Existence for the sliding Plateau problem

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Introduction to Plateau

There are many Plateau problems ("minimize area" given "topological constraints"):

- With parameterizations (Radó, Douglas)
- With sets: Reifenberg (homology), sliding (deformations), Harrison-Pugh (linking)
- With currents, chains, varifolds : Federer, Fleming, De Giorgi, Almgren, Allard, ...

I will only mention very briefly as comparisons and insist on "sliding".

Many have existence results, but some are still resisting: size-minimizing currents, sliding minimizers.
Regularity results, especially near the boundary, are often very incomplete.

Plateau with sets (1)

General notation: we work in $\mathbb{R}^n$, with a given boundary set Γ.

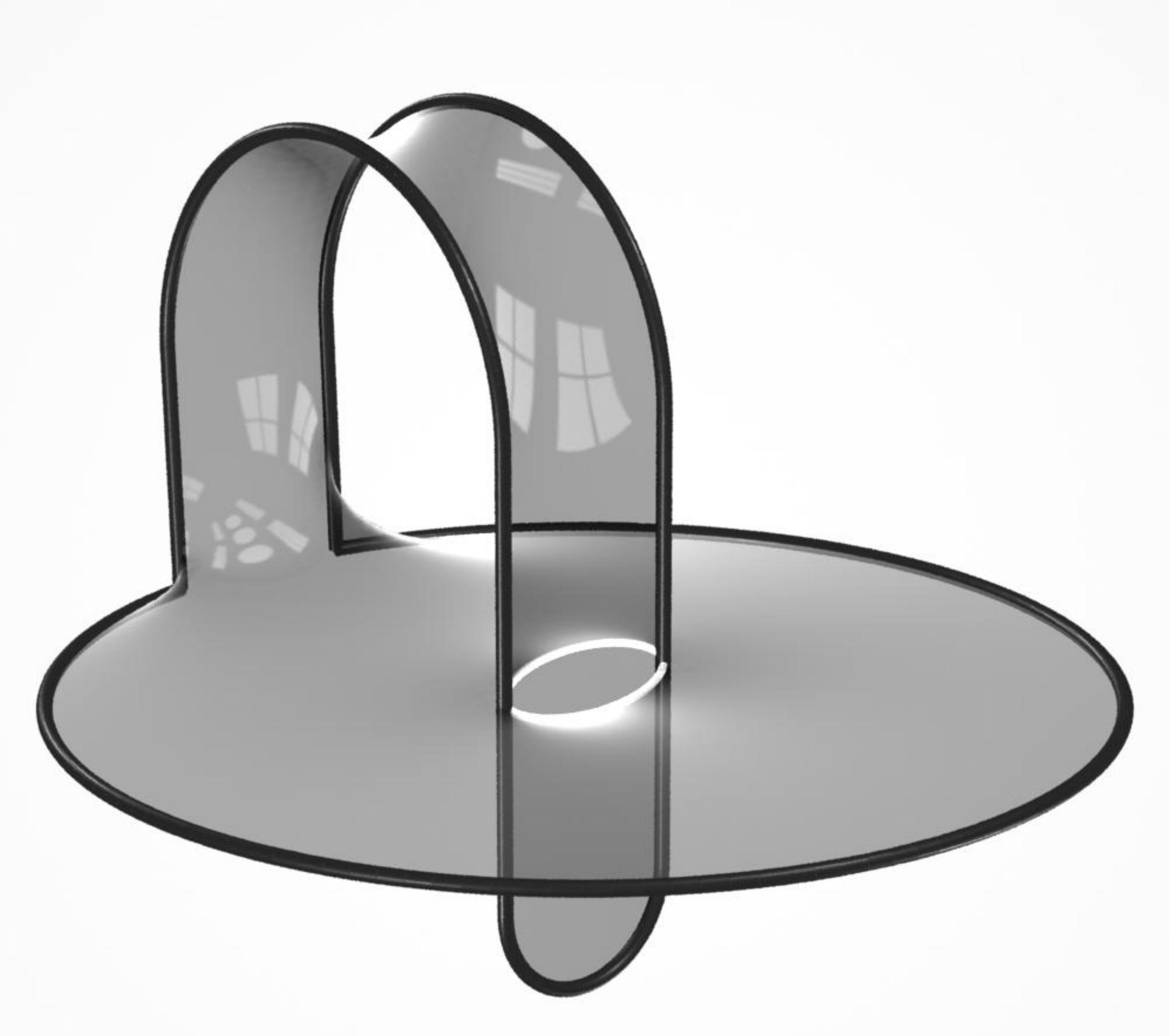
We look for a closed set $E \subset \mathbb{R}^n$, which minimizes the Hausdorff measure $\mathcal{H}^d(E)$ under some topological constraints. We often say "bounded by Γ" but many meanings are possible.

Very soon, here, $d = 2$ and Γ is composed of smooth curves.

First an ancestor (Radó-Douglas): Γ is a loop, and we look for $E = f(\mathbb{D})$ for f such that $f(\partial\mathbb{D})$ parameterizes the loop.
Surface is computed through $\int_{\mathbb{D}} Jac(f)(x)dx$.

Looks very bad because parameterizations for a minimizing sequence could go crazy.
But beautiful results ($d = 2$) using conformal parameterizations to gain compactness.
Yet difficulties with different shapes (why $\mathbb{D}$?) and importantly injectivity (should we count multiplicity?).

Things happen when soap films cross



[Thanks to John Sullivan]
[Same problem with mass-minimizing currents]

Two words about Reifenberg

Here we minimize $\mathcal{H}^d(E)$ under a homology constraint.
For instance pick a group G, some elements of the homology of dimension $d-1$ in Γ (with coefficients in G), and minimize $\mathcal{H}^d(E)$ under the constraint that these elements become 0 when we embed Γ in E.

Quite general existence results (with Cech homology):
E. Reifenberg, F. Almgren, Y. Fang,
Simpler proofs by C. De Lellis - F. Ghiraldin - F. Maggi ;
G. De Phllipis - A. De Rosa- F. Ghiraldin ;
Y. Fang - S. Kolasiński ; C. Labourie.

Looks like size minimizing currents.

Amusing that the results change with G, orientability is sometimes an issue, but beautiful and solutions that look like soap.

Sliding Plateau problem

Finally my preferred.
Definitions in general, existence theorems only for $d = 2$.
We are given $\Gamma \subset \mathbb{R}^n$ and $E_0 \subset \mathbb{R}^n$ compact (say).
The competitors (the class $\mathcal{E}(E_0)$) are the closed sets $E \subset \mathbb{R}^n$ that are deformations of E_0 through mappings that preserve Γ.
That is, $E \in \mathcal{E}(E_0)$ when $E = \varphi_1(E_0)$ where $\{\varphi_t\}, 0 \leq t \leq 1$ is s.t.:
- $(x, t) \to \varphi_t(x) : E_0 \times [0, 1] \to \mathbb{R}^n$ is continuous
- $\varphi_0(x) = x$ for all x
- φ_1 is Lipschitz [not necessarily needed]
- $\varphi_t(x) \in \Gamma$ when $x \in E_0 \cap \Gamma$. [the sliding condition]

Think that $d = 2$, Γ is a finite collection of closed curves, and E_0 is a rubber sheet attached to Γ like a shower curtain.

Of course minimizers for $\mathcal{H}^d$ in $\mathcal{E}(E_0)$ depend on E_0, and stupid choices of E_0 yield trivial Plateau problems.

Definition: a <u>sliding minimal set</u> is a closed set E that minimizes $\mathcal{H}^d$ in the class $\mathcal{E}(E)$.

Status of the Sliding Plateau Problem (SSP)

The SPP is tempting: Flexible problem; minimizers really look like real soap films, Reifenberg minimizers or Size minimizing currents give sliding minimizers.

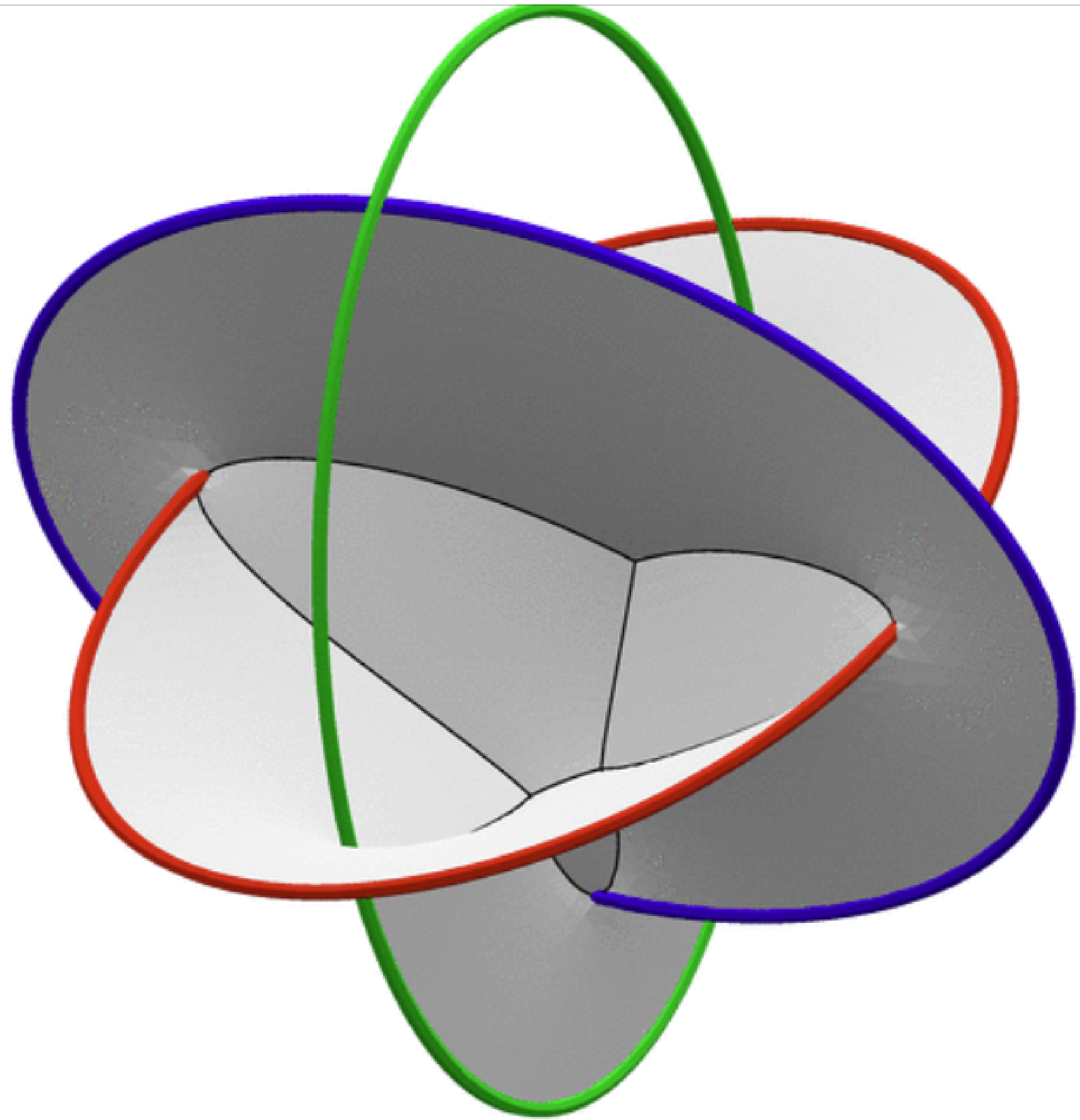
But in general: no existence result for the SPP, and only vague regularity results.
Even for $d = 2$ and Γ is a finite union of smooth loops: no general existence result; some interior regularity (Jean Taylor, etc.); incomplete description at the boundary.
But (Fang and Kolasiński), existence and regularity for tubular boundaries $\Gamma \subset \mathbb{R}^3$! As for this result, regularity first (as in the next picture) and existence follows (by the argument below).

The worst singularity (for tubes)

The smoothness of E is as suggested here, due to a small number of possible blow-up limits on Γ.

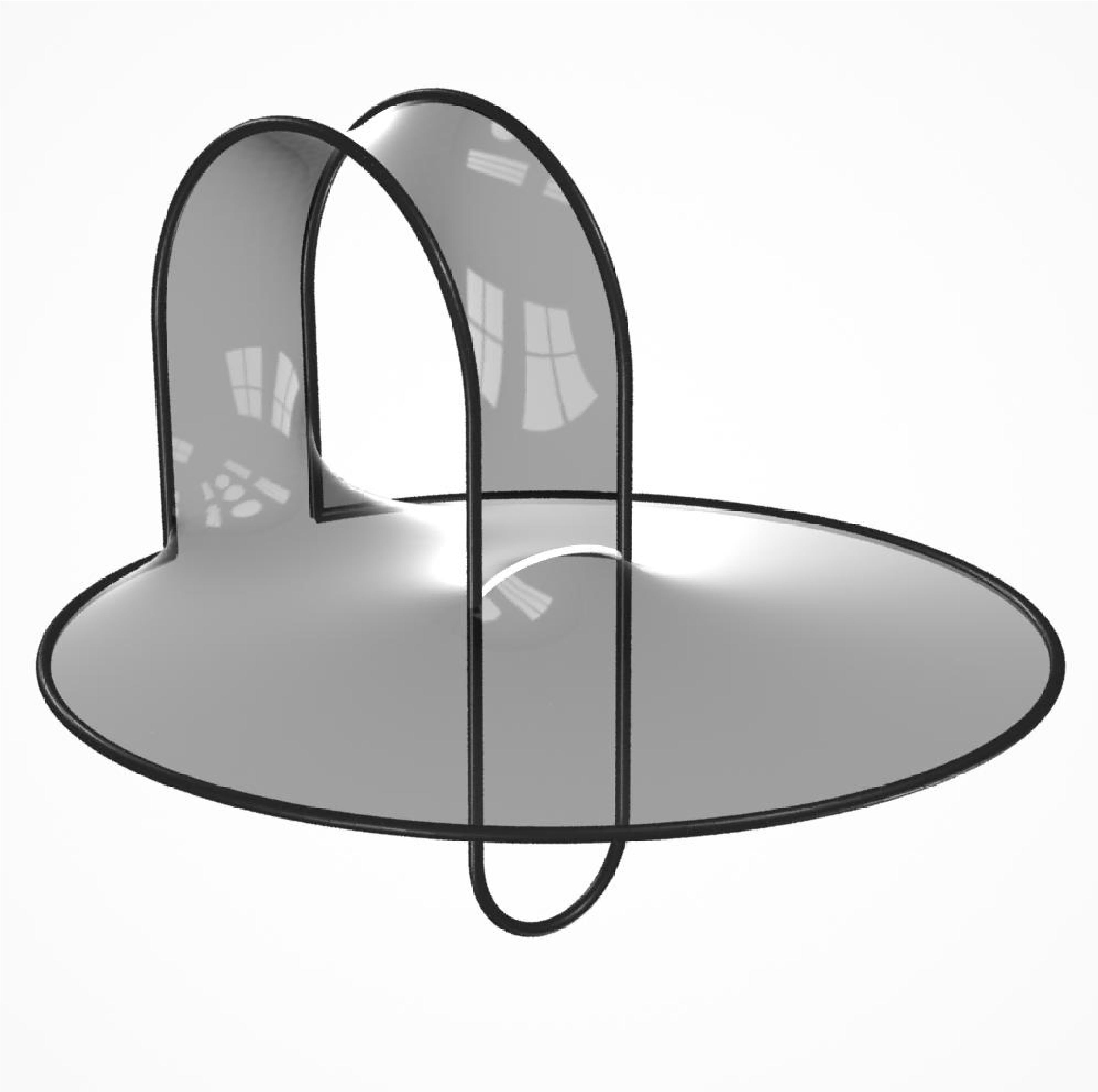
[Picture by John Sullivan]

Another case when E leaves the tube Γ



[Picture by Ken Brakke]

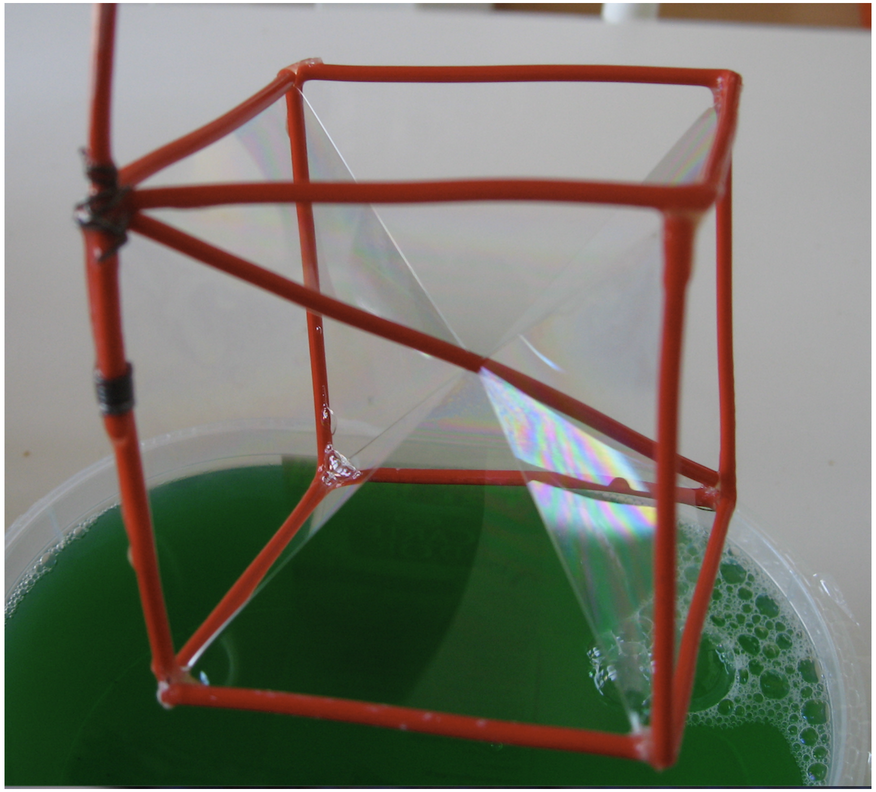
Yet another example



What happens when the thickness tends to 0?

- What are the singularities of the sliding minimal set at the boundary (even in $d = 2$, $n = 3$)?
- How does E leave Γ (we know for tubes, but does this help?)

One example (X. Liang) of probable sliding minimal cone in $\mathbb{R}^3$:



A (not so bright) existence result for the SPP

Theorem (with Camille Labourie)

Take for Γ a finite union of disjoint smooth loops in $\mathbb{R}^n$. Suppose Γ has a good access to the complement of the convex hull of Γ. Let $E_0 \subset \mathbb{R}^n$ compact. Then there is a set $\mathcal{E}(E_0)$ such that $\mathcal{H}^2(E)$ is minimal.

Simple example of good access: $n = 3$ and Γ is contained in the boundary of K.

There is an old theorem of Morgan with these last conditions, for size minimizing currents. Quite different proof (I think), but the starting point is to control a set that contains minimal sets.

Some flexibility in the access condition. Also $\mathcal{H}^d$ can be replaced by slightly different functionals [advertisement for almost minimal sets].

Good Access

The point of K is that we need find a minimizing sequence $\{E_k\}$ such that $E_k \subset K$. Convex hull works for $\mathcal{H}^2$. There is some flexibility but possibly hard to organize.

Good access means: for every blow-up limit L_0 of Γ at a point $x_0 \in \Gamma$, and every blow-up blow-up limit K_0 of K at x_0, if e_1, e_2, e_3 are three unit vectors that are orthogonal to L_0 such that $e_1 + e_2 + e_3 = 0$ and e_0 is a unit vector in L_0, for each $t > 0$ at least one of the $e_0 + te_i$ lies outside of K_0.

Main point of the condition: no blow-up limit of a limit of the E_k has a $\mathbb{Y}$-singularity with a spine contained in L_0.

Idea of proof (1): find a minimizing sequence

Let Γ and E_0 be given. We start with a sequence $\{E_k\}$ so that $\mathcal{H}^2(E_k)$ tends to $\inf_{E \in \mathcal{E}(E_0)} \mathcal{H}^d(E)$. [$d = 2$ is only needed later]

Usual attempt: take a subsequence that converges.

But the only topology on sets where this automatically happens with any $\{E_k\}$ is "Hausdorff limits". And then the limit E_∞ could be anything.

Reifenberg (and later Feuvrier, Fang, and others): modify E_k so that the limit E_∞ has a chance of being a minimizer.

Two main difficulties: prove that $\mathcal{H}^d(E_\infty) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k)$, and prove that $E_\infty \in \mathcal{E}(E_0)$.

Idea of proof (2): weak limits of measures.

In the Reifenberg tradition: we modify the E_k so that they have an extra property, like quasiminimality, so that

$$\mathcal{H}^d(E_\infty) \leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k).$$

Idea of De Lellis - Ghiraldin - Maggi, then also used by G. De Phllipis - De Rosa - Ghiraldin: consider $\mu_k = \mathbb{1}_{E_k} \mathcal{H}^d_{|E_k}$, and take a weak limit μ_∞ of the μ_k. Then use μ_k to find a minimizer.

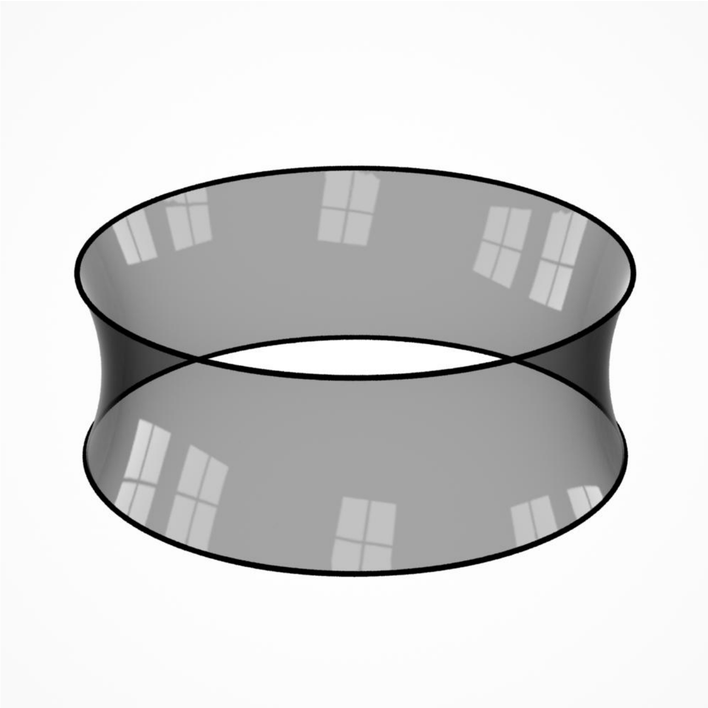
What works well: use the minimizing properties of $\{E_k\}$ to prove lower semicontinuity for the μ_k, and eventually that $\mu_\infty = \mathcal{H}^d_{|E_\infty}$ for some sliding minimal set E_∞.

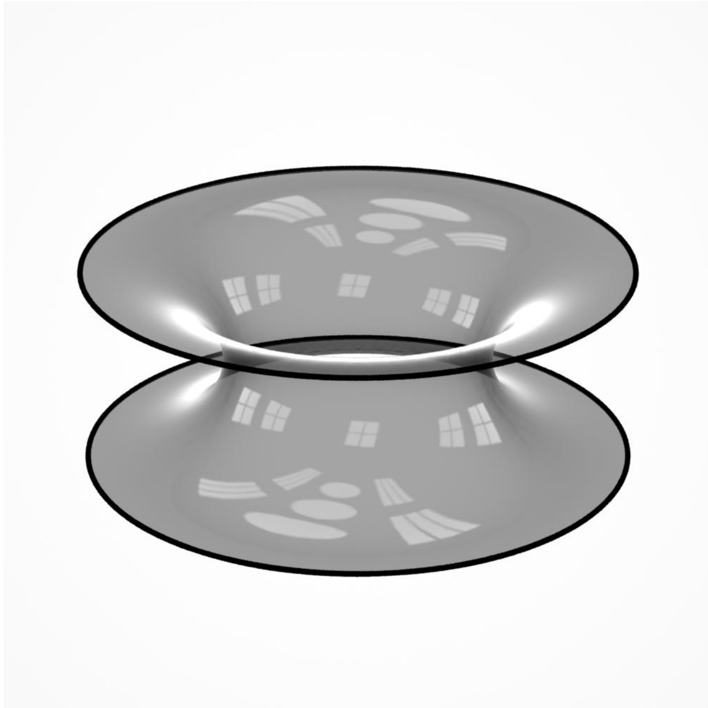
Variant by C. Labourie: use less precise assumptions and a limiting theorem for almost minimal sets, to get a similar result with almost minimal limits. Morte flexible and uses less strong theorems about minimal sets.

But does $E_\infty \in \mathcal{E}(E_0)$? [No in general!]

Wires and topology (1)

What happens if E_0 is the thin catenoid on the right?





The soap film will become the union E_∞ of two disks, plus, for the Sliding Condition, a wire that connects them.
Without the wire, $E_\infty \notin \mathcal{E}(E_0)$. So for the SPP, we need to add something to E_∞.
Incidentally, the SPP no longer describes soap films so well in this example!

Wires and topology (2)

For the method of Reifenberg, . . . , Fang, they manage to keep track of the wires in the haircutting construction.

For the Harrison-Pugh linking problem (with limits of the μ_k), I think one proves that the linking condition is not affected by losing the wires in the limit.

For (with limits of the μ_k), Labourie shows that this does not affect the Reifenberg conditions either.

Here with the SPP, we need to keep the wires, because we want a parameterization of our competitor by E_0.

Looks again like the Radó-Douglas problem: each E_k has a parameterization by E_0, and we want to parameterize the limit. Here we'll use the fact that (under strong assumptions) E_∞ is nice.

Existence (3)

This is where we use a description of E_∞, coming from the fact that it is Sliding minimal (or almost minimal!).

Alas, this is only available so far
- when $d = 2$ and away from the boundaries (J. Taylor + GD, and then existence result of V. Feuvrier),
- when $d = 2$, $n = 3$, for tubular (or smooth oriented $2d$-) boundaries Γ (Fang, Fang-Kolasiński),
- when $d = 2$ and with our accessibility condition on K and Γ (D., D.-Labourie).

That is, we use the accessibility condition to prove that E_∞ has no blow-up with a piece of type $\mathbb{Y}$ with a spine in the tangent direction of Γ; then we have a correct local description of E_∞ near Γ.

We use this description of E_∞ near Γ to construct a contraction $\pi : E_\infty^{(\varepsilon)} \to E_\infty$ (defined in an ε-neihborhood of E_∞).

Existence (4)

We extend π to a mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$.

By construction, π is the endpoint of an acceptable deformation (as in the defnition of $\mathcal{E}(E_0)$).

So $\widehat{E}_\infty = \pi(E_k) \in \mathcal{E}(E_0)$. Is this the minimizer we wanted?

At least $\pi(E_\infty^{(\varepsilon)}) \subset E_\infty$, so $\mathcal{H}^2(\pi(E_\infty^{(\varepsilon)})) \leq \mathcal{H}^2(E_\infty)$...

But maybe $\mathcal{H}^d(\pi(E_k \setminus E_\infty^{(\varepsilon)}))$ is large and compensates?

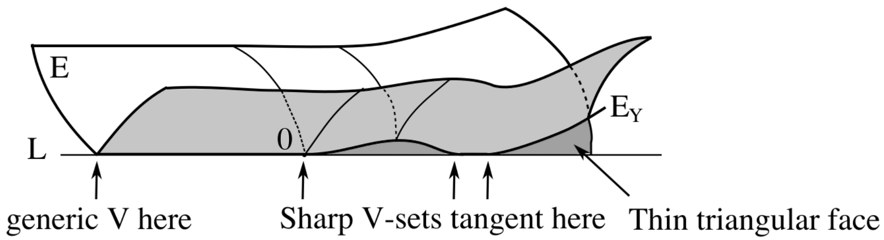
Yes but $\mathcal{H}^d(E_k \setminus E_\infty^{(\varepsilon)})$ tends to 0 (by definition of μ_∞).

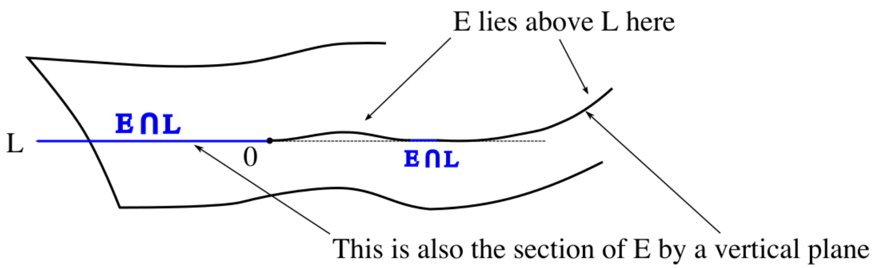
So we first do a Federer-Fleming projection in the complement of $E_\infty^{(\varepsilon/2)}$, that kills $\mathcal{H}^d(E_k \setminus E_\infty^{(\varepsilon)})$. And then $\mathcal{H}^d(\pi(E_k \setminus E_\infty^{(\varepsilon)})) = 0$.

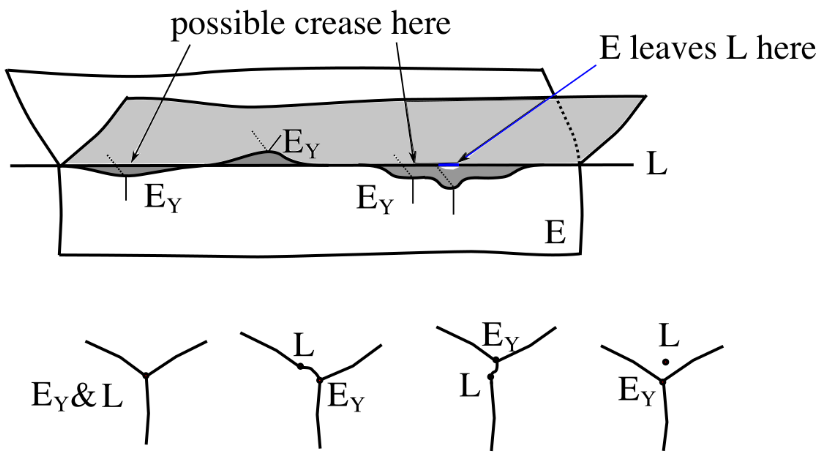
We don't care if the projection increases the mass of E_k in $E_\infty^{(\varepsilon)}$, because this part gets mapped to a subset of E_∞.

$\square$

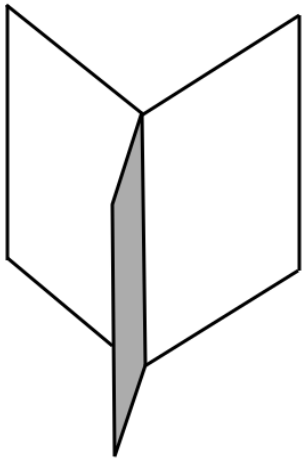
Pictures of Sliding minimal sets near Γ



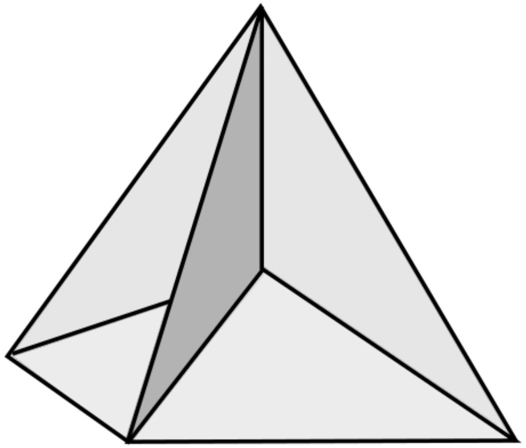




Other pictures



A set of type Y



A set of type T