D-modules

Damien Simon*

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Use these notes at your own risk, they are an extended version of the two talks I gave. I don't claim the proofs are anywhere near optimal nor original, it is just how I understand things.

Please reach out by email or in person if you have any questions or remarks, I'll be happy to discuss. If you happen to have the answer to any of the question I left pending, please reach out !

All references used have been given during the talks or are up on the website.

Conventions and recollections

If X is a topological space, we use capital letters U, V, W to denote open sets. We try to keep Ω and related notations for differential forms. Sheaves and notations related to sheaves are denoted by capital italic letters, $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{O}$... We denote by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})$$

the *sheaf Hom* of $\mathcal{F}$ and $\mathcal{G}$, and we denote by

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

the set of homomorphism of sheaves between $\mathcal{F}$ and $\mathcal{G}$. Since the sheaf Hom $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ can be confusing, we recall it is defined on every open U as

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

so a section $D \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ is the data for every $W \subset V \subset U$ of maps $D(V)$ and $D(W)$ compatible with restriction i.e. making the following diagram commute

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{D(V)} & \mathcal{G}(V) \\ \downarrow \mathrm{res}_W^V & & \downarrow \mathrm{res}_W^V \\ \mathcal{F}(W) & \xrightarrow{D(W)} & \mathcal{G}(W) \end{array}.$$

*Email : damien.simon@universite-paris-saclay.fr / simondamien69@orange.fr

If we have $V \subset U$ then from $D \in \mathcal{H}om(F,G)(U)$ we get a section $D_{|V} \in \mathcal{H}om(F,G)(V)$ defined for all $W \subset V \subset U$ by

$$D_{|V}(W) = D(W) : \mathcal{F}(W) \longrightarrow \mathcal{G}(W).$$

If our topological space comes with a sheaf of associative algebras say $\mathcal{R}$ then we may consider sheaves of left modules over $\mathcal{R}$. Sheaves of right modules over $\mathcal{R}$ are by definition sheaves of left modules over the opposite ring $\mathcal{R}^{\mathrm{op}}$. The corresponding linearities will be indicated in notations as subscripts. For instance if X is a complex manifold we will consider the sheaves of rings $\underline{\mathbb{C}}$ (or simply $\mathbb{C}$) of locally constant holomorphic functions which is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}_X$. Both are sheaves of commutative rings, and for instance

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{F},\mathcal{G}) \text{ (resp. } \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$$

denotes the $\mathbb{C}$-linear (resp. $\mathcal{O}_X$-linear) sheaf Hom between two sheaves of $\mathbb{C}$-vector spaces (resp. $\mathcal{O}_X$-modules) $\mathcal{F}$ and $\mathcal{G}$ on X. Later, after having properly defined the sheaf of differential operators $\mathcal{D}_X$, we will deal with sheaves of left and right $\mathcal{D}_X$-modules.

To compute in algebras of differential operators, we use the multi-index notation. If $n \in \mathbb{N}$, we typically use $\alpha \in \mathbb{Z}^n$ to denote a multi-index. That is $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$. If $z = (z_1, ..., z_n)$ (thought as coordinates on an open U of $\mathbb{C}^n$) then

$$\partial_z^\alpha = \partial_{z_1}^{\alpha_1} ... \partial_{z_n}^{\alpha_n} = \frac{\partial^{\alpha_1 + ... \alpha_n}}{\partial_{z_1}^{\alpha_1} ... \partial_{z_n}^{\alpha_n}}.$$

It is very useful in explicit computation to use the following convention

$$\partial_z^m = 0 \text{ if } m < 0.$$

We recall a few useful formula concerning Lie brackets, Let A be an associative $\mathbb{C}$-algebra, it is naturally a Lie algebra by means of the bracket given by the commutator

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y \in A$. As such it satisfies the Jacobi identity, for all $x, y, z \in A$, we have

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

It is easily checked that

$$[x, yz] = [x, y]z + y[x, z],$$
$$[xy, z] = [x, z]y + x[y, z],$$

in particular if y commutes with $[x, y]$, i.e. if $[y, [x, y]] = 0$, we have

$$[x, y^2] = [x, y]y + y[x, y] = 2y[x, y]$$

and more generally if $[y, [x, y]] = 0$ then for all $n \in \mathbb{N}$

$$[x, y^n] = ny^{n-1}[x, y].$$

1 The sheaf of differential operators

Let X be an n-dimensional complex manifold and $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. For each $x \in X$, we have an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}_X \simeq \mathbb{C}\{z_1, ..., z_n\}$$

whenever we choose local coordinates $(z_1, ..., z_n)$ around x where $\mathbb{C}\{z_1, ..., z_n\}$ is the ring of power series in n variables with a nonzero convergence radius.

We want to define what the sheaf of differential operators $\mathcal{D}_X$ on X. Because it is a sheaf of operators, it should be a subsheaf of

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

First, it is clear that $\mathcal{O}_X$ is a subsheaf of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$, we see this by defining the map

$$m : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X),$$

by defining for each open set U the map

$$m(U) : \mathcal{O}_X(U) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

that assigns to a $f \in \mathcal{O}_X(U)$ the morphism of sheaves defined for all $V \subset U$ and all $g \in \mathcal{O}_X(V)$ by

$$m(U)(f)(V)(g) = f\big|_V \cdot g \in \mathcal{O}_X(V).$$

It is clear that m is injective.

Let's turn to operators that are slightly more complicated. Denote by Θ_X the sheaf of vector fields. We can think of vector fields as derivations, it is either a definition or a property, depending on one's background. For any $\mathcal{O}_X$-module $\mathcal{F}$ we can define

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F}) \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F})$$

to be the subsheaf given on every open U by

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F})(U) = \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{F}_{|U}) \,|\, D(V)(f \cdot g) = fD(V)(g) + gD(V)(f) \text{ for all } V \subset U \text{ and all } f, g \in \mathcal{O}_X(V)\}.$$

In particular, again it is either a definition or a property, we have

$$\Theta_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Note that this can be rewritten

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)(U) = \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] = m(V)(D(V)(f)) \text{ in } \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_V, \mathcal{O}_V) \text{ for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}.$$

It can be convenient to think that the Lie derivative construction realizes this isomorphism, i.e. whenever we pick a vector field X on an open U, and a function $f \in \mathcal{O}_X(U)$, we can differentiate f along X at each $p \in U$ by following the flow. That yields a map of sheaves of $\mathcal{O}_X$-modules

$$\Theta_X \longrightarrow \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

that turns out to be an isomorphism. From now one, **we think of vector fields as derivations**, that is we have an equality

$$\Theta_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Remark 1. *We are not saying that for a given open set U there exists an isomorphism between $\mathbb{C}$-linear derivations of the $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ and vector fields on U. We can think of $\mathbb{P}^1_{\mathbb{C}}$ where there is a fair amount of global vector fields but no non-trivial derivations of global holomorphic functions (as they are constant functions).*

Question 1. *Can one formulate a sufficient condition so that it holds ?*

This motivates our first definition of $\mathcal{D}_X$, we look at the assignement

$$U \longrightarrow \langle \mathcal{O}_X(U), \Theta_X(U) \rangle \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

where $\langle \mathcal{O}_X(U), \Theta(U) \rangle$ denotes the $\mathbb{C}$-algebra generated by $\mathcal{O}_X(U)$ and $\Theta(U)$ in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$. This clearly yields a presheaf, but it needs not be a sheaf. We denote by

$$\langle \mathcal{O}_X, \Theta_X \rangle$$

its sheafification. The following Lemma shows that sheafification is not too "harmful" in our context.

Lemma 1. *Let X be a topological space, let $\mathcal{G}$ be a sheaf on X. Let $\mathcal{F}$ be a pre-subsheaf of $\mathcal{G}$. Then $\mathcal{F}^{sh}$ can be described on every open U as*

$$\begin{aligned}\mathcal{F}^{sh}(U) &= \{g \in \mathcal{G}(U) \,|\, g_x \in \mathcal{F}_x \text{ for all } x \in U\} \\ &= \{g \in \mathcal{G}(U) \,|\, \text{For all } x \in U \text{ there exists } x \in V \subset U \text{ such that } g_{|V} \in \mathcal{F}(V)\} \\ &= \{g \in \mathcal{G}(U) \,|\, \text{There exists a covering } U = \bigcup_i U_i \text{ such that for all } i \in I,\ g_{|U_i} \in \mathcal{F}(U_i)\}.\end{aligned}$$

In particular, it is clear in that case that $\mathcal{F}$ is a sub-presheaf of $\mathcal{F}^{sh}$ which is itself a subsheaf of $\mathcal{G}$.

Proof. When you glue the sections in some bigger sheaf, which you can always do, it doesn't have to remain in the presheaf, precisely because it is not a sheaf. But clearly it is in the presheaf locally, so you add all the people obtained by this procedure and can show this is enough. □

We get the following explicit description, for every U an element $D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$ belongs to $\langle \mathcal{O}_X, \Theta_X \rangle(U)$ if and only if around every point $x \in U$ a neighboorhood V such that $D_{|V} \in \langle \mathcal{O}_X(V), \Theta_X(V) \rangle$.

Remark 2. *For some open sets U, typically with infinitely many connected components, it may happen that*

$$\langle \mathcal{O}_X(U), \Theta_X(U)\rangle \neq \langle \mathcal{O}_X, \Theta_X\rangle(U).$$

It may also happen for "non-trivial" reasons.

Question 2. *Find a "non-trivial" counterexample.*

Proposition 1. *Let U be an open of X and $D \in \langle \mathcal{O}_X, \Theta_X\rangle(U)$. Then for all $x \in U$ there exists a neighborhood $V \subset U$ of x that can be chosen to be a coordinate chart*

$$z = (z_1, ..., z_n) : V \longrightarrow \mathbb{C}^n$$

such that

$$D_{|V} = \sum_\alpha m(V)(a_\alpha(z))\partial_z^\alpha = \sum_\alpha a_\alpha(z)\partial_z^\alpha$$

for some ***uniquely determined*** *functions $a_\alpha(z) \in \mathcal{O}_{\mathbb{C}^n}(V)$.*

Proof. The existence of such a V is clear and we leave the reader to check the unicity of the writing. Because V is a coordinate chart, it is clear that elements of $\Theta_X(V)$ are given by elements of the form

$$\sum_{i=1}^n m(V)(f_i(z))\partial_{z_i},$$

where the $(f_i(z))_{1\leq i\leq n}$ are holomorphic functions on V. So it is clearly enough by linearity to show for instance that an element of the form

$$m(V)(f(z))\partial_{z_1}^2 m(V)(g(z))\partial_{z_2} \;\; f, g \in \mathcal{O}_X(V)$$

can be written in the correct way. It is a straightforward computation, we just move the partial derivatives to the right :

$$\begin{aligned} f(z)\partial_{z_1}^2 g(z)\partial_{z_2} &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)[\partial_{z_1}^2, g(z)]\partial_{z_2} \\ &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\partial_{z_1}\frac{\partial g}{\partial z_1}(z)\partial_{z_2} \\ &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\frac{\partial^2 g}{\partial z_1^2}(z)\partial_{z_2}. \end{aligned}$$

□

Using this definition, we could define the order of $D \in \langle \mathcal{O}_X, \Theta_X\rangle(U)$ at $x \in U$ to be the maximal order of derivative that appears when restricted to a local chart around x and check this doesn't depend on the chart.

We'll give another description of the sheaf of differential operators that will make the order clearly coordinate-independant.

To do so, we define recursively for sub-presheaves $\mathcal{D}_X^{\leq m}$ of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ on each U via the conditions

$$\begin{aligned}
\mathcal{D}_X^{\leq -1}(U) &= 0, \\
\mathcal{D}_X^{\leq 0}(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] = 0 \\
&\quad \text{for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}, \\
\mathcal{D}_X^{\leq m}(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] \in \mathcal{D}_X^{\leq m-1}(V) \\
&\quad \text{for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}, \text{ if } m \geq 0.
\end{aligned}$$

They turn out to be subsheaves. It is clear by definition that

$$0 = \mathcal{D}_X^{\leq -1} \subset \mathcal{D}_X^{\leq 0} \subset \mathcal{D}_X^{\leq 1} \subset \ldots .$$

The first two steps can be very explicitely described as follows.

Proposition 2. *The image of the map of $\mathcal{O}_X$-modules*

$$m : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

lies in $\mathcal{D}_X^{\leq 0}$. Conversely evaluating at the constant function equal to 1 yields a map of $\mathcal{O}_X$-modules

$$ev_1 : \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{O}_X.$$

These two maps are inverses of each other, in particular the map m identifies $\mathcal{O}_X$ with $\mathcal{D}_X^{\leq 0}$.

Proof. Let's be explicit about how ev_1 is defined. Let U be an open and $D \in \mathcal{D}_X^{\leq 0}(U)$, we can evaluate $D(U)$ on the constant function $1_U \in \mathcal{O}_X(U)$ which yields $D(U)(1_U) \in \mathcal{O}_X(U)$, it is easy to see it defines a map of sheaves of $\mathcal{O}_X$-modules

$$\mathrm{ev}_1 : \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{O}_X.$$

I claim that it is straightforward to check $m \circ \mathrm{ev}_1 = \mathrm{id}_{\mathcal{D}_X^{\leq 0}}$ and $\mathrm{ev}_1 \circ m = \mathrm{id}_{\mathcal{O}_X}$.[1] □

Let's now turn to the next part of our filtration. Now that we know $\mathcal{D}_X^{\leq 0} \simeq \mathcal{O}_X$, it is straightforward from the definition of Θ_X to check that

$$\Theta_X \subset \mathcal{D}_X^{\leq 1}.$$

Let $U \subset X$ and $D \in \mathcal{D}_X^{\leq 1}(U)$, by our definition, for all $V \subset U$ and $f \in \mathcal{O}_X(V)$ we have

$$[D(V), m(V)(f)] \in \mathcal{D}^{\leq 0}(V).$$

[1] This is not any deeper than checking the following fact : let R be a commutative $\mathbb{C}$-algebra and $M \in \mathrm{End}_{\mathbb{C}}(R)$, if M commutes with all operators of multiplication $m_f : r \in R \mapsto f \cdot r \in R$ for $f \in R$ then $M = m_{M(1)}$ and conversely $m_f = m_{m_f(1)}$ for all $f \in R$.

But we have just shown that $\mathcal{D}_X^{\leq 0} \simeq \mathcal{O}_X$. Explicitely,

$$[D(V), m(V)(f)] = m(V)([D(V), m(V)(f)](1_V)) = m(V)(D(V)(f) - fD(V)(1_V)),$$

hence we have built a morphism of $\mathcal{O}_X$-modules

$$p : \mathcal{D}_X^{\leq 1} \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{D}_X^{\leq 0}) = \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

that (all identifications done and being loose on the open sets) associates to a $D \in \mathcal{D}_X^{\leq 1}$ the map $f \in \mathcal{O}_X \mapsto [D, f](1) \in \mathcal{O}_X$. We can say a bit more, let $f, g \in \mathcal{O}_X(V)$ then

$$\begin{aligned}[D(V), m(V)(fg)] &= [D(V), m(V)(f)m(V)(g)] \\ &= [D(V), m(V)(f)]m(V)(g) + m(V)(g)[D(V), m(v)(f)]\end{aligned}$$

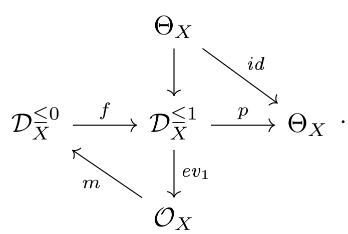
that is exactly the same as saying, loosely :

$$p(D)(fg) = p(D)(f)g + fp(D)(g)$$

i.e. the image of p lies in $\Theta_X \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

Remark 3. *This whole discussion can feel very formal, but we are just saying that from the order one differential operator $z^4 \frac{\partial}{\partial z} + 12z$ to recover the operator $z^4 \frac{\partial}{\partial z}$ it is enough to do the bracket with the operator of multiplication by f, which yields the operator of multiplication by $z^4 \frac{\partial f}{\partial z}$ and evaluate on the unit.*

Proposition 3. *The following diagram of sheaves of $\mathcal{O}_X$-modules is commutative with exact rows and columns*



In particular the short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{D}_X^{\leq 1} \xrightarrow{p} \Theta_X \longrightarrow 0,$$

and

$$0 \longrightarrow \Theta_X \longrightarrow \mathcal{D}_X^{\leq 1} \xrightarrow{ev_1} \mathcal{O}_X \longrightarrow 0$$

are split and after identifying $\mathcal{O}_X$ with $\mathcal{D}_X^{\leq 0}$ via m we have, as sheaves of $\mathcal{O}_X$-modules, the equality

$$\mathcal{D}_X^{\leq 1} = \mathcal{D}_X^{\leq 0} \oplus \Theta_X = \mathcal{O}_X \oplus \Theta_X.$$

Proof. Exercice. □

Now that we have a bit more intuition as to why this inductive definition makes sense, we define the following presheaf

$$\mathcal{D}_X^{pre} : U \longmapsto \bigcup_{n\geq 0} \mathcal{D}_X^{\leq n}(U).$$

Proposition 4. *For all $m, m' \geq 0$ and all open $U \subset X$ we have*

$$\mathcal{D}_X^{\leq m}(U) \circ \mathcal{D}_X^{\leq m'}(U) \subset \mathcal{D}_X^{\leq m+m'}(U),$$
$$[\mathcal{D}_X^{\leq m}(U), \mathcal{D}_X^{\leq m'}(U)] \subset \mathcal{D}_X^{\leq m+m'-1}(U).$$

In particular, $\mathcal{D}_X^{\leq 1}$ is a sheaf of Lie algebras and $D_X^{pre}(U) = \bigcup_{m>0} \mathcal{D}_X^{\leq m}(U)$ is an $\mathcal{O}_X(U)$-filtered algebra that is quasi-commutative, that is to say such that the graded $\mathcal{O}_X(U)$-algebra

$$\bigoplus_{m\geq 0} D_X^{\leq m}(U)/D_X^{\leq m-1}(U)$$

is commutative.

Proof. Exercice, this follows either from the explicit formulas for the principal symbol or the Jacobi identity. □

We define $\mathcal{D}_X$ as the sheafification of $\mathcal{D}_X^{pre}$. To put it "explicitely" :

$$\begin{aligned}\mathcal{D}_X(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) | \text{For all } x \in U, \text{ there exists } m_x \in \mathbb{N} \text{ such that } D_x \in (\mathcal{D}_X^{\leq m_x})_x\} \\ &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) | \text{For all } x \in U, \text{ there exists } m_x \in \mathbb{N} \text{ and } x \in V \subset U \text{ s.t. } D_{|V} \in \mathcal{D}_X^{\leq m_x}(V)\}.\end{aligned}$$

We will write loosely

$$\mathcal{D}_X = \bigcup_{m\geq 0} \mathcal{D}_X^{\leq m}.$$

It follows from the previous proposition that for all $m, m' \in \mathbb{N}$ we have well defined multiplication maps of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{D}_X^{\leq m} \times \mathcal{D}_X^{\leq m'} \longrightarrow \mathcal{D}_X^{\leq m+m'}$$

that make

$$\mathcal{D}_X = \bigcup_{m\geq 0} \mathcal{D}_X^{\leq m}$$

into a sheaf of $\mathbb{Z}$-filtrered associative $\mathcal{O}_X$-algebras.

Remark 4. *The order of a differential operator on an open U i.e. of an element of $\mathcal{D}_X(U)$ is only defined locally on U. In fact let $D \in \mathcal{D}_X(U)$, then for all $x \in U$ by assumption there exists a unique $m_x \in \mathbb{N}$ such that $D_x \in (\mathcal{D}_X^{\leq m_x})_{n_x}$ and $D_x \notin (\mathcal{D}_X^{\leq m_x - 1})_x$. We say that m_x is the order of P at x. We'll see later that the order defines a locally constant function in particular, sheafification doesn't change anything on a connected open set U i.e. we have*

$$\mathcal{D}_X(U) = \bigcup_{m\geq 0} \mathcal{D}_X(U) = \mathcal{D}_X^{pre}(U).$$

We now explain how to derive a commutative object from $\mathcal{D}_X$. Define for each $m \in \mathbb{N}$ the sheaf of $\mathcal{O}_X$-modules $\mathrm{Gr}^m(\mathcal{D}_X)$ to be the quotient sheaf of $\mathcal{D}_X^{\leq m}$ by its subsheaf $\mathcal{D}_X^{\leq m-1}$. By definition it fits in an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{D}_X^{\leq m-1} \longrightarrow \mathcal{D}_X^{\leq m} \longrightarrow \mathrm{Gr}^m(\mathcal{D}_X) \longrightarrow 0.$$

We recall that by definition $\mathrm{Gr}^m(\mathcal{D}_X) = \mathcal{D}_X^{\leq m}/\mathcal{D}_X^{\leq m-1}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{D}_X^{\leq m}(U)/\mathcal{D}_X^{\leq m-1}(U).$$

Remark 5. *For an open set U, it might happen that*

$$Gr^m(\mathcal{D}_X)(U) \neq \mathcal{D}^{\leq m}(U)/\mathcal{D}^{\leq m}(U).$$

This fact has to do with sheaf cohomology as we'll see later.

We can define $\mathrm{Gr}(\mathcal{D}_X)$ to be the sheaf associated with the presheaf of $\mathcal{O}_X$-modules

$$U \mapsto \bigoplus_{m\geq 0} \mathcal{D}^{\leq m}(U)/\mathcal{D}^{\leq m-1}(U).$$

So that we have, and there is a little something to check I think, as sheaves of $\mathcal{O}_X$-modules, the equality

$$\mathrm{Gr}(\mathcal{D}_X) = \bigoplus_{m\geq 0} \mathrm{Gr}^m(\mathcal{D}_X).$$

Proposition 5. *For all pair of integers $m, m' \in \mathbb{Z}$, the multiplication map*

$$\mathcal{D}_X^{\leq m} \times \mathcal{D}_X^{\leq m'} \longrightarrow \mathcal{D}_X^{\leq m+m'}$$

defines a map of sheaves of $\mathcal{O}_X$-modules

$$Gr^m(\mathcal{D}_X) \times Gr^{m'}(\mathcal{D}_X) \longrightarrow Gr^{m+m'}(\mathcal{D}_X).$$

*These maps in turn define a structure of a sheaf of a $\mathbb{Z}$-graded **commutative** $\mathcal{O}_X$-algebra on*

$$Gr(\mathcal{D}_X) = \bigoplus_{m\geq 0} Gr^m(\mathcal{D}_X).$$

Remark 6. *Again, for an open set U, it might happen that*

$$Gr(\mathcal{D}_X)(U) \neq \bigoplus_{m\geq 0} \mathcal{D}_X^{\leq m}(U)/\mathcal{D}_X^{\leq m-1}(U).$$

It can happen for stupid reasons, for instance if U has an infinite number of connected components then it can clearly almost never work because of the infinite direct sum. And it can happen for deeper reasons, that have to do with the previous remark and are related to sheaf cohomology.

Question 3. *When U is connected, do we need to sheafify the infinite direct sum ? That is do we have the equality*

$$Gr(\mathcal{D}_X)(U) = \bigoplus_{m \geq 0} Gr^m(\mathcal{D}_X)(U) \ ?$$

Answer : No need to sheafify

The following theorem shows that our two definitions of differential operators coincide. In particular the order of a differential operator near a point is a well-defined notion that is independant of coordinates. It will also allows us to describe explicitely the associated graded sheaf.

Theorem 1. *The equality of sheaves of $\mathcal{O}_X$-modules*

$$\mathcal{D}_X = \langle \mathcal{O}_X, \Theta_X \rangle$$

holds inside of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. We start by the easy inclusion, let $D \in \langle \mathcal{O}_X, \Theta_X \rangle(U)$, then for each $x \in U$ there exists $x \in V \subset U$ such that

$$D_{|V} \in \langle \mathcal{O}_X(V), \Theta_X(V) \rangle$$

so in particular, since $\mathcal{O}_X = \mathcal{D}_X^{\leq 0}(V)$ and $\Theta_X(V) \subset \mathcal{D}_X^{\leq 1}(V)$ then it is clear from Proposition 4 that there exists $n_x \in \mathbb{N}$ such that

$$D_{|V} \in \mathcal{D}_X^{\leq n_x}(V)$$

and so $D \in \mathcal{D}_X(U)$. The reverse inclusion is more subtle, let $D \in \mathcal{D}_X(U)$ then for each $x \in U$ there exists $x \in V \subset U$ and $n_x \in \mathbb{N}$ such that

$$D_{|V} \in \mathcal{D}_X^{\leq n_x}(V),$$

if we show that there exists an open neighbourhood $W \subset V$ of x such that $\mathcal{D}_X^{\leq n_x}(W) \subset \langle \mathcal{O}_X(W), \Theta_X(W) \rangle$ we are done. To do so, we may shrink V as we please and assume for instance that we are on a coordinate chart mapping x to 0

$$(z_1, ..., z_n) : V \longrightarrow \mathbb{C}^n,$$

so the proof reduces to that of the following Lemma.

Lemma 2. *Let V an open neighbourhood of 0 in $\mathbb{C}^n$. Then for all $m \in \mathbb{N}$ and all $D \in \mathcal{D}_{\mathbb{C}^n}^{\leq m}(V)$ there exists an neighbourhood $W \subset V$ of 0 such that*

$$D_{|W} \in \langle \mathcal{O}_{\mathbb{C}^n}(W), \Theta_{\mathbb{C}^n}(W) \rangle.$$

Proof. Let $D \in \mathcal{D}_{\mathbb{C}^n}^{\leq m}(V) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_V, \mathcal{O}_V)$, looking at the stalk at 0, it defines a $\mathbb{C}$-linear morphism

$$D_0 : \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow \mathcal{O}_{\mathbb{C}^n,0}.$$

The following equality holds in $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})$

$$[...[[D_0, f_1], f_2, ..., f_{m+1}]] \dots] = 0,$$

for all $f_1, ..., f_{m+1} \in \mathcal{O}_{\mathbb{C}^n,0}$. But since

$$\mathcal{O}_{\mathbb{C}^n,0} \simeq \mathbb{C}\{z_1, ..., z_n\},$$

it is enough to show the following Lemma

Lemma 3. *Let $m \in \mathbb{N}$ and $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $P_1, ..., P_{m+1} \in \mathbb{C}\{z_1, ..., z_n\}$ we have the following equality in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$*

$$[...[[D, P_1], P_2], ..., P_{m+1}] = 0.$$

Denote by $\mathcal{D}^{\leq m}(\mathbb{C}\{z_1, ..., z_n\})$ the set of all such D. Then there exists a family $(a_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}\{z_1, ..., z_n\}$ such that

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial_z^\alpha$$

and $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| > m$.

Proof. We proceed by induction on m. If $m = 0$ this is easy (we've seen it in a more general context, recall the proof of the equality of sheaves $\mathcal{D}_X^{\leq 0} = \mathcal{O}_X$). Let $m \in \mathbb{N}$ such that the statement holds and let $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $P_1, ..., P_{m+2} \in \mathbb{C}\{z_1, ..., z_n\}$ we have

$$[...[[D, P_1], P_2], ..., P_{m+2}] = 0.$$

Then clearly if we set for all $i \in \{1, ..., n\}$,

$$D_i = [D, z_i] \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$$

then $D_i \in \mathcal{D}^{\leq m}(\mathbb{C}\{z_1, ..., z_n\})$ and moreover

$$[D_i, z_j] = [[D, z_i], z_j] = [[D, z_j], z_i] = [D_j, z_i]$$

for all $i, j \in \{1, ..., n\}$. Assume momentaneously that there exists

$$\widetilde{D} \in \mathcal{D}^{\leq m+1}(\mathbb{C}\{z_1, ..., z_n\})$$

such that for all $i \in \{1, ..., n\}$ we have

$$[\widetilde{D}, z_i] = D_i.$$

Then

$$[D - \widetilde{D}, z_i] = [D, z_i] - [\widetilde{D}, z_i] = D_i - D_i = 0$$

for all $i \in \{1, ..., n\}$. It follows from Lemma 4 below that $D - D' \in \mathcal{D}^{\leq 0}(\mathbb{C}\{z_1, ..., z_n\}) = \mathbb{C}\{z_1, ..., z_n\}$ then we are done.

We come back to the proof of the existence of $\widetilde{D}$. By induction hypothesis we know that for all $i \in \{1, ..., n\}$ there exists $a_\alpha^i \in \mathbb{C}\{z_1, ..., z_n\}$ such that

$$D_i = [D, z_i] = \sum_{|\alpha| \leq m} a_\alpha^i \partial_z^\alpha.$$

Now forget about D, it is a general fact of differential operators that if a family $(D_i)_{1 \leq i \leq n}$ of differential operators of order $\leq m$ satisfy[2]

$$[D_i, z_j] = [D_j, z_i]$$

then they come from an operator of degree at most one more, i.e. there exists $\widetilde{D}$ of order $\leq m+1$ such that

$$[\widetilde{D}, z_i] = D_i.$$

We look for $\widetilde{D}$ in the form

$$\widetilde{D} = \sum_{|\alpha| \leq m+1} A_\alpha(z) \partial_z^\alpha.$$

The following equality of operators hold

$$[\partial_z^\alpha, z_i] = \alpha_i \partial_z^{\alpha - e_i},$$

where $e_i = (0, ..., \overset{i}{1}, ..., 0)$, and hence

$$[\widetilde{D}, z_i] = \sum_{|\alpha| \leq m+1} \alpha_i A_\alpha(z) \partial_z^{\alpha - e_i}$$

now this is equal to $D_i = \sum_{|\alpha| \leq m} a_\alpha^i(z) \partial_z^\alpha = \sum_\alpha a_{\alpha - e_i}^i \partial_z^{\alpha - e_i}$ if and only if for all α we have

$$\alpha_i A_\alpha(z) = a_{\alpha - e_i}^i(z).$$

So for each α such that $|\alpha| > 0$ i.e. such that there exists $i \in \{1, ..., n\}$ such that $\alpha_i \neq 0$ we set

$$A_\alpha(z) = \frac{1}{\alpha_i} a_{\alpha - e_i}^i(z).$$

To show this doesn't depend on i we need to show that for each $|\alpha| \leq m$ such that $\alpha_i \neq 0$ and $\alpha_j \neq 0$ we have

$$\alpha_j a_{\alpha - e_i}^i = \alpha_i a_{\alpha - e_j}^j$$

but this follows from the equality

$$[D_i, z_j] = [D_j, z_i].$$

[2] This condition looks "Fourier dual" to a Schwarzian integrability condition if $m = 1$, is there a bigger picture ?

In fact, if $i \neq j$ we have

$$[D_i, z_j] = \sum_{\alpha} \alpha_j a^i_\alpha(z) \partial_z^{\alpha - e_j} \overset{\alpha' = \alpha + e_i}{=} \sum_{\alpha'} \alpha'_j a^i_{\alpha' - e_i} \partial_z^{\alpha' - e_j - e_i},$$

$$[D_j, z_i] = \sum_{\alpha} \alpha_i a^j_\alpha(z) \partial_z^{\alpha - e_i} \overset{\alpha' = \alpha + e_j}{=} \sum_{\alpha'} \alpha'_i a^j_{\alpha' - e_j} \partial_z^{\alpha' - e_i - e_j},$$

and that yields the desired equality looking at the term corresponding to a given α' such that $\alpha'_i \neq 0$ and $\alpha'_j \neq 0$ because for such an α', we have

$$\alpha' - e_i - e_j \geq 0.$$

□

□

□

Lemma 4. *Let $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $i \in \{1, ..., n\}$ we have*

$$[D, z_i] = 0,$$

then for all $f \in \mathbb{C}\{z_1, ..., z_n\}$,

$$[D, f] = 0.$$

Hence $D \in \mathcal{D}^{\leq 0}(\mathbb{C}\{z_1, ..., z_n\}) = \mathbb{C}\{z_1, ..., z_n\}$.

Proof. Here's an indication on how to prove it :

Any element of $f \in \mathbb{C}\{z_1, ..., z_n\}$ can be written (in a highly non unique way when $n \geq 2$)

$$f = f(0) + z_1 f_1 + ... + z_n f_n$$

so

$$[D, f] = [D, f_1] z_1 + ... + [D, f_n] z_n$$

and iterating there is a problem... because it should map into

$$\bigcap_{m > 0} (z_1, ..., z_n)^m = 0$$

so this is the zero map. □

Proposition 6. *Let U be an open set of X and $D \in \mathcal{D}_X(U)$ then for all $x \in X$ there exists a unique $m_x \in \mathbb{N}$ called the order of D at x such that*

$$D_x \in (\mathcal{D}_X^{\leq m_x})_x \setminus (\mathcal{D}_X^{\leq m_x - 1})_x.$$

The function

$$x \in U \longmapsto m_x \in \mathbb{N}$$

is locally constant and hence constant if U is connected. Consequently for any connected open U, we have the equality of $\mathcal{O}_X(U)$-algebras

$$\mathcal{D}_X(U) = \bigcup_{m \geq 0} \mathcal{D}_X^{\leq m}(U) = \mathcal{D}_X^{pre}(U).$$

Moreover for all choice of coordinates on a connected open neighborhood V of x say

$$z : V \longrightarrow \mathbb{C}^n$$

there exists uniquely determined functions $a_\alpha \in \mathcal{O}_{\mathbb{C}^n}(z(V))$ such that in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_V, \mathcal{O}_V)$ we have

$$D_{|V} = \sum_{|\alpha| \leq m_x} a_\alpha(z)\partial_z^\alpha.$$

Proof. The existence of such an m_x is clear by definition. Fix coordinates on some connected open neighborhood V of x. By assumption there exists an open W^x, which we may choose connected and inside of V, such that

$$D_{|W} \in \mathcal{D}_X^{\leq m_x}(W^x) - \mathcal{D}_X^{\leq m_x - 1}(W^x).$$

With our choice of coordinates and thanks to the previous theorem, up to maybe shrinking W^x, there exists $a_\alpha^{W^x} \in \mathcal{O}_{\mathbb{C}^n}(z(W^x))$ such that

$$D_{|W^x} = \sum_{\alpha \leq m_x} a_\alpha^{W^x}(z)\partial_z^\alpha.$$

The same holds for any point $y \in V$, namely there exists a connected open $W^y \subset V$ containing y such that there exists $a_\alpha^{W^x} \in \mathcal{O}_{\mathbb{C}^n}(z(W^y))$ such that

$$D_{|W^y} = \sum_{\alpha \leq m_y} a_\alpha^{W^y}(z)\partial_z^\alpha.$$

Now for all $x, y \in V$ such that

$$W^x \cap W^y \neq \emptyset,$$

which happens for instance when $y \in W^x$, we have the equality in $\mathcal{D}_X(W^x \cap W^y)$

$$D_{|W^x \cap W^y} = D_{|W^y \cap W^x}$$

that is in particular

$$\sum_{\alpha \leq m_x} a_\alpha^{W^x}(z)\big|_{W^x \cap W^y}\partial_z^\alpha = \sum_{\alpha \leq m_y} a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}\partial_z^\alpha.$$

so using unicity of the writing we get that for all α :

$$a_\alpha^{W^x}(z)\big|_{W^x \cap W^y} = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}$$

In particular for all α such that $|\alpha| > m_x$ the previous equality becomes

$$0 = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}.$$

Now by analytic continuation because W^y is connected and $W^y \cap W^x \neq \emptyset$ we get that $a_\alpha^{W^y}(z) = 0$ and hence $m_y \leq m_x$ so by symmetry $m_x = m_y$ and hence the fact that the order is constant on W^x and hence locally constant. We have an open covering

$$V = \bigcup_{x \in V} W^x$$

and because V is connected we have $m_x = m_y = m$ for all $x, y \in V$. For all α such that $|\alpha| \leq m$ we have by unicity again

$$a_\alpha^{W^x}(z)\big|_{W^x \cap W^y} = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}.$$

So for each α they glue to a certain holomorphic function $a_\alpha \in \mathcal{O}_X(V)$ such that for all $x \in V$ the following equality holds in $\mathcal{O}_X(W^x)$

$$a_\alpha(z)\big|_{W^x} = a_\alpha^{W^x}(z).$$

Now it is clear from the fact that $\mathcal{D}_X$ is a sheaf that

$$D_{|V} = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_z^\alpha.$$

$\square$

Remark 7. *We note that this proves the expected fact that for all $n \geq 0$ the associative $\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$-algebra of global differential operators on $\mathbb{C}^n$*

$$\mathcal{D}_{\mathbb{C}^n}(\mathbb{C}^n)$$

consists of all the elements of the form

$$\sum_\alpha a_\alpha(z) \partial_z^\alpha.$$

In particular for each $m \in \mathbb{N}$, the sheaf of $\mathcal{O}_X$-modules $\mathcal{D}_X^{\leq m}$ is a locally free of finite rank i.e. corresponds to a finite dimensional vector bundle. Explicitely for each connected coordinate chart U in coordinates $z_1, ..., z_n$ we have an isomorphism of $\mathcal{O}_U$-modules

$$\mathcal{D}_U^{\leq m} \simeq \bigoplus_{|\alpha| \leq m} \mathcal{O}_U \partial_z^\alpha.$$

In fact for every (possibly non connected) coordinate chart U in coordinates $z_1, ..., z_n$, we have

$$\mathcal{D}_U \simeq \bigoplus_\alpha \mathcal{O}_U \partial_z^\alpha$$

where the right hand side is the appropriately defined (=sheafified) infinite direct sum of sheaves. On every connected open set $V \subset U$, analytic continuation ensures that

$$\mathcal{D}_X(V) = \bigoplus_\alpha \mathcal{O}_X(V)\partial_z^\alpha.$$

So maybe one motivation for saying everything in terms of sheaves is to have a flexible setting that allows one to speak of $\mathcal{D}_X$, an infinite dimensional vector bundle, without every thinking of its transition maps as "infinite dimensional" matrices.

We can now describe the sheaf $\mathrm{Gr}(\mathcal{D}_X)$.

Proposition 7. *We have the equality of sheaves of $\mathcal{O}_X$-modules*

$$Gr^0(\mathcal{D}_X) = \mathcal{O}_X, Gr^1(\mathcal{D}_X) = \Theta_X.$$

That defines a map of sheaves of graded commutative $\mathcal{O}_X$-algebras

$$Sym_{\mathcal{O}_X}(\Theta_X) \longrightarrow Gr(\mathcal{D}_X).$$

which is an isomorphism. In particular, for all $m \geq 0$ we have an isomorphism of sheaves of $\mathcal{O}_X$-modules

$$Sym^m_{\mathcal{O}_X}(\Theta_X) \longrightarrow Gr^m(\mathcal{D}_X).$$

Sketch of proof. The construction of the map is clear by playing around with universal properties, to see this is an isomorphism one can work locally and then it is clear in local coordinates. □

Remark 8. *Give the interpretation in terms of the cotangent bundle. Sym in the notation stands for symmetric, funnily enough, it stands for (principal) symbol at the same time, so it's a really good notation.*

2 Differential operators on the projective line

Let $\mathbb{P}^1_{\mathbb{C}} = \mathbb{P}^1$ be the projective line over the complex number. It is covered by two charts

$$\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}_z,$$
$$\mathbb{P}^1 \setminus \{0\} = \mathbb{C}_w,$$

and on the intersection, i.e. on $\mathbb{P}^1 \setminus \{0, \infty\}$ the change of coordinate is given by

$$z \in \mathbb{C}^*_z \longrightarrow \frac{1}{w} \in \mathbb{C}^*_w.$$

In the previous section we defined the sheaf $\mathcal{D}_{\mathbb{P}^1}$ of differential operators on $\mathbb{P}^1$. Here we want to describe the global differential operators on the projective line

$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1),$$

as a $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$-algebra.

First recall that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C}$ because any holomorphic function $\mathbb{P}^1 \longrightarrow \mathbb{C}$ needs to be bounded and hence constant. So really what we are describing is a (a priori non-commutative) $\mathbb{C}$-algebra.

Because $\mathbb{P}^1$ is connected, the order of a global differential operator is well defined, that is, as $\mathbb{C}$-algebras we have

$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1) = \bigcup_{m \geq 0} \mathcal{D}^{\leq m}_{\mathbb{P}^1}(\mathbb{P}^1).$$

We have the following equality of $\mathbb{C}$-vector spaces

$$\mathcal{D}^{\leq 1}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \oplus \Theta_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C} \oplus \Theta_{\mathbb{P}^1}(\mathbb{P}^1).$$

where $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ is the Lie algebra of global vector fields on $\mathbb{P}^1$. Let us first compute $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.

Lemma 5. *The Lie algebra $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ is 3-dimensional with basis f, h, e satisfying*

$$\begin{aligned} [h, f] &= -2f, \\ [e, f] &= h, \\ [h, e] &= 2e. \end{aligned}$$

That is, there is an isomorphism of Lie algebra

$$\Theta_{\mathbb{P}^1}(\mathbb{P}^1) \simeq \mathfrak{sl}_2.$$

Proof. Let $\theta \in \Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ then by definition,

$$\begin{aligned} \theta_{|\mathbb{C}_z} &= a(z)\partial_z, \\ \theta_{|\mathbb{C}_w} &= b(w)\partial_w, \end{aligned}$$

for some $a \in \mathcal{O}_{\mathbb{C}_z}(\mathbb{C}_z)$ and $b \in \mathcal{O}_{\mathbb{C}_w}(\mathbb{C}_w)$. Now because θ is globally defined it means that the vector fields coincide on the intersection i.e. on they agree on

$$\mathbb{C}_z^* \simeq \mathbb{P}^1 \setminus \{0, \infty\} \simeq \mathbb{C}_w^*.$$

In the coordinate w that means we have, as vector fields on $\mathbb{C}_w^*$ the equality

$$a(\frac{1}{w})(-w^2)\partial w = b(w)\partial w.$$

That is, for all $w \in \mathbb{C}_w^*$ we have

$$b(w) = -w^2 a(\frac{1}{w})$$

which implies that f, and g are both polynomials of order at most 2. That is we can write

$$\begin{aligned} a(z) &= a_0 + a_1 z + a_2 z^2, \ a_0, a_1, a_2 \in \mathbb{C}, \\ b(w) &= b_0 + b_1 w + b_2 w^2, \ b_0, b_1, b_2 \in \mathbb{C}, \end{aligned}$$

and the equality $b(w) = -w^2 a(\frac{1}{w})$ implies

$$b_0 + b_1 w + b_2 w^2 = -w^2 a_0 + -af_1 - a_2$$

i.e.

$$\begin{aligned} b_0 &= -a_2 \\ b_1 &= -a_1 \\ b_2 &= -a_0. \end{aligned}$$

So a $\mathbb{C}$-basis of the Lie algebra of global vector fields is for instance, symbolically given by the vector fields

$$\begin{aligned} e &= -\partial_z = w^2 \partial_w, \\ h &= -2z\partial_z = 2w\partial_w, \\ f &= z^2 \partial_z = -\partial_w. \end{aligned}$$

Then one easily sees that (by computing on each chart) that they satisfy the commutation relation of $\mathfrak{sl}_2$. □

Remark 9. *Another way to see it is to say that SL_2 acts on $\mathbb{P}^1$ and so by differentiating the action $\mathfrak{sl}_2$ lies in $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.*

So far we have described the global differential operators of order ≤ 1, we have that

$$\mathcal{D}^{\leq 1}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C} \oplus \mathfrak{sl}_2.$$

It is harder to deal with higher order differential operators by looking explicitely at the change of coordinates. In fact, fix an operator of order 2 on each chart

$$\begin{aligned} D_z &= a_0(z) + a_1(z)\partial_z + a_2(z)\partial_z^2, \\ D_w &= b_0(w) + b_1(w)\partial_w + b_2(w)\partial_w^2, \end{aligned}$$

with holomorphic coefficients. They glue if and only if they coincide on the intersection after change of coordinates, that is, if and only if

$$\begin{aligned} & a_0(\frac{1}{w}) + a_1(\frac{1}{w})(-w^2)\partial_w + a_2(\frac{1}{w})(-w^2)\partial_w(-w^2)\partial_w \\ & \quad = a_0(\frac{1}{w}) - w^2 a_1(\frac{1}{w})\partial_w - w^2 a_2(\frac{1}{w})(-w^2)\partial_w\partial_w - w^2 a_2(\frac{1}{w})(-2w)\partial_w \\ & \quad = a_0(\frac{1}{w}) - w^2 a_1(\frac{1}{w})\partial_w - w^2 a_2(\frac{1}{w})(-w^2)\partial_w\partial_w - w^2 a_2(\frac{1}{w})(-2w)\partial_w \\ & \qquad = b_0(w) + b_1(w)\partial_w + b_2(w)\partial_w^2. \end{aligned}$$

That is to say if and only if for all $w \in \mathbb{C}_w^*$ we have

$$b_0(w) = a_0(\frac{1}{w}), \tag{1}$$

$$b_1(w) = -w^2 a_1(\frac{1}{w}) + 2w^3 a_2(\frac{1}{w}), \tag{2}$$

$$b_2(w) = w^4 a_2(\frac{1}{w}). \tag{3}$$

This is still manageable, and maybe this is explicit enough for many purposes. Notice that the top coefficients a_2 and b_2 behave "nicely" i.e. the glueing conditions look familiar. It is a general feature of the principal symbol of a differential operator, the condition (3) is nothing by the glueing condition of the tensor product of the tangent sheaf with itself, namely this is nothing but us explicitely looking at the sheaf equality

$$\mathrm{Gr}^2(\mathcal{D}_{\mathbb{P}^1}) = \mathrm{Sym}^2(\Theta_X) = \Theta_X \otimes \Theta_X,$$

that we already proved !

Hopefully this serves as a motivation to show that the tools we introduced before are there to help us make **explicit computations**. The additional algebraic input will be the representation theory of the Lie algebra $\mathfrak{sl}_2$.

Let's begin, because $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is an associative algebra, there exists a morphism of associative algebras

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1),$$

where $\mathcal{U}(\mathfrak{sl}_2)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. This is defined via generators and relations as the quotient of the tensor algebra

$$\mathcal{T}(\mathfrak{sl}_2) = \bigoplus_{n \geq 0} (\mathfrak{sl}_2)^{\otimes n} = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathfrak{sl}_2) \oplus ...$$

obtained by imposing the defining relations of $\mathfrak{sl}_2$ i.e. asking that

$$h \otimes f - f \otimes h = -2f,$$
$$e \otimes f - f \otimes e = h,$$
$$h \otimes e - e \otimes h = 2e.$$

The image of φ inside of $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is the $\mathbb{C}$-subalgebra generated by $\mathfrak{sl}_2$ i.e. the subalgebra of global differential operators generated by global vector fields.

By definition, $\mathcal{T}(\mathfrak{sl}_2)$ is a filtered algebra, so we can define a filtration on its quotient $\mathcal{U}(\mathfrak{sl}_2)$ by mean of that filtration. Precisely we define for all $m \in \mathbb{Z}$

$$\mathcal{U}(\mathfrak{sl}_2)^{\leq m} = \pi(\mathcal{T}^{\leq m}(\mathfrak{sl}_2))$$

where $\pi : \mathcal{T}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2)$ is the quotient map. Recall the following Theorem in the case of the Lie algebra $\mathfrak{sl}_2$, it explicitely describes the graded algebra associated with the previously described filtration

Theorem 2 (Poincaré-Birkhoff-Witt). *The following family of elements of $\mathcal{U}(\mathfrak{sl}_2)$*

$$(f^\alpha h^\beta e^\gamma)_{\alpha,\beta,\gamma\in\mathbb{N}}$$

form a $\mathbb{C}$-basis of $\mathcal{U}(\mathfrak{sl}_2)$. More precisely, for each $m \in \mathbb{N}$, the family of elements of $\mathcal{U}(\mathfrak{sl}_2)$

$$(f^\alpha h^\beta e^\gamma)_{\alpha+\beta+\gamma\leq m}$$

forms a basis of $\mathcal{U}^{\leq m}(\mathfrak{sl}_2)$. The associated graded algebra of $\mathcal{U}(\mathfrak{sl}_2)$ is isomorphic to the symmetric algebra on the generators (of degree one) f, h and e. That is to say there exists an isomorphism of graded $\mathbb{C}$-algebras

$$Sym(\mathfrak{sl}_2) = \mathbb{C}[f,h,e] \simeq Gr(\mathcal{U}(\mathfrak{sl}_2)).$$

The map of associative algebras $\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ previously defined is compatible with the filtrations on both side.

We are now in following general setting, we have a filtered map φ between two filtered associative algebras. And to check if it is an isomorphism (between non-commutative objects), it is enough to check if it is an isomorphism at the level of the associated graded algebras (that so happen to be commutative here).

Hence, before applying this general idea, we first need to describe the associated graded of the associative algebra

$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1) = \bigcup_{m\geq 0} \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1).$$

For each $m \in \mathbb{N}$, by the very definition of the sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules $\mathrm{Gr}^{m+1}\mathcal{D}_{\mathbb{P}^1}$, we have a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m} \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1} \longrightarrow \mathrm{Gr}^{m+1}\mathcal{D}_{\mathbb{P}^1} \longrightarrow 0,$$

for $m = 0$ this is an exact sequence that we have already encountered

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq 1} \longrightarrow \Theta_{\mathbb{P}^1} \longrightarrow 0.$$

In general, we have explained that for all $m \geq 0$ we have the equality of sheaves of $\mathcal{O}_X$-modules

$$\mathrm{Gr}^m(\mathcal{D}_{\mathbb{P}^1}) = \mathrm{Sym}^m(\Theta_{\mathbb{P}^1}).$$

Lemma 6. *Let $m \geq 0$, then we have the equalities of sheaves of $\mathcal{O}_X$-modules*

$$Sym^n(\Theta_{\mathbb{P}^1}) = \Theta_{\mathbb{P}^1}^{\otimes m}.$$

In particular we have,

$$\dim_{\mathbb{C}}(\Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1)) = 2m + 1.$$

Proof. Because Θ_X is a locally free sheaf of rank 1 i.e. a line bundle, its symmetric and tensor powers are equal and are again locally free sheaves of rank r. This is not any deeper than the fact that

$$S^m_R(R) = R^{\otimes m} = R \otimes_R ... \otimes_R R = R$$

for any ring R. So we are left to describe what line bundle is $\Theta_X^{\otimes m}$. Let's be very explicit, clearly when restricting to say the open U_z, the multiplication of $\mathcal{O}_{U_z}$ yields an isomorphism between the presheaf on U_z

$$U \longmapsto \Theta_{U_z}^{\otimes m} \simeq \mathcal{O}_{U_z}^{\otimes m}$$

and the sheaf $\mathcal{O}_{U_z}$. This says that we don't need to sheafify our definition of (iterated) tensor product when looking at open sets contained in U_z or U_w. Now because $\Theta_{\mathbb{P}^1}^{\otimes m}$ is a sheaf, to give a section of $\mathbb{P}^1$ is the same as giving sections on U_z and U_w that transform accordingly under change of coordinates. A section on U_z is, because by the previous remark we don't need to sheafify,

$$f_1(z)\partial_z \otimes f_2(z)\partial_z \otimes ... \otimes f_m(z)\partial_z = f_1(z)...f_m(z)(\partial_z \otimes ... \otimes \partial_z) \simeq f_1(z)...f_m(z)$$

where $f_1, ..., f_m \in \mathcal{O}_{U_z}(U_z)$. For each $i \in \{1, ..., m\}$, the vector field $f_i(z)\partial_z$ written in coordinate w becomes $-w^2 f_i(\frac{1}{w})\partial_w$.

Summing up we are looking for holomorphic functions $f_1, ..., f_m \in \mathcal{O}_{U_z}(U_z)$ (resp. $g_1, ..., g_m \in \mathcal{O}_{U_w}(U_w)$) that satisfy for each $w \in \mathbb{C}^*$ the equality

$$(-1)^m w^{2m} f_1(\frac{1}{w})...f_m(\frac{1}{w}) = g_1(w)...g_m(w).$$

Now it is trivial to note that when $f_1, ..., f_m$ describes $\mathcal{O}_{U_z}(U_z)$ then so does their product F, and the same goes with $g_1, ..., g_m$ by calling the product G. So global sections of $\Theta_{\mathbb{P}^1}^{\otimes m}$ are in bijection with holomorphic functions F and G satisfying for all $w \in \mathbb{C}^*$ the equality

$$(-1)^m w^{2m} F(\frac{1}{w}) = G(w),$$

i.e. polynomials of degree at most $2m$. Clearly this is a vector space of degree $2m + 1$. □

Remark 10. *For those who know about the family of sheaves $\mathcal{O}_{\mathbb{P}^1}(m)$ on $\mathbb{P}^1$ for $m \in \mathbb{Z}$, it is clear that $\Theta_{\mathbb{P}^1} = \mathcal{O}(2)$ and we are of course just iterating the fact that*

$$\mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m') = \mathcal{O}_{\mathbb{P}^1}(m + m').$$

Seeing tensor products of sheaves for the first time can be confusing. In particular these sheaves give an example as to why we need to sheafify the definition of tensor products. For instance the global sections of $\Theta_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \Theta_{\mathbb{P}^1}$ is a vector space of dimension 5 as we explained. But it is not equal to the "naive guess" (i.e the value of the presheaf) which is equal to $\Theta_{\mathbb{P}^1}(\mathbb{P}^1) \otimes_{\mathbb{C}} \Theta_{\mathbb{P}^1}(\mathbb{P}^1)$, a vector space of dimension $3 * 3 = 9$! Looking at the proof, it really boils down to *the following fact, when multiplying two polynomials f_1, f_2 of degree ≤ 2 in the variable z, one always gets a polynomial F of degree ≤ 4. But a polynomial of degree ≤ 4 may not be written uniquely in that form*

$$z^2 = z \cdot z = z^2 \cdot 1 = 1 \cdot z^2.$$

It is good for these kind of issues to keep in mind the example of the tensor product of Ω^1 (the sheaf of differential 1-forms = the cotangent bundle = $\mathcal{O}(-2)$) and Θ_X (the sheaf of vector fields = the tangent bundle = $\mathcal{O}(2)$) which gives $\mathcal{O}_X$ although there is no globally defined holomorphic 1-form on $\mathbb{P}^1$.

Hence we have an exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m} \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1} \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m} \longrightarrow 0$$

looking at global sections yields the exact sequence of $\mathbb{C}$-vector spaces ($= \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)-$ modules)

$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

There is no zero on the right, and this is not a mistake ! Taking global sections is a left exact operation, in a sense this is the reason why one needs to sheafify the definition of quotients (see Remark 5). The very definition of sheaf cohomology shows that there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow \;& \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1) \\ \longrightarrow \;& \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \\ \longrightarrow \;& \mathrm{H}^2(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^2(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^2(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \\ \longrightarrow \;& \ldots \end{aligned}$$

For dimension reasons, the cohomology of the sheaves we consider vanishes starting from degree 2. That is we have an exact sequence of $\mathbb{C}$-vector spaces

$$\begin{aligned} 0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1) & \\ \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \longrightarrow 0. & \quad (*) \end{aligned}$$

We can compute the first cohomology groups of the line bundles that appear.

Lemma 7. *We have*

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0,$$

and more generally for all $m \geq 0$, we have

$$H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) = 0.$$

Proof. Omitted. □

From there we can show

Proposition 8. *For all $m \geq 0$,*

$$H^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) = 0$$

and

$$Gr^m(\mathcal{D}_{\mathbb{P}^1})(\mathbb{P}^1) = \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

In particular

$$\dim_{\mathbb{C}}(Gr^m(\mathcal{D}_{\mathbb{P}^1})(\mathbb{P}^1)) = 2m + 1.$$

Proof. It is clear by induction on m, the previous lemma and the long exact sequence (*). The computation of the dimension what done in Lemma 6. □

Let's come back to our problem, we want to check if

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism of filtered associative algebras. Passing to the graded objects we get a graded morphism

$$\mathrm{Gr}(\varphi) : \mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)) \simeq \mathbb{C}[f, h, e] \longrightarrow \mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)) = \bigoplus_{m \geq 0} \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

To check if it is an isomorphism we can check degree by degree

$$\begin{aligned}
&\mathrm{Gr}^0(\varphi) : \mathbb{C} \longrightarrow \mathbb{C}, \\
&\mathrm{Gr}^1(\varphi) : f\mathbb{C} \oplus h\mathbb{C} \oplus e\mathbb{C} \longrightarrow \Theta_{\mathbb{P}^1}(\mathbb{P}^1), \\
&\mathrm{Gr}^2(\varphi) : f^2\mathbb{C} \oplus h^2\mathbb{C} \oplus e^2\mathbb{C} \oplus fe\mathbb{C} \oplus fh\mathbb{C} \oplus he\mathbb{C} \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes 2}(\mathbb{P}^1).
\end{aligned}$$

But there clearly is a problem in degree 2 because the dimensions don't match, the dimension of the left hand side is 6 while on the right we computed it to be 5. This can mean two things, either we have been too optimistic and missed a relation in degree 2 in $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ and moding out by this relation will give the correct answer. Or that there are too much relation of order higher than one between the vector fields inside of $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ and it is not even true that $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is generated by $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ and $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.

Question 4. *Can one formulate a sufficient condition on X so that $\mathcal{D}_X(X)$ is generated as an associative algebra by $\mathcal{O}_X(X)$ and $\Theta_X(X)$?*

Let's try and find a relation, it is natural from the perspective of Lie theory to look at the following element, known as the Casimir element

$$C = \frac{1}{2}h^2 + ef + fe \in \mathcal{U}(\mathfrak{sl}_2),$$

its importance lies in the fact that it generates the center of $\mathcal{U}(\mathfrak{sl}_2)$. It corresponds to the vector field , in the z-coordinate it corresponds to

$$\begin{aligned} C &= \frac{1}{2}(-2z\partial_z)(-2z\partial_z) - \partial_z z^2\partial_z + z^2\partial_z(-\partial_z) \\ &= 2z\partial_z z\partial_z - \partial_z z^2\partial_z - z^2\partial_z\partial_z \\ &= 2z^2\partial_z\partial_z + 2z\partial_z - z^2\partial_z\partial_z - 2z\partial_z - z^2\partial_z\partial_z \\ &= 0. \end{aligned}$$

and a similar computation shows it too in the w-coordinate.

Remark 11. *This could have been expected because algebras of differential operators typically don't have central elements ? Can I prove it thought ?*

We have shown that the map of filtered associative algebras

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

factors through the quotient $\mathcal{U}(\mathfrak{sl}_2)_0$ of $\mathcal{U}(\mathfrak{sl}_2)$ by the two-sided ideal generated by the Casimir element. Hence we have defined a morphism of filtered associative algebras

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1).$$

The result is the following, the rest of this section is devoted to proving it.

Theorem 3. *The morphism of associative algebras*

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 = \mathcal{U}(\mathfrak{sl}_2)/((\frac{1}{2}h^2 + ef + fe)\mathcal{U}(\mathfrak{sl}_2)) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism.

Both side carry an action of $\mathfrak{sl}_2$, on the left it comes from the adjoint action and on the right it comes from the inclusion

$$\mathfrak{sl}_2 = \Theta_{\mathbb{P}^1}(\mathbb{P}^1) \subset \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

by having $x \in \mathfrak{sl}_2$ acting as $[x, \cdot]$ on $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$. It is clear from the fact that φ is a morphism of associative algebra that it is a morphism of $\mathfrak{sl}_2$-modules with respect to these structures. Moreover it is clear that $C\mathcal{U}(\mathfrak{sl}_2)$ is a $\mathfrak{sl}_2$-submodule of $\mathcal{U}(\mathfrak{sl}_2)$. For all $x \in \mathfrak{sl}_2$ and all $y \in \mathcal{U}(\mathfrak{sl}_2)$ we have

$$[x, Cy] = C[x, y]$$

because C is central. Hence $C\mathcal{U}(\mathfrak{sl}_2)$ is isomorphic to $\mathcal{U}(\mathfrak{sl}_2)$ as an $\mathfrak{sl}_2$-module.

First let's show that φ is surjective. For all $m \in \mathbb{N}$, the vector $e^m \in \mathcal{U}(\mathfrak{sl}_2)$ is clearly an highest weight vector of weight $2m$. It is mapped by φ to the differential operator $e^m \in \mathrm{Sym}^m(\Theta_{\mathbb{P}^1})(\mathbb{P}^1)$ which in the z-chart reads

$$(-1)^m\partial_z^m,$$

in particular it is nonzero. As such it is an highest weight vector of weight $2m$ so the $\mathfrak{sl}_2$-submodule generated by it is of dimension $2m+1$. For dimension reasons we see that it is the whole of $\mathrm{Sym}^n(\Theta_{\mathbb{P}^1})(\mathbb{P}^1)$, that proving the surjectivity of φ and hence of $\widetilde{\varphi}$.

Injectivity is more subtle. We have a surjective map between two semisimple $\mathfrak{sl}_2$-modules, so to conclude we have an isomorphism it is enough to explain why both side have the same decomposition in simple $\mathfrak{sl}_2$-modules. To show this, we'll make use of a certain notion of character, it measures the "size" of a graded vector space with finite dimensional graded pieces. To be more precise, there are two gradations in our setting, one is given by the weight with respect to the action of $\mathrm{ad}(h)$ and the other is given by the degree. On then enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$, the degree gradation is the usual grading of the polynomial algebra

$$\mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)) = \mathbb{C}[f, h, e]$$

by requiring that $\deg(f) = \deg(h) = \deg(e) = 1$ (Recall that as $\mathbb{C}$-vector spaces, $\mathcal{U}(\mathfrak{sl}_2) \simeq \mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)))$. On the right hand side, it is the gradation given by the order of the differential operator. For a $\mathbb{Z}$-bigraded $\mathbb{C}$-vector spaces $V = \bigoplus_{m,m' \in \mathbb{Z}} V_{m,m'}$ such that each $V_{m,m'}$ is of finite dimension, we define its character by

$$\mathrm{ch}(V) = \sum_{m,m'} \dim_{\mathbb{C}}(V_{m,m'}) x^m t^{m'} \in \mathbb{Z}[[x^{\pm 1}, t^{\pm 1}]].$$

So the variable x indicates the weight gradation and the variable t the degree gradation.

Remark 12. *Be careful that in what follows, whenever we take the character of a graded vector space, it is is assumed to have finite dimensional graded pieces with respect to our gradation.*

Proposition 9. *For any short exact sequence of $\mathbb{Z}^2$-graded modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have

$$ch(B) = ch(A) + ch(C).$$

And for all pair A, B of $\mathbb{Z}^2$-graded vector spaces, we have

$$ch(A \otimes_{\mathbb{C}} B) = ch(A) ch(B).$$

For any family of graded vector spaces $(A_m)_{m \in \mathbb{N}}$, we have

$$ch(\bigoplus_{m \in \mathbb{N}} A_m) = \sum_{m \in \mathbb{N}} ch(A_m).$$

Proof. Omitted. □

For each $k \in \mathbb{N}$, we denote by $L(k)$ the simple representation of $\mathfrak{sl}_2$ of highest weight k. Its character with respect to the $\mathbb{Z}$-gradation given by the h-weights is given by

$$\mathrm{ch}(L(k)) = x^{-k} + x^{-k+2} + ... + x^{k-2} + x^k = \frac{x^{k+1} - x^{-k-1}}{x - x^{-1}}.$$

We have explained before that as $\mathfrak{sl}_2$-modules we have

$$\mathrm{Sym}^m(\Theta_{\mathbb{P}^1})(\mathbb{P}^1) = L(2m),$$

so

$$\mathrm{ch}(\mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1))) = \mathrm{ch}(\bigoplus_{m\in\mathbb{N}} L(2m)) = \sum_{m\geq 0} \mathrm{ch}(L(2m)) = \sum_{m\geq 0} \frac{x^{2m+1} - x^{-2m-1}}{x - x^{-1}} t^m.$$

To compute the character of the enveloping algebra side, notice we have an exact sequence of $\mathfrak{sl}_2$-modules

$$0 \longrightarrow C\mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow 0.$$

It is an exact sequence of $\mathbb{Z}^2$-graded modules, so we have the equality

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)) - \mathrm{ch}(C\mathcal{U}(\mathfrak{sl}_2)).$$

As we noticed $\mathcal{U}(\mathfrak{sl}_2)$ and $C\mathcal{U}(\mathfrak{sl}_2)$ are isomorphic as $\mathfrak{sl}_2$-modules, and only the degree is shifted by this isomorphism, so we have

$$\mathrm{ch}(C\mathcal{U}(\mathfrak{sl}_2)) = t^2 \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)).$$

That readily implies the equality

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = (1 - t^2)\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)).$$

Now it is clear from the equality of graded vector spaces

$$\mathcal{U}(\mathfrak{sl}_2) = \mathbb{C}[f, h, e] = \mathbb{C}[f] \otimes \mathbb{C}[h] \otimes \mathbb{C}[e]$$

that

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)) = \mathrm{ch}(\mathbb{C}[f])\mathrm{ch}(\mathbb{C}[h])\mathrm{ch}(\mathbb{C}[e]) = \frac{1}{1 - x^{-2}t} \cdot \frac{1}{1 - t} \cdot \frac{1}{1 - x^2 t}.$$

Putting everything together we have

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \frac{1 - t^2}{(1 - x^{-2}t)(1 - t)(1 - x^2 t)}.$$

Now it is enough to check that

$$\frac{1 - t^2}{(1 - x^{-2}t)(1 - t)(1 - x^2 t)} = \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \mathrm{ch}(\mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1))) = \sum_{m\geq 0} \frac{x^{2m+1} - x^{-2m-1}}{x - x^{-1}} t^m$$

to decide if

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism of associative algebras.

It is indeed the case

Lemma 8. *The following equality*

$$\frac{1-t^2}{(1-x^{-2}t)(1-t)(1-x^2t)} = \sum_{m\geq 0} \frac{x^{2m+1}-x^{-2m-1}}{x-x^{-1}} t^m$$

holds in $\mathbb{Z}[[x^{\pm 1}, t]]$.

Proof. We have

$$\begin{aligned}
\sum_{m\geq 0} \frac{x^{2m+1}-x^{-2m-1}}{x-x^{-1}} t^m &= \frac{1}{x-x^{-1}} \left(x \cdot \sum_{m\geq 0} (x^2t)^m - x^{-1} \cdot \sum_{m\geq 0} (x^{-2}t)^m \right) \\
&= \frac{1}{x-x^{-1}} \left(x \cdot \frac{1}{1-x^2t} - x^{-1} \cdot \frac{1}{1-x^{-2}t} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x}{1-x^2t} - \frac{x^{-1}}{1-x^{-2}t} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x(1-x^{-2}t) - x^{-1}(1-x^2t)}{(1-x^2t)(1-x^{-2}t)} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x - x^{-1} - x^{-1}t + x^1 t}{(1-x^2t)(1-x^{-2}t)} \right) \\
&= \frac{x-x^{-1}}{x-x^{-1}} \frac{1+t}{(1-x^2t)(1-x^{-2}t)} \\
&= \frac{(1+t)(1-t)}{(1-x^2t)(1-x^{-2}t)(1-t)} = \frac{1-t^2}{(1-x^2t)(1-x^{-2}t)(1-t)}.
\end{aligned}$$

□

Remark 13. *Note that we have essentially proven the following equalities of* $\mathfrak{sl}_2$*-modules, that describe the decomposition of* $\mathcal{U}(\mathfrak{sl}_2)$ *and* $\mathcal{U}(\mathfrak{sl}_2)_0$ *in simple* $\mathfrak{sl}_2$*-modules under the adjoint action*

$$\begin{aligned}
\mathcal{U}(\mathfrak{sl}_2) &= \bigoplus_{k\geq 0} \mathbb{C}[\frac{1}{2}h^2 + ef + fe] \otimes L(2k), \\
\mathcal{U}(\mathfrak{sl}_2)_0 &= \bigoplus_{k\geq 0} L(2k).
\end{aligned}$$

Question 5. *How does that generalizes to a simple Lie algebra* $\mathfrak{g}$ *?*

3 Modules over the sheaf $\mathcal{D}_X$

Let $\mathcal{M}$ be a sheaf of abelian group on X. The sheaf $\mathcal{D}_X$ is a sheaf of (non-commutative) rings on X. A structure of a left $\mathcal{D}_X$-module is the data for each for each open U of X of a structure of a left $\mathcal{D}_X(U)$-module on $\mathcal{M}(U)$, compatible with the restriction maps. A structure of a right $\mathcal{D}_X$-module on $\mathcal{M}$

is the structure of left $\mathcal{D}_X^{\text{op}}$-module on $\mathcal{M}$. From now on, $\mathcal{D}_X$-module means left $\mathcal{D}_X$-module and $\mathcal{D}_X^{\text{op}}$-module means right $\mathcal{D}_X$-module. Let

$$\text{Mod}(\mathcal{D}_X) \text{ (resp.Mod}(\mathcal{D}_X^{\text{op}}))$$

denote the category of left $\mathcal{D}_X$-modules (resp. right $\mathcal{D}_X$-modules).

Recall that we have a map of sheaves

$$m : \mathcal{O}_X \longrightarrow \mathcal{D}_X$$

so by restriction, a $\mathcal{D}_X$-module is a $\mathcal{O}_X$-module. Another way to put it is that a $\mathcal{D}_X$-module is a $\mathcal{O}_X$-module with an extra datum. To understand what precisely is that datum we adopt a generator and relations point of view on the sheaf of rings $\mathcal{D}_X$.

Lemma 9. *Let $\mathcal{R}$ be a sheaf of rings on X, and*

$$\iota : \mathcal{O}_X \longrightarrow \mathcal{R}$$
$$\nabla : \Theta_X \longrightarrow \mathcal{R}$$

be two sheaf morphisms such that

1. *$\iota : \mathcal{O}_X \longrightarrow \mathcal{R}$ is a morphism of sheaves of rings, making $\mathcal{R}$ into a $\mathcal{O}_X$-module,*

2. *$\nabla : \Theta_X \longrightarrow \mathcal{R}$ is left $\mathcal{O}_X$-linear, i.e. for all open U, $f \in \mathcal{O}_X(U)$ and $\theta \in \Theta_X(U)$ we have*

$$\nabla(U)(f\theta) = \iota(U)(f)\nabla(U)(\theta).$$

3. *The Leibniz rule holds for all open U, i.e. for all $\theta \in \Theta_X(U)$ and all $f \in \mathcal{O}_X(U)$ we have the equality*

$$[\nabla(U)(\theta), \iota(U)(f)] = \iota(U)(\theta(f)).$$

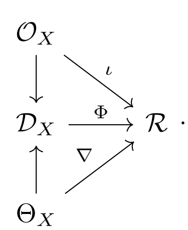
4. *$\nabla : \Theta_X \longrightarrow \mathcal{R}$ is a morphism of sheaves of Lie algebra, where Lie brackets in $\mathcal{R}$ are commutators i.e. for all open U, for all $\theta, \theta' \in \mathcal{O}_X(U)$ we have the equality*

$$[\nabla(U)(\theta), \nabla(U)(\theta')] = \nabla(U)([\theta, \theta']).$$

Then there exists a unique morphism of sheaves of rings

$$\Phi : \mathcal{D}_X \longrightarrow R$$

such that the following diagram of sheaves commutes



Proof. We start by proving the result in the case where X is isomorphic to an open of $\mathbb{C}^n$ in coordinates $z_1, ..., z_n$. In that case

$$\mathcal{D}_X = \bigoplus_{\alpha} \mathcal{O}_X \partial_z^\alpha.$$

Let U an open set of X. To define Φ it is enough to define it on every connected component of U so we may as well assume U itself is connected. So we must define a ring morphism

$$\Phi(U) : \mathcal{D}_X(U) = \bigoplus_{\alpha} \mathcal{O}_X(U)\partial_z^\alpha \longrightarrow \mathcal{R}(U).$$

Our hypothesis forces the unicity, for every almost finite family $a_\alpha \in \mathcal{O}_X(U)$ we have no choice but to set

$$\Phi(U)(\sum_\alpha a_\alpha(z)\partial_z^\alpha) = \sum_\alpha \iota(U)(a_\alpha(z))\nabla(U)(\partial_z)^\alpha.$$

Now it is an exercice to check this is indeed a morphism of rings.

Back to the general case, we cover X by some open coordinate charts $(U_i)_{i\in I}$ where I is a set. Defining a morphism of sheaves of rings

$$\Phi : \mathcal{D}_X \longrightarrow \mathcal{R}$$

is the same as defining for each $i \in I$ a morphism of sheaves of rings

$$\Phi_i : \mathcal{D}_{U_i} \longrightarrow \mathcal{R}_{|U_i}$$

such that for all $i, j \in I$ we have the equality

$$\Phi_i\big|_{U_i\cap U_j} = \Phi_j\big|_{U_i\cap U_j}.$$

The morphism Φ_i that we are looking for exist and are unique by what we already proved. Moreover the equality

$$\Phi_i\big|_{U_i\cap U_j} = \Phi_j\big|_{U_j\cap U_i}$$

follows formally from the unicity already proven since $U_i \cap U_j$ is again a coordinate chart. Hence we get a well defined morphism of rings

$$\Phi : \mathcal{D}_X \longrightarrow \mathcal{R}.$$

The fact that the diagram

$$\begin{array}{ccc} \mathcal{O}_{U_i} & & \\ \downarrow & \overset{\iota}{\searrow} & \\ \mathcal{D}_{U_i} & \xrightarrow{\Phi} & \mathcal{R}_{|U_i} \\ \uparrow & \overset{\nabla}{\nearrow} & \\ \Theta_{U_i} & & \end{array}.$$

commutes for each $i \in I$ implies that the diagram

$$\begin{array}{ccc} \mathcal{O}_X & & \\ \downarrow & \overset{\iota}{\searrow} & \\ \mathcal{D}_X & \xrightarrow{\Phi} & \mathcal{R} \;\cdot \\ \uparrow & \underset{\nabla}{\nearrow} & \\ \Theta_X & & \end{array}$$

commutes. □

Proposition 10. *Let $\mathcal{M}$ be a sheaf of abelian groups. Let it be a $\mathcal{O}_X$-module via a map of sheaves*

$$\iota : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M}).$$

Endowing $\mathcal{M}$ with the structure of a $\mathcal{D}_X$-module extending this $\mathcal{O}_X$-module structure is equivalent to giving a morphism of sheaves

$$\nabla : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M})$$

satisfying the following equalities in the ring $(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_{|U}, \mathcal{M}_{|U}), \circ)$ for every open set U

$$\begin{aligned} \nabla(U)(f\theta) &= \iota(U)(f) \circ \nabla(U)(\theta), \ \textit{(Left } \mathcal{O}_X\textit{-linearity)} \\ \nabla(U)(\theta) \circ \iota(U)(f) &= \iota(U)(\theta(f)) + \iota(U)(f) \circ \nabla(U)(\theta), \ \textit{(Leibniz rule)} \\ \nabla(U)([\theta, \theta']) &= [\nabla(U)(\theta), \nabla(U)(\theta')], \ \textit{(Integrability condition)} \end{aligned}$$

for all $f \in \mathcal{O}_X(U)$, $\theta, \theta' \in \Theta_X(U)$.

The datum of ∇ satisfying the previous conditions is equivalent to the data of a map of sheaves

$$d^{\nabla} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

satisfying the following conditions.

The following Leibniz rule holds for every open set U

$$d^{\nabla}(U)(fs) = f d^{\nabla}(U)(s) + df \otimes s$$

for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{M}(U)$ where

$$d : \mathcal{O}_X \longrightarrow \Omega^1_X$$

is the total differential[3]*.*

The map d^{∇} defines a map of sheaves denoted again by the same symbol

$$d^{\nabla} : \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

[3] Note that technically $df \otimes s$ belongs to $\Omega^1_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)$, but we see it in $(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M})(U)$ via the canonical map from a presheaf to its sheafification.

uniquely determined by the condition that for all open U and all $\omega \in \Omega^1_X(U)$, $s \in \mathcal{M}(U)$ we have the equality[4]

$$d^{\nabla}(U)(\omega \otimes s) = d\omega \otimes s - \omega \wedge d^{\nabla}(U)s$$

where $d : \Omega^1_X \longrightarrow \Omega^2_X$ is the usual exterior derivative. The integrability condition then translates as

$$d^{\nabla} \circ d^{\nabla} = 0$$

as a morphism of sheaves from $\mathcal{M}$ to $\Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M}$.

The morphism ∇ and d^{∇} are deduced from one another by the requirement that for all open U and all $\theta \in \Theta_X(U)$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}_{|U} & \xrightarrow{d^{\nabla}_{|U}} & \Omega^1_U \otimes_{\mathcal{O}_U} \mathcal{M}_{|U} \\ & {\scriptstyle \nabla(U)(\theta)} \searrow \quad \swarrow {\scriptstyle ev_\theta \otimes id} & \\ & \mathcal{M}_{|U} & \end{array} .$$

Proof. Let

$$\iota : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M})$$

be the morphism defining the $\mathcal{O}_X$-module structure on $\mathcal{M}$. Extending this structure to a $\mathcal{D}_X$-module is equivalent to giving a map of sheaves $\Phi : \mathcal{D}_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{O}_X & & \\ \downarrow & \overset{\iota}{\searrow} & \\ \mathcal{D}_X & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathcal{M}) \end{array} .$$

In view of the previous Lemma, this is exactly the data of a morphism of sheaves

$$\nabla : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M})$$

satisfying the conditions of the proposition. Now assume we are given such a

$$\nabla : \Theta_X \longrightarrow \mathcal{H}om_Z(\mathcal{M}, \mathcal{M}),$$

and cover $X = \bigcup_{i \in I} U_i$ with coordinate charts for some set I. To define d^{∇} it is enough to define some d^{∇}_i on each U_i, so we fix one, say in coordinates $z^i_1, ..., z^i_n$.

[4] Again here we make a confusion between the sheaf and the associated presheaf. So to speak, the pure tensor $\omega \otimes s$ lives in the tensor product presheaf and our condition uniquely defines a morphism of presheaves :

$$(\mathrm{d}^{\nabla})^{\mathrm{pre}} : (\Omega^1_X \otimes \mathcal{M})^{pre} \longrightarrow (\Omega^2_X \otimes \mathcal{M})^{pre}$$

which in turns uniquely defines the sought after morphism of sheaves.

The relation between $\nabla_{|U_i}$ and d_i^{∇} is that for all $\theta \in \Theta_X(U_i) = \bigoplus \mathcal{O}_X(U_i)\partial_{z_i}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}_{|U_i} & \xrightarrow{\mathrm{d}_i^{\nabla}} & \Omega^1_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{M}_{|U_i} \simeq \bigoplus_{j=1}^n \mathrm{d}z_j^i \otimes \mathcal{M}_{|U_i} \\ & {}_{\nabla(U_i)(\theta)}\searrow \quad \swarrow_{\mathrm{ev}_\theta \otimes \mathrm{id}} & \\ & \mathcal{M}_{|U_i} & \end{array} .$$

Evaluating, for each open $V \subset U_i$, for θ ranging through $\{\partial_{z_1^i}, ..., \partial_{z_n^i}\}$ shows that we have no choices for the formula of $\mathrm{d}_i^{\nabla}(V)$, it has to be for each $s \in \mathcal{M}(V)$

$$\mathrm{d}_i^{\nabla}(V)(s) = \sum_{j=1}^{n} \mathrm{d}z_j^i \otimes \nabla(U_i)(\partial_{z_j^i})(V)(s).$$

The unicity we have just proven (recall the technique used in the proof of Lemma 9) shows that the d_i^{∇} have to coincide on the double intersections and hence glue to a morphism of sheaves

$$d^{\nabla} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

We have to check that d^{∇} satisfies the Leibniz rule and the integrability condition. Let U be an open set, $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{M}(U)$. Let

$$\mathrm{sh} : (\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M})^{pre} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

be the canonical map of sheaf between a presheaf and its sheafification. We have to show that in $(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M})(U)$ we have the equality

$$\mathrm{d}^{\nabla}(U)(fs) = f\mathrm{d}^{\nabla}(U)(s) + \mathrm{sh}(U)(\mathrm{d}f \otimes s).$$

Because $(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a sheaf it is enough to check that on an open cover $U = \bigcup_i U_i$ by coordinate charts in coordinates $z_1^i, ..., z_n^i$ for some set I. Fix $i \in I$ then after restricting, the previous equality becomes

$$\mathrm{d}^{\nabla}(U_i)(f_{|U_i} s_{|U_i}) = f_{|U_i}\mathrm{d}^{\nabla}(U_i)(s_{|U_i}) + \mathrm{sh}(U_i)(\mathrm{d}f_{|U_i} \otimes s_{|U_i}),$$

but now because on U_i, the sheaf $\Omega^1_{U_i}$ is free, we need not to sheafify, in other terms

$$\mathrm{sh}(U_i) : \Omega^1_X(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{M}(U_i) \longrightarrow (\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M})(U_i)$$

is an isomorphism. So we are reduced to computations in local coordinates, forgetting about the cover by the U_i, we can assume U has coordinates $z_1, ..., z_n$.

We compute

$$
\begin{aligned}
\mathrm{d}^{\nabla}(U)(fs) &= \sum_{j=1}^{n} \mathrm{d}z_j \otimes \nabla(U)(\partial_{z_j})(U)(fs) \\
&= \sum_{j=1}^{n} \mathrm{d}z_j \otimes \left(\frac{\partial f}{\partial z_j} s + f\nabla(U)(\partial_{z_j})(U)(s) \right) \\
&= \sum_{j=1}^{n} \mathrm{d}z_j \otimes \left(\frac{\partial f}{\partial z_j} s \right) + f \left(\sum_{j=1}^{n} \mathrm{d}z_j \otimes \nabla(U)(\partial_{z_j})(U)(s) \right) \\
&= \left(\sum_{j=1}^{n} \mathrm{d}z_j \frac{\partial f}{\partial z_j} \right) \otimes s + f\mathrm{d}^{\nabla}(U)(s) \\
&= \mathrm{d}f \otimes s + f\mathrm{d}^{\nabla}(U)(s).
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{d}^{\nabla}(U)(\mathrm{d}^{\nabla}(U)(s)) &= \mathrm{d}^{\nabla}(U)(\sum_{j=1}^{n} \mathrm{d}z_j \otimes \nabla(U)(\partial_{z_j})(U)(s)) \\
&= \sum_{j=1}^{n} \mathrm{d}^{\nabla}(U)(\mathrm{d}z_j \otimes \nabla(U)(\partial_{z_j})(U)(s)) \\
&= \sum_{j=1}^{n} \mathrm{d}(\mathrm{d}z_j) \otimes \nabla(U)(\partial_{z_j})(U)(s) - \mathrm{d}z_j \wedge \mathrm{d}^{\nabla}(U)(\nabla(U)(\partial_{z_j})(U)(s)) \\
&= -\sum_{j=1}^{n}\sum_{k=1}^{n} (\mathrm{d}z_j \wedge \mathrm{d}z_k) \otimes (\nabla(U)(\partial_{z_k})(U) \circ \nabla(U)(\partial z_j)(U))(s) \\
&= -\sum_{1 \leq j < k \leq n} (\mathrm{d}z_j \wedge \mathrm{d}z_k) \otimes [\nabla(U)(\partial_{z_k}), \nabla(U)(\partial_{z_j}))](U)(s)
\end{aligned}
$$

but for all $1 \leq j, k \leq n$ we have the equality in $\mathrm{Hom}(\mathcal{M}_{|U}, \mathcal{M}_{|U})$

$$
[\nabla(U)(\partial_{z_k}), \nabla(U)(\partial_{z_j})] = \nabla(U)([\partial_{z_k}, \partial_{z_k}]) = \nabla(U)(0) = 0
$$

which shows that each term of the previous sum vanishes.

Now if one is given a $\mathcal{O}_X$-module $\mathcal{M}$ and a map of sheaves d^{∇} satisfying the conditions of the Proposition, we leave the reader to check that by defining for every open U and every $\theta \in \Theta_X(U)$

$$
\nabla(U)(\theta) = (\mathrm{ev}_\theta \otimes \mathrm{id}) \circ \mathrm{d}^{\nabla}(U) \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_{|U}, \mathcal{M}_{|U})
$$

we indeed get a map

$$
\nabla : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M})
$$

satisfying all our requirements. I didn't check... □

Lemma 10. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two sheaves of $\mathcal{D}_X$-modules i.e $\mathcal{O}_X$-modules with connections ∇ and ∇'. Let*

$$\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$$

be a morphism of sheaves of $\mathcal{O}_X$-modules. Then φ is a morphism of $\mathcal{D}_X$-modules if and only if for all open set U and all $\theta \in \Theta_X(U)$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(U) & \xrightarrow{\varphi(U)} & \mathcal{M}'(U) \\ \big\downarrow{\scriptstyle \nabla(U)(\theta)(U)} & & \big\downarrow{\scriptstyle \nabla'(U)(\theta)(U)} \\ \mathcal{M}(U) & \xrightarrow{\varphi(U)} & \mathcal{M}'(U) \end{array}.$$

Proof. Because

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M}')$$

is a subsheaf of

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$$

to check if $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$ is $\mathcal{D}_X$-linear it is enough to check it locally. Cover X with coordinate charts $(U_i)_{i \in I}$ where I is a set. It is enough to check that each $\varphi_{|U_i}$ is $\mathcal{D}_{U_i}$-linear. But for every (connected) open subset U of some coordinate chart U_i it is true that

$$\mathcal{D}_X(U) = \langle \mathcal{O}_X(U), \Theta_X(U) \rangle$$

so it is clear from our hypothesis that $\varphi(U) : \mathcal{M}(U) \longrightarrow \mathcal{M}'(U)$ is $\mathcal{D}_X(U)$-linear, and the result follows. The converse is clear. □

Remark 14. *Implicitely we are using the following fact to reduce to the case of a connected open set U. If a topological space X is written as the disjoint union of its connected components $(X_j)_{j \in J}$ for some set J, if all the X_j are open*[5]*, then for any sheaf of (possibly non-commutative) rings $\mathcal{R}$ on X we have an isomorphism of rings*

$$\mathcal{R}(X) = \prod_{j \in J} \mathcal{R}(X_j).$$

If $\mathcal{F}$ is a sheaf of abelian groups on X then as abelian groups

$$\mathcal{F}(X) = \prod_{j \in J} \mathcal{F}(X_j).$$

If moreover $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules then

$$\mathcal{F}(X) = \prod_{j \in J} \mathcal{F}(X_j)$$

[5] This is clearly satisfied in the cases we care about here.

as a $\mathcal{R}(X)$-module with the natural action coming from the decomposition of the ring

$$\mathcal{R}(X) = \prod_{j \in J} \mathcal{R}(X_j)$$

as a product of rings.

Remark 15. *Recall that for a general open set U, we have*

$$\mathcal{D}_X(U) \neq \langle \mathcal{O}_X(U), \Theta_X(U) \rangle$$

*as subalgebras of $(\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U), \circ)$. So we are not saying that for each open U, one can test $\mathcal{D}_X(U)$-linearity by checking it on $\mathcal{O}_X(U)$ (part of the hypothesis) and on $\Theta_X(U)$ (the commutativity on the diagram). We are really saying that if we have the "naive" linearities (i.e. with respect to $\mathcal{O}_X(U), \Theta_X(U)$) on **every open** U then we get the "strong" linearity (i.e. with respect to $\mathcal{D}_X(U)$) on **every open** U.*

Lemma 11. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module, let*

$$d^{\nabla} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

be the associated integrable connection. For all $m \geq 0$, we can define a $\mathbb{C}$-linear morphism of sheaves

$$d^{\nabla} : \Omega^m_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \Omega^{m+1}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

via the requirement that for all open U, all $m \in \mathcal{M}(U)$ and $\omega \in \Omega^m_X(U)$, we have

$$d^{\nabla}(U)(\omega \otimes m) = d\omega \otimes m + (-1)^m \omega \wedge d^{\nabla}(U)(m)$$

then

$$DR_X(\mathcal{M}) = \mathcal{M} \xrightarrow{d^{\nabla}} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{d^{\nabla}} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow ... \longrightarrow \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0$$

*is a complex of sheaves of vector spaces, called the **De Rham complex** of the $\mathcal{D}_X$-module $\mathcal{M}$.*

Proof. The fact that the composition of the first two d^{∇} is zero is exactly the integrability condition. A local computation in the same spirit shows that it is a complex. I didn't check □

Lemma 12. *Let $\mathcal{M}$ be a $\mathcal{O}_X$-module and let*

$$\nabla : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$$
$$\nabla' : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$$

or equivalently

$$d^{\nabla} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$
$$d^{\nabla'} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

be two structure of $\mathcal{D}_X$-module on $\mathcal{M}$. Then

$$d^{\nabla} - d^{\nabla'} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

or equivalently for all open $V \subset U$ and all $\theta \in \Theta_X(U)$, $f \in \mathcal{O}_X(V)$ and $m \in \mathcal{M}(V)$ we have the equality

$$(\nabla - \nabla')(U)(\theta)(V)(fm) = f(\nabla - \nabla')(U)(\theta)(V)(m).$$

Proof. Let U be an open, $m \in \mathcal{M}(U)$ and $f \in \mathcal{O}_X(U)$ then

$$\begin{aligned} \mathrm{d}^{\nabla}(U)(fm) &= f\mathrm{d}^{\nabla}(U)(m) + \mathrm{d}f \otimes m \\ \mathrm{d}^{\nabla'}(U)(fm) &= f\mathrm{d}^{\nabla'}(U)(m) + \mathrm{d}f \otimes m \end{aligned}$$

so we have

$$(\mathrm{d}^{\nabla} - \mathrm{d}^{\nabla'})(U)(fm) = f(\mathrm{d}^{\nabla} - \mathrm{d}^{\nabla'})(U)(m)$$

which shows the $\mathcal{O}_X$-linearity. □

We can answer the following basic question : what are the structure of $\mathcal{D}_X$-module on the structure sheaf $\mathcal{O}_X$ extending the natural $\mathcal{O}_X$-module structure on $\mathcal{O}_X$? First there is an "obvious" one, given by the natural action of $\mathcal{D}_X$ on $\mathcal{O}_X$. It corresponds to the connection given by the total derivative

$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \Omega^1_X.$$

Let $\mathrm{d}^{\nabla} : \mathcal{O}_X \longrightarrow \Omega^1_X$ be another such structure. By the previous Lemma, the map $\mathrm{d}^{\nabla} - \mathrm{d}$ is a $\mathcal{O}_X$-linear sheaf morphism from $\mathcal{O}_X$ to Ω^1_X, this is nothing but a global 1-form $\omega \in \Omega^1_X(X)$. Precisely for all open set U and $f \in \mathcal{O}_X(U)$ we have

$$\mathrm{d}^{\nabla}(U)(f) = f\omega_{|U} + \mathrm{d}f$$

where $\omega = \mathrm{d}^{\nabla}(X)(1_X)$. Conversely one can check that if $\omega \in \Omega^1_X(X)$ is a global 1-form then

$$\mathrm{d}^{\omega} = \omega + \mathrm{d}$$

defines an integrable connection on $\mathcal{O}_X$ via the previous formula if and only if ω is closed i.e. $d\omega = 0$. It follows from the fact that for all U and all $f \in \mathcal{O}_X(U)$ then we have in $\Omega^2_X(U)$

$$\begin{aligned} \mathrm{d}^{\omega}(\mathrm{d}^{\omega}(f)) &= \mathrm{d}^{\omega}(f\omega \otimes 1 + \mathrm{d}f \otimes 1) \\ &= \mathrm{d}(f\omega) \otimes 1 - f\omega \wedge \mathrm{d}^{\omega}(1) + \mathrm{d}^2 f \otimes 1 - \mathrm{d}f \wedge \mathrm{d}^{\omega}(1) \\ &= \mathrm{d}f \wedge \omega + f\mathrm{d}(\omega) - f\omega \wedge \omega + 0 - \mathrm{d}f \wedge \omega \\ &= f\mathrm{d}(\omega). \end{aligned}$$

but it is required that $\mathrm{d}^{\omega} \circ \mathrm{d}^{\omega} = 0$ which in turn implies $\mathrm{d}(\omega) = 0$.

Let us fix $\omega, \omega' \in \Omega^1_X(X)$, what does it mean for the two $\mathcal{D}_X$-module structure on $\mathcal{O}_X$ given by d^{ω} and $\mathrm{d}^{\omega'}$ (with corresponding connections ∇^{ω} and $\nabla^{\omega'}$) to be isomorphic ?

It means that there exists an $\mathcal{O}_X$-linear isomorphism of sheaves

$$\varphi : \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

such that (recall Lemma 10) for all open set U and all $\theta \in \Theta_X(U)$ and $g \in \mathcal{O}_X(U)$ we have the equality

$$\varphi(U)(\nabla(U)(\theta)(U)(g)) = \nabla'(U)(\theta)(U)(\varphi(U)(g))$$

or without refering to the open set U which we will start doing from now on, the equality becomes

$$\varphi(\nabla^{\omega}(\theta)(g)) = \nabla^{\omega'}(\theta)(\varphi(g)).$$

Because φ is $\mathcal{O}_X$-linear it is nothing but the multiplication by a global function $f \in \mathcal{O}_X(X)$, and because φ is an isomorphism, f is a never vanishing global function. So the previous equality becomes

$$f_{|U}\nabla^{\omega}(\theta)(g) = \nabla^{\omega'}(\theta)(f_{|U}g)$$

which really is the equality for all open U, all function $g \in \mathcal{O}_X(U)$ and all vector field $\theta \in \Theta_X(U)$

$$f_{|U}(g\omega_{|U}(\theta)) + f_{|U}\theta(g) = f_{|U}g\omega'_{|U}(\theta) + \theta(f_{|U}g) = f_{|U}g\omega'_{|U}(\theta) + \theta(f_{|U})g + f_{|U}\theta(g)$$

which implies

$$f_{|U}g(\omega_{|U} - \omega'_{|U})(\theta) = g\theta(f) = g(\mathrm{d}f)(\theta).$$

in particular for $g = 1_U$ and since θ is an arbitrary vector field we obtain the equality in $\Omega^1_X(U)$

$$(f(\omega - \omega'))_{|U} = \mathrm{d}(f_{|U}).$$

But because the morphism of sheaves of multiplication by f is an isomorphism, $f \in \mathcal{O}^*_X(X)$ that is, it never vanishes. The previous equality then becomes the equality in $\Omega^1_X(X)$:

$$(\omega - \omega') = \frac{\mathrm{d}f}{f}.$$

So the structure of $\mathcal{D}_X$-module on $\mathcal{O}_X$ extending that of $\mathcal{O}_X$ i.e. integrable connections on the trivial bundle $X \times \mathbb{C}$ up to isomorphism are in bijection with the quotient of the closed 1-form modulo the subgroup consisting of the forms that are **log-exact** i.e. can be written $\frac{\mathrm{d}f}{f}$ for some global never vanishing function $f \in \mathcal{O}^*_X(X)$.

In the analytic setting, we have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \overset{\exp}{\longrightarrow} \mathcal{O}^*_X \longrightarrow 0.$$

We define a morphism of sheaves of abelian groups

$$\mathrm{d}_{\log} : f \in \mathcal{O}^*_X \longrightarrow \frac{\mathrm{d}f}{f} \in \Omega^1_X,$$

then we have a commutative diagram of sheaves of abelian groups

$$\begin{array}{ccccccc}
\mathcal{O}_X & \xrightarrow{\mathrm{d}} & \Omega_X^1 & \xrightarrow{\mathrm{d}} & \Omega_X^2 & \xrightarrow{\mathrm{d}} & \ldots \\
\downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\
\mathcal{O}_X^* & \xrightarrow{\mathrm{d}_{\log}} & \Omega_X^1 & \xrightarrow{\mathrm{d}} & \Omega_X^2 & \xrightarrow{\mathrm{d}} & \ldots
\end{array}.$$

So it is clear that an exact form is log-exact, if a global form $\omega \in \Omega_X^1(X)$ can be written $\omega = \mathrm{d}u$ for a global function $u \in \mathcal{O}_X(X)$ then letting $f = \exp(u) \in \mathcal{O}_X^*(X)$ we have

$$\mathrm{d}(\exp(f)) = \exp(f)\mathrm{d}f = \exp(f)\omega$$

and dividing by $\exp(f)$ shows the claim. A 1-form can be log-exact without being exact, think of $X = \mathbb{C}^*$, then $z \in \mathcal{O}_{\mathbb{C}^*}^*(\mathbb{C}^*)$ and set $\omega = \frac{\mathrm{d}z}{z}$.

Now let us see what a connection on a trivial vector bundle is in general, let $m \in \mathbb{N}$, there is a canonical connection on $\mathcal{O}_X^{\oplus m}$ given by

$$\mathrm{d}^{\oplus m} : \mathcal{O}_X^{\oplus m} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus m} = (\Omega_X^1)^{\oplus m}$$

and so any connection $\mathrm{d}^\nabla : \mathcal{O}_X^{\oplus m} \longrightarrow (\Omega_X^1)^{\oplus m}$ is the sum of $d^{\oplus m}$ and of a morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{\oplus m} \longrightarrow (\Omega_X^1)^{\oplus m}$$

i.e. of a square matrix $M \in \mathrm{M}_m(\Omega_X^1(X))$ of size m whose coefficients are global 1-forms. For instance if $m = 2$ and $X = \mathbb{C}^n$ then a connection d^∇ on $\mathcal{O}_X^{\oplus 2}$ can be written

$$\mathrm{d}^\nabla = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \mathrm{d} \end{pmatrix} - \begin{pmatrix} \omega_{1,1}(z) & \omega_{1,2}(z) \\ \omega_{2,1}(z) & \omega_{2,2}(z) \end{pmatrix}$$

in the sense that for all $\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \in \mathcal{O}_X^{\oplus 2}(U)$ then

$$\mathrm{d}^\nabla(U) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \begin{pmatrix} \mathrm{d}f \\ \mathrm{d}g \end{pmatrix} - \begin{pmatrix} \omega_{1,1}(z)f(z) + \omega_{1,2}(z)g(z) \\ \omega_{2,1}(z)f(z) + \omega_{2,2}(z)g(z) \end{pmatrix}$$

Let us write the coordinates of $\mathbb{C}^n$ as $z = (z_1, ...z_n)$, then if we write

$$\omega_{1,1} = \sum_{i=1}^n a_{1,1}^i(z)\mathrm{d}z_i, \ \omega_{1,2} = \sum_{i=1}^n a_{1,2}^i(z)\mathrm{d}z_i$$
$$\omega_{2,1} = \sum_{i=1}^n a_{2,1}^i(z)\mathrm{d}z_i, \ \omega_{2,2} = \sum_{i=1}^n a_{2,2}^i(z)\mathrm{d}z_i$$

then an element $\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \in \mathcal{O}_X^{\oplus 2}(U)$ is in the kernel of $\mathrm{d}^\nabla(U)$ if and only if

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \frac{\partial f}{\partial z_i}(z)\mathrm{d}z_i \\ \sum_{i=1}^n \frac{\partial g}{\partial z_i}(z)\mathrm{d}z_i \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^n (a_{1,1}^i(z)f(z) + a_{1,2}^i(z)g(z))\mathrm{d}z_i \\ \sum_{i=1}^n (a_{2,1}^i(z)f(z) + a_{2,2}^i(z)g(z))\mathrm{d}z_i \end{pmatrix}$$

Which is the system of differential equations given for all $1 \leq i \leq n$ by

$$\begin{cases} \frac{\partial f}{\partial z_i} = a_{1,1}^i(z)f(z) + a_{1,2}^i(z)g(z) \\ \frac{\partial g}{\partial z_i} = a_{2,1}^i(z)f(z) + a_{2,2}^i(z)g(z) \end{cases}$$

equivalently if we introduce for all $1 \leq i \leq n$,

$$A_i = \begin{pmatrix} a_{1,1}^i(z) & a_{1,2}^i(z) \\ a_{2,1}^i(z) & a_{2,2}^i(z) \end{pmatrix},$$

and write

$$P_i = \begin{pmatrix} \partial_{z_i} & 0 \\ 0 & \partial_{z_i} \end{pmatrix}$$

then the solutions of our system is precisely the intersection of the sheaf kernel of the morphisms

$$\nabla(U)(\partial_{z_i}) = P_i - A_i : \mathcal{O}_X^{\oplus 2}(U) \longrightarrow \mathcal{O}_X^{\oplus 2}(U).$$

In term of those matrices, the connection is flat if and only if for all $1 \leq i, j \leq n$ we have the equality

$$\frac{\partial A_i}{\partial z_j} - \frac{\partial A_j}{\partial z_i} + [A_i, A_j] = 0.$$

which is really nothing but the integrability condition

$$[\nabla(\partial_{z_i}), \nabla(\partial_{z_j})] = \nabla([\partial_{z_i}, \partial_{z_j}]) = \nabla(0) = 0$$

since

$$\begin{aligned} [\nabla(\partial_{z_i}), \nabla(\partial_{z_j})] = [P_i - A_i, P_j - A_j] &= [P_i, P_j] - [P_i, A_j] - [A_i, P_j] + [A_i, A_j] \\ &= -\frac{\partial A_j}{\partial z_i} + \frac{\partial A_i}{\partial z_j} + [A_i, A_j]. \end{aligned}$$

In turn this condition ensures that we can integrate (=solve) locally our system of partial differential equations, this was shown to be necessary by the symmetry of second order derivatives, which is nothing but the vanishing $[\partial_{z_i}, \partial_{z_j}] = 0$. We'll see more about how the sheaf of solutions behave in Thomas' talks.

4 Coherence

We have seen previously that $\mathcal{D}_X$-modules are nothing but $\mathcal{O}_X$-modules with a connection. In differential geometry, vector bundles play an important role. If $E \longrightarrow X$ is a vector bundle on X, its sheaf of sections $\mathcal{E}$ is naturally an $\mathcal{O}_X$-module. Most of the examples we saw are of this form :

- The sheaf $\mathcal{O}_X$ is the sheaf of sections of the trivial bundle $X \times \mathbb{C} \longrightarrow X$;
- The tangent sheaf Θ_X is the sheaf of sections of the tangent bundle $TX \longrightarrow X$;
- The cotangent sheaf Ω^1_X is the sheaf of sections of the cotangent bundle $T^*X \longrightarrow X$;
- For all $m \geq 0$, the sheaf of differential operators $\mathcal{D}_X^{\leq m}$ of degree $\leq m$ is the sheaf of sections How is it called in differential geometry ? ;
- If one makes sense of infinite dimensional vector bundles, then $\mathcal{D}_X$ is the sheaf of sections of such an object.

So if all the examples of sheaves of $\mathcal{O}_X$-modules we have in mind really come from (possibly infinite-dimensional) vector bundles, why should we care ?

The issue with vector bundles is that they don't form a well-behaved category for our use (i.e. they for instance don't form an abelian category). This is easily seen from the following example, let $f \in \mathcal{O}_X(X)$ be a global function and look at the map of vector bundles

$$f : (x, z) \in X \times \mathbb{C} \longrightarrow (x, f(x)z) \in X \times \mathbb{C},$$

for each $x \in X$, it defines a map on the corresponding fiber

$$f(x) : z \in \mathbb{C} \longrightarrow f(x)z \in \mathbb{C}$$

but for each $x \in X$ such that f vanishes, this map is zero ! So if there was such a thing as the cokernel of f and if it was a vector bundle, it would be reasonnable to expect it to have a zero fiber at x if $f(x) \neq 0$ and a fiber $\mathbb{C}$ at the point x where $f(x) = 0$. This cannot be a vector bundle because the dimensions of the fibers "jump" for a sufficiently non-trivial function f. If we look at this exemple from the perspective of $\mathcal{O}_X$-modules, we are looking at a map of sheaves we have already encountered before

$$m(X)(f) : \mathcal{O}_X \longrightarrow \mathcal{O}_X.$$

For simplicity, let us assume that X is a curve and $f \neq 0$, let us denote by $Z(f) \subset X$ the set of zeroes of the function f, this is nothing but a possibly infinite disjoint union of isolated points, the cokernel of this map is described by the exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \overset{m(X)(f)}{\longrightarrow} \mathcal{O}_X \longrightarrow \bigoplus_{x \in Z(f)} \delta_x \longrightarrow 0$$

where for $x \in X$, δ_x is the skyscraper sheaf[6] at x with fiber $\mathbb{C}$. If X is not a curve this cokernel is nothing but (the pushforward of) the structure sheaf of the hypersurface defined by f.

[6]If U is an open not containing x, $\delta_x(U) = 0$ and if U contains x we define $\delta_x(U) = \mathbb{C}$, an element $g \in \mathcal{O}_X(U)$ acts on $\delta_x(U) = \mathbb{C}$ via multiplication by $\mathrm{ev}_x(g_x) = g(x) \in \mathbb{C}$.

We now have a motivation as to why we need not only to work with vector bundles but with $\mathcal{O}_X$-modules, now among $\mathcal{O}_X$-modules, there is a "well behaved" subcategory consisting of coherent modules which we introduce now.

We will momentaneously need a more general setting. Let $\mathcal{R}$ be a sheaf of (possibly non-commutative) rings on a topological space X and $\mathcal{F}$ be a sheaf of (left) $\mathcal{R}$-modules.

1. The sheaf $\mathcal{F}$ is said to be **locally finitely generated** if for each $x \in X$ there exists an open neighborhood U_x of x and $m \in \mathbb{N}$ such that we have an exact sequence of sheaves of $\mathcal{R}_{U_x}$-modules

$$\mathcal{R}_{U_x}^{\oplus m} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0.$$

   This is equivalent to saying that there exists $s^1, ..., s^m \in \mathcal{F}(U_x)$ such that for all $y \in U_x$, the corresponding germs

$$s_y^1, ..., s_y^m \in \mathcal{F}_y$$

   span the stalk $\mathcal{F}_y$ as a $\mathcal{R}_y$-module.

2. The sheaf $\mathcal{F}$ is said to be **locally finitely presented** if for each $x \in X$ there exists an open neighborhood U_x of x and $m_0, m_1 \in \mathbb{N}$ such that we have an exact sequence of sheaves of $\mathcal{R}_{U_x}$-modules

$$\mathcal{R}_{U_x}^{\oplus m_1} \longrightarrow \mathcal{R}_{U_x}^{\oplus m_0} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0$$

   This is equivalent to saying that there exists $s^1, ..., s^{m_0} \in \mathcal{F}(U_x)$ such that for all $y \in U_x$, the corresponding germs

$$s_y^1, ..., s_y^{m_0} \in \mathcal{F}_y$$

   span the stalk $\mathcal{F}_y$ as a $\mathcal{R}_y$-module. And that moreover there exists a matrix with coefficients in $\mathcal{R}(U_x)$

$$(f_j^i) \in \mathrm{M}_{m_0,m_1}(\mathcal{R}(U_x))$$

   such that for all $y \in U_x$ we have an exact sequence of $\mathcal{R}_y$-modules

$$\mathcal{R}_y^{\oplus m_1} \overset{\mathrm{M}_y = ((f_j^i)_y)}{\longrightarrow} \mathcal{R}_y^{\oplus m_0} \longrightarrow \mathcal{F}_y \longrightarrow 0.$$

   That is to say the relations between the generators $s_y^1, ..., s_y^{m_0}$ are controlled by the matrix M_y i.e. for all $r_y^1, ..., r_y^{m_0} \in \mathcal{R}_y$ we have the equality in $\mathcal{F}_y$

$$\sum_{i=1}^{m_0} r_y^i s_y^i = 0$$

   if and only if

$$\begin{bmatrix} r_y^1 \\ r_y^2 \\ \vdots \\ r_y^{m_0} \end{bmatrix} \in \mathrm{Im}(M_y) \text{ i.e. } \begin{bmatrix} r_y^1 \\ r_y^2 \\ \vdots \\ r_y^{m_0} \end{bmatrix} = [(f_j^i)_y] \begin{bmatrix} s_y^1 \\ s_y^2 \\ \vdots \\ s_y^{m_1} \end{bmatrix} \text{ for some } s_y^1, ..., s_y^{m_1} \in \mathcal{R}_y.$$

Another way to put it is that the "sheaf of relations" defined as the sheaf kernel of the map

$$\mathcal{R}_{U_x}^{\oplus m_0} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0$$

is itself locally finitely generated.

3. The sheaf $\mathcal{F}$ is said to be **locally free** if it is locally finitely generated without relations i.e. for all $x \in X$ there exists an open neighborhood U_x and $m \in \mathbb{N}$ such that we have a short exact sequence

$$0 \longrightarrow \mathcal{R}_{U_x}^{\oplus m} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0,$$

i.e. $\mathcal{F}$ is locally isomorphic to a direct sum of copies of $\mathcal{R}$. This is the same as saying that $\mathcal{F}$ is locally finitely generated and that the "sheaf of relations" is zero.

Lemma 13. *Let $\mathcal{F}$ be a locally finitely generated sheaf of $\mathcal{R}$-modules. Let $x \in X$ be such that $\mathcal{F}_x = 0$, then there exists an open neighborhood V of x such that $\mathcal{F}_{|V} = 0$.*

Proof. Let $x \in U$, by assumption there exists an open neighborhood U_x and $m \in \mathbb{N}$ and $s^1, ..., s^m \in \mathcal{F}(U_x)$ that define a surjective morphism of sheaves of $\mathcal{R}_{U_x}$-modules :

$$\mathcal{R}_{U_x}^{\oplus m} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0.$$

Now because $\mathcal{F}_x = 0$ there exists for each $i \in \{1, ..., m\}$ an open neighborhood U_x^i of x such that

$$s^i_{|U_x^i} = 0$$

Letting

$$V = \bigcap_{i=1}^{m} U_i^x$$

which is clearly open because the intersection is finite, it is clear that $\mathcal{F}_{|V} = 0$. ☐

Proposition 11. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{R}$-modules. Let $x \in X$ and let*

$$\varphi_x : \mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})_x \longrightarrow \mathrm{Hom}_{\mathcal{R}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

be the canonical morphism.

1. *If $\mathcal{F}$ is locally finitely generated, then φ_x is injective.*

2. *If $\mathcal{F}$ is locally finitely presented, then φ_x is an isomorphism.*

Proof. Omitted. ☐

Let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. Then $\mathcal{F}$ is said to be a **coherent sheaf** of $\mathcal{R}$-modules if the following conditions hold.

1. The sheaf of $\mathcal{R}$-modules $\mathcal{F}$ is locally finitely generated.

2. For all open set U, any localy finitely generated $\mathcal{R}_U$-submodule of $\mathcal{F}_{|U}$ is locally finitely presented.

A sheaf of rings $\mathcal{R}$ is said to be coherent if it is as a left module over itself.

The notion of coherence has two major advantages.

- It allows one to pass from stalk-properties to local properties. It is roughly a principle of analytic continuation for the sheaves themselves à ref
- It is a robust notion in the sense that coherent sheaves arrange themselves as an abelian subcategory of the category of all sheaves of $\mathcal{R}$-modules. They moreover satisfy a "two out of three" property. à ref

Theorem 4. *Let $\mathcal{F}, \mathcal{F}'$ and $\mathcal{F}''$ be sheaves of $\mathcal{R}$-modules fitting in an exact sequence*

$$0 \longrightarrow \mathcal{F}^{'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

then if two out of the three are coherent, so is the third.

Proof. Omitted. □

Lemma 14. *If $\mathcal{F}$ and $\mathcal{F}'$ are coherent $\mathcal{R}$-modules and*

$$\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$$

is a morphism of sheaves then $Ker(\varphi), Im(\varphi)$ and $Coker(\varphi)$ are coherent $\mathcal{R}$-modules.

Lemma 15. *Let $\mathcal{F}, \mathcal{F}'$ and $\mathcal{F}''$ be sheaves of coherent $\mathcal{R}$-modules then if we are given morphism of sheaves*

$$\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

such that for a certain $x \in X$ the corresponding sequence is exact at x i.e.

$$\mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x$$

then there exists an open neighborhood U of x such that

$$\mathcal{F}^{'}_{|U} \longrightarrow \mathcal{F}_{|U} \longrightarrow \mathcal{F}''_{|U}$$

is an exact sequence.

Lemma 16. *Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{R}$-modules then $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is a locally free $\mathcal{R}_{X,x}$-module for all $x \in X$.*

When the base ring $\mathcal{R}$ is coherent itself, the notion of coherent module is a bit easier to handle

Lemma 17. *Let $\mathcal{R}$ be a coherent sheaf of rings. Let $\mathcal{F}$ be a $\mathcal{R}$-module then $\mathcal{F}$ is coherent if and only if it is locally finitely presented.*

Proof. The direct implication is clear from the definition of coherence.

Conversely assume that $\mathcal{F}$ is locally finitely presented. We want to show that $\mathcal{F}$ is coherent. Clearly it is locally finitely generated. Let U be an open set and $\mathcal{G}_{|U}$ be a subsheaf of $\mathcal{F}_{|U}$ that is locally finitely generated, we have to show that it is locally finitely presented. Let $x \in U$, because $\mathcal{F}_x$ is locally finitely presented, there is an open neighborhood $U_x \subset U$ and $m_0, m_1 \in \mathbb{N}$ such that we have an exact sequence of $\mathcal{R}_{U_x}$-modules

$$\mathcal{R}_{U_x}^{\oplus m_1} \longrightarrow \mathcal{R}_{U_x}^{\oplus m_0} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0$$

But we see from the previous proposition that the sheaf $\mathcal{F}_{|U_x}$ is coherent as a cokernel of a map between coherent sheaves. Hence $\mathcal{G}_{|U_x}$ being a locally finitely generated $\mathcal{R}_{U_x}$-submodule of $\mathcal{F}_{|U_x}$ is locally of finite presentation hence so is $\mathcal{G}_{|U}$ because x was arbitrary. □

Theorem 5 (Oka)**.** *Let X be a complex manifold. Then the sheaf $\mathcal{O}_X$ of holomorphic functions on X is a coherent ring.*

Theorem 6. *Let X be a complex manifold. The sheaf of differential operators $\mathcal{D}_X$ is a coherent ring.*

Remark 16. *In fact more is true, on a complex manifold X, both $\mathcal{O}_X$ and $\mathcal{D}_X$ are Noetherian rings (see the Appendix A of Kashiwara's book for the definition).*

Proposition 12. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module that is coherent as a $\mathcal{O}_X$-modules. Then $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of finite rank.*

Proof. Let us write $\nabla : \Theta_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$ the connection corresponding to the structure of $\mathcal{D}_X$-module on $\mathcal{M}$. To show that $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of finite rank, it is enough to show that for each $x \in X$, $\mathcal{M}_x$ is a free $\mathcal{O}_{X,x}$-module of finite rank. Because $\mathcal{M}$ is $\mathcal{O}_X$-coherent it is $\mathcal{O}_X$-locally finitely generated and it is clear from there that $\mathcal{M}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module. Denote by $\mathbf{m}_x$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$, the evaluation map at x yields an exact sequence of $\mathbb{C}$-vector space

$$0 \longrightarrow \mathbf{m}_x \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\mathrm{ev}_x} \mathbb{C} \longrightarrow 0.$$

The geometric fiber of the coherent sheaf $\mathcal{M}$ at x is defined to be the $\mathbb{C}$-vector space

$$\mathcal{M}(x) = \mathcal{M}_x / \mathbf{m}_x \mathcal{M}_x = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} / \mathbf{m}_x.$$

Because $\mathcal{M}_x$ is finitely generated as a $\mathcal{O}_x$-module then $\mathcal{M}(x)$ is a finite dimensional $\mathbb{C}$-vector space. Let $s^1(x), ..., s^m(x)$ be a $\mathbb{C}$-basis of $\mathcal{M}_x$, and choose some lift $s_x^1, ..., s_x^m \in \mathcal{M}_{X,x}$. It follows from a form of Nakayama's lemma that $s_x^1, ..., s_x^m$ span $\mathcal{M}_x$ as a $\mathcal{O}_{X,x}$-module.

Assume for a contradiction that we have a relation between those generators in $\mathcal{M}_x$ given by

$$\sum_{i=1}^{m} f_x^i s_x^i = 0 \qquad (*)$$

where $(f_x^i)_{1\leq i\leq m} \in \mathcal{O}_{X,x}$. Evaluating at x this equality we obtain the following equality in the $\mathbb{C}$-vector space $\mathcal{M}(x)$

$$\sum_{i=1}^{m} f^i(x)s^i(x) = 0$$

which implies, since the $(s^i(x))_{1\leq i\leq m}$ form a basis of $\mathcal{M}(x)$, that for all $i = 1, ..., m$ we have

$$f^i(x) = 0 \text{ i.e. } f_x^i \in \mathbf{m}_x.$$

Let $\theta_x \in \Theta_{X,x}$, applying the differential operator θ_x to equation (*) we obtain the equality in $\mathcal{M}_x$

$$\sum_{i=1}^{m} \theta_x(f_x^i)s_x^i + \sum_{i=1}^{m} f_x^i \nabla_{\theta_x}(s_x^i) = 0.$$

Now because the $(s_x^i)_{1\leq i\leq m}$ span $\mathcal{M}_x$ as a $\mathcal{O}_{X,x}$-module, there exists for each $i = 1, ..., m$ some $g^j(\theta_x) \in \mathcal{O}_{X,x}$ such that

$$\nabla_{\theta_x}(s_x^i) = \sum_{j=1}^{m} g^j(\theta_x)s_x^j,$$

hence the previous equality becomes the equality in $\mathcal{M}_x$

$$0 = \sum_{i=1}^{m} \theta_x(f_x^i)s_x^i + \sum_{i=1}^{m} f_x^i \sum_{j=1}^{m} g^j(\theta_x)s_x^j = \sum_{i=1}^{m} \left(\theta_x(f_x^i) + g_x^i \sum_{j=1}^{m} f_x^j \right) s_x^i \qquad (**)$$

so evaluating at x we obtain the equality in the $\mathbb{C}$-vector space $\mathcal{M}(x)$

$$0 = \sum_{i=1}^{m} \left(\theta_x(f_x^i)(x) + g^i(x) \sum_{j=1}^{m} f^j(x) \right) s^i(x) = \sum_{i=1}^{m} \theta_x(f_x^i)(x)s^i(x)$$

hence as before for every $i = 1, ..., m$ we have the equality of complex numbers

$$0 = \theta_x(f_x^i)(x)$$

and because θ was arbitrary, this shows that $f_x^i \in \mathbf{m}^2$. Now pick $\theta' \in \Theta_x$ and apply it to equation (**), we obtain an equality of the form

$$0 = \sum_{i=1}^{m} \left(\theta'_x\theta_x(f_x^i) + \theta'_x(g_x^i) \sum_{j=1}^{m} f_x^j + g_x^i \sum_{j=1}^{m} \theta'_x(f_x^j) \right) s_x^i$$
$$+ \sum_{i=1}^{m} \left(\theta_x(f_x^i) + g_x^i \sum_{j=1}^{m} f_x^j \right) \nabla_{\theta'_x}(s_x^i). \qquad (***)$$

Evaluating once more at x, we obtain the equality in the $\mathbb{C}$-vector space $\mathcal{M}(x)$

$$0 = \sum_{i=1}^{m} \theta'(\theta(f^i))(x) s^i(x)$$

which in turns yield the equality of complex numbers

$$\theta'(\theta(f^i))(x) = 0$$

for all $i = 1, ..., m$ and all $\theta_x, \theta'_x \in \Theta_x$ which shows that $f^i_x \in \mathbf{m}^3$. An inductive argument based on the previous idea now shows that for all $i = 1, ..., m$ we have

$$f^i_x \in \bigcap_{k \geq 0} \mathbf{m}^k = \{0\}.$$

So there can be no relations between generators of $\mathcal{M}_x$ we have chosen i.e. the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$ is free ! $\square$