Absolutely continuous harmonic measure on Cantor sets

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with

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and by
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1. two statements

Goal of the lecture: mainly two results that concern harmonic/elliptic measure on Cantor sets in the plane.
Results, then comments, then some ideas of proofs.

Theorem (D. - Mayboroda)

Let K be the Garnett-Ivanov Cantor set of dimension 1 in $\mathbb{R}^2$. There is a measurable function $a : \mathbb{R}^2 \setminus K \to [1, C]$ such that

$$C^{-1} H^1(E) \leq \omega_L(E) \leq C H^1(E) \quad \text{for } E \subset K.$$

where ω_L denotes the elliptic measure on K associated to the operator $L = \operatorname{div} a\nabla$.

We take $\omega_L = \omega_L^\infty$ with pole at ∞ for simplicity.
New also for elliptic matrices A, but we like the fact that we can take $A = a\, I_2$ scalar. Notice that $L = \Delta$ fails.
Maybe later in the lecture: variant by Polina Perstneva on snowflakes $K \subset \mathbb{R}^2$ (and there $A = a\, I_2$ matters).

The Garnett-Ivanov 1-dimensional Cantor set

$K = \bigcap_{k \geq 0} K_k$, suggested by the picture.

K_k is composed of 4^k squares of size 4^{-k}

A natural measure μ on K gives the same mass 4^{-k} to each square of K_k. And then $\mu = cH^1_{|K}$.

K is totally unrectifiable: $\mu(E \cap \Gamma) = 0$ for every curve Γ with finite length

One dimensional, NTA complement.

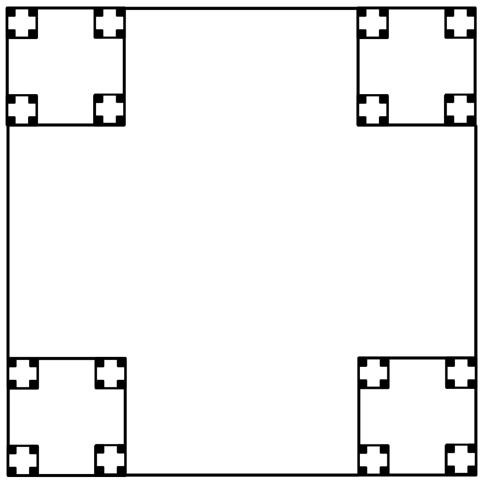


Figure: The set K_3 (three generations of the construction of K)

Theorem 2

Theorem (G. D. - C. Jeznach - A. Julia)

For $0 < d < 0.4$, there is an Ahlfors regular d-dimensional Cantor set K in $\mathbb{R}^2$ such that

$$C^{-1}H^d(E) \leq \omega_L(E) \leq CH^d(E) \quad \textit{for } E \subset K,$$

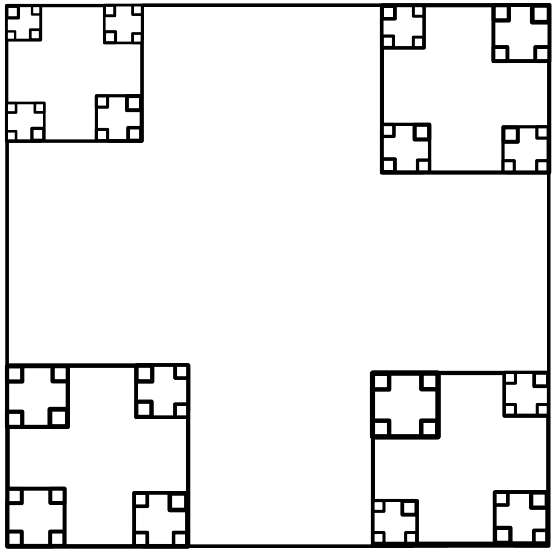
where ω_Δ is harmonic measure on K associated to the Laplacian.

Ahlfors regular means that there is a constant $C \geq 1$ such that

$$C^{-1}r^d \leq H^d(K \cap B(x,r)) \leq C^{-1}r^d \text{ for } x \in K \text{ and } r < \mathrm{diam}(K).$$

In fact, K is a perturbation and a bilipschitz image of the self-similar d-dimensional analogue of the Garnett-Ivanov set.
When $d < 0.249$, we can even get $K \subset \mathbb{R}$ as a perturbation of a "middle n^{th}" Cantor set.
A theorem of Tolsa says that for $K \subset \mathbb{R}$, $d \geq 1/2$ is impossible.
And $d = 1$ is impossible for any unrectifiable K.

The asymmetric Cantor set of Theorem 2

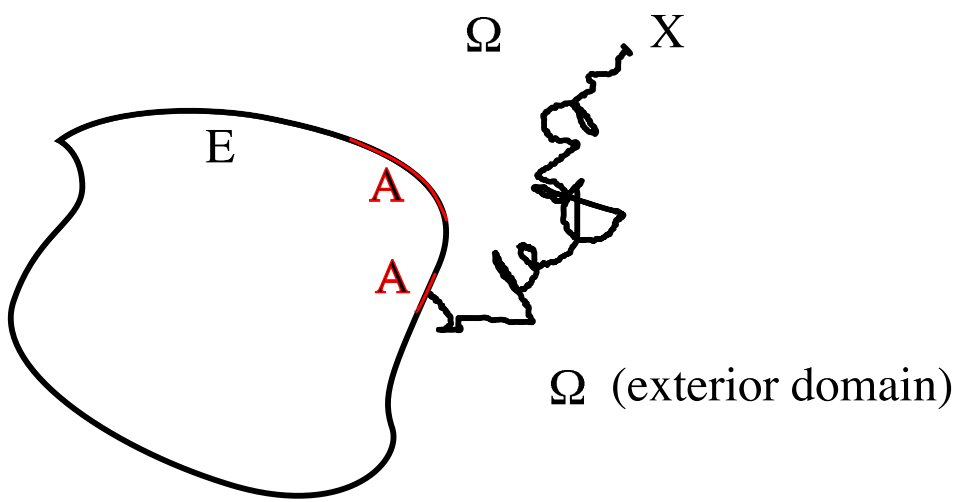


A picture of the Cantor set in Theorem 2,
with 3 generations, exagerated differences, and d larger than real.

2. Definitions. Harmonic measure

First the Brownian path definition of the harmonic measure ω^X in a nice domain $\Omega \subset \mathbb{R}^n$. Say $K = \partial\Omega$ is bounded and smooth and Ω is the unbounded component of $\mathbb{R}^n \setminus K$.

We define the harmonic measure ω^X (centered at $X \in \Omega$) by this: For $A \subset E$, $\omega^X(A)$ is the probability for a Brownian trajectory starting from X to lie in A the first time it hits E.



Rather easy from the mean value property: $\omega^X(A)$ is a harmonic function of $X \in \Omega$; hence by Harnack its size depends nicely on X.

Definition with the Dirichlet problem

The above is intuitive, but requires some work and appropriate assumptions to write it down. And in practice we use the definition based on the Dirichlet problem:
Justification: for $A \subset \partial\Omega$, $\omega^X(A)$ is harmonic. Believable: it "tends to" $\mathbb{1}_A$ on $\partial\Omega$. And $\omega^X(A) = \int_{\partial\Omega} \mathbb{1}_A(\xi) d\omega^X(\xi)$.

Definition 2: for each $g \in \mathcal{C}(\partial\Omega)$ (that is, continuous on $\partial\Omega$), there is a unique continuous extension f of g to $\overline{\Omega}$ which is harmonic in Ω. (Still assume Ω is "regular" enough). Then for each $X \in \Omega$, $g \to f(X)$ is a continuous linear form on $\mathcal{C}(\partial\Omega)$ and by the Riesz Theorem there is a finite measure ω^X on $\partial\Omega$ such that

$$f(X) = \int_{\partial\Omega} g(\xi) d\omega^X(\xi) \quad \text{for } g \in \mathcal{C}(\partial\Omega).$$

It is even a probability measure, by the maximum principle.
Coincides with the previous defn (with $g = \mathbb{1}_A$) in the good cases.
Often easier to manipulate. And to generalize:

Elliptic operators

This was for harmonic functions and the Laplacian $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j{}^2}$. But the definitions work also for some other elliptic operators. Let us consider only the operators in divergence form:

$$L = \operatorname{div} A\nabla \tag{1}$$

where $A = A(X)$ is an $n \times n$ real matrix (measurable in X), and we require the usual boundedness property

$$|A(X)| \leq C \qquad \text{for } X \in \Omega \tag{2}$$

and ellipticity condition

$$\langle A(x)\xi, \xi\rangle \geq C^{-1}|\xi|^2 \qquad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \tag{3}$$

Then it is possible to define elliptic measure ω_L^X as above, but with solutions of $Lf = 0$. We should rather call ω_L^X elliptic measure.

Motivations 1: Positive absolute continuity results

Often $K = \partial\Omega$ comes with a natural measure μ: surface measure for smooth (or Lipschitz) domains, $\mathcal{H}^d_{|K}$ for the Cantor sets above.

When are ω and μ absolutely continuous to each other?

The notion does not depend on X; thus we take $\omega = \omega^\infty$.

Answer yes when K and L are both nice. Main symbolic example:

Dahlberg 77: When $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and $L = \Delta$, ω and μ are mutually absolutely continuous, with a Muckenhoupt A_∞ density.

Some connectedness for Ω is needed: typically one sided NTA: Ω contains corkscrew points and Harnack chains. We pass.

Next: does it work for other good sets and operators?

Motivations 2: Extensions of Dahlberg

Does this generalize?

• To **more general domains, yes**. Many results; the typical conditions for an Ahlfors regular boundary $\partial\Omega$ of co-dimension 1 are:

- [Uniform] rectifiability of $\partial\Omega$;
- one-sided NonTangential Access from Ω (or slightly weaker)

And then one gets that ω and $\mu = \mathcal{H}^{n-1}_{|\partial\Omega}$ are mutually absolutely continuous [with an A_∞ density].

• Even results for mere absolutely continuous and rectifiability

• **Extensions to elliptic operators** $L = \mathrm{div}\, A\nabla$, A sufficiently close to constant (Dahlberg-Kenig-Pipher and WDKP conditions, all stated with Carleson measures).

• Some extensions to **higher co-dimensional boundaries** too.

Many contributors: Azzam, Martell, Mayboroda, Jerison, Hofmann, Lacey, Mourgolgou, Semmes, Tolsa, Toro, Volberg, Zhao...

Counterexamples: bad A and why we like scalar matrices

Recall $L = \operatorname{div} A\nabla$, where A is always assumed to be elliptic.

Traditional counterexamples for bad operators: Modica-Mortola; Caffarelli-Fabes-Kenig: Even on the half plane, when A is badly behaved, ω_L may be singular with respect to μ.

Nice fact here: if $\psi : \Omega_0 \to \Omega$ is quasiconformal and $u : \Omega \to \mathbb{R}$ is harmonic, then $u \circ \psi$ satisfies $Lu = 0$ for some elliptic $L = \operatorname{div} A\nabla$. But ψ may distort distances a lot and not preserve abs. continuity.

Are there nice examples with $L = \operatorname{div} a\nabla$?

Certainly the MM and CFK and examples from QC mappings are not like that.

$L = \operatorname{div} a\nabla$ is more about the (locally isotropic) geometry associated to the distance coming from a weight w through

$$\mathrm{dist}_w(X, Y) = \inf_{\Gamma \text{ from } X \text{ to } Y} \int_\Gamma w(x)^{1/n} d\mathcal{H}^1(x).$$

Counterexamples 2: good operators L but bad sets K

Beautiful (and hard) converse results by [AHMMT] and subsequent: in codimension 1, the (A_∞) absolute continuity of ω_L, L close enough to Δ, essentially implies the (uniform) rectifiability of $\partial\Omega$.

For $0 < d \leq 1$, if K is a self-similar Cantor set in the plane, then ω (for Δ) is singular, and even carried by a subset of dimension $< d$ of K.
Not easy [Carleson 85, Batakis, Volberg, ...]

But surprisingly...

Theorem 1: bad set, bad operator, and good elliptic measure;

Theorem 2: bad set, good operator Δ but wrong dimension, good elliptic measure.

Comment if it helps: the Brownian motion, with the adapted drift or the adapted boundary, goes uniformly to the boundary.

Common points and the Green function

Main actor: the Green function (follows ideas from Azzam and DM, DLM).

Main psychological progress (for me at least): usually the Green function $G = G^\infty$ is impossible to compute, but not if you first choose it!

We also had an example of explicit Green functions with the DEM magic case in large co-dimensions, where G^α is equal to the adapted smooth distance D_α.

Anyway, for Theorem 1 it is enough to check that

$$C^{-1}\,\mathrm{dist}(X,K) \leq G^\infty(X) \leq C\,\mathrm{dist}(X,K)$$

[Think of the smooth case where the density of ω is $\frac{\partial G}{\partial n}$].

Proof of Theorem 1: Pairs of conjugated functions

[Not sure this is the right term.]
We want to construct G on $\Omega = \mathbb{R}^2 \setminus K$, and then show that $\operatorname{div} a\nabla G = 0$ for some elliptic function a.

We will use another function R, such that $\nabla R \perp \nabla G$ everywhere (but we do the construction locally, where both gradients are $\neq 0$).

A computation shows that if $\nabla R \perp \nabla G$, then $\operatorname{div} a\nabla G = 0$ with

$$a(x) = |\nabla R| / |\nabla G|.$$

So we shall draw the level curves of G and the level curves of R (they are orthogonal), check that $|\nabla R|/|\nabla G|$, or the distances between curves, stay under control, and we are done!

We'll use a fractal construction: this way we can be sure that $a = |\nabla R|/|\nabla G|$ and a^{-1} stay bounded, if we have a control in a fundamental domain and if we can glue correctly.

A fundamental domain

We cut $\mathbb{R}^2 \setminus K$ into annular regions.
The fundamental region (in grey) is the A_0 bounded by the exterior circle ∂B_0 and the four small green circles.
"Enough" to construct G and R in A_0, and then, by symmetry, in the smaller A_{00} (one eighth of A_0).

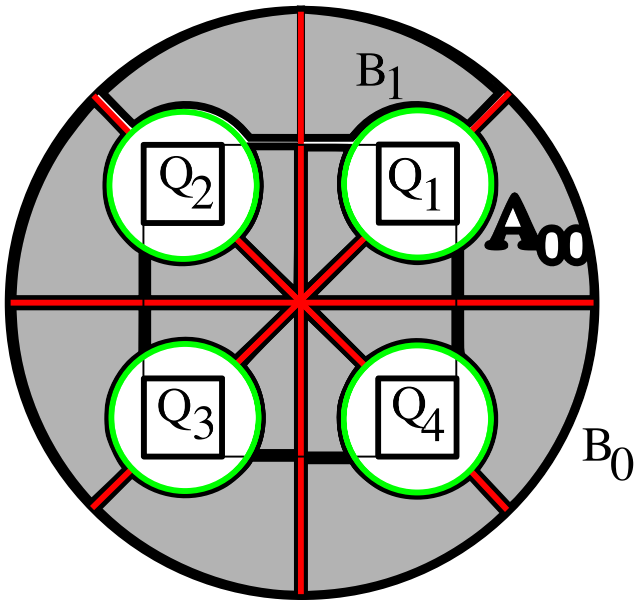


Figure: The cubes Q_j of generation 1, the balls B_0 (large) and B_1 (small), the annulus A_0 (in grey) and a fundamental piece A_{00} (one eighth of A_0)

We draw the red and green curves in A_0

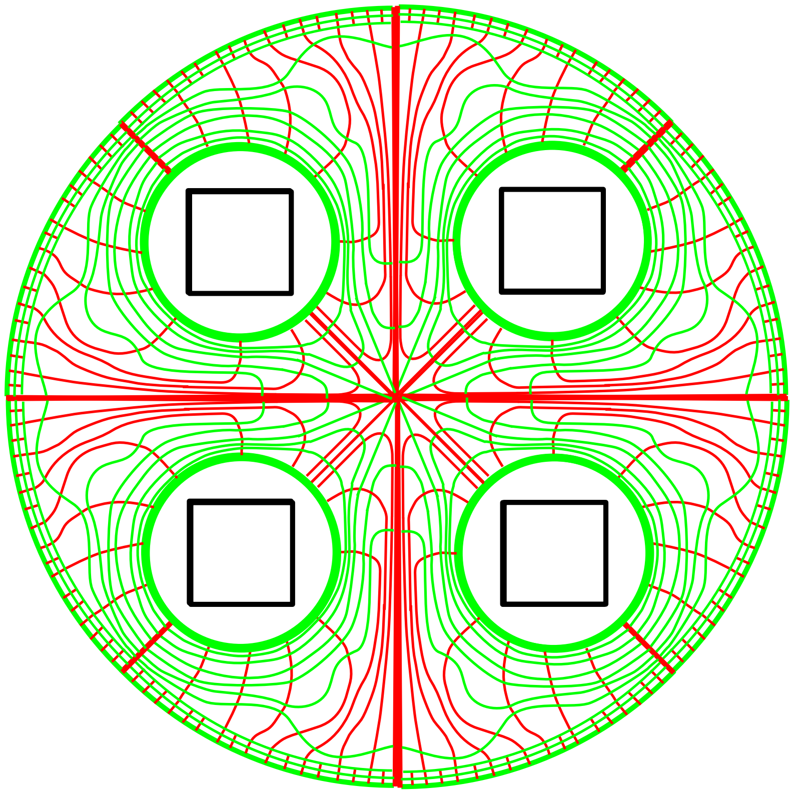


Figure: The level and gradient lines of G in A_0. Mind the symmetry. Also, it is fair that the green curves surround K (recall that $G = 0$ on K) and the red curves go towards K.

We prepare for gluing

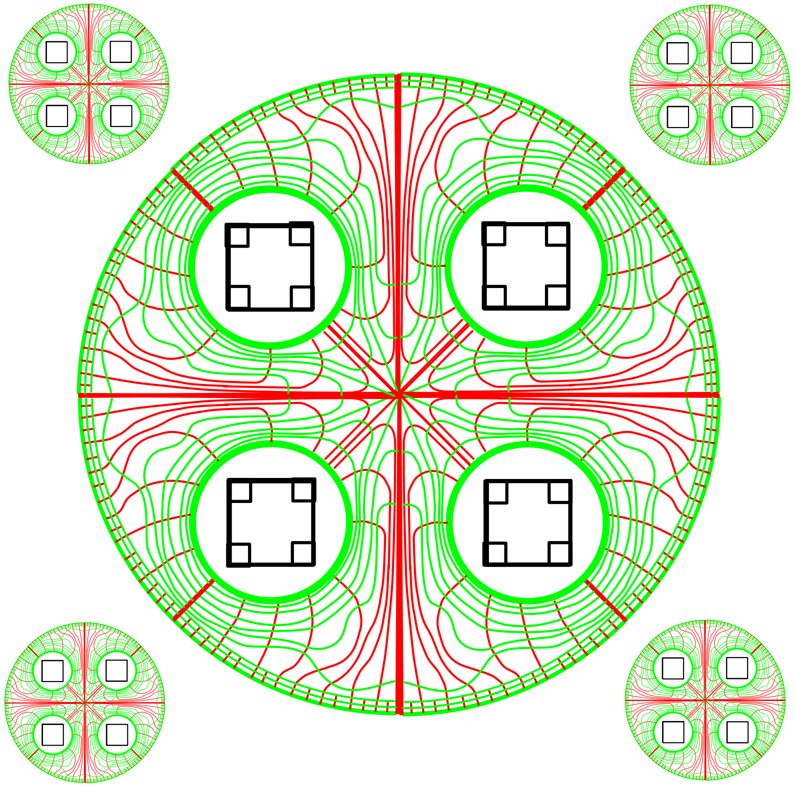


Figure: We prepare four, 4 times smaller copies of the same picture, to be put in the main holes

We glue the next generation

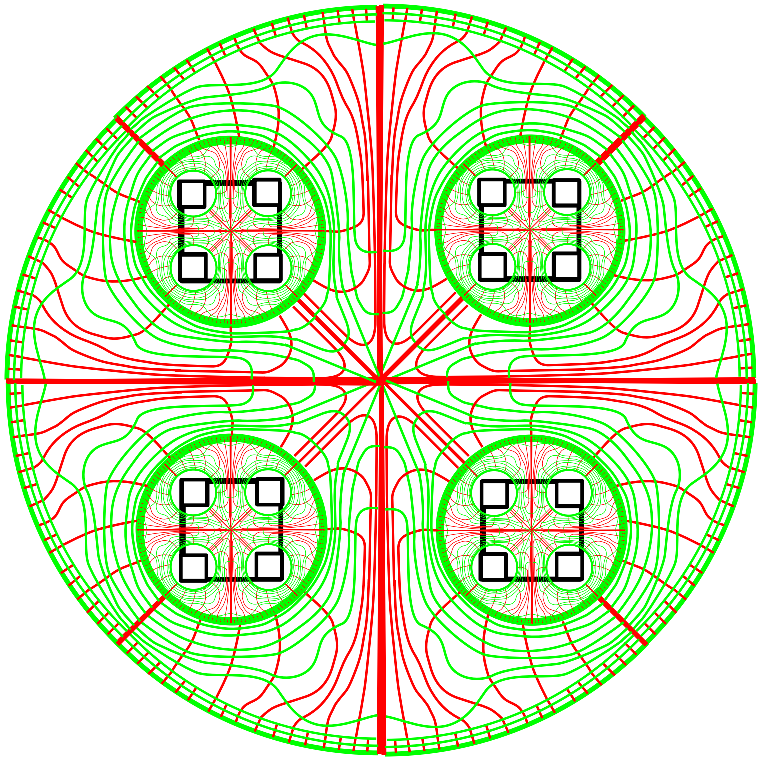


Figure: The level and gradient lines of G on a larger region than A_0, completed by self-similarity

... And so on. The fractal construction allows uniform estimate. Important additional constraint to get uniform bounds on $a(x) = |\nabla R|/|\nabla G|$:
The end of red curve that starts along the first large green circle runs along the four smaller green circles at constant speed.

Some other examples

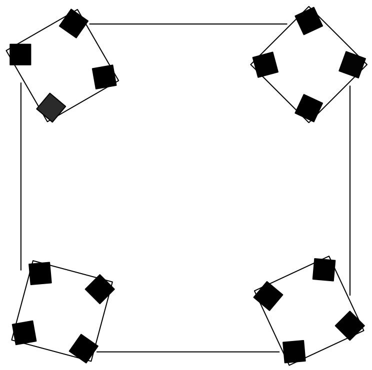


Figure: The third iteration of a rotating version of the Cantor set

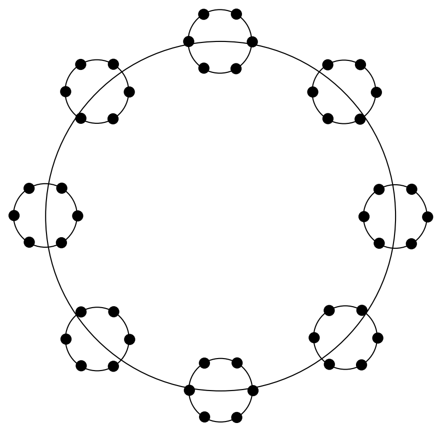


Figure: The third iteration of a variable scale/multiplicity analogue of K; there is no point in trying to draw polygones in this case

And snowflakes (with $A = aI$) by Polina Perstneva.

Ideas for Theorem 2

How to control the Green function G^∞ on the complement of our Cantor set $K \subset \mathbb{R}^2$ of small dimension (constructed on purpose)?

Usual definition for K: For $n \geq 0$, construct K_n, composed of 4^n squares $Q_j = Q_j^n$, $j \in J(n)$ of size r^n, and take the limit.
Here r is small because $r^d \leq 1/4$.
One way to describe the self-similar set K_0 of dimension d is by nested squares (as above), or by a parameterization
$F_0 : E = 4^{\mathbb{N}} \to \mathbb{R}^2$. Choose four points e_1, e_2, e_3, e_4 of $\partial B(0,1)$, on the diagonal, and for $\varepsilon = (\varepsilon_k)_{k\in\mathbb{N}}$, set

$$F_0(\varepsilon) = \sum_k r^k e_{\varepsilon_k}.$$

Now for K we take

$$F(\varepsilon) = \sum_k r^k \lambda_k(\varepsilon) e_{\varepsilon_k},$$

with $\lambda_k(\varepsilon) \in [1,2]$ that depends only on $\varepsilon_0, \dots \varepsilon_{k-1}$.

Idea 2

That is, when we construct the 4 children of the square Q of generation k, we place the next cubes at distance $\lambda_Q r^d$ from the center x_Q.

$$F(\varepsilon) = \sum_k r^k \lambda_k(\varepsilon) e_{\varepsilon_k},$$

Easy to check: $K = F(4^{\mathbb{N}})$ is a bi-lipschitz image of K_0, and Ahlfors regular of dimension d.

We have a natural measure μ_k on K_n, such that $\mu_n(Q) = 4^{-n}$ for each cube Q of generation n.
And the natural limit μ of the μ_n on K.
Here is a natural harmonic function g: set, for $z \in \mathbb{R}^2 \setminus K$,

$$g(z) = \mu_n * \ln(|\cdot|)(z) = \int_K \ln(|z-x|) d\mu(x).$$

Idea 3

$$g(z) = \mu_n * \ln(|\cdot|)(z) = \int_K \ln(|z-x|)d\mu(x).$$

The integral converges because μ is Ahlfors regular.
At ∞, $g(z) \sim \ln(|z|)$, which not bad.
It would be great if we had $g(z) = 0$ on K, but of course this won't happen.
It will be equally good if g is a constant c on K, because then $G = g - c$ is the Green function! so that is what we aim for, if K is chosen well.

Missing piece, which I won't do: check that

$$g(z) - c \simeq \mathrm{dist}(z, K)^d,$$

and then conclude using the relation between $G(z)$ at a corkscrew point and harmonic measure of the corresponding disk.

We discretize

Call $\mathcal{Q}_n$ the set of cubes of generation n, and x_Q the center of $Q \in \mathcal{Q}_n$. Then set

$$g_n(z) = \mu_n * \ln(|\cdot|)(z) = 4^{-n} \sum_{Q \in \mathcal{Q}_n} \ln(|z - x_Q|). \tag{1}$$

We want to arrange things so that g_n is almost constant on K_n (or the union of the circles centered on the x_Q and radius r^n, say). Something like

$$\sup_{K_n} g_n(z) - \inf_{K_n} g_n(z) \leq C4^{-n} \tag{2}$$

The main question: assuming (2) at generation n, how do we arrange (2) at generation $n+1$.

That is, how do we choose the λ_Q to make the oscillation of g_{n+1} smaller?

Idea 5

Take d and r very small. This way, g_n is roughly constant near each cube $Q \in \mathcal{Q}_n$, and the differences between cubes Q is not large.
Call Q_j the four children of Q.
Then write $g_{n+1}(z) - g_n(z)$ for points z near Q.
There are a few terms, that are not small, but are roughly the same on all the ∂Q_j across Q, so we we don't care, they feed the constant.
And the main term is something like

$$\begin{aligned}\delta_n(z) &= 4^{-n-1} \sum_j \left[\ln(|z - x_{Q_j}|) - \ln(|z - x_Q|) \right] \\ &= 4^{-n-1} \sum_j \ln\left(\frac{|z - x_{Q_j}|}{|z - x_Q|}. \right)\end{aligned} \tag{3}$$

Idea 6

$$\delta_n(z) = 4^{-n-1} \sum_j \ln\left(\frac{|z - x_{Q_j}|}{|z - x_Q|}\right)$$

For z in a circle of fixed small radius around x_{Q_j}, $|z - x_{Q_j}|$ is always the same (across the whole set), while

$$|z - x_Q| \simeq |x_{Q_i} - x_Q| = c\lambda_Q r^n.$$

That's it. We are adding essentially equal terms, minus $4^{-n-1}\ln(\lambda_Q)$.

If g_n was larger than average near Q, we take λ_Q small. Otherwise, we take λ_Q larger. This allows us to add a varying constant of size $4^{-n-1}\ln(\lambda_Q)$ to g_n near Q, which turns out to be enough to compensate variations of the averages of g_n among the Q.

Last comments about the proof

Taking r and d small simplifies the proof: the scales are more and more independent, and the extra errors are smaller. Then we sort of optimized.

The effect of increasing the distances $|x_{Q_j} - x_Q| \sim \lambda_Q r^n$ is to increase the chance that a Brownian path that passes nearby will land on the Q_ℓ^{n+1}. But we are lucky that we din't need to evaluate the absorption probabilities and we can sum potentials instead.

Again, once we know that $g \equiv c$ on K, we can estimate the Green function $G^\infty = g - c$, and then use G to estimate ω.
For instance we can estimate $\nabla g = \mu * \frac{1}{z}$ near K.

We could do other shapes (for instance, $K \subset \mathbb{R}$), but squares seem to be nice.

Snowflakes by Polina Perstneva

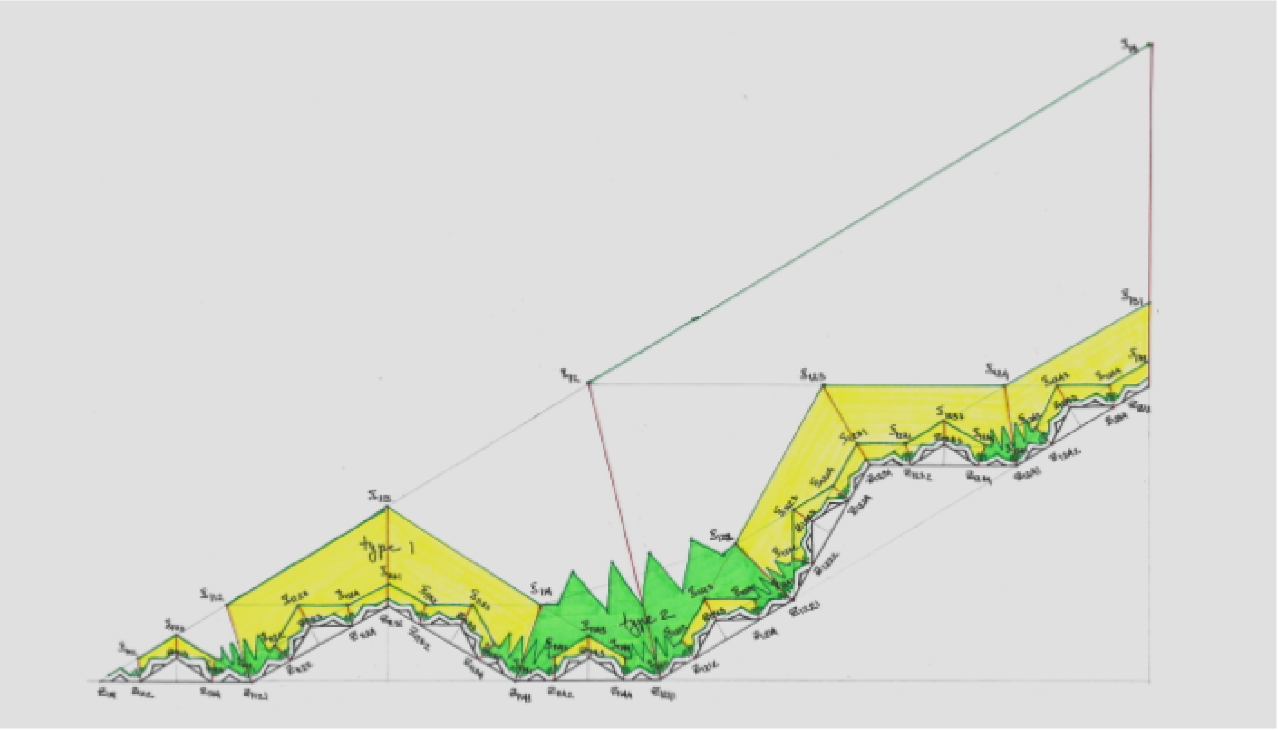


Figure: Cutting the domain above into puzzle pieces

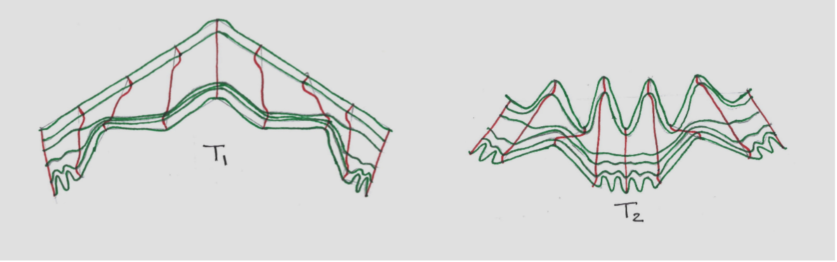
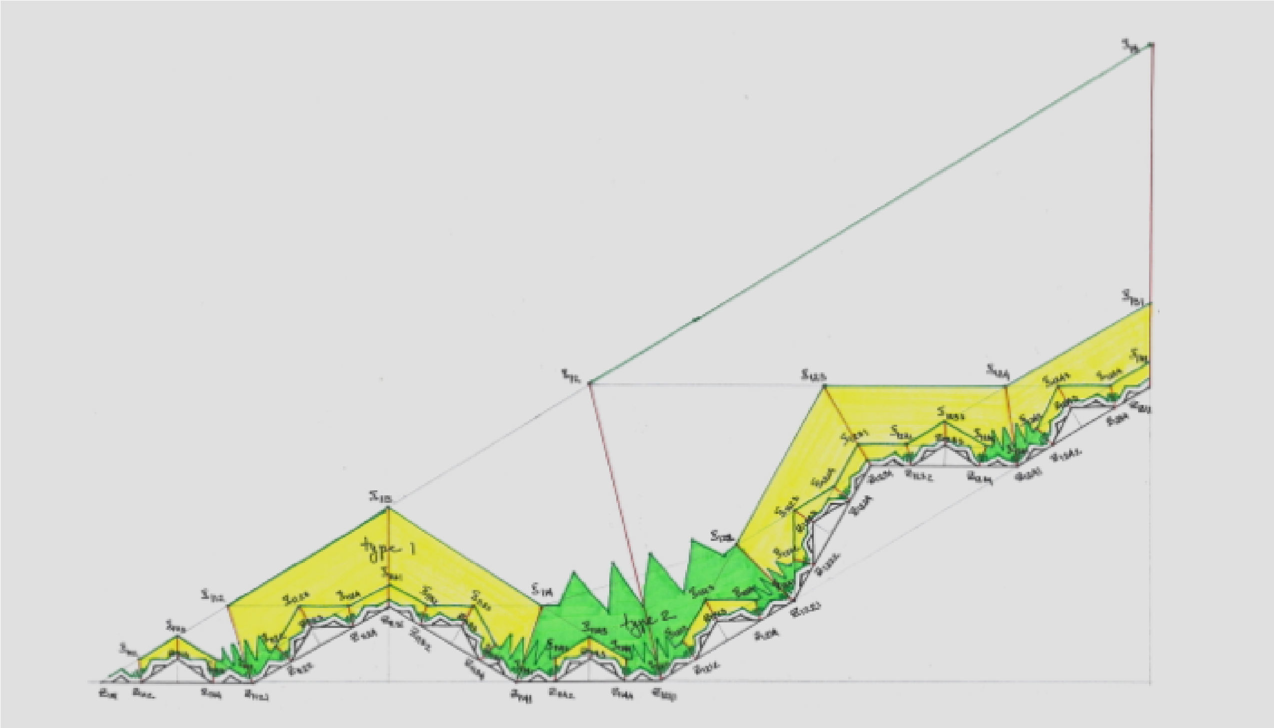


Figure: Filling of two pieces of puzzle by red and green curves

... Thanks!

Snowflakes by Polina Perstneva



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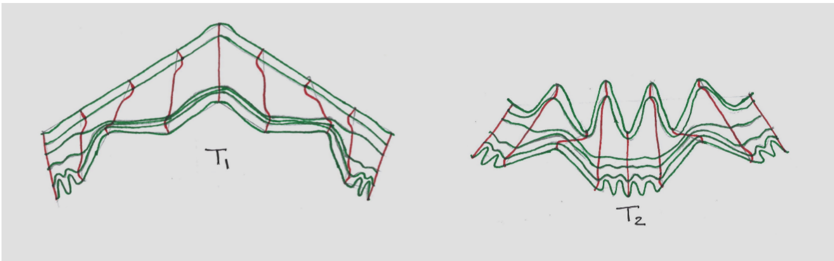

Figure: Filling of two pieces of puzzle by red and green curves

... Thanks!