Part I: Dirichlet and Robin Harmonic measures

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Summer school Hangzhou July 2025

Introduction post-lectures: I do not proofread the text, so please forgive all mistakes. There should be many.

There should be a shorter part on Minimal sets of dimension 2.
In these lectures, only a limited number of topics (those that I understand), and in Parts I and II, we'll insist on relations to the regularity of the boundary. Sorry If I repeat things that you know.

Part I: Harmonic measure. 1.Brownian motion intuition

We start with an intuitive definition of Harmonic measure.
Our notation: $\Omega \subset \mathbb{R}^n$ is a bounded domain, we give ourselves $X \in \Omega$ (a pole) and we "define" a measure ω^X on $\partial\Omega$ by

$$\omega^X(E) \text{ is the probability that a Brownian path starting from } X \text{ will lie in } E \text{ the first time it hits the boundary } \partial\Omega \tag{1}$$

Comment:

• Possibly hard to define, depending on Ω.

• If $\partial\Omega$ is too thin, the Brownian path will never hit it. But for most of our cases, a Wiener criterion will say that it hits.

• Easy to see, with a stopping time argument, that $X \mapsto \omega^X(E)$ satisfies the mean value property, and hence is harmonic.
[That is $⨍_{\partial B(X,r)} \omega^Y(E) = \omega^X(E)$ for r small.]

• And to imagine that $\omega^X(E)$ tends to $\mathbb{1}_E$ when X tends to $\partial\Omega$.

2. Harmonic measure by Dirichlet Problem

Now the definition that we use. Suppose Ω is such that for all continuous f defined on Ω there is a unique harmonic function u_f, with a continuous extension on $\overline{\Omega}$ such that $u_f = f$ on $\partial\Omega$.

[That is, $\Delta u_f = 0$ on Ω and $u_f = f$ on $\partial\Omega$.]

Notice that by the maximum principle (to be checked but),

$$\sup_{X\in\Omega} u_f(X) = \sup_{\partial\Omega} f(\xi).$$

Then by Riesz, for each $X \in \Omega$ there is a probability measure ω^X on $\partial\Omega$ such that

$$u_f(X) = \int_{\partial\Omega} f(\xi) d\omega^X(\xi). \tag{2}$$

Our main questions: Is all this defined? What does ω^X look like? Is it supported on a small set? What geometric properties of Ω matter for this?

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3. Simple examples: the ball and upper half space

When $\Omega = B(0,1) \subset \mathbb{R}^n$, ω^X can be computed: it is the absolutely continuous measure $P(X,y)d\sigma(y)$ whose density (the Poisson kernel) is given by

$$P(X,y) = \frac{1-|X|^2}{\omega_{n-1}|X-y|^n} \quad \text{for } X \in B(0,1) \text{ and } y \in \partial B(0,1). \tag{3}$$

Here $\omega_{n-1} = \sigma(\partial B(0,1)$ (so that $\int_y P(X,y)d\sigma(y) = 1$). Similarly for the upper half space $\mathbb{R}^n_+$, and with $X = (x,t)$ (with $t > 0$) and $x, y \in \mathbb{R}^{n-1}$,

$$P(X,y) = \frac{c_n t}{(t^2 + |x-y|^2)^{n/2}} \tag{4}$$

Comment: We can check that this works, or use Fourier.

By (48) applied to $B(X, 9\,\mathrm{dist}(x,\partial\Omega)/10$, harmonic functions are (locally) smooth on Ω. And also for $u \geq 0$ harmonic on Ω, and $X, Y \in \Omega$ such that $|X-Y| \leq \mathrm{dist}(X,\partial\Omega)/2$,

$$C_n^{-1}u(X) \leq u(Y) \leq C_n u(X) \qquad \text{(Harnack inequality).} \tag{5}$$

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Simple examples (2): simply connected domains in $\mathbb{R}^2$

Let $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ and $\psi : \mathbb{D} \to \Omega$ a conformal mapping. Suppose $\psi(0) = X \in \Omega$. Observe that u is harmonic on Ω iff $u \circ \psi$ is harmonic on $\mathbb{D}$. If $\partial\Omega$ is a Jordan curve, ψ extends to a homeomorphism $\psi : \overline{\mathbb{D}} \to \overline{\Omega}$. And for f continuous on $\partial\Omega$, its continuous harmonic extension is $v \circ \psi^{-1}$, where v is the harmonic extension of $u \circ \psi$. Thus

$$u(X) : \fint_{\partial\mathbb{D}} u \circ \psi \tag{6}$$

F. and M. Riesz proved that if $\partial\Omega$ is a (Jordan) curve with finite length, then ψ^{-1} exists a.e. and that (6) becomes

$$u(X) = c \int_{\partial\Omega} u(\xi)|(\psi^{-1})'|(\xi) d\ell(\xi). \tag{7}$$

That is, in this case ω^X is absolutely continuous with respect to σ, with density $|(\psi^{-1})'|$.

Comment: All the ω^X, $X \in \Omega$, have the same behavior: compose with a conformal mapping of $\mathbb{D}$. In general, true by Harnack!

A delicate example : a self-similar Cantor set

Let $K \subset \mathbb{R}^2$ be a four-corner Cantor set of dimension $d \in (0, 2)$.
See next page.

Let σ be the natural probability measure on K.

Set $\Omega = \mathbb{R}^2 \setminus K$. Then (Carleson, I think)

$$\text{for all } X \in \Omega,\ \omega^X \perp \sigma. \tag{8}$$

That is, there is a set $E \subset K$ such that $\sigma(E) = 0$ and $\omega^X(E) = 1$.

Subtle proof, but in short Brownian paths like to land on the corners. And for non self-similar Cantor set, the story can be different (see later).

The Garnett-Ivanov 1-dimensional Cantor set (d=1)

$K = \bigcap_{k\geq 0} K_k$, suggested by the picture.

K_k is composed of 4^k squares of size 4^{-k}.

A natural measure μ on K gives the same mass 4^{-k} to each square of K_k. And then $\mu = cH^1_{|K}$.

K is totally unrectifiable: $\mu(E \cap \Gamma) = 0$ for every curve Γ with finite length.

This set one-dimensional Ahlfors regular, with a one-sided NTA complement.

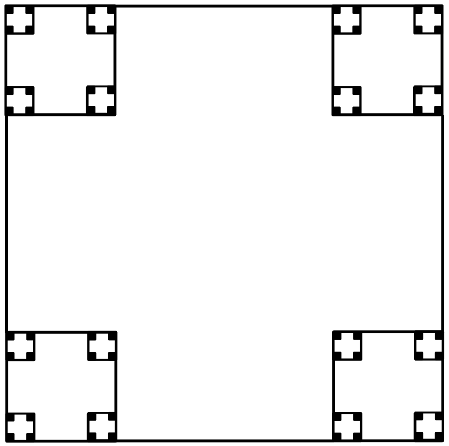

Figure: The set K_3 (three generations of the construction of K).
Then just continue and only keep the dust at the limit.

4. Rapid overview of the situation (more later)

For Ω such that $\partial\Omega$ is of dimension $n-1$ (for instance, Ahlfors regular), we know that the absolute continuity of ω^X with respect to $\sigma = \mathcal{H}^{n-1}{}_{|\partial\Omega}$ is a matter of:

- quantitative connectedness of Ω
- Rectifiability.

Results of (among others) Riesz, Lavrentiev, Makarov, Jones, Wolff ($n = 2$);
Dahlberg (for Lipschitz domains), and many others (David-Jerison, Semmes, Hofmann, Martell, ... ($n \geq 3$);
And these results are essentially optimal: Azzzam-Hofmann-Lacey-Martell-Mayboroda-Mourgoglou-Tolsa-Volberg.

Geometric conditions 1: AR or mixed dimension boundaries

For all our results $\Omega \subset \mathbb{R}^n$ will have some minimal regularity.
Either $E = \partial\Omega$ is Ahlfors-regular of some dimension $d < n$, i.e., there is a measure σ supported on E such that

$$C^{-1} r^d \leq \sigma(B(x,r)) \leq C r^d \quad \text{for } x \in E \text{ and } 0 < r < \dim(E). \tag{9}$$

For instance, Lipschitz graphs, or the Cantor sets above. For harmonic functions, we will take $d > n-2$ because (for instance) otherwise the brownian paths will not see E.
Or more general, and often convenient, E has mixed dimensions: there is a doubling measure σ supported on E such that in addition, with a lower bound d on the moral local dimension:

$$\sigma(B(x,tr)) \leq C t^d \sigma(B(x,r)) \quad \text{for } x \in E,\ 0 < r < \dim(E), \text{ and } 0 < t < 1. \tag{10}$$

[for standard elliptic operators, we ask $d > n-2$]. Doubling means

$$\sigma(B(x,2r)) \leq C\sigma(B(x,r)) \quad \text{for } x \in E \text{ and } 0 < r < \dim(E) \tag{11}$$

Geometric conditions 2: One-sided NTA = uniform

Easy to see that for w^X to be tame, Ω should have some uniform connectedness property. Here we will always assume one-sided NTA (non-tangential access), i.e., Ω is a uniform domain. We say that Ω is 1-NTA when it has corkscrew balls and Harnack chains:

• There exists $C > 1$ such that: for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$, there is $Y \in \Omega$ such that $B(Y, C^{-1}r) \subset \Omega \cap B(x, r)$;

• For $A > 1$ there is an integer $N = N(A) > 1$ such that: if $X, Y \in \Omega$ are such that $|X - Y| \leq A \min(\delta(X), \delta(Y))$, then there is a Harnack chain of length N that goes from X to Y.

Here $\delta(X) = \operatorname{dist}(X, \partial\Omega)$. A harnack chain is a chain of balls B_j, $0 \leq j \leq N$, such that $B_{j+1} \cap B_j \neq \emptyset$ for $0 \leq j \leq N-1$, and $2B_j \subset \Omega$. It connects X to Y when $X \in B_0$ and $Y \in B_N$.

Comments: Useful, because if $u \geq 0$ is harmonic on Ω, then by Harnack, $u(Y) \leq C(N(A))u(X)$. In particular we have upper and lower bounds on u on compact sets $K \subset\subset \Omega$.

Some times one can do with even less: recent works with just Corkscrew balls.

We never assume corckscrew balls in $\mathbb{R}^n \setminus \overline{\Omega}$ though.

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Pictures for 1-NTA (here the domain is outside)

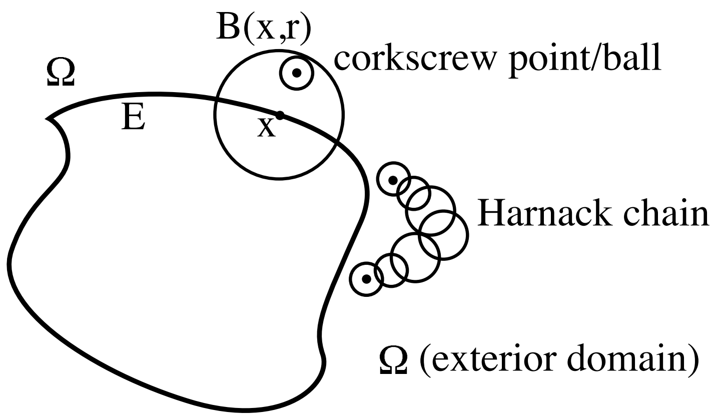

Figure: A corkscew ball (top) and a Harnack chain between two points (right). The domain is outside.

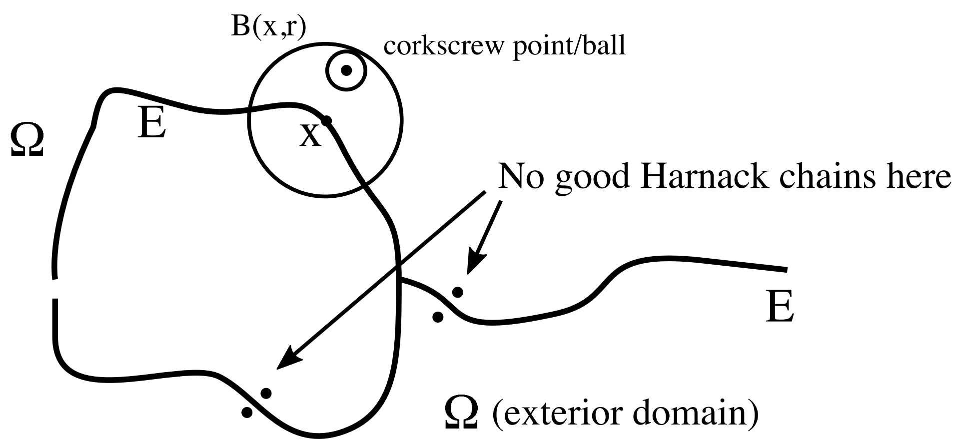

Figure: A situation with bad 1-NTA constants

A_∞ mutual absolute continuity

If $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, $X \in \Omega \setminus B(x, 2r)$, and $E \subset \Delta := \partial\Omega \cap B(x, r)$, we say that $\omega \in A_\infty(\sigma)$ when we always have that

$$\sigma(E) \leq \delta\sigma(\Delta) \implies \omega^X(E) \leq \varepsilon\omega^X(\Delta). \tag{12}$$

Other equivalent definitions exist, including with other quantifiers. This implies estimates, like

$$C^{-1}\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{\alpha} \leq \frac{\omega^X(E)}{\omega^X(\Delta)} \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{\beta}. \tag{13}$$

Thus we require a uniform estimate on the various balls $B(x, r)$ and X (adapted to the situation).
Here we assume σ doubling, not a priori the ω^X. But this follows.
Stronger than just doubling: E could be a complicated set in Δ, not just a ball. Also implies that $\sigma \in A_\infty(\omega)$ too.

Rapid overview for the Laplacian (again)

Here are some statements:

• Dahlberg 77: If $\Omega = \{(x,t) \in \mathbb{R}^{n+1} ; \, t > A(x)\}$ for some Lipschitz $A : \mathbb{R}^n \to \mathbb{R}$, then $\omega \in A_\infty(\sigma)$, where σ is the forward image of dx.
• Many authors: If $\Omega \subset \mathbb{R}^n$ is one-sided NTA and $\partial\Omega$ is uniformly rectifiable of dimension $n-1$, then $\omega \in A_\infty(\sigma)$, where $\sigma = \mathcal{H}^{n-1}_{|\partial\Omega}$.
• And [AzHoLaMaMaMoToVo] there is a converse of the type 1-NTA $+A_\infty \implies UR$ (even true with less uniform assumptions).

Proofs always connected some way to L^2-boundedness of singular integral operators on $\partial\Omega$. We'll discuss small parts later.

Comment about α and β, or L^q integrability of $\frac{d\omega}{d\sigma}$ (mystery except for Dahlberg).

What about other operators?

We'll study solutions of other operators

$$L = -\mathrm{div} A \nabla \tag{14}$$

where $A = A(X)$ is a $n \times n$ matrix-valued function on Ω, with real valued coefficients (we won't do complex). And for the moment, A is bounded elliptic which means that there are constants $C, c > 0$ such that for every $X \in \Omega$,

$$||A(X)|| \leq C; \tag{15}$$

that is for "bounded" and usually we would use the operator norm on the set of matrices, which means that

$$||A(X)|| = \sup \left\{ |\langle A(X)\xi, \zeta \rangle| \, ; \, \xi \in \mathbb{R}^n ; ||\xi|| \leq 1 \text{ and } \zeta \in \mathbb{R}^n ; ||\zeta|| \leq 1 \right\} \tag{16}$$

and (for elliptic) that for $X \in \Omega$ and $\xi \in \mathbb{R}^n$

$$\langle A(X)\xi, \xi \rangle \geq c||\xi||^2. \tag{17}$$

Other operators (2)

... But we want A to be measurable, not more regular (than bounded elliptic). Which means that we'll need to be careful when we define solutions of $Lu = -\mathrm{div} A \nabla u = 0$. And we'll see that solutions are no as smooth as harmonic functions!
Here is an interesting example. Suppose $u : \widetilde{\Omega} \to \mathbb{R}$ is harmonic, $\Phi : \Omega \to \widetilde{\Omega}$ is a change of variable, and let $v = u \circ \Phi$. Does v satisfy an equation? We've seen that v is harmonic when Φ is conformal, but conformal mappings are rare! A computation shows that $Lv = 0$, where A is given by

$$A = A(X) = \det(M)^{-2} M^t M \,, \tag{18}$$

where $M = M(X)$ is the matrix of $D\Phi(X)$ and $\det(M)$ is its determinant.
If M is an isometry or a dilation, $A = I$. And if M has bounded distortion, i.e. $\sup_{||\xi||=1} ||M\xi|| \leq K \inf_{||\xi||=1} ||M\xi||$, we get that A is bounded elliptic...

Other operators (3)

... That is, quasiconformal (QC) mappings preserve the class of bounded elliptic mappings.

This gives a few examples, as for instance the image by Φ^{-1} of the smooth $\omega_{\widetilde{\Omega}}$ on $\widetilde{\Omega}$ may be more singular because of measure distortion!
Note: a typical, rather simple QC mapping is the radial dilation given by $\Phi(x) = |x|^{\alpha} \frac{x}{|x|}$, $\alpha > 0$, which is more (or less) singular than the identity at the origin. Certainly QC mappings are wilder than conformal mappings, and as a consequence v is not as smooth as u.
Note: lots of interesting elliptic operators do not come from a QC mapping, so this was only an example. Typical use: description of a non isotropic or even just variable material (conductivity, etc). Probably later, the Dahlberg-Kenig-Pipher conditions.

Next (after some analysis/geometry): definition of (weak) solutions, existence, regularity.

Our main tool Poincaré & traces. 1. The Hilbert space W

Needed for our definition of weak solution and proof of existence, which will be based on the space

$$W = W^{1,2}(\Omega) = \{f \in L^2_{loc(\Omega)}\,;\, \nabla f \in L^2(\Omega)\}. \tag{19}$$

Actually, I should write $\dot{W}$ because I mean the homogeneous space (for the moment), with the norm $||f||_W = \left(\int_\Omega |\nabla f|^2\right)^{1/2}$.
This is only a Hilbert space of functions defined modulo an additive constant (and we feel better here assuming that Ω is connected). This won't cause problems. And ∇f is the distribution derivative (or gradient, I will not be careful here), which we assume is given by an L^2 function. Thus the coordinate $\partial_j f = \frac{\partial f}{\partial x_j}$ is such that

$$\int_\Omega f \partial_j \varphi = -\int_\Omega \varphi \partial_j f \tag{20}$$

for every $\varphi \in C_c^\infty(\Omega)$. We'll work with this beautiful space for some time. There may be a weighted variant later.

Poincaré 2: the basic local Poincaré estimate

We'll need to know that for $f \in W^{1,2}(\Omega)$ and $B = B(X, r) \subset \Omega$,

$$\fint_B |f - m_B f|^2 \leq C r^2 \fint_B |\nabla f|^2. \quad \text{Here } m_B f = \fint_B f. \tag{21}$$

There are lots of better variants, for instance we have the same estimate for $\left(\fint_B |f - m_B f|^p\right)^{2/p}$ for some $p > 2$.
Notice that adding a constant to f changes nothing.

Easy proof: first check that test functions are (locally) dense, then check (21) for test functions f. Write $f(x) - f(y)$, $x, y \in B$, as an average of $\int_\Gamma \nabla f$ on a collection of paths in B from x to y, then estimate brutally some multiple integral. We'll see a variant soon.

Typical consequence: for $B_1, B_2 \subset \Omega$,

$$|m_{B_1} f - m_{B_2} f| \leq C(B_1, B_2)\, ||f||_W, \tag{22}$$

where $C(B_1, B_2)$ can be computed in terms of a Harnack chain from B_1 to B_2 (and you could guess its homogeneity).

Traces 1

And now consider $\Delta = B(x, r)$ centered on $\partial\Omega$ and pick a ball $D \subset \Omega \cap B(x, r)$, with radius $C^{-1}r$, a corkscrew ball.
Say f is smooth to simplify. Then

$$\fint_{\Delta} |f - m_D f|^2 d\sigma \leq C r^2 r^{\beta} \fint_{CB\cap\Omega} |\nabla f(X)|^2 \delta(X)^{-\beta} dX \qquad (23)$$

provided that $d > n - 2 - \beta$ and $\sigma \in AR(d)$. Even true with mixed dimension boundary.

The blue part will be useful when we study $\partial\Omega$ of higher co-dimensions.

Proof by 1-NTA (you may construct nontangential access regions that connect $x \in \partial\Omega$ to D (and stay in $\Omega \cap CB$)), Hölder, and for instance (21) or (22) on strings of balls.

Below is a toy computation in the upper half space; otherwise look at the mixed Astérisque book.

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Traces 2 (proof of (23) in the upper half space)

Consider $H = \mathbb{R}^n_+$ is the upper half space in $\mathbb{R}^n$, or even the complement of $\mathbb{R}^d$ in $\mathbb{R}^n$, $d < n-1$. So $\partial H \in AR(d)$.
Take $B = B(0,r)$ and $\Delta = B \cap \partial H$.
For $x \in \Delta$ and $k \geq 0$, call $Q_k(x)$ a cube of size $r_k = 2^{-k} r$ centered above x at distance r_k. And $f_k(x) = m_{Q_k(x)} f$.
We want to evaluate $||f_{k+1} - f_k||_2^2$. First

$$|f_{k+1}(x) - f_k(x)|^2 \leq C r_k^2 \fint_{R_k(x)} |\nabla f|^2 \leq C r_k^{2-n} \int_{R_k(x)} |\nabla f|^2 \quad (24)$$

for some slightly larger $R_k(x) \supset Q_k(x) \cup Q_{k+1}(x)$. Cover Δ by essentially disjoint cubes S of size $C^{-1} r_k$ (of measure $\sigma(S) \sim r_k^d$), and sum over S. This yields

$$\int_\Delta |f_{k+1}(x) - f_k(x)|^2 \leq C r_k^d \sum_S r_k^{2-n} \int_{R_k(S)} |\nabla f|^2 \leq C r_k^{d+2-n} \int_{A_k} |\nabla f|^2 \quad (25)$$

with $A_k = \{X \in B(0, Cr)\,;\, r_{k+1} \leq \mathrm{dist}(X, \partial H) \leq 4 r^{k+1}\}$.

Traces 3 (end of proof)

Recall (and add the harmless blue)

$$\int_{\Delta} |f_{k+1}(x) - f_k(x)|^2 \leq C r_k^{d+2-n} \int_{A_k} |\nabla f|^2 \leq C r_k^{d+2-n+\beta} \int_{A_k} |\nabla f|^2 \delta(X)^{-\beta}.$$

If $d + 2 - n + \beta > 0$ the coefficient tends to 0, we can take the root and sum, and we get

$$\sum_k ||f_{k+1}(x) - f_k(x)||_{L^2(\Delta)} \leq C r^{(d+2-n+\beta)/2} ||\nabla f(X)||_{L^2(B(0,Cr),\delta^{-\beta} dX)}, \tag{26}$$

as needed for (23). In fact, this is enough to define a correct trace $\mathrm{Tr}\, f \in L^2_{loc}(\sigma)$.

In general, replace direct contact by a short Harnack chain; the homogeneities stay the same.

We can see that when $n - d \geq 2$, we need the weight $\delta^{-\beta}$ because otherwise $\int |\nabla f|^2$ is not enough to control f on ∂H. Even true when f is harmonic.

Traces and extensions (More about the trace)

In fact when Ω is 1-NTA, $\partial\Omega \in AR(d)$, and

$$W = \{f\, ;\, \int_\Omega |\nabla f|^2 \delta(X)^{-\beta} < +\infty\}$$

(with $d + 2 - n + \beta > 0$), we can identify the space of traces $\mathrm{Tr}\,(f); f \in W)$ as the space H defined by semi-norm

$$||g||_H = \int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{2d+2+\beta-n}} d\sigma(x) d\sigma(y). \qquad (27)$$

This is a Gagliardo semi-norm (logical to use, on a metric space)
See Astérisque for the generalization to mixed dimensions.
The numerology seems strange, but at least when $n = d + 1$ and $\beta = 0$ (the classic case) we recover $d + 1$ which corresponds to the usual $H^{1/2}(\partial\Omega)$.
We even know H is the right space, because there is a bounded extension operator $\mathrm{Ext} : H \to W$ such that $\mathrm{Tr} \circ \mathrm{Ext} = I$.

Definition of weak solutions

Time to say what is a weak solution of $Lu = 0$ in Ω: it is a function $u \in W$ such that

$$a(u, \varphi) := \int_\Omega \langle A\nabla u, \nabla \varphi \rangle = 0 \tag{28}$$

for every $\varphi \in C_c^\infty(\Omega)$. Well defined because $u \in W$ and $A \in L^\infty$.
This is the local condition in Ω. We would get this if everything was smooth, by an integration by part:

$$\int_\Omega (divA\nabla u)\varphi = -\int_\Omega \langle A\nabla u, \nabla \varphi \rangle + \text{ vanishing boundary terms.}$$

Then we can add conditions at the boundary, like $\mathrm{Tr}\,(u) = f$ at the boundary (or $Rob(u) = 0$, see later).
General idea: this is easy to define, then weak solutions exist and have some regularity (if L is elliptic). And then they are strong solutions, at least when $L = -\Delta$.

Existence by Lax-Milgram

Assume 1-NTA and $\partial\Omega \in AR(d)$ with $d > n-2$, for instance.

Theorem

For $g \in H$, there is a unique weak solution $u \in W$ of $Lu = 0$ such that $\mathrm{Tr}\,(u) = g$.

Proof: consider the following Lax-Milgram situation.
Set $W_0 = \{u \in W \,;\, \mathrm{Tr}\,(u) = 0\}$. This is a real Hilbert space, because $\mathrm{Tr}\,: W \to H$ is continuous, and there is a single $u \in W$ with $||u||_W = 0$ and $\mathrm{Tr}\,(u) = 0$.
Set $a(u, v) := \int_\Omega \langle A\nabla u, \nabla v\rangle$. This is a continuous bilinear forem on W_0, and it is even **accretive** because

$$a(u, u) = \int_\Omega \langle A\nabla u, \nabla u\rangle \geq c \int_\Omega |\nabla u|^2$$

for $u \in W_0$, by (17). We also need a linear form on W_0, so we take $G \in W$ such that $\mathrm{Tr}\,(G) = g$, and set $L(\varphi) = a(G, \varphi)$ for $\varphi \in W_0$.

Existence by Lax-Milgram (2) and variational definition

Lax-Milgram says that we can find $\omega \in W_0$ such that

$$L(\varphi) = a(w, \varphi) \text{ for all } \varphi \in W$$

That is, $a(G, \varphi) = a(w, \varphi)$, for al φ, i.e., $u = G - w$ is a weak solution of $Lu = 0$, and its trace is $g = \mathrm{Tr}\,(G)$, as needed. $\square$

Comment: So easy! But then we need to deal with weak solutions.

Comment: When $A(X)$ is a symmetric matrix for a.e. $X \in \Omega$, then u is also the unique minimizer in W of the energy

$$E(u) = a(u, u) = \int_\Omega \langle A\nabla u, \nabla u\rangle \tag{29}$$

with the constraint that $\mathrm{Tr}\,(u) = g$.

Hint: First, we can write $u = G + w$ and then

$$E(u) = E(G + w) = E(w) + 2\int_\Omega \langle A\nabla G, \nabla w\rangle + C$$

so we minimize a strictly convex function of v, on a vector space. Easy to see that $E(u) \gg E(G)$ when w has a large norm, so minimizers u_0 exist. And then the Lagrange equation (try $u_0 + \lambda\varphi$, $\varphi \in \mathcal{C}_c^\infty(\Omega)$) exactly says that $Lu_0 = 0$. Hence $u = u_0$.

Hölder Regularity of weak solutions (1/3)

We did not define $\omega^X_{L,\Omega}$ yet. There is a long but classical road, and we'll skip details. We have all the ingredients (with Poincaré, the stronger version in L^q, the trace and the extension). Here are key words:

- Caccioppoli estimates (we'll see some): local improvement of the regularity in smaller balls
- Moser estimates, which use the above, and embeddings, to prove local improvements of L^p bounds, all the way to L^∞. So $u \in L^\infty_{loc}$.
- Similar bounds for balls centered on $\partial\Omega$.
- Oscillation decay: if u is a solution in $B = \Omega \cap B(x,r)$, with $u = 0$ on $\partial\Omega \cap B(x,r)$, and setting $osc(u,B) = \sup_B u - \inf_B(u)$, we get that $osc(u, \Omega \cap B(x,r/2) \leq (1-\eta) osc(u,B)$. This leads to
- the comparizon principle (also called Harnack up to the boundary): if u, v are two positive solutions of $Lu = Lv = 0$ in $\Omega \cap B(x,r)$, $x \in \partial\Omega$, with $u(\xi) = v(\xi)$ at a corkscrew point for B, then
$$C^{-1} \leq \frac{u}{v} \leq C \text{ on } \Omega \cap B(x,r/2).$$

Caccioppoli (1)

More on the Caccioppoli inequality because it is an important tool. Suppose $u \geq 0$ is a solution in $\Omega \cap B(x,(1+\eta)r)$, and $u = 0$ on $\partial\Omega \cap B(x,(1+\eta)r)$. Then

$$\int_{\Omega \cap B(x,r)} |\nabla u|^2 \leq C\eta^{-2} r^{-2} \int_{\Omega \cap B(x,(1+\eta)r)} u^2 \tag{30}$$

Idea of variational proof (in the symmetric case). We try the competitor $v = \chi u$, where χ is a cut-off function that vanishes in $B(x,r)$. We save $\int_{\Omega \cap B(x,r)} \langle \nabla u, \nabla u \rangle \sim \int_{\Omega \cap B(x,r)} |\nabla u|^2$, and we lose something coming from $\nabla \chi$ in the annulus.
Well, we can also try $u + t(v-u)$, t small, and still get some information. And then the information is the same as with the definition of a weak solution. Hence the proof below:

Caccioppoli (2)

Let u be as in the statement. Set $I = \int_\Omega \chi^2 |\nabla u|^2$. Notice that

$$I \leq C \int_\Omega \chi^2 \langle A\nabla u, \nabla u\rangle \tag{31}$$

by ellipticity. Test (28) against the function $\varphi = u\chi^2$, where now $\chi = 1$ in $B(x, r)$ and $\chi = 0$ in $\mathbb{R}^n \setminus B(x, (1+\eta)r)$.
Thus $\nabla\varphi = \chi^2 \nabla u + 2\chi u \nabla\chi$, and (28) becomes

$$\int_\Omega \langle A\nabla u, \chi^2 \nabla u\rangle + 2\int_\Omega \langle A\nabla u, \chi u \nabla\chi\rangle = \int_\Omega \langle A\nabla u, \nabla\varphi\rangle = 0. \tag{32}$$

So

$$I \leq -2C \int_\Omega \langle A\nabla u, \chi u \nabla\chi\rangle \leq 2C \int_\Omega |A|\, |\nabla u| \chi u |\nabla\chi|. \tag{33}$$

By Cauchy-Schwarz and the boundedness of A,

$$I \leq C\Big\{\int_\Omega \chi^2 |\nabla u|^2\Big\}^{1/2} \Big\{\int_\Omega u^2 |\nabla\chi|^2\Big\}^{1/2} \leq C I^{1/2} \int_{B(x,(1+\eta)r} u^2 |\nabla\chi|^2 \tag{34}$$

and (30) follows because $|\nabla\chi| \leq C(\eta r)^{-1}$.

Harmonic measure is finally defined

At this stage, we have weak solutions, and it is possible to prove (with the ingredients above) that, say, if Ω is 1-NTA and $\partial\Omega \in AR(d)$, $d > n-2$, the solution of $Lu = 0$ with $\mathrm{Tr}\,(u) = f$ is Hölder-continuous on $\overline{\Omega}$, as soon as f is.

By this (and the maximum principle), $Lu = 0$ & $\mathrm{Tr}\,(u) = f$ also has a continuous solution when f is continuous on $\partial\Omega$ (recall Ω is bounded). This goes further than just for $f \in H$.

Then we define $\omega^X = \omega^X_{L,\Omega}$ as above: $u_f(X) = \int_{\partial\Omega} f(\xi) d\omega^X(\xi)$.

Here ω^X is doubling ($\omega^X(B(x,2r) \leq C\omega^X(B(x,r)$ for all x, r, X), but maybe it is not A_∞. For instance, when $L = -\Delta$ but $\partial\Omega$ is a regular Cantor set, or $\Omega = \mathbb{R}^n_+$ but A is a "bad" elliptic matrix. Classical counterexamples of Modica-Mortola 80 and Caffarelli-Fabes-Kenig 81.
So A_∞ holds only for special combinations of Ω and L. As in Dahlberg 77, or Fefferman-Kenig-Pipher 91. The two facts are related (changes of variable).

Techniques for proving A_∞

For Dahlberg 77, If I recall, Maximal functions and integrations by parts. And the vertical direction is important.
To get to chord-arc surfaces (i.e., $\partial\Omega \in AR(n-1)$, 1-NTA, and corkscrew in the complement), but $L = -\Delta$ use either "big pieces of Lipschitz graphs" and the maximum principle (D-Jerison) or "Corona decompositions" (Semmes).
For general uniformly rectifiable boundaries (see later) with $\partial\Omega \in AR(n-1)$ and 1-NTA, more geometry and corona decompositions (Hoffman-Martell).
For more general operators, Carleson conditions on the oscillation of coefficients (DKP), square functions, and more harmonic analysis. Lipschitz domains (and a bit more) can also be reduced to the half space by good parameterizations, with Carleson estimates on their distortion.
Approximation results (Poggi, Cao, Hidalgo-Palencia, Martell, ...): two operators whose coefficients are close enough (in terms of Carleson measures) behave the same way on Ω.

Token for more (1/5)

Comment Post-lectures : token means that at some point I thought I could speak about this if I had too much time. This rarely happenned.

Concerning rectifiability: Dahlberg's example is Lipschitz graphs.

Then we finally have the case of 1-NTA domains with uniformly rectifiable Ahlfors-regular boundaries if co-dimension 1.
Def of UR by "big pieces of bilipschitz images of $\mathbb{R}^d$".
Example: bilipschitz images of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$.
(Unbounded) chord-arc curves in $\mathbb{R}^2$.

Big pieces and harmonic (not elliptic for a long time) measure.

The $\beta(x, r)$ numbers of P. Jones and Carleson measure estimates.
Existence for Dirichlet in L^p spaces. Lipschitz graphs (B2) vs chord-arc curves (only A_∞ by Jerison–Jones-Zinsmeister). Why?

Token for more (2): Carleson measures and Dahlberg-Kenig-Pipher conditions

Say $E = \partial\Omega \in AR(d)$. A Carleson measure μ on Ω is ... such that

$$\mu(\Omega \cap B(x,r)) \leq Cr^d \quad \text{for } x \in \partial\Omega \text{ and } 0 < r < \mathrm{diam}(\partial\Omega). \tag{35}$$

A Carleson measure μ on $E \times [0, \mathrm{diam}(E))$ (the set of balls centered on E) is defined by

$$\mu(B(x,r) \times (0,r)) \leq Cr^d \quad \text{for } x \in \partial\Omega \text{ and } 0 < r < \mathrm{diam}(\partial\Omega).$$

For our DKP conditions on a matrix A, we compute a distance $\gamma(x,r)$ between A and constant matrices in a Whitney box

$$W(x,r) = \left\{X \in \Omega \cap B(x,r)\,;\, \delta(X) \geq C^{-1}r\right\},$$

and then we require that

$$\mu = \gamma(x,r)^2 d\sigma(x)\frac{dr}{r} \quad \text{be a Carleson measure} \tag{36}$$

(comment on the infinite "invariant" measure $d\sigma(x)\frac{dr}{r}$).

Dahlberg-Kenig-Pipher conditions...

Classical choices of γ:

$$\gamma_s(x,r) = \inf_{A_0(x,r)} \sup_{X\in W(x,r)} |A(X) - A_0| \tag{37}$$

with a supremum taken over constant elliptic matrices $A_0 = A_0(x,r)$. Or even the stronger

$$\gamma_{vs}(x,r) = \delta(X) \sup_{X\in W(x,r)} |\nabla A(X)|. \tag{38}$$

And I personally like the weak

$$\gamma_{vs}(x,r) = \inf_{A_0(x,r)} \left\{ \fint_{W(x,r)} |A(X) - A_0|^2 \right\}^{1/2} \tag{39}$$

Choice related to dependence on parameters. Controls the same thing on tents. [DLM] Arch. Rational Mech. Anal.

Good parameterizations of Lipschitz graphs and the DKP condition

CASSC, big pieces of (flat) Lipschitz graphs, and weak DKP coefficients.

Higher codimensions

Started with S. Mayboroda, mostly for fun.
Can we extend the classical codimension results ($\partial\Omega \in AR(n-1)$) to larger co-dimensions ($\partial\Omega \in AR(d)$ with any $d < n-1$)?
One way we found: use $L = -\mathrm{div} A \nabla$, but with $A(X) = \delta(X)^{-\beta}\widetilde{A}(X)$, with $\widetilde{A}$ bounded elliptic and $n-d-2 < \beta < n-d$ (central value at $n-d-1$, $\gamma = 0$ in the classical case).
The method above, with the weighted Sobolev space $W = \left\{ f \in L^1_{loc}(\Omega)\,;\, \int_\Omega |\nabla f|^2 \delta(X)^{-\beta} < +\infty \right\}$, gives Hölder continuous, weak solutions, and doubling elliptic measures ω_L^X.

Next [D.-Mayboroda-Feneuil] there is a (class of) smoother operators, like the Laplacian when $n-2 < d < 1$, and for which $\omega_L^X \in A_\infty$ as soon as $\partial\Omega$ is uniformly rectifiable.

Other devlopments with Engelstein, Li, ...

Token for more higher co-dimensions

Our class of “degenerate” operators when $d < n-1$:
$L = -\mathrm{div} A\nabla$, where $\delta(X)^{n-d-1+\tau} A(X)$ is elliptic (any $\tau \in (0,1)$ works).

Our replacement for Δ: $L_\beta = -\mathrm{div} D_\beta^{-(n-d-1)}\nabla$, with

$$D = D_\beta(X) = \left\{ \int_{\partial\Omega} |y - X|^{-(d+\beta)} d\sigma(y) \right\}^{-1/\beta}, \qquad (40)$$

with any $\sigma \in AR(d)$ on $\partial\Omega$ and any $\beta > 0$.
Turns out to give A_∞ elliptic measure when $\partial\Omega \in UR$.

Special (magic) case when $d + \beta = n - 2$ (possible when $d < n-2$). We claim that $L(D_\beta) = 0$ (and hence, D_β turns out to be the Green function for L on Ω.
Indeed, $V = D^{-\beta}$ is harmonic on Ω and $\nabla D = \nabla(V^{-1/\beta}) = (-1/\beta)V^{-1-1/\beta}\nabla V = -cD^{\beta(1+1/\beta)}\nabla V = -cD^{\beta+1}\nabla V$. So $L(D_\beta) = -\mathrm{div} D_\beta^{-(n-d-1)}\nabla D_\beta = \mathrm{div} D_\beta^0 \nabla V = c\Delta V = 0$ because $-(n-d-1)+\beta+1 = -n+d+2+\beta = 0$.

The Green function

Take any Ω and L as above. For $X \in \Omega$, there is a function $G^X = G^X_{\Omega,L}$, whose main properties are:

- G^X is a nonnegative solution of $Lu = 0$ in $\Omega \setminus \{X\}$, with
- $\mathrm{Tr}\,(G^X) = 0$ on $\partial\Omega$, and
- Some normalization near X (formally, $Lu = \delta_X$, a Dirac mass).

Traditionally, used to recover solutions of $Lu = f$ by integrating against $f(X)dX$. But, thanks to the comparison principle, G^X gives the local boundary behavior of any nonnegative solution with a vanishing trace!

And we found out (after Azzam) that A_∞ is related to the good approximation of the G^X by multiples of the distance to $\partial\Omega$.

Often we like better G^∞, some limit of G^X when X goes to ∞.

But of course hard to compute, except in some very simple examples (ball, half space) or when the example is constructed from G^X.

The Green function is our friend (1)

Two or three examples to convince you that we can use the Green function G.

Usually G is impossible to compute. Except if we decide about G first and construct everything around!

Baby example: Find x-independent elliptic operators L on $H = \{(x,t)\,;\, t > 0\}$ such that $f(t)$ is the Green function with pole at ∞. Here $f \geq 0$ is given, with $f(0) = 0$.
Take $A = A(t)I$, and $Lu = 0$ means that $\mathrm{div} A(t)\frac{\partial f}{dt} e_n = 0$, i.e., $A'(t)f'(t) + A(t)f''(t) = 0$. Easy to check that $A(t) = \frac{C}{f'(t)}$ works, with elliptic coefficients as soon as $C^{-1} \leq f'(t) \leq C$.
This gives easy examples where G is not so close to affine functions (when A is not too close to I).

Green friend (2): Good Green function on the Cantor set

Here K is the Garnett-Ivanov Cantor set of dimension 1 in the plane, and $\Omega = \mathbb{R}^2 \setminus K$.

Theorem (D.-Mayboroda)

There is an elliptic operator $L = -\mathrm{div}\, a(X)\,\nabla$ on Ω such that $\omega_{\Omega,L}^{\infty} = \sigma$. Here $\sigma = c\mathcal{H}^1_{|K}$ is the natural probability measure on K.

A priori surprising because K was known to be bad for harmonic measure (but for Δ).

So $A = aI$ is isotropic but it cannot be close to I or constant coefficients in the DKP sense.

I think isotropy is nice; $L = -\mathrm{div}\, a(X)\,\nabla$ should be connected to the geometry of $\mathbb{R}^2$ with the distance

$$\mathrm{dist}_w(X,Y) = \inf_{\Gamma \text{ from } X \text{ to } Y} \int_\Gamma w(x)^{1/2} d\mathcal{H}^1(x).$$

coming from a weight.

Green friend (3): proof on the GI Cantor set

We construct G by pictures: we draw the level sets of $G = G^\infty$ and its conjugate function R (in red).
The red curves are orthogonal to the green ones, and are also the gradient lines of G.
Finding G and R amount to labeling these curves.

By algebra, we can reconstruct the equation from G and R, assuming their level sets are orthogonal: we can take

$$a(X) = \frac{|\nabla R(X)|}{|\nabla G(X)|} \tag{41}$$

So A is elliptic as soon as $C^{-1} \leq \frac{|\nabla R|}{|\nabla G|} \leq C$, which can also be seen from the level lines, and here will be ensured by the self-similarity of the construction.

Green friend (4): proof on the GI Cantor set (2)

Also we'll make sure that

$$C^{-1}\,\mathrm{dist}(X,K) \le G^{\infty}(X) \le C\,\mathrm{dist}(X,K) \qquad (42)$$

This is enough to show that $\omega_L^{\infty} \sim \sigma$: Think of the smooth case where the density of ω is $\frac{\partial G}{\partial n}$. But in fact we get equality as announced.

In the next slides we show the construction by pictures. We need to glue the level lines correctly to make sure the level lines are well distributed and G satisfies the equation. The fractal invariance will do the rest.

A fundamental domain

We cut $\mathbb{R}^2 \setminus K$ into annular regions.
The fundamental region (in grey) is the A_0 bounded by the exterior circle ∂B_0 and the four small green circles.
"Enough" to construct G and R in A_0, and then, by symmetry, in the smaller A_{00} (one eighth of A_0).

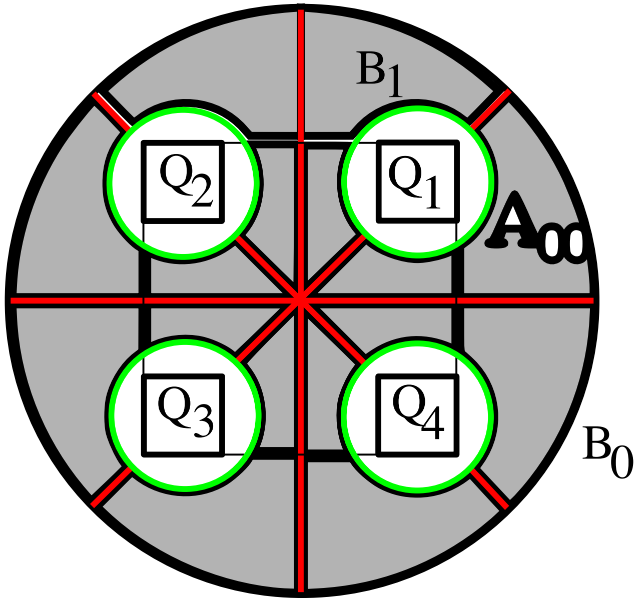


Figure: The cubes Q_j of generation 1, the balls B_0 (large) and B_1 (small), the annulus A_0 (in grey) and a fundamental piece A_{00} (one eighth of A_0)

We draw the red and green curves in A_0

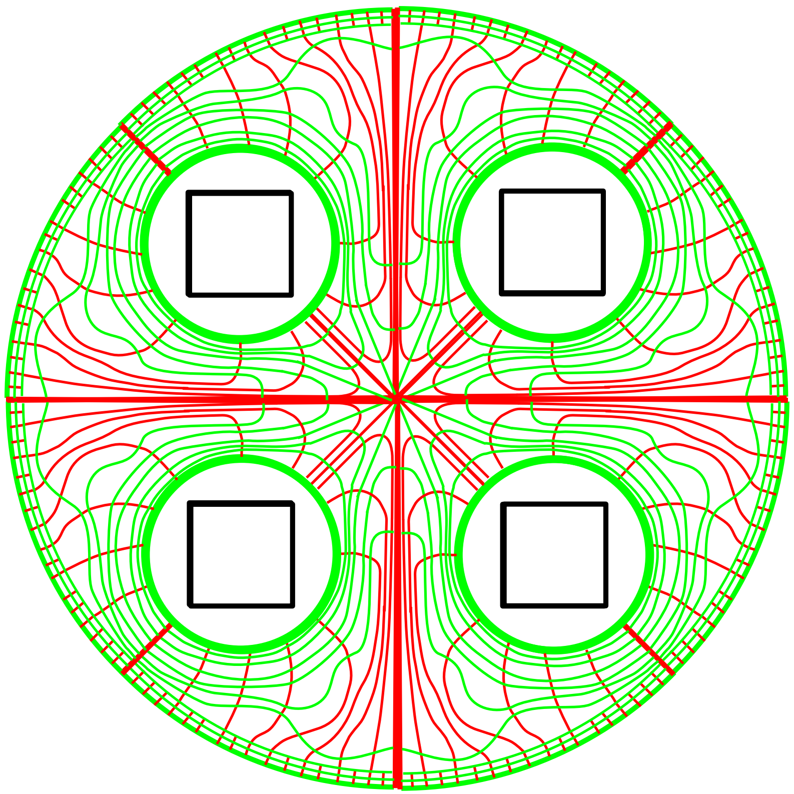


Figure: The level and gradient lines of G in A_0. Mind the symmetry. Also, it is fair that the green curves surround K (recall that $G = 0$ on K) and the red curves go towards K.
Finally the critical point at the center is forced by the geometry, but there G behaves like $Re(z^4)$ which is harmonic.

We prepare for gluing

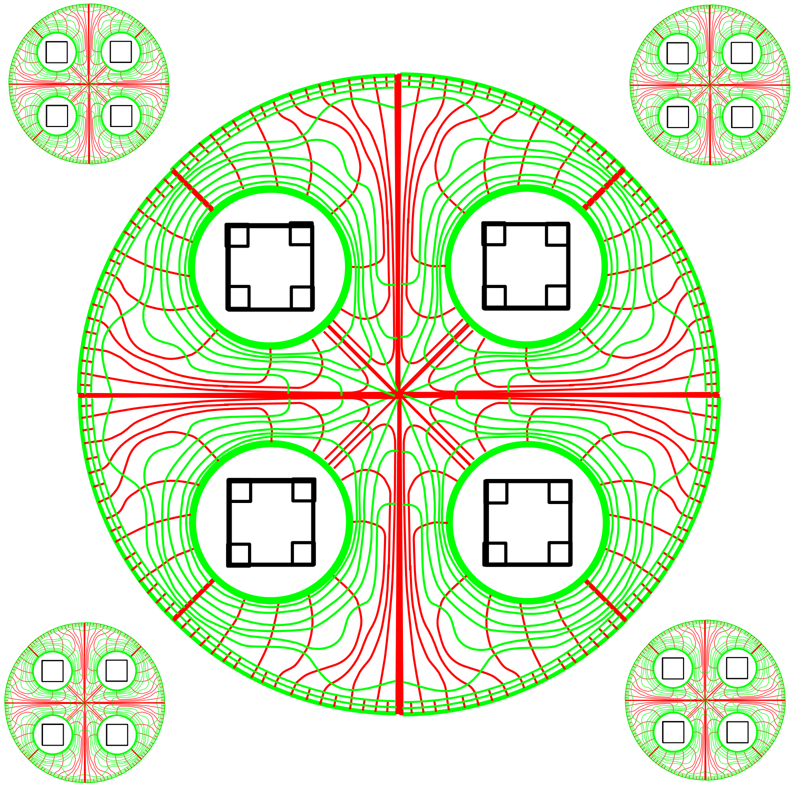


Figure: We prepare four, 4 times smaller copies of the same picture, to be put in the main holes

We glue the next generation

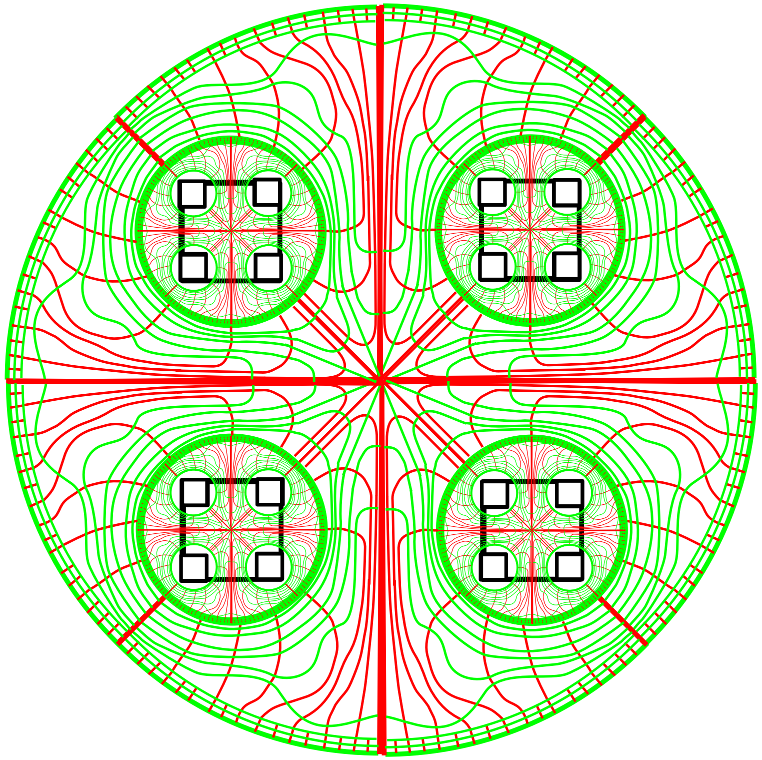


Figure: The level and gradient lines of G on a larger region than A_0, completed by self-similarity

... And so on. The fractal construction allows uniform estimate. An important additional constraint to get uniform bounds on $a(x) = |\nabla R|/|\nabla G|$ is that the end of red curve that starts along the first large green circle runs along the four smaller green circles at constant speed.

Some other examples

There is some limited flexibility in the example above (rotations, different dimensions), but not much.

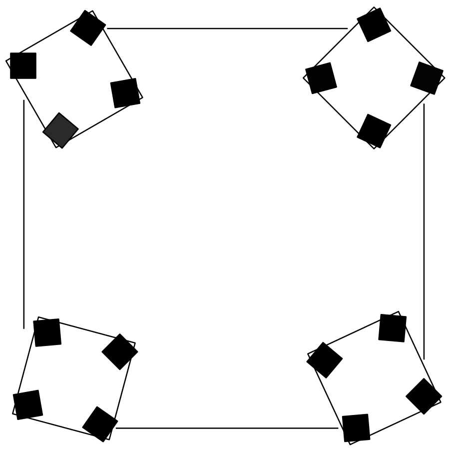


Figure: The third iteration of a rotating version of the Cantor set

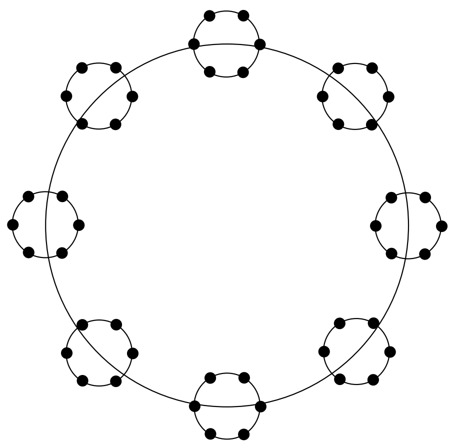


Figure: The third iteration of a variable scale/multiplicity analogue of K

Elliptic measure on snowflakes (1) [P. Perstneva]

This time Ω is one of the components of the complement of a snowflake Γ in the plane, as below or closer to a line.

Theorem (P. Perstneva)

There is an (isotropic) elliptic operator $L = -\operatorname{div} a(X)\nabla$ on Ω such that $C^{-1}\sigma \le \omega_{\Omega,L}^{\infty} \le C\sigma$.
Here $\sigma = c\mathcal{H}^s_{|\Gamma}$ is the natural Hausdorff measure on Γ, snowflake of dimension $s > 1$.

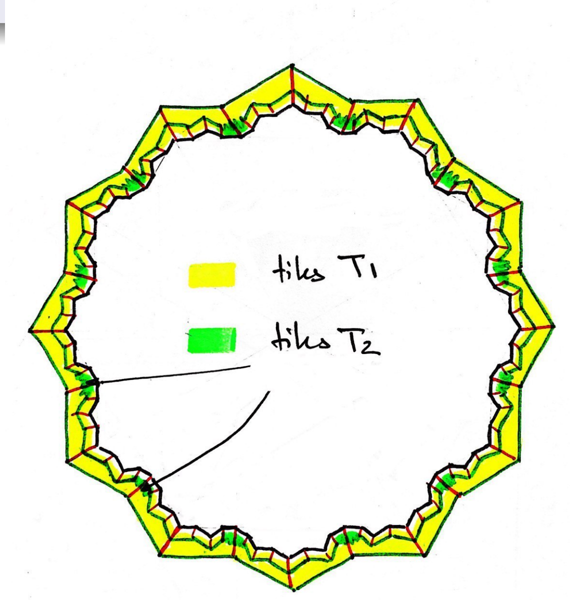


Figure: The snowflake Γ can be seen at the center; the rest is the beginning of a covering of Ω by tiles.

Elliptic measure on snowflakes (2)

As before she constructs a Green function.
To ensure that the level lines of G and R stay at comparable distances, she also uses selfsimilarity, and the first step is to cover Ω by similar tiles, of only 2 or 3 types. Here are partial coverings:

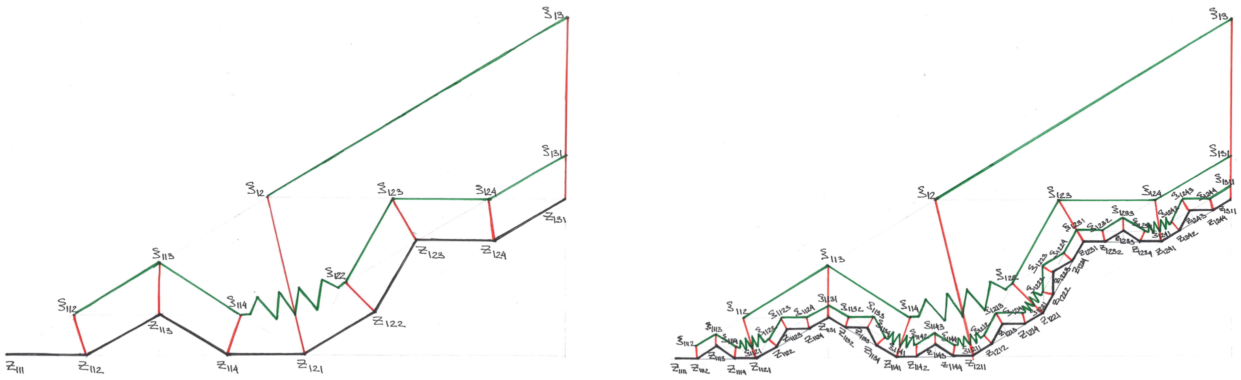
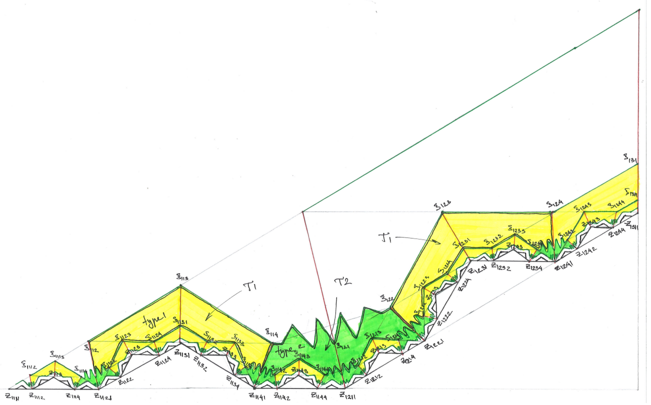


Figure: Two local pictures of tilings above, and the last one below, with colored tiles.



Elliptic measure on snowflakes (3)

And now Green and Red level lines inside the tiles, that will connect efficiently.

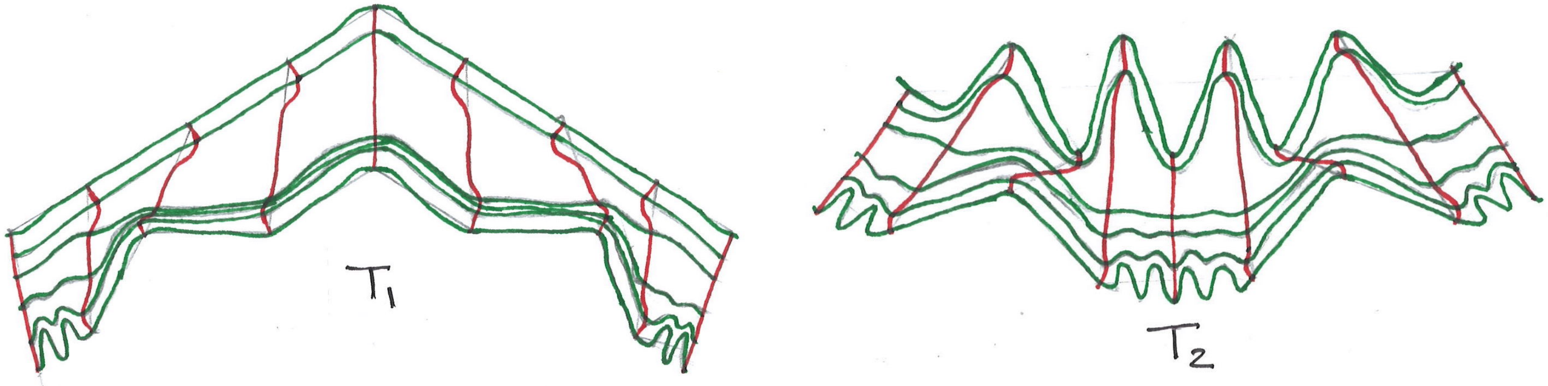


Figure: Level sets in the two types of tiles.

Left-Right symmetry, and constant speed of arrival for the red curves, are important for global control.
A few different models of snowflakes are possible, but for the moment, not yet general Reifenberg-flat curves because symmetry is used.
Higher dimensional $\mathbb{R}^n$ would be interesting, but...

More Green: A.C. Harmonic measure on Cantor sets in $\mathbb{R}^2$

Finally a counterexample where the harmonic measure (for Δ) on a Cantor set in the plane is proportional to $\mathcal{H}^d$. Here K is a (non self-similar) Cantor set in the plane, and $\Omega = \mathbb{R}^2 \setminus K$.

Theorem (G. D. - C. Jeznach - A. Julia)

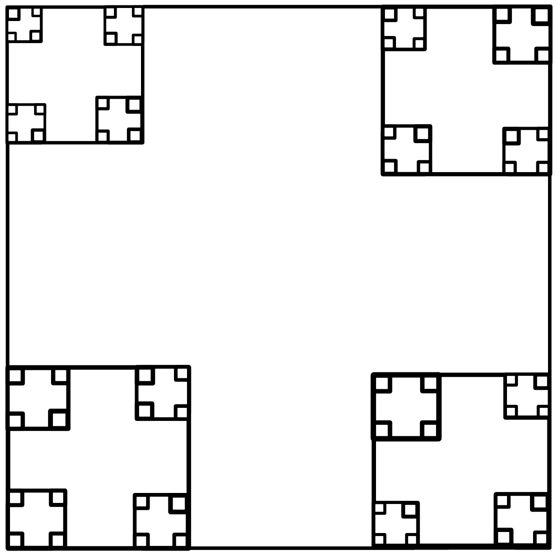
For $0 < d < 0.4$, there is an Ahlfors regular d-dimensional Cantor set K in $\mathbb{R}^2$ such that

$$C^{-1}H^d(E) \leq \omega_{\Delta,\Omega}(E) \leq CH^d(E) \quad \text{for } E \subset K,$$

where ω_Δ is harmonic measure on K associated to the Laplacian.

Most people (A. Volberg) expected the opposite.
Impossible for $d = 1$ ($K = \partial\Omega$ should be rectifiable!)
Again made possible thanks to a friendly Green function.

The asymmetric Cantor set of the DJJ Theorem



A picture of the Cantor set in the DJJ Theorem,
with 3 generations, exagerate differences, and d larger than real.

The size of the various squares is adjusted along the proof, so that G looks like what we want.

Ideas for the proof of DJJ (1)

How to control the Green function G^∞ on the complement of our Cantor set $K \subset \mathbb{R}^2$ of small dimension (constructed on purpose)?

Usual definition for K: For $n \geq 0$, construct K_n, composed of 4^n squares $Q_j = Q_j^n$, $j \in J(n)$ of size r^n, and take the limit.
Here r is small because $r^d \leq 1/4$.
One way to describe the self-similar set K_0 of dimension d is by nested squares (as above), or by a parameterization $F_0 : E = 4^{\mathbb{N}} \to \mathbb{R}^2$. Choose four points e_1, e_2, e_3, e_4 of $\partial B(0,1)$, on the diagonal, and for $\varepsilon = (\varepsilon_k)_{k\in\mathbb{N}}$, set

$$F_0(\varepsilon) = \sum_k r^k e_{\varepsilon_k}.$$

Now for K we take

$$F(\varepsilon) = \sum_k r^k \lambda_k(\varepsilon) e_{\varepsilon_k},$$

with $\lambda_k(\varepsilon) \in [1,2]$ that depends only on $\varepsilon_0, \ldots \varepsilon_{k-1}$.

Ideas for the proof (2)

That is, when we construct the 4 children of the square Q of generation k, we place the next cubes at distance $\lambda_Q r^d$ from the center x_Q.

$$F(\varepsilon) = \sum_k r^k \lambda_k(\varepsilon) e_{\varepsilon_k},$$

Easy to check: $K = F(4^{\mathbb{N}})$ is a bi-lipschitz image of K_0, and Ahlfors regular of dimension d.

We have a natural measure μ_k on K_n, such that $\mu_n(Q) = 4^{-n}$ for each cube Q of generation n.
And the natural limit μ of the μ_n on K.
Here is a natural harmonic function g: set, for $z \in \mathbb{R}^2 \setminus K$,

$$g(z) = \mu_n * \ln(|\cdot|)(z) = \int_K \ln(|z-x|) d\mu(x).$$

Ideas for the proof (3)

$$g(z) = \mu_n * \ln(|\cdot|)(z) = \int_K \ln(|z-x|)d\mu(x).$$

The integral converges because μ is Ahlfors regular.
At ∞, $g(z) \sim \ln(|z|)$, which not bad.
It would be great if we had $g(z) = 0$ on K, but of course this won't happen.
It will be equally good if g is a constant c on K, because then $G = g - c$ is the Green function! so that is what we aim for, if K is chosen well.

Missing piece, which I won't do: check that

$$g(z) - c \simeq \mathrm{dist}(z, K)^d,$$

and then conclude using the relation between $G(z)$ at a corkscrew point and harmonic measure of the corresponding disk.

Ideas for the proof (4): we discretize

Call $\mathcal{Q}_n$ the set of cubes of generation n, and x_Q the center of $Q \in \mathcal{Q}_n$. Then set

$$g_n(z) = \mu_n * \ln(|\cdot|)(z) = 4^{-n} \sum_{Q \in \mathcal{Q}_n} \ln(|z - x_Q|). \tag{43}$$

We want to arrange things so that g_n is almost constant on K_n (or the union of the circles centered on the x_Q and radius r^n, say). Something like

$$\sup_{K_n} g_n(z) - \inf_{K_n} g_n(z) \leq C4^{-n} \tag{44}$$

The main question: assuming (44) at generation n, how do we arrange (44) at generation $n+1$.

That is, how do we choose the λ_Q to make the oscillation of g_{n+1} smaller?

Ideas for the proof (5)

Take d and r very small. This way, g_n is roughly constant near each cube $Q \in \mathcal{Q}_n$, and the differences between cubes Q is not large.
Call Q_j the four children of Q.
Then write $g_{n+1}(z) - g_n(z)$ for points z near Q.
There are a few terms, that are not small, but are roughly the same on all the ∂Q_j across Q, so we we don't care, they feed the constant.
And the main term is something like

$$\begin{aligned} \delta_n(z) &= 4^{-n-1} \sum_j \left[\ln(|z - x_{Q_j}|) - \ln(|z - x_Q|) \right] \\ &= 4^{-n-1} \sum_j \ln\left(\frac{|z - x_{Q_j}|}{|z - x_Q|}. \right) \end{aligned} \tag{45}$$

Ideas for the proof (6)

$$\delta_n(z) = 4^{-n-1} \sum_j \ln\left(\frac{|z - x_{Q_j}|}{|z - x_Q|}\right)$$

For z in a circle of fixed small radius around x_{Q_j}, $|z - x_{Q_j}|$ is always the same (across the whole set), while

$$|z - x_Q| \simeq |x_{Q_i} - x_Q| = c\lambda_Q r^n.$$

That's it. We are adding essentially equal terms, minus $4^{-n-1}\ln(\lambda_Q)$.
If g_n was larger than average near Q, we take λ_Q small. Otherwise, we take λ_Q larger. This allows us to add a varying constant of size $4^{-n-1}\ln(\lambda_Q)$ to g_n near Q, which turns out to be enough to compensate variations of the averages of g_n among the Q.

Last comments about the proof

Taking r and d small simplifies the proof: the scales are more and more independent, and the extra errors are smaller. Then we sort of optimized.

The effect of increasing the distances $|x_{Q_j} - x_Q| \sim \lambda_Q r^n$ is to increase the chance that a Brownian path that passes nearby will land on the Q_ℓ^{n+1}. But we are lucky that we din't need to evaluate the absorption probabilities and we can sum potentials instead.

Again, once we know that $g \equiv c$ on K, we can estimate the Green function $G^\infty = g - c$, and then use G to estimate ω.
For instance we can estimate $\nabla g = \mu * \frac{1}{z}$ near K.

We could do other shapes (for instance, $K \subset \mathbb{R}$), but squares seem to be nice.

II. The Robin boundary problem – first the strong definition

To simplify (and avoid talking about the conormal derivative), we take $L = -\Delta$. And also Ω bounded, with 1-NTA.
In the smooth case of codimension 1, the Robin problem is

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ Rob_a(u) := \dfrac{1}{a}\dfrac{\partial u}{\partial n} + \mathrm{Tr}\,(u) = f & \text{on } \partial\Omega. \end{cases} \tag{46}$$

Here $\frac{\partial u}{\partial n}$ is the outwards normal derivative and $a > 0$ is a constant.
$Tr(u)$ is the trace on $\partial\Omega$.
Often we work locally with u given on $\Omega \setminus B(x, r)$ and $f = 0$.

$a = 0$ corresponds to Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$;
$a = +\infty$ corresponds to the Dirichlet boundary conditions above.

Comment: we chose the outwards normal, so that typically $u > 0$ on $\partial\Omega$ and u is larger on Ω than on $\partial\Omega$.

Trivial examples, with $f = 0$: $u(x, t) = t + \frac{1}{a}$ on $\mathbb{R}^2_+$;
$u(x) = \frac{1}{a} - \log(|x|)$ on $\Omega = B(0, 1) \subset \mathbb{R}^2$.

Comment: $L = \mathrm{div} A\nabla$, A bounded elliptic, is allowed too.

Why Robin?

A natural condition. Introduced by Fourier and studied by Robin.

• Example of the temperature u in a room Ω. It is harmonic inside (if equilibrium);
- Dirichlet, $u = f$ on $\partial\Omega$ corresponds to prescribing the temperature on the walls.
- Neumann, $\frac{\partial u}{\partial n} = 0$, corresponds to perfect insulation (put a source for the Green function).
And in real life, there is some transmission, Robin corresponds to a transmission proportional to the temperature.

• Example of the deep lung, u is the concentration of oxygen: imagine mostly diffusion (less convection), and a describes the (small) absorption rate of oxygen along the walls.
Why does a fractal shape for the lungs really helps (a small)?

• Variants with a potential fractal boundary: catalysis, electrodes.

Lungs are fractal

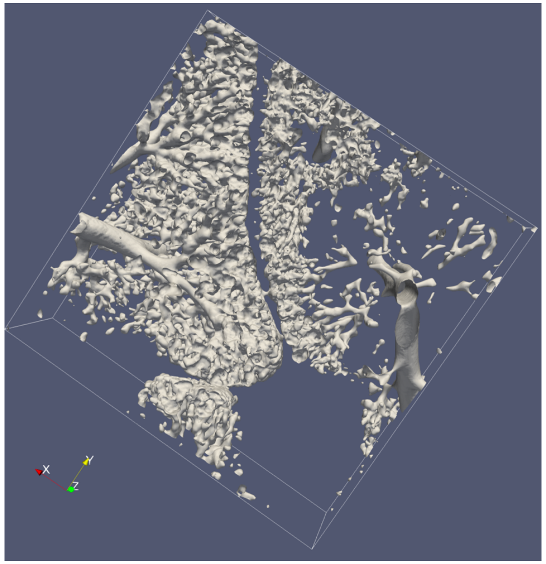
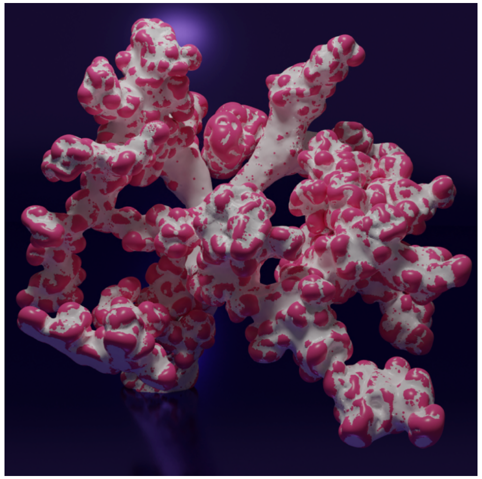
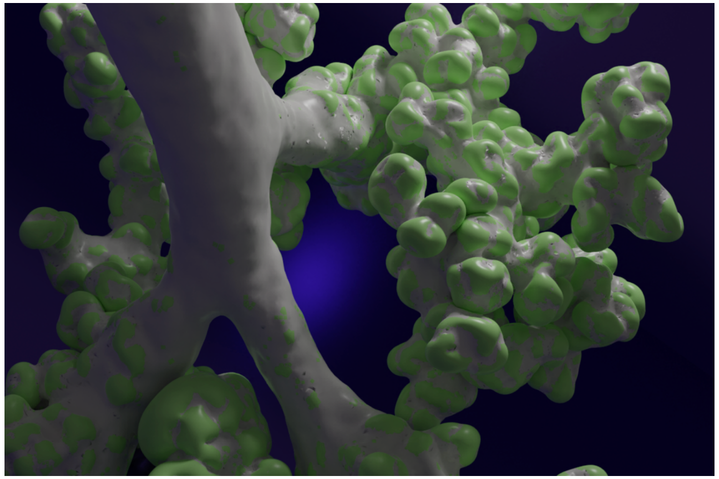


Figure: Pictures of rat lungs by tomography performed at the Grenoble Syncrotron. Credits: S. Bayat, H. Leclerc, S. Martin, B. Maury, B. Semin.





Robin harmonic measure

Assume for the moment that $\partial\Omega$ is sufficiently smooth of co-dimension 1 so that everything is well defined.
Take a pole $X \in \Omega$. Define the Robin harmonic measure ω^X_{Rob} by $\omega^X_{Rob}(E) = u_E(X)$, where for $E \subset \partial\Omega$, u_E solves (at least formally)

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ Rob_a(u) := \dfrac{1}{a}\dfrac{\partial u}{\partial n} + u = \mathbb{1}_E & \text{on } \partial\Omega. \end{cases} \tag{47}$$

Or ω^X_{Rob} is the probability measure on $\partial\Omega$ such that the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ Rob_a(u) = f & \text{on } \partial\Omega. \end{cases} \tag{48}$$

is given, for $f \in C(\partial\Omega)$, by

$$u(X) = \int_{\xi\in\partial\Omega} f(\xi)\, d\omega^X_{Rob}(\xi). \tag{49}$$

[Same as for the Dirichlet harmonic measure ω^X_{Dir} above, where we would require $u = f$ on $\partial\Omega$. We'll need to construct all this.]

Brownian interpretation

In the Dirichlet case, we think of $\omega^X_{Dir}(E)$ as the probability that a Brownian particle starting at X first exits Ω through a point of E.

For $\omega^X_{Rob}(E)$, think of a Brownian particle that starts from X, and each time it hits $\partial\Omega$, has a certain "probability" (small if a is small) of being absorbed. And if not we start it again from where it is, and continue playing until the particle is absorbed.

In fact, easier to define discretely, with random walks; otherwise one would try to use local time in $\partial\Omega$ of Brownian motion in $\overline{\Omega}$.

Main question for us:
• Define the Robin problem and $\omega^X_{Rob}(E)$ in a general enough context.
• find out where on $\partial\Omega$ is $\omega^X_{Rob}(E)$ supported and how regular it is. (is it A_∞?)

Comment: People like the Robin problem a lot, but apparently not the Robin harmonic measure, especially for irregular boundaries!

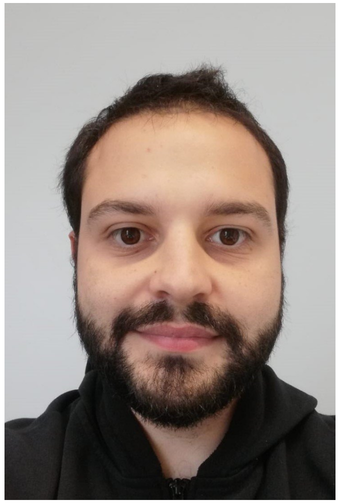
The team

Figure: Stefano Decio

Max Engelstein

Marcel Filoche

Figure: Svitlana Mayboroda



Marco Michetti

And thanks to Anna Rozanova-Pierrat (similar results, discussions) and Jill Pipher (help).

Comments before we start

Do we expect the regularity ω^X to depend on Ω as before? How?

Does the choice of the specific elliptic operator L matter?

Can Robin harmonic measure live on a set of smaller dimension than $\partial\Omega$?

Do we expect a phase transition when a changes?

As we'll see, the same basic tools as for Dirichlet are used (Poincaré, traces, extensions).
Having used boundaries of dimension $d \neq n-1$ helped us.
Specific new lemmas will be needed too.

Maybe later: back to the lungs

Definition of a weak solution

In what follows, we take Ω bounded and assume that $\partial\Omega$ comes equipped with a "natural" doubling measure σ.
For instance, $\partial\Omega \in AR(d)$ and $\sigma = \mathcal{H}^d_{|\partial\Omega}$; in general, we allow "mixed dimensions" with a doubling measure σ that behaves in a "more-than $(n-2)$-dimensional" way.
A (weak) solution to the Robin problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{1}{a}\dfrac{\partial u}{\partial n} + u = f & \text{on } \partial\Omega. \end{cases} \tag{50}$$

is a function $u \in W^{1,2}(\Omega) = \{u \in L^2(\Omega)\,;\, \nabla u \in L^2(\Omega)\}$ (we now use the usual Sobolev space) such that

$$\frac{1}{a}\int_\Omega \nabla u \cdot \nabla\varphi + \int_{\partial\Omega} Trace(u)\varphi d\sigma = \int_{\partial\Omega} f\varphi d\sigma \tag{51}$$

for all test functions $\varphi \in C^1_c(\mathbb{R}^n)$.
[and to get that $\Delta u = 0$ on Ω, we would just consider $\varphi \in C^1_c(\Omega)$.]

Comments on the definition of weak solution

$$\frac{1}{a}\int_\Omega \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} Trace(u)\varphi d\sigma = \int_{\partial\Omega} f\varphi d\sigma \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

First, restricting to $\varphi \in C_c^1(\mathbb{R}^n)$ gives that u is harmonic in Ω. Next, if everything is smooth (and σ is the surface measure), integrating by parts yields

$$\int_\Omega \nabla u \cdot \nabla \varphi = -\int_{\partial\Omega} \varphi \frac{\partial u}{\partial n} d\sigma \tag{52}$$

because $\Delta u = 0$. Whence we get $\int_{\partial\Omega}\left[\frac{1}{a}\frac{\partial u}{\partial n} + u\right]\varphi d\sigma = \int_{\partial\Omega} f\varphi d\sigma$ for all φ, which fits with (50).

Notice that $\frac{\partial u}{\partial n}$ makes no sense in general, but for u harmonic, the product $\frac{\partial u}{\partial n} d\sigma$ will make sense (weakly by Partial Integration).

Notice that $\mathrm{Tr}\,(u)$ is defined (and lies in $H \subset L^2(d\sigma)$), see above.

Weak solution exist

And now we follow the same route as for Dirichlet! Take (Ω, σ) as above, now with Ω bounded.

Theorem

For every $f \in L^2(\sigma)$, there is a unique weak solution to (51). *In addition $||u||_W \leq C||f||_{L^2(\sigma)}$.*

Ingredients: Poincaré, traces, extensions as above. Also, the fact that Ω is an extension domain for $W^{1,2}$ functions, because it is uniform [Jones].
Proof by Lax-Milgram. This time we use the accretive form

$$a(u, v) = \int_\Omega \langle A\nabla u, \nabla v\rangle + \int_{\partial\Omega} \mathrm{Tr}\,(u)\mathrm{Tr}\,(v)d\sigma.$$

And the bounded linear form $v \mapsto L(v) = \int_{\partial\Omega} \mathrm{Tr}\,(v)fd\sigma$. There is a unique $u \in W$ such that $a(u, v) = L(v)$ for all $v \in W$.
The theorem follows.

Weak solution are Hölder-continuous

Here again we prove versions of Caccioppoli (almost the same proof), Moser, oscillation decay (we need to prove it for Neumann too), comparison principle, and finally we get that...

Theorem

If $f \in L^2(\sigma)$ is Hölder continuous, then the weak solution to (51) is also Hölder continuous on the whole $\overline{\Omega}$.

Some proofs need to be adapted.
We want our constants not to depend on a.
Have to find proofs that work both for (vanishing) Dirichlet ($u = 0$ on $\partial\Omega$) and Neumann ($\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$).
Homogeneity is strange (see later).

Corollary

We can define the Robin harmonic measure ω^X_{Rob} as hinted before.

As before, this and the existence of a Green function follow from the above. And ω^X_{Rob} turns out to be doubling.

The "density lemma"

Here is an example of a new ingredient called density lemma, relative to oscillation. Variants (and the name) known before us.

Lemma

Suppose $0 \leq u \leq 1$ is a solution in $B(x,r) \cap \Omega$, with $Rob(u) = 0$ on $\partial\Omega \cap B(x,r)$. Then we have $0 \leq u \leq (1-\eta)$ on $\Omega \cap B(x,r/2)$ or $\eta \leq u \leq 1$ on $\Omega \cap B(x,r/2)$.

Here $\eta > 0$ does not depend on a or u (just on the geometric constants). For Dirichlet, we would have the first option, but for instance if $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, we need to leave the two options.

Rough idea of the proof. An interesting case is when $\frac{\partial u}{\partial n} = 0$ and $u \leq \frac{1}{2}$ on most of $B('x,r) \cap \Omega$, and we prove the second option. Idea when $A = I$ (otherwise adapt): u minimizes $\int_{\Omega \cap B(x,r)} |\nabla u|^2$ given its values on $\partial B(x,r)$. Try to truncate u by above, at various levels, do the accounting, and choose the best level by Chebyshev. To be checked (different proof).

Different scale invariance/Neumann condition

An important feature of the problem. Annoying at first but very useful to know: our problem is not scale invariant.

When u is a solution of $\mathrm{Rob_a(u)} = 0$ on $\partial\Omega$, i.e., when $\frac{1}{a}\frac{\partial u}{\partial n} + u = 0$, then the function v given by

$$v(x) = u(\lambda^{-1}x) \text{ solves } \mathrm{Rob_{\lambda^{-1}a}(v)} = 0 \text{ on } \partial(\lambda\Omega).$$

And/or, for physicists, the constant a scales like $\frac{1}{\mathrm{length}}$.

Our results will have to acknowledge this. Said in other words, at small scales we expect u to look more like a Neumann solution (small a), and at large scales like a Dirichlet solution (large a).

And, for symmetric operators at least, $\frac{\partial u}{\partial n} = 0$ just says that u minimizes the energy $\int_\Omega \langle A\nabla u, \nabla u\rangle$. Thus globally u would be constant (see later).

Locally it still sees the geometry, but not so much. The Brownian particles bounce all the time, but maybe their presence is not uniform?

A statement finally (1)

Recall our assumptions:

• Ω is bounded and 1-NTA (corkscrew points and Harnack chains)

• $\partial\Omega$ is the support of a doubling measure σ, which is "mixed of dimension $d > n-2$": there is $C \geq 1$ such that (as in (10))

$$\sigma(B(x,tr)) \leq Ct^d \sigma(B(x,r)) \text{ for } x \in E,\ 0 < r < \dim(E), \text{ and } 0 < t < 1. \tag{53}$$

This allows different behaviors at different scales.
Example: $\partial\Omega \in AR(d)$ and $\sigma = \mathcal{H}^d_{|\partial\Omega}$.

• $L = -\mathrm{div} A\nabla$ is elliptic (or stick to $L = -\Delta$).

Thus bad, unrectifiable sets, non-integer dimensions, and "bad" coefficients A are allowed.

A statement finally (2)

Theorem (DDEFMM)

With the assumptions above, ω_{Rob} is A_∞, with the precise linear A_∞ estimate for small radii: for $x \in \partial\Omega$, $0 < r < \mathrm{diam}(\Omega)$ such that

$$a\sigma(B(x,r))r^{2-n} \leq 1 \tag{54}$$

and $X \in \Omega \setminus B(x, 2r)$, then for all $E \subset \Delta = \partial\Omega \cap B(x,r)$,

$$C^{-1}\frac{\sigma(E)}{\sigma(\Delta)} \leq \frac{\omega^X_{Rob}(E)}{\omega^X_{Rob}(\Delta)} \leq C\frac{\sigma(E)}{\sigma(\Delta)}. \tag{55}$$

Good for small scales. For large scales, we do not get better than the estimates for G_{Dir}, which in the best cases (UR of codimension 1 and good coefficients A) are

$$C^{-1}\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\alpha \leq \frac{\omega^X_{Rob}(E)}{\omega^X_{Rob}(\Delta)} \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\beta \tag{56}$$

for some $\alpha, \beta > 0$ that usually one does not know precisely.

A statement (3): more comments

• Thus, no phase transition, and optimal bounds at small scales (the two measures are essentially proportional on small balls).
• Much more than A_∞ (but at small scales that depend on a).

• (54) defines the scale r at which we switch from Neumann mode to Dirichlet mode (given x, r is essentially unique, by (53)).

• So the Brownian motion goes all over the place, essentially uniformly at small scales. Not true for Dirichlet, and the lung is probably right to be fractal because a is small.

• In the computations, a and σ go together, only the product $a\sigma$ counts. And if we want slightly variable coefficients a, just multiply σ by a density!
• More general cases under way (where a may be very small in some places, or with nonlinear Robin conditions $\frac{\partial u}{\partial n} = F(u)$).

Mostly for my fun: a variational definition of ω_{Rob}^X

We do the computation in co-dimension 1, with a smooth boundary, and $L = -\Delta$. Call σ the surface measure.

Let $E \subset \partial\Omega$. Minimize (for the given Robin constant $0 < a < +\infty$)

$$J(u) = \frac{1}{a}\mathcal{E}(u) + \int_{\partial\Omega} u^2 d\sigma - 2\int_E u d\sigma, \quad \text{with } \mathcal{E}(u) = \int_\Omega |\nabla u|^2. \tag{57}$$

Not so hard to prove that a unique minimizer u_E exists, by convexity, Poincaré, and the existence of a nice trace.

Next, the minimizer $u = u_E$ is harmonic on Ω.

We can integrate by parts to compute that $\mathcal{E}(u) = \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\sigma$.

By Lagrange (i.e., expand $J(u + tv)$ and differentiate at $t = 0$), $Rob(u) = \mathbb{1}_E$ on $\partial\Omega$. So

$$\omega_{Rob}^X(E) = u_E(X). \tag{58}$$

Variational definition of ω^X_{Rob} (2)

Or (more directly), we can compare the variational and weak definitions. Since $u = u_E$ minimizes

$$J(u) = \frac{1}{a}\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2 d\sigma - 2\int_E u d\sigma, \tag{59}$$

the linear term in t of $J(u + t\varphi)$ is null for alll $\varphi \in C^1_c(\mathbb{R}^n)$, i.e.,

$$\frac{2}{a}\int_\Omega \nabla u \cdot \nabla\varphi + 2\int_{\partial\Omega} u\varphi d\sigma - 2\int_E \varphi d\sigma = 0 \tag{60}$$

which is exactly the definition of "u is a weak solution of $\Delta u = 0$, with $Rob(u) = \mathbb{1}_E$".

Even if we did not know about weak solutions, the fact that J is well defined and has minimizers for rough $\partial\Omega$ and σ would be a hint that some definition of ω^X_{Rob} should exist, at least through (58).

Mutual absolute continuity by the calculus of variations

Here is a simple argument for the mutual absolute continuity of ω^X and σ (but qualitative only) by the calculus of variation, also assuming the symmetry of the coefficient matrix A).

Assume $A = I$ for simplicity.
Recall that $\omega^X_{Rob}(E) = u_E(X)$ where $u = u_E$ minimizes

$$J(u) = \frac{1}{a}\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2 d\sigma - 2\int_E u d\sigma,$$

We want to show that $\omega^X(E) = 0$ if and only if $\sigma(E) = 0$.

• If $\sigma(E) = 0$, then $J \geq 0$, the minimum is for $u \equiv 0$, and then $\omega^X_{Rob}(E) = u(X) = 0$.

• If $\omega^X_{Rob}(E) = 0$, then $u_E = 0$ everywhere on Ω by nonnegativity and Harnack, and so $J(u) \geq 0$ for all u. But if $\sigma(E) > 0$, taking $u = c$, where c is a very small constant, gives $J(u) < 0$.
So $\sigma(E) = 0$.

This was easy! We could probably make this quantitative, but painfully and we prefer the proof, with the Green function (below).

Proof with the Robin Green function (1)

The Robin Green function is a nonnegative function $G^X_{Rob}(Y) = G_{Rob}(X, Y)$, which satisfies $Lu = 0$ in $\Omega \setminus \{X\}$ and

$$Rob_a(G^X_{Rob}) = 0 \quad \text{on } \partial\Omega \tag{61}$$

and has a normalized singularity at X.
Existence and some regularity for G_{Rob} is a little bit as usual, once we know that solutions are Hölder-continuous.
We will use the fact that morally $\frac{\partial G}{\partial n} = -aG$ at the boundary while (traditionally) the density of ω^X is $\frac{\partial G}{\partial n}$, here equal to $-aG$. In fact we have the nice formula

$$\omega^X_{Rob}(E) = a \int_E G_{Rob}(X, y) d\sigma(y) \tag{62}$$

for $E \subset \partial\Omega$, obtained by algebraic manipulations.

Proof with the Robin Green function (2)

Recall that

$$\omega^X_{Rob}(E) = a \int_E G_{Rob}(X, y) d\sigma(y)$$

for $E \subset \Delta = \partial\Omega \cap B(x, r)$ if, say, $X \in \Omega B(x, 2r)$.

We compare $G_{Rob}(X, y)$ with $G_{Rob}(X, Y_B)$, where Y_B is a corkscrew point for $B(x, r)$. With (54) and by a variant of the density lemma,

$$G_{Rob}(X, y) \geq C^{-1} G_{Rob}(X, Y_B) \quad \text{for } y \in \Delta.$$

[To be checked, the proof in the paper is different]

By local Hölder continuity and Harnack, we also get that

$$G_{Rob}(X, y) \leq C G_{Rob}(X, Y_B).$$

Then

$$\omega^X_{Rob}(E) \sim a G_{Rob}(X, Y_B) \sigma(E).$$

This is true also with $E = \Delta$; compare and get $\dfrac{\omega^X_{Rob}(E)}{\omega^X_{Rob}(\Delta)} \sim \dfrac{\sigma(E)}{\sigma(\Delta)}$,

as needed.

Comparison between the Green functions (1): the splitting scale

Claim: the behavior of the Robin Green function G_{Rob} near $\partial\Omega$ is not so mysterious: at small (Neumann) scales it is nearly constant; at large scales (Dirichlet) it is equivalent to G_{Dir}.

What are these scales? For $x \in \partial\Omega$, consider the quantity

$$I(x,r) = a\sigma(B(x,r))r^{2-n} \tag{63}$$

of (54). Due to the $\geq d$-dimensional behavior of σ ($d > n-2$), this quantity is essentially increasing:

$$I(x,tr) \leq Ct^{d+2-n} I(x,r) \quad \text{for } 0 \leq t \leq 1.$$

Thus for x given, there is a $r(x)$ such that $I(x,r) \leq C$ for $r \leq r(x)$ and $I(x,r) \geq C^{-1}$ for $r > r_0$ (minor modifications when $a\sigma(\partial\Omega) < \mathrm{diam}(\Omega)^{2-n}$).

We expect Neumann behavior at scales smaller than the splitting scale $r(x)$, and Dirichlet behavior above.

When $\sigma \in AR(d)$, this yields $r(x)^{d+2-n} \sim a^{-1}$.

When in addition $d = n-1$, this yields $ar(x) \sim 1$.

Comparison between Green (3): token for two lemmas

The Balance lemma for (nearly) Neumann functions

Approximation by Dirichlet when a is large.

PART III: Minimal sets of dimension 2 in $\mathbb{R}^2$

A few questions and pictures concerning the sliding Plateau problem in $\mathbb{R}^3$: finding sets E bounded by a (smooth) curve $\Gamma \subset \mathbb{R}^2$ and that minimize (or almost minimize) $\mathcal{H}^2(E)$.

Because of potential new progress by Camillo De Lellis and Federico Glaudo, I will insist on a "preliminary" question: what is the list of minimal cones of dimension 2 bounded by a line?

No final result yet, but I will show pictures

Very rapid introduction to the Plateau problem

There are many Plateau problems ("minimize area" given "topological constraints"):

- With parameterizations (Radó, Douglas)
- With sets: Reifenberg (homology), sliding (deformations), Harrison-Pugh (linking)
- With currents, chains, varifolds : Federer, Fleming, De Giorgi, Almgren, Allard, ...

I will only mention very briefly as comparisons and insist on the one I prefer: "sliding minimizers".

Many have existence results, but some are still resisting: size-minimizing currents, sliding minimizers.
Regularity results, especially near the boundary, are often very incomplete.

Plateau with sets (1)

General notation: we work in $\mathbb{R}^n$, with a given boundary set Γ.

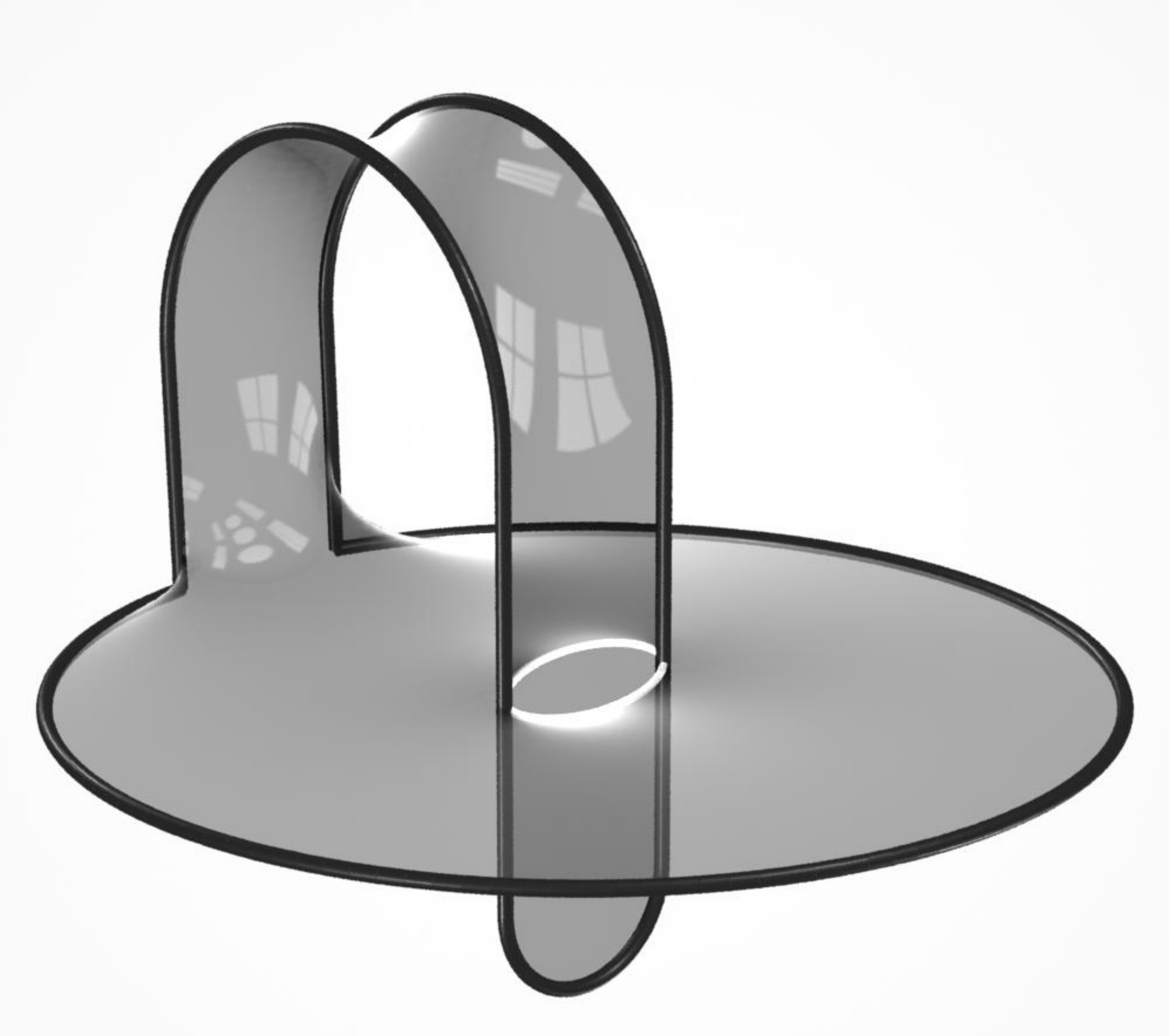
We look for a closed set $E \subset \mathbb{R}^n$, which minimizes the Hausdorff measure $\mathcal{H}^d(E)$ under some topological constraints. We often say "bounded by Γ" but many meanings are possible.

Very soon today, $d = 2$ and Γ is composed of smooth curves.

First an ancestor (Radó-Douglas): Γ is a loop, and we look for $E = f(\mathbb{D})$ for f such that $f(\partial\mathbb{D})$ parameterizes the loop.
Surface is computed through $\int_{\mathbb{D}} Jac(f)(x)dx$.

Looks hard because parameterizations for a minimizing sequence could go crazy.
But beautiful results ($d = 2$) using conformal parameterizations to gain compactness.
Even so, difficulties with different shapes (why parameterize by $\mathbb{D}$?) and importantly injectivity (should we count multiplicity?).

Things happen when soap films cross



[Thanks to John Sullivan]
[Same problem with mass-minimizing currents]

Two words about Reifenberg (because I like it!)

Here we minimize $\mathcal{H}^d(E)$ under a homology constraint.
For instance pick a group G, some elements of the homology of dimension $d-1$ in Γ (with coefficients in G), and minimize $\mathcal{H}^d(E)$ under the constraint that these elements become 0 when we embed Γ in E.

Quite general existence results (with Cech homology):
E. Reifenberg, F. Almgren, Y. Fang,
Simpler proofs by C. De Lellis - F. Ghiraldin - F. Maggi ;
G. De Phllipis - A. De Rosa- F. Ghiraldin ;
Y. Fang - S. Kolasiński ; C. Labourie.

Looks like size minimizing currents.

Amusing that the results change with G, orientability is sometimes an issue, but beautiful and solutions that look like soap.

Sliding Plateau problem

Finally here is my preferred notion.
Definitions in general, then we take $d = 2$.
We are given $\Gamma \subset \mathbb{R}^n$ and $E_0 \subset \mathbb{R}^n$ compact (say).
The competitors (the class $\mathcal{E}(E_0)$) are the closed sets $E \subset \mathbb{R}^n$ that are deformations of E_0 through mappings that preserve Γ.
That is, $E \in \mathcal{E}(E_0)$ when $E = \varphi_1(E_0)$ where $\{\varphi_t\}, 0 \leq t \leq 1$ is s.t.:
- $(x, t) \to \varphi_t(x) : E_0 \times [0, 1] \to \mathbb{R}^n$ is continuous
- $\varphi_0(x) = x$ for all x
- φ_1 is Lipschitz [not necessarily needed]
- $\varphi_t(x) \in \Gamma$ when $x \in E_0 \cap \Gamma$. [the sliding condition]

Think that $d = 2$, Γ is a finite collection of closed curves, and E_0 is a rubber sheet attached to Γ like a shower curtain.

Of course minimizers for $\mathcal{H}^d$ in $\mathcal{E}(E_0)$ depend on E_0, and stupid choices of E_0 yield trivial Plateau problems.

Definition: a <u>sliding minimal set</u> is a closed set E that minimizes $\mathcal{H}^d$ in the class $\mathcal{E}(E)$ defined by E itself.

Status of the Sliding Plateau Problem (SPP)

The SPP is tempting: Flexible problem; minimizers really look like real soap films, Reifenberg minimizers or Size minimizing currents give sliding minimizers.

But in general: no existence result for the SPP, and only vague regularity results.

A little more for $d = 2$ and Γ is a finite union of smooth loops, but no general existence result; some interior regularity (Jean Taylor, etc.), and an incomplete description at the boundary.

But (Fang and Kolasiński), existence and regularity for tubular boundaries $\Gamma \subset \mathbb{R}^3$! For this result, regularity first (as in the next picture) and existence follows (by the argument below).

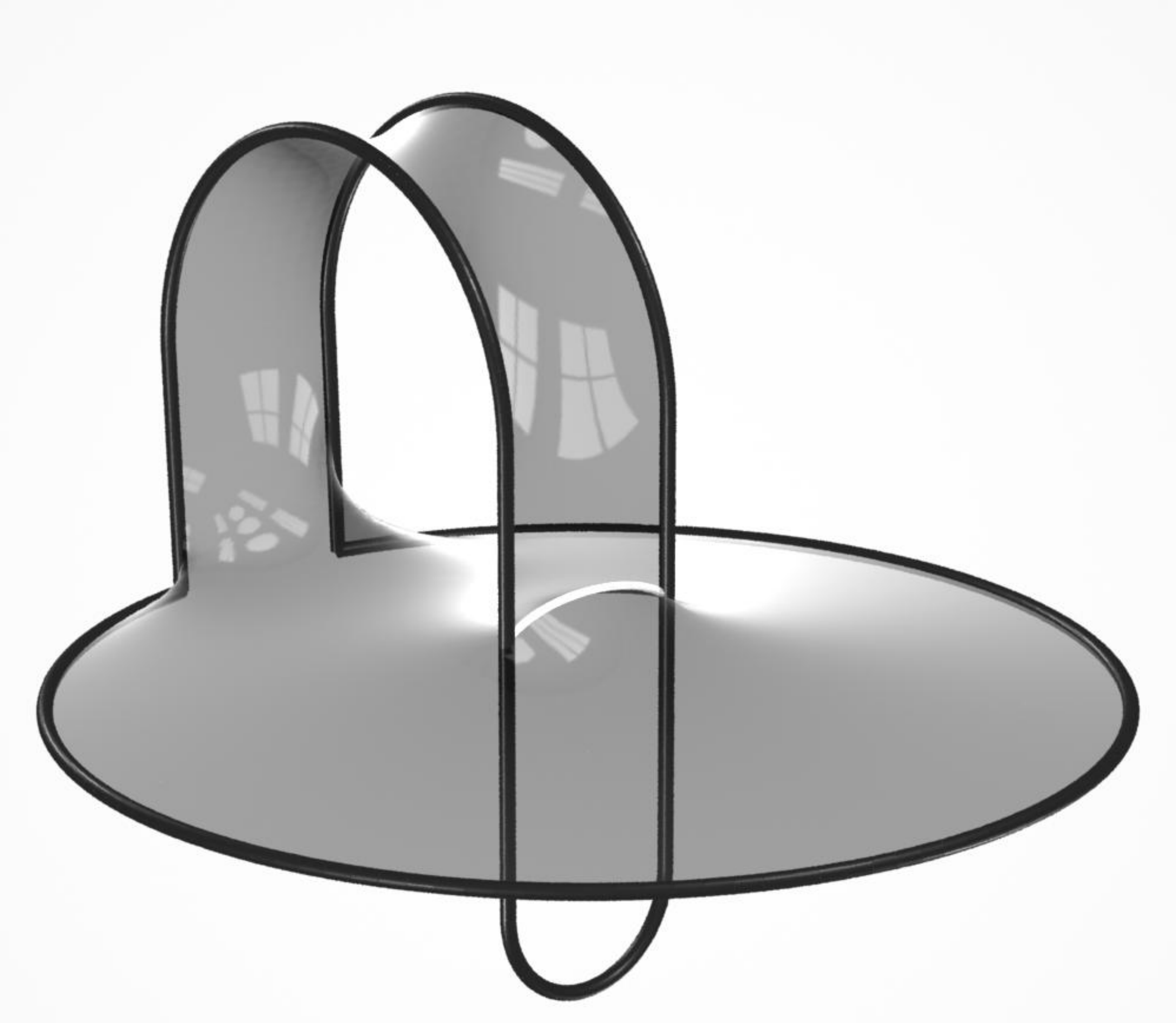
The worst singularity (for tubes)

The smoothness of E (bounded by a tube or surface) is as suggested here, due to a small number of possible blow-up limits

[Picture by John Sullivan]

Question: what happens to E when the diameter of the tube tends to 0?

A typical example where E departs from a (thick) wire



Here there is no soap near the bottom part. What happens for very thin tubes? [Again picture by John Sullivan]

A plan for proving existence, SPP

Alas, not a conclusive plan yet. But it is tempting to try this.

Let E_0 and Γ be given.

Use a minimizing sequence $\{E_k\}$, $E_k = \varphi_{k,1}(E_0)$.

Take a subsequence where $\{E_k\}$ converges (Hausdorff) to E_∞.

Then (done), E_∞ is a Sliding minimal set.

But is it a sliding deformation of E_0?

So prove a regularity result for E_∞, which says we can slidong-deform a neighborhood of E_∞ onto E_∞. To be done!

Then we could conclude.

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Sliding minimal cones

For our program, a first step is a description of the sliding minimal cones (centered on Γ), when Γ is a line.
This is because there the density

$$\theta(x,r) = \lim_{r\to+\infty} r^{-2}\mathcal{H}^2(E\cap B(x,r))$$

is monotone nondecreasing, for $x \in E \cap \Gamma$.

And blow-up limits at $x \in E \cap \Gamma$ are sliding minimal (with a line).
And they are even cones, because sliding minimal sets with constant density at x are cones centered at x.

Thus it is enough to enumerate the minimal cones (sliding, for instance) and then prove a regularity result near each type of cone.

Partial list of minimal cones centered at 0

With a sliding boundary L, a line through 0.

• All the J. Taylor cones, which are already minimal without L: sets $\mathbb{P}$ (planes) , $\mathbb{Y}$ (three half planes making $2\pi/3$ angles along a line L', $\mathbb{T}$ (cone over the edges of a regular tetraedron centered at 0.

• The sets $\mathbb{H}$: a half plane bounded by L.

• The sets $\mathbb{V}$: two half planes bounded by L, with angle $\theta \in [2\pi/3, \pi]$.

All known to be minimal (I think)

• ...And some "new ones"

Partial list of minimal cones (2)

With a sliding boundary L, a line through 0. Here are two ones, believed to be minimal:

• The (cone over the edges of a) cube, with the long diagonal L (X. Liang). Believed to be minimal, tested against soap and I think Surface evolver.

• Probably neighbors of this one, with L passing through an edge near opposite vertices.

• The Thin Fish (Camillo De Lellis and Federico Glaudo). Believed to be minimal, tested against soap and maybe Surface evolver.

• A long list of candidates (that verify the required angle conditions) obtained by C. De Lellis and F. Glaudo. Maybe the list is complete by now, but still to be checked. Some tested against Surface Evolver/soap. All believed not to be minimal (except the above).

Token for a slide show

OpenScad pictures and 3D prints done at Institut Charpak (Orsay);

Nice cones printed by Michael Niedermeyer et Doerte Rueweler from Passau University, pictures by Camillo De Lellis and Federico Glaudo.

What are we missing for the SPP?

For getting existence and some regularity for the SPP, not much more would be needed:

"Merely" prove a local description of E when it is close enough to a $\mathbb{Y}$, with $L' = L$.
For instance, a metric condition on some unions of 8 geodesics on the sphere would suffice (too hard for me so far).

Then even near exotic cones (like the $\mathbb{T}$ or the Thin Fish) we would still get something, because exotic points are isolated.

A small consolation result with C. Labourie, where we make sure to exclude these $\mathbb{Y}$-sets from the limits: existence when the set Γ of curves is traced on the boundary of a convex set $C \subset \mathbb{R}^3$.
In fact a bit more general, but not much. And compare with an old result of F. Morgan for size minimizers.