Collapsing in polygonal dynamics

Based on `arXiv:2507.16432`

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Polygons in projective spaces

Definition (Polygon)

Let $n, d \in \mathbb{N}^*$. An ***n*-gon** P in $\mathbb{P}^d$ is the data of:

- a sequence of **vertices** $(p_i)_{i\in\mathbb{Z}}$ in $\mathbb{P}^d$,
- a **monodromy** $M \in \mathbb{P}\mathrm{GL}_{d+1}$,

such that $\forall i \in \mathbb{Z}$:

- $p_{i+n} = Mp_i$, *(M-periodicity)*
- $(p_i, \dots, p_{i+d+1})$ forms a projective frame. *(Non-degeneracy)*

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P is said be **closed** if $M = \mathrm{Id}$ and **twisted** otherwise.
Their sets are respectively denoted by $\tilde{\mathcal{P}}_n \subset \tilde{\mathcal{T}}_n$ (varieties).

Example of polygons (in $\mathbb{P}^2(\mathbb{R})$)

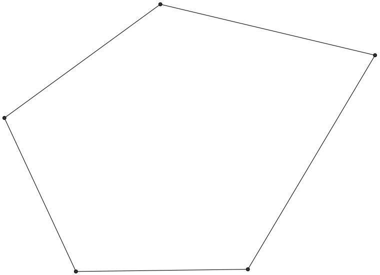


Figure: A closed pentagon $\in \tilde{\mathcal{P}}_5$.

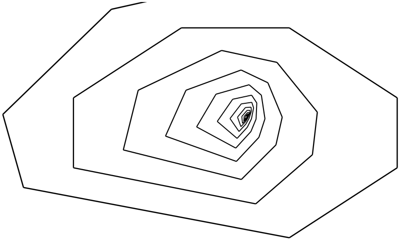


Figure: A twisted heptagon $\in \tilde{\mathcal{T}}_7$.

Polygonal dynamics

Definition

A **polygonal transformation** is a birational map

$$T : \tilde{\mathcal{T}}_n \dashrightarrow \tilde{\mathcal{T}}_n$$

which commutes with the following action of $\mathbb{P}\mathrm{GL}_{d+1}$:

$$A \cdot P = ((Ap_i), AMA^{-1}), \quad A \in \mathbb{P}\mathrm{GL}_{d+1}.$$

T preserves monodromies, so it preserves $\tilde{\mathcal{P}}_n$.

Example in $\mathbb{P}^2$: the pentagram map (Schwartz, 1992)

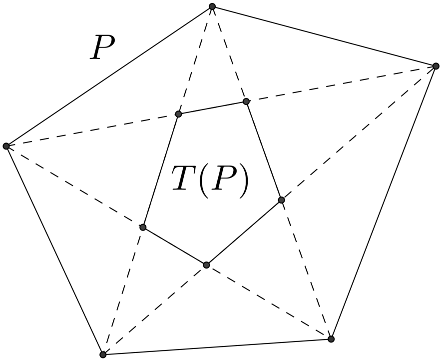


Figure: The pentagram map sends the pentagon P to $T(P)$.

Examples in $\mathbb{P}^1$ (coming from discrete holomorphic functions)

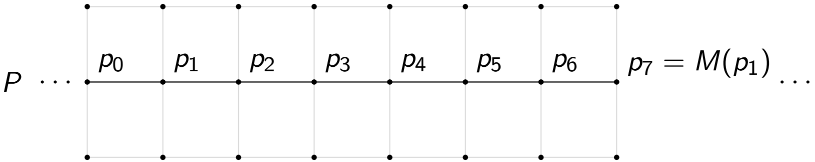


Figure: Flat cross-ratio dynamics.

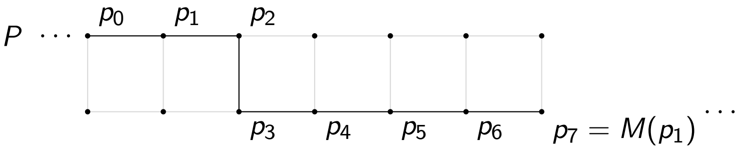
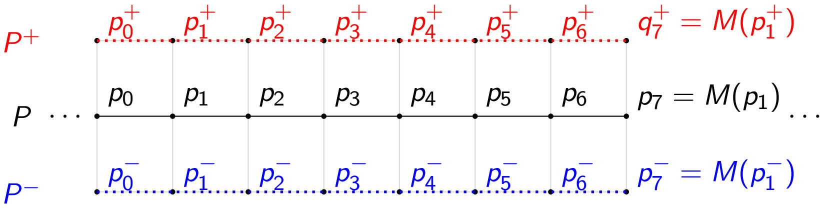


Figure: Staircase cross-ratio dynamics.



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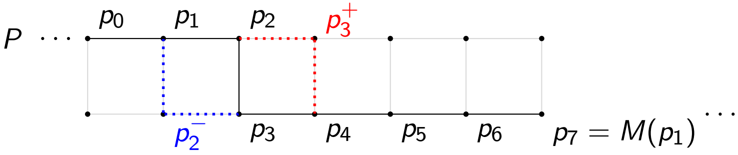

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Previous works about collapsing for the pentagram map

Schwartz 1992

If P is a convex closed polygon in $\mathbb{P}^2(\mathbb{R})$, then its iterates under the pentagram map collapse towards a point q.

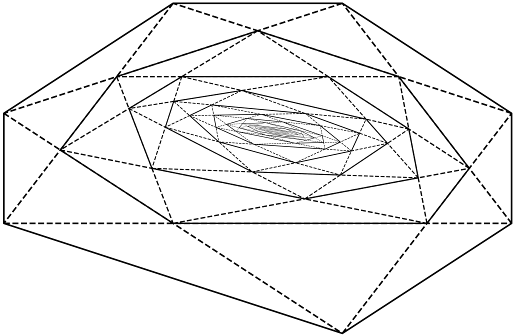

Figure: Collapsing of a convex heptagon.

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Glick 2020

The collapse point q is a fixed point of a linear operator G_P of $\mathbb{R}^3$, depending algebraically on P.

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Arnold-Arreche 2024, Schwartz 2026

Reappearance of G_P in other polygonal dynamics.

Questions

What about the pentagram map on:

- non-convex polygons ?
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Numerical experiments indicate that the collapsing happens in the past/future towards fixed points of G_P, for most polygonal dynamics.

Goal of this talk: prove it !

Moduli spaces

Definition

The **moduli spaces** of closed and twisted n-gons are respectively

$$\mathcal{P}_n := \tilde{\mathcal{P}}_n/\mathbb{P}\mathrm{GL}_{d+1} \quad \subset \quad \mathcal{T}_n := \tilde{\mathcal{T}}_n/\mathbb{P}\mathrm{GL}_{d+1}.$$

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Proposition (S. 2026)

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Proposition (S. 2026)

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Proof:

Coordinate system given by "generalized cross-ratios"

$$(c_{i,j})_{1\leq j\leq d}^{1\leq i\leq n}.$$

Dynamics on the moduli space

Let T be any polygonal transformation. Since it commutes with the action of $\mathbb{P}\mathrm{GL}_{d+1}$, it passes to the moduli space.

$$\begin{tikzcd}
\tilde{\mathcal{T}}_n \arrow[r, dashed, "T"] \arrow[d] & \tilde{\mathcal{T}}_n \arrow[d] \\
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Often, the dynamics of T is integrable on the moduli space.

Scaling symmetry

Let F be any field and consider a polygonal dynamics on $\mathbb{P}^d(F)$.

Definition

A **scaling symmetry** is an action of the multiplicative group F^* on $\mathcal{T}_n$

$$\begin{pmatrix} F^* \times \mathcal{T}_n & \to & \mathcal{T}_n \\ (t, (c_{i,j})) & \mapsto & (c_{i,j}(t)) \end{pmatrix}$$

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Use: Deform algebraicly a polygon $P = (p, M)$ into a family

$$P(t) = (p(t), M(t)).$$

Infinitesimal monodromy (=Glick's operator)

Generalization of previous work for the pentagram map.[1]

Definition

Let T be a polygonal dynamics in $\mathbb{P}^d$ admitting an scaling symmetry, and P be a closed polygon. Its **infinitesimal monodromy** is the formal derivative

$$M'(1) \in \mathfrak{pgl}_{d+1}.$$

It is well defined and invariant through the dynamics.

[1] Quinton Aboud and Anton Izosimov. "The Limit Point of the Pentagram Map and Infinitesimal Monodromy". In: *International Mathematics Research Notices* 2022.7 (Mar. 2022), pp. 5383–5397.

Infinitesimal monodromy (=Glick's operator)

Theorem 1 (S. 2026)

Let T be a polygonal dynamics in $\mathbb{P}^d$ admitting a scaling symmetry, and $P = (p, M)$ be a closed polygon with cross-ratio coordinates $(c_{i,j})$.
Then, its infinitesimal monodromy is

$$M'(1) : p \mapsto \sum_{i=1}^{n} \sum_{j=0}^{d} \frac{c'_{i,j+1}(1)}{c_{i,j+1}(1)} \frac{\det(p_i, \ldots, p_{i+j-1}, p, p_{i+j+1}, \ldots, p_{i+d})}{\det(p_i, \ldots, p_{i+d})} p_{i+j}.$$

Proof:
Write explicitly $M(t)$ and perform tedious computations.

Collapsing of polygons

Proved when the orbit is periodic on the moduli space (there are examples).

Theorem 2 (S. 2026)

Let F be an algebraicly closed field, complete for the metric coming from a non-trivial valuation (for instance $\mathbb{C}$). Let T be a polygonal transformation taking place in $\mathbb{P}^d(F)$.
Then for almost every n-gon $P = (p, M)$ whose dynamics is periodic on the moduli space, the vertices of P collapse towards a fixed point of M.

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Proof:

Periodic orbit on the moduli space.
$\implies$ Iteration of some $A_P \in \mathbb{P}\mathrm{GL}_{d+1}(F)$ on the initial space.
$\implies$ North-South dynamics (generically).

Collapsing of polygons

Conjecture (S. 2025)

Theorem 2 holds even if the dynamics isn't periodic on the moduli space.

Idea of proof:
For integrable dynamics on the moduli space, use the quasi-periodic motion to mimic the proof of Theorem 2.

Limit point for closed polygons ?

Theorem 2 states that the limit point is fixed by the monodromy M, which generically have $d + 1$ of them.
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Question

What are the "true fixed points of the identity" ?

Limit point and $M'(1)$

Answer to the reappearance of Glick's operator (=infinitesimal monodromy), which we compute using Theorem 1.

Theorem 3 (S. 2026)

For closed polygons, the limit point from Theorem 2 is a fixed point of the infinitesimal monodromy $M'(1)$.

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Theorem 3 (S. 2026)

For closed polygons, the limit point from Theorem 2 is a fixed point of the infinitesimal monodromy $M'(1)$.

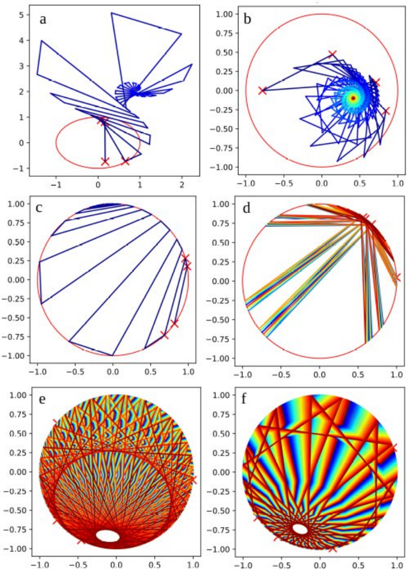
Proof:

Use the scaling symmetry to "open up" the closed polygon P into a family of twisted ones $P(t)$. By Theorem 2, each $P(t)$ collapses to $q(t)$, which is a fixed point of $M(t)$. Observe that

$$M(t) = \mathrm{Id} + (t-1)M'(1) + o(t-1),$$

so by continuity, $q(1)$ is a fixed point of $M'(1)$.

Thank for having listened to me !



Any questions ?

(← Random orbits for the staircase cross-ratio dynamics in $\mathbb{P}^1(\mathbb{C})$.)