbb{P}^M$, il n'est pas contenu dans Λ_p, donc le sous-schéma $V \times \mathbb{P}^1$ de $\mathbb{P}^M \times \mathbb{P}^1$ n'est pas contenu dans le diviseur $\mathcal{I}_p$. On peut donc considérer l'hypersurface de $V \times \mathbb{P}^1$:

$$\mathcal{H}_p := \mathcal{I}_p \cap (V \times \mathbb{P}^1).$$

Cette hypersurface est définie par l'annulation de la restriction à $V \times \mathbb{P}^1$ d'une section non nulle du fibré en droites $\mathcal{O}_{\mathbb{P}^M}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ sur $\mathbb{P}^M \times \mathbb{P}^1$, donc par l'annulation d'une section du fibré en droites $\mathcal{O}_V(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ sur $V \times \mathbb{P}^1$, élément non nul du sous-espace vectoriel :

$$W := \operatorname{Im}\big(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \hookrightarrow H^0(V, \mathcal{O}_V(1))\big) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \tag{A.6.2}$$

de l'espace de sections :

$$H^0(V, \mathcal{O}_V(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(V \times \mathbb{P}^1, \mathcal{O}_V(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)).$$

3. Les diviseurs de $\mathbb{P}^M \times \mathbb{P}^1$ dans le complémentaire $|L_0| - \Omega$ sont précisément les diviseurs de la forme $H \times \mathbb{P}^1 + \mathbb{P}^M \times \{x\}$, où H (resp. x) désigne un hyperplan de $\mathbb{P}^M$ (resp. un point de $\mathbb{P}^1$).

LEMME A.6.1. *Les deux assertions suivantes sont équivalentes :*

(i) Le sous-espace projectif Λ_p *de* $\mathbb{P}^M$ *est transverse à* V*.*

(ii) L'hypersurface $\mathcal{H}_p$ *est lisse dans* $V \times \mathbb{P}^1$*.*

DÉMONSTRATION. Soit (y, x) un point de $\mathcal{H}_p - ((V \cap \Lambda_p) \times \mathbb{P}^1)$. Le point y dans V est dans l'hyperplan projectif $p(x)$, mais il n'est pas dans le centre Λ_p, on a donc que l'espace tangent algébrique $T_{\mathcal{H}_p,(y,x)}$ ne contient pas la droite $T_{\{y\}\times\mathbb{P}^1,(y,x)}$, donc que ce n'est pas tout $T_{V\times\mathbb{P}^1,(y,x)}$. Donc $\mathcal{H}_p$ est lisse en (y, x).

L'assertion (ii) est donc équivalente à :

(ii') : L'hypersurface $\mathcal{H}_p$ est lisse en tout point de $\mathcal{H}_p \cap ((V \cap \Lambda_p) \times \mathbb{P}^1)$.

Il suffit donc de montrer que (i) et (ii') sont équivalentes. On va montrer le résultat plus fin suivant : pour tout point y de $V \cap \Lambda_p$, les deux assertions suivantes sont équivalentes :

$(\mathrm{i_y})$: Le sous-espace projectif Λ_p intersecte transversalement V en y.

$(\mathrm{ii_y})$: Pour tout x dans $\mathbb{P}^1$, l'hypersurface $\mathcal{H}_p$ est lisse en (y, x).

Soit y un tel point, et soit x un point de $\mathbb{P}^1$. Notons H_x l'hyperplan $p(x)$ dans $\mathbb{P}^M$.

Comme y est dans Λ_p, l'hypersurface $\mathcal{H}_p$ contient tout $\{y\} \times \mathbb{P}^1$, en particulier, l'espace tangent $T_{\mathcal{H}_p,(y,x)}$ contient la droite $T_{\{y\}\times\mathbb{P}^1,(y,x)}$. L'hypersurface $\mathcal{H}_p$ est lisse en (y, x) si et seulement si l'espace tangent $T_{\mathcal{H}_p,(y,x)}$ est un hyperplan de $T_{V\times\mathbb{P}^1,(y,x)}$, donc si et seulement s'il ne contient pas le supplémentaire $T_{V\times\{x\},(y,x)}$ de la droite $T_{\{y\}\times\mathbb{P}^1,(y,x)}$, c'est-à-dire si et seulement si l'espace tangent $T_{H_x,y}$ dans $T_{\mathbb{P}^M,y}$ ne contient pas l'espace tangent $T_{V,y}$.

Donc l'assertion $(\mathrm{ii_y})$ est vraie si et seulement si pour tout point x dans $\mathbb{P}^1$, l'espace tangent $T_{H_x,y}$ dans $T_{\mathbb{P}^M,y}$ ne contient pas l'espace tangent $T_{V,y}$. Comme Λ_p est l'intersection de n'importe quel couple d'hyperplans distincts dans le pinceau $p(\mathbb{P}^1)$, cela revient à dire que l'espace tangent $T_{V\cap\Lambda_p,y}$ est de codimension 2 dans $T_{V,y}$, c'est-à-dire que l'assertion $(\mathrm{i_y})$ est vraie. □

Lorsque les conditions du Lemme A.6.1 sont satisfaites, l'image inverse $\widetilde{V}_p := \nu_p^{-1}(V)$ de V par l'éclatement $\nu_p : \widetilde{\mathbb{P}}_p^M \to \mathbb{P}^M$ s'identifie à l'éclatement dans V de sa sous-variété $V \cap \Lambda_p$, lisse de codimension 2, et l'isomorphisme (A.6.1) se restreint en un isomorphisme de variétés lisses :

$$(\nu_{p|\widetilde{V}_p}, \pi_{p|\widetilde{V}_p}) : \widetilde{V}_p \xrightarrow{\sim} \mathcal{H}_p. \tag{A.6.3}$$

Rappelons que le pinceau

$$p : \mathbb{P}^1 \longrightarrow \mathbb{P}^{M\vee}$$

d'hyperplans de $\mathbb{P}^M$ est appelé *pinceau de Lefschetz relativement à la sous-variété* V lorsque le centre Λ_p du pinceau est transverse à V (et donc $\tilde{V}_p := \nu_p^{-1}(V)$ est lisse), que le morphisme

$$\pi_{p|\tilde{V}_p} : \tilde{V}_p \longrightarrow \mathbb{P}^1$$

a un nombre fini de points critiques, tous non dégénérés, et qu'il y en a au plus un dans chaque fibre.

D'après le lemme A.6.1 et l'isomorphisme (A.6.3), ces conditions sont satisfaites si et seulement si le diviseur $\mathcal{H}_p$ dans $V \times \mathbb{P}^1$ est lisse, si le morphisme

$$\mathrm{pr}_{2|\mathcal{H}_p} : \mathcal{H}_p \hookrightarrow V \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

a un nombre fini de points critiques, qui sont tous non dégénérés, et s'il y en a au plus un dans chaque fibre.

L'existence d'un tel diviseur, et donc d'un pinceau de Lefschetz, découle aussitôt des théorèmes A.1.1 et A.1.2 appliqués à la variété projective lisse :

$$X := V \times \mathbb{P}^1,$$

au fibré en droites sur X :

$$L := L_{0|V\times\mathbb{P}^1} = \mathcal{O}_V(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1),$$

au sous-espace vectoriel W de $H^0(X, L)$ défini par (A.6.2), et au morphisme surjectif lisse :

$$\pi := \mathrm{pr}_2 : X = V \times \mathbb{P}^1 \longrightarrow C := \mathbb{P}^1.$$

Ces théorèmes établissent en fait l'existence d'un ouvert Zariski dense de pinceaux de Lefschetz relativement à V.

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