
RIGIDITY OF GROUP VON NEUMANN ALGEBRAS

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These are notes from a talk delivered to a general mathematical audience. The goal was to introduce group von Neumann algebras and some questions concerning them, with a focus on rigidity phenomena. No prior knowledge of operator algebras is assumed.

1 GROUP VON NEUMANN ALGEBRAS

1.1 THE DEFINITION

Von Neumann algebras were introduced (under the name of *rings of operators*) by J. VON NEUMANN in [29], and systematically studied by F. J. MURRAY and J. VON NEUMANN in [21], as part of a program to ground the emerging quantum mechanics on a firm mathematical basis. Of course, one of the first concerns was to produce various different examples of such objects so as to explore their structure. Very naturally, groups were enrolled in the story through the following construction which goes back to [21, 22]. Throughout, Γ will denote a countable discrete group with neutral element e , and we consider the Hilbert space

$$\ell^2(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{C} \mid \sum_{g \in \Gamma} |f(g)|^2 < +\infty \right\},$$

with its canonical orthonormal basis $(\delta_g)_{g \in \Gamma}$. The group acts on it by translation : for $g \in \Gamma$, let $\lambda(g)$ be the operator defined by

$$\lambda(g)\delta_h = \delta_{gh}.$$

This operator simply permutes the basis vectors, hence it is *unitary*, and one easily checks that $\lambda(gh) = \lambda(g)\lambda(h)$. In other words,

$$\lambda : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma))$$

is a group homomorphism into the group of unitary operators, called the *left regular representation* of Γ . We can now consider the linear span of these operators inside the algebra $\mathcal{B}(\ell^2(\Gamma))$ of all bounded operators on $\ell^2(\Gamma)$. It consists of the finite linear combinations

$$\sum_{g \in \Gamma} a_g \lambda(g)$$

with $a_g \in \mathbb{C}$. Since $\lambda(g)\lambda(h) = \lambda(gh)$ and $\lambda(g)^* = \lambda(g^{-1})$, it is a $*$ -algebra, called the *group algebra* $\mathbb{C}[\Gamma]$, and it is a purely algebraic object. To get an analytic object, we have to take a closure, and there are several topologies available on $\mathcal{B}(\ell^2(\Gamma))$ with which one can work. The one which we will be interested in is the following.

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DEFINITION 1.1. A net $(x_i)_i$ of operators on a Hilbert space H converges to x in the *weak operator topology* if for all $\xi, \eta \in H$,

$$\langle x_i \xi, \eta \rangle \longrightarrow \langle x \xi, \eta \rangle.$$

DEFINITION 1.2. The *group von Neumann algebra* of Γ is the weak operator closure

$$L(\Gamma) = \overline{\text{Span}\{\lambda(g) \mid g \in \Gamma\}}^{wo} \subset \mathcal{B}(\ell^2(\Gamma)).$$

It is useful to have a concrete way of thinking about elements of $L(\Gamma)$. If $x \in L(\Gamma)$, we can look at the vector $x\delta_e \in \ell^2(\Gamma)$ and expand it in the canonical basis :

$$x\delta_e = \sum_{g \in \Gamma} \hat{x}(g)\delta_g.$$

These “Fourier coefficients” $\hat{x}(g)$ in fact completely determine x .

Proposition 1.3. *Let $x, y \in L(\Gamma)$. If $\hat{x}(g) = \hat{y}(g)$ for all $g \in \Gamma$, then $x = y$.*

Proof. Let us write the action of the operator $\lambda(g)$ in a slightly different way: for a function $f \in \ell^2(\Gamma)$, $\lambda(g)f$ sends an element h to $f(g^{-1}h)$. Indeed, the formula holds for δ_k :

$$\delta_k(g^{-1}h) = \delta_{gk}(h)$$

and passes to linear combinations and limits. This suggests defining another function $\rho(g)f$ sending h to $f(hg)$. This defines a bounded operator $\rho(g)$ on $\ell^2(\Gamma)$, with the property that

$$\rho(g)\lambda(h) = \lambda(h)\rho(g)$$

for all $g, h \in \Gamma$. Taking linear combinations and limits, the same commutation relation holds with $\lambda(h)$ replaced by an element of $L(\Gamma)$. Therefore, we have for any $g \in \Gamma$,

$$\begin{aligned} x\delta_g &= x\rho(g^{-1})\delta_e \\ &= \rho(g^{-1})x\delta_e \\ &= \rho(g^{-1})\left(\sum_{h \in \Gamma} \hat{x}(h)\delta_h\right) \\ &= \rho(g^{-1})\left(\sum_{h \in \Gamma} \hat{y}(h)\delta_h\right) \\ &= \rho(g^{-1})y\delta_e \\ &= y\rho(g^{-1})\delta_e \\ &= y\delta_g. \end{aligned}$$

The two bounded operators x and y therefore coincide on a Hilbert basis, hence are equal. ■

Based on this, one should think of x as the formal series

$$x = \sum_{g \in \Gamma} \hat{x}(g)\lambda(g),$$

keeping in mind that this series does not converge in any reasonable operator sense in general, just as the Fourier series of a bounded function need not converge uniformly.

In fact, the Fourier analogy can be pushed further when Γ is abelian. Recall that in that case, the *Pontryagin dual* $\hat{\Gamma}$ is the set of all group homomorphisms $\chi : \Gamma \rightarrow \mathbb{T}$, where $\mathbb{T}$ is the unit circle (see for instance [20]). It is a compact abelian group for pointwise multiplication and pointwise convergence, and we endow it with its normalised Haar measure μ . For instance, $\hat{\mathbb{Z}} \simeq \mathbb{T}$ through $\chi \mapsto \chi(1)$, and μ is then the normalised Lebesgue measure on the circle.

Proposition 1.4. *Let Γ be an abelian group. Then, $L(\Gamma) \simeq L^\infty(\widehat{\Gamma}, \mu)$, where the latter acts on $L^2(\widehat{\Gamma}, \mu)$ by multiplication.*

Proof. For $g \in \Gamma$, let e_g be the evaluation function $\chi \mapsto \chi(g)$ on $\widehat{\Gamma}$. By Pontryagin duality, these are exactly the characters of the compact group $\widehat{\Gamma}$, hence they form an orthonormal basis of $L^2(\widehat{\Gamma}, \mu)$. As a consequence, the *Fourier transform*

$$\mathcal{F} : \ell^2(\Gamma) \rightarrow L^2(\widehat{\Gamma}, \mu), \quad \delta_g \mapsto e_g$$

is a unitary isomorphism. Moreover, $e_g e_h = e_{gh}$, so that $\mathcal{F} \lambda(g) \mathcal{F}^*$ is the operator of multiplication by e_g . Therefore, $\mathcal{F} L(\Gamma) \mathcal{F}^*$ is the weak closure of the algebra of “trigonometric polynomials”, i.e. linear combinations of characters, acting by multiplication. By the STONE–WEIERSTRASS THEOREM, these are uniformly dense in $C(\widehat{\Gamma})$, and $C(\widehat{\Gamma})$ is weakly dense in $L^\infty(\widehat{\Gamma}, \mu)$, for instance by LUSIN’S THEOREM. Since $L^\infty(\widehat{\Gamma}, \mu)$ is itself weakly closed in $\mathcal{B}(L^2(\widehat{\Gamma}, \mu))$ (see for instance [27, Thm III.1.2]), the result follows. ■

Under this isomorphism, the Fourier coefficients $\widehat{x}(g)$ of an element $x \in L(\Gamma)$ are exactly the Fourier coefficients of the corresponding function on $\widehat{\Gamma}$. For $\Gamma = \mathbb{Z}$, we recover the classical theory of Fourier series : $L(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$, and $\lambda(n)$ corresponds to the function $z \mapsto z^n$.

We can in fact go further and use Proposition 1.4 to identify the von Neumann algebras of all infinite abelian groups. Indeed, if Γ is infinite, then $\widehat{\Gamma}$ is an infinite compact group, so that its Haar measure has no atoms². Moreover, $\widehat{\Gamma}$ is metrisable because Γ is countable, hence it is a *standard Borel space*. Now, measure theory tells us that all standard probability spaces without atoms are isomorphic as measure spaces to $[0, 1]$ with the Lebesgue measure (see for instance [17, Theorem 17.41]). As a consequence,

$$L(\mathbb{Z}) \simeq L(\mathbb{Z}^d) \simeq L\left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}\right) \simeq L^\infty([0, 1]).$$

In other words, the von Neumann algebra of an infinite abelian group remembers nothing but the fact that the group is infinite and abelian.

1.2 THE QUESTION

This leads us to the following motivating problem :

Question. *Given two non-isomorphic groups Γ and Γ' , are $L(\Gamma)$ and $L(\Gamma')$ non-isomorphic?*

We have already seen that the answer is no for infinite abelian groups. It is also easy to understand what happens for finite groups. First, observe that because $\lambda(g)\delta_e = \delta_g$, the elements $\lambda(g)$ are linearly independent, hence $L(\Gamma)$ is finite-dimensional if and only if Γ is finite, in which case $L(\Gamma) = \mathbb{C}[\Gamma]$. In that case, the structure is completely described by representation theory. Recall that a finite group has finitely many irreducible representations up to equivalence, which we denote by $\pi_1, \dots, \pi_k$, with dimensions $d_1, \dots, d_k$.

Proposition 1.5. *Let Γ be a finite group. Then,*

$$L(\Gamma) \simeq \bigoplus_{i=1}^k M_{d_i}(\mathbb{C}).$$

²If a point had measure $c > 0$, then by invariance so would every point, and $n > 1/c$ distinct points would have total measure larger than $\mu(\widehat{\Gamma}) = 1$.

Proof. Each π_i extends linearly to $\mathbb{C}[\Gamma]$, and this yields a $*$ -homomorphism

$$\begin{aligned}\Phi : \mathbb{C}[\Gamma] &\rightarrow \bigoplus_{i=1}^k M_{d_i}(\mathbb{C}) \\ x &\mapsto (\pi_1(x), \dots, \pi_k(x)).\end{aligned}$$

Since the left regular representation decomposes as a direct sum in which each π_i appears d_i times, $\Phi(x) = 0$ implies that x acts by 0 on $\ell^2(\Gamma)$, i.e. $x = 0$. Thus Φ is injective, and it is surjective because both sides have dimension $|\Gamma| = d_1^2 + \dots + d_k^2$. ■

In other words, the von Neumann algebra of a finite group only remembers the dimensions of its irreducible representations. For instance, all abelian groups of order n have von Neumann algebra $\mathbb{C}^n$, and the dihedral group D_4 and the quaternion group Q_8 , which both have four one-dimensional and one two-dimensional irreducible representations, satisfy

$$L(D_4) \simeq \mathbb{C}^4 \oplus M_2(\mathbb{C}) \simeq L(Q_8)$$

and can therefore not be distinguished by their von Neumann algebras.

The temporary conclusion of this is that if we want to be able to distinguish von Neumann algebras of groups, we should require them to be infinite and non-abelian in a sufficiently strong sense. It has quickly been recognised that the correct notion is the following one.

DEFINITION 1.6. A group Γ has *infinite conjugacy classes*, or is *i.c.c.*, if for every $g \neq e$, the conjugacy class $\{hgh^{-1} \mid h \in \Gamma\}$ is infinite³.

Let us give some examples.

Example 1.7. Let $2 \leq N \leq +\infty$. Then, the free group $\mathbb{F}_N$ is i.c.c. Indeed, recall that every element of $\mathbb{F}_N$ can be written in a unique way as a *reduced word*, i.e. a word in the free generators and their inverses in which no letter is immediately followed by its own inverse. Let $g \neq e$, let $\ell \geq 1$ be the length of its reduced form, and let x and y be respectively the first and last letters of this reduced form. Because $N \geq 2$, there are at least four letters, so that we can choose a letter s such that $s \neq x^{-1}$ and $s \neq y$. Then, for every $n \in \mathbb{N}$, the word $s^n g s^{-n}$ is reduced. Its length is therefore $\ell + 2n$, so that the elements $s^n g s^{-n}$ for $n \in \mathbb{N}$ are pairwise distinct, and the conjugacy class of g is infinite.

Example 1.8. Let S_∞ be the group of all permutations σ of $\{1, 2, 3, \dots\}$ which are *finitely supported*, i.e. such that the *support* $\text{supp}(\sigma) = \{i \mid \sigma(i) \neq i\}$ is finite. Then, S_∞ is i.c.c. Indeed, let $\sigma \neq e$ and pick i such that $j = \sigma(i) \neq i$. Note that $j \in \text{supp}(\sigma)$ as well, since $\sigma(j) = j = \sigma(i)$ would contradict injectivity. For every $k \notin \text{supp}(\sigma)$, let τ_k be the transposition exchanging j and k . Since i is different from both j and k , we have $\tau_k(i) = i$, hence

$$\begin{aligned}\tau_k \sigma \tau_k^{-1}(i) &= \tau_k \sigma(i) \\ &= \tau_k(j) \\ &= k.\end{aligned}$$

As a consequence, the conjugates $\tau_k \sigma \tau_k^{-1}$ for $k \notin \text{supp}(\sigma)$ are pairwise distinct, and since $\text{supp}(\sigma)$ is finite, there are infinitely many of them.

What makes being i.c.c. interesting is that it has a simple characterisation at the level of the von Neumann algebra.

³Note that in particular a non-trivial i.c.c. group is infinite.

Proposition 1.9. *A non-trivial group Γ is i.c.c. if and only if $L(\Gamma)$ is infinite-dimensional and has trivial centre, i.e.*

$$\mathcal{Z}(L(\Gamma)) = \{x \in L(\Gamma) \mid xy = yx \text{ for all } y \in L(\Gamma)\}$$

is reduced to $\mathbb{C}1$.

Proof. Assume first that Γ is not i.c.c., so that there exists $g \neq e$ with a finite conjugacy class C . Then, the element

$$z = \sum_{h \in C} \lambda(h)$$

belongs to $L(\Gamma)$, and since

$$\lambda(k)z\lambda(k)^{-1} = \sum_{h \in C} \lambda(khk^{-1}) = z$$

for all $k \in \Gamma$, it commutes with all the generators, hence with all of $L(\Gamma)$ by weak continuity. But it is not a multiple of the identity because

$$z\delta_e = \sum_{h \in C} \delta_h$$

is not proportional to δ_e . Thus $L(\Gamma)$ has non-trivial centre.

Conversely, assume that Γ is i.c.c. Then, it is infinite so that $L(\Gamma)$ is infinite-dimensional. Moreover, if z is central, then writing the equality

$$\lambda(g)z\lambda(g)^{-1}\delta_e = z\delta_e$$

in terms of Fourier coefficients yields

$$\widehat{z}(ghg^{-1}) = \widehat{z}(h)$$

for all $g, h \in \Gamma$. Hence, $\widehat{z}$ is constant on conjugacy classes. But it is also square-summable, so that it must vanish on all infinite conjugacy classes. Since Γ is i.c.c., this means that $\widehat{z}$ is supported on $\{e\}$, hence

$$z = \widehat{z}(e) \cdot 1 \in \mathbb{C}1.$$

■

For an infinite i.c.c. group, $L(\Gamma)$ is called a II_1 -factor. We have mentioned free groups as basic examples of i.c.c. groups, and the following question is natural.

Question. *Do we have $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m)$ for $n \neq m$?*

This is the *free group factor problem*, and it is one of the major open problems in the theory. It motivated D.-V. VOICULESCU to develop *free probability theory* [28], a non-commutative analogue of probability theory in which generators of the free groups play the role that independent random variables play in classical probability. Building on this theory, K. DYKEMA [11] and F. RĂDULESCU [26] independently proved the following surprising dichotomy.

THEOREM 1.10 (DYKEMA, RĂDULESCU) Exactly one of the following holds :

- All the factors $L(\mathbb{F}_n)$ for $2 \leq n \leq +\infty$ are isomorphic;
- All the factors $L(\mathbb{F}_n)$ for $2 \leq n \leq +\infty$ are pairwise non-isomorphic.

An AI model claims a proof that all these are isomorphic [24].

2 DID YOU SAY RIGID ?

2.1 THE AMENABILITY ISSUE

To understand why we are looking for extra properties that could enable us to distinguish group von Neumann algebras, it is enlightening to see first which property prevents any such distinction. It is one of the most fundamental notions of group theory, called *amenability*. There are many equivalent definitions of it, and we will use the one which is best suited to the approximation properties which will enter the picture later on. Let us start by recalling an elementary notion. We refer the reader to [3, Sec 2.5] for details.

DEFINITION 2.1. A function $\varphi : \Gamma \rightarrow \mathbb{C}$ is *positive definite* if for all $g_1, \dots, g_n \in \Gamma$, the matrix

$$[\varphi(g_i^{-1}g_j)]_{1 \leq i, j \leq n}$$

is positive semidefinite.

The typical example is a *coefficient* of a unitary representation : if $\pi : \Gamma \rightarrow \mathcal{U}(H)$ is a unitary representation and $\xi \in H$, then $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ is positive definite, since

$$[\varphi(g_i^{-1}g_j)]_{1 \leq i, j \leq n} = [\langle \pi(g_j)\xi, \pi(g_i)\xi \rangle]_{1 \leq i, j \leq n}$$

is the Gram matrix of the vectors $\pi(g_1)\xi, \dots, \pi(g_n)\xi$, hence is positive semidefinite. Conversely, every positive definite function is of this form (this is the so-called *GNS construction*).

DEFINITION 2.2. A group Γ is *amenable* if there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of finitely supported positive definite functions on Γ such that $\varphi_n(e) = 1$ for all n and

$$\varphi_n(g) \xrightarrow{n \rightarrow +\infty} 1$$

for every $g \in \Gamma$.

Finite groups are obviously amenable (take $\varphi_n = 1$), and the fundamental example of an infinite amenable group is $\mathbb{Z}$.

Example 2.3. For $n \in \mathbb{N}$, consider the unit vector

$$\xi_n = (n+1)^{-1/2} \sum_{k=0}^n \delta_k \in \ell^2(\mathbb{Z})$$

and the associated coefficient of the left regular representation

$$\varphi_n(k) = \langle \lambda(k)\xi_n, \xi_n \rangle = \frac{|[0, n] \cap [k, k+n]|}{n+1} = \begin{cases} 1 - \frac{|k|}{n+1} & \text{if } |k| \leq n \\ 0 & \text{otherwise.} \end{cases}$$

These functions are positive definite, finitely supported, satisfy $\varphi_n(0) = 1$, and converge pointwise to 1. Hence, $\mathbb{Z}$ is amenable. Note that the functions φ_n are old friends in disguise : they are the Fourier coefficients of the *Fejér kernel*

$$K_n(z) = \sum_{|k| \leq n} \left(1 - \frac{|k|}{n+1}\right) z^k = \frac{1}{n+1} \left| \sum_{k=0}^n z^k \right|^2.$$

Another easy example is the group S_∞ : it is the increasing union of the finite subgroups S_n of permutations of $\{1, \dots, n\}$, and the indicator function of a subgroup H is always positive definite : it is the coefficient $\langle \pi(g)\delta_H, \delta_H \rangle$ of the unitary representation π of Γ on $\ell^2(\Gamma/H)$ by translation, where δ_H is the point mass at the trivial coset, so that $\varphi_n = \mathbf{1}_{S_n}$ does the job. More generally, all abelian groups and all solvable groups (such as the group $\mathbb{Z}^2 \rtimes \mathbb{Z}$ appearing below) are amenable, while the free groups $\mathbb{F}_N$ for $N \geq 2$ are not, see for instance [3, Ex 2.6.7].

Amenability, though extremely useful in many situations, is as bad as commutativity for the problem we are interested in, due to the following combination of results of F. J. MURRAY and J. VON NEUMANN [22] (uniqueness of the hyperfinite II_1 -factor) and of A. CONNES [5] (amenable i.c.c. groups yield the hyperfinite II_1 -factor).

THEOREM 2.4 (MURRAY–VON NEUMANN, CONNES) Let Γ and Γ' be two infinite i.c.c. amenable groups. Then,

$$L(\Gamma) \simeq L(\Gamma').$$

More precisely, all these factors are isomorphic to the so-called *hyperfinite II_1 -factor* R , which can be obtained as a suitable limit of the matrix algebras $M_{2^n}(\mathbb{C})$. Once again, the von Neumann algebra forgets almost everything : for instance,

$$L(S_\infty) \simeq L(\mathbb{Z}^2 \rtimes \mathbb{Z}),$$

where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ through the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

2.2 PROPERTY (T)

Property (T) was introduced by D. KAZHDAN in [16] as a rigidity property of unitary representations of groups. The original definition has an equivalent statement in terms of positive definite functions (see [2] for the equivalence), which is the one we will use.

DEFINITION 2.5. A group Γ has *property (T)* if every sequence $(\varphi_n)_{n \in \mathbb{N}}$ of positive definite functions on Γ such that $\varphi_n(e) = 1$ and which converges pointwise to 1 in fact converges *uniformly* to 1 on Γ .

In other words, a group has property (T) if being close to 1 on a finite set forces a positive definite function to be close to 1 everywhere. The comparison with the definition of amenability is striking : amenability requires the existence of positive definite functions which converge pointwise to 1 while vanishing outside a finite set, which is as far from uniform convergence as possible. Here are some basic examples and non-examples.

- Finite groups have property (T), since pointwise and uniform convergence are then the same thing.
- Infinite amenable groups never have property (T) : the functions φ_n from the definition of amenability converge pointwise to 1, but $\sup_{g \in \Gamma} |1 - \varphi_n(g)| = 1$ for all n because φ_n is finitely supported and Γ is infinite.
- The free groups $\mathbb{F}_N$ do not have property (T). Indeed, let $|g|$ denote the length of the reduced word representing $g \in \mathbb{F}_N$, which is also the distance from e to g in the Cayley graph of $\mathbb{F}_N$ with respect to its free generators. This Cayley graph is a regular tree, and a result of U. HAAGERUP [13] asserts that for every $t > 0$, the function

$$\varphi_t(g) = e^{-t|g|}$$

is positive definite. The idea is that the distance between two vertices of a tree is the number of edges separating them, which allows one to write it as $\|c(x) - c(y)\|^2$ for a suitable map c from the vertices to a Hilbert space, and a classical theorem of I. SCHOENBERG then ensures that $e^{-t\|c(x) - c(y)\|^2}$ is a positive definite kernel (see [3, Sec 12.2] for details). Now $\varphi_t(e) = 1$ and $\varphi_t \rightarrow 1$ pointwise as $t \rightarrow 0$, but for each t , $\varphi_t(g) \rightarrow 0$ as $|g| \rightarrow +\infty$, so that the convergence is not uniform. This property of $\mathbb{F}_N$ (the existence of positive definite functions converging pointwise to 1 and vanishing at infinity) is called the *Haagerup property*, and it is a negation of property (T) for infinite groups, weaker than amenability.

Note also that if Γ has property (T) and $q : \Gamma \rightarrow \Gamma'$ is a surjective group homomorphism, then Γ' has property (T) as well, since $\varphi \circ q$ is positive definite on Γ whenever φ is positive definite on Γ' , and $\varphi_n \circ q$ converges uniformly if and only if φ_n does. The fundamental examples are the following ones.

THEOREM 2.6 (KAZHDAN) For $n \geq 3$, the group $\mathrm{SL}_n(\mathbb{Z})$ has property (T), and so does its quotient $\mathrm{PSL}_n(\mathbb{Z})$.

More generally, lattices in simple Lie groups of real rank at least 2 have property (T), and so do lattices in the rank one groups $\mathrm{Sp}(n, 1)$ ⁴ for $n \geq 2$ by a result of B. KOSTANT [18]. On the other hand, $\mathrm{SL}_2(\mathbb{Z})$ does not have property (T), since it contains a free group of finite index. We refer the reader to [2] for a comprehensive treatment of property (T), including proofs of all the assertions above.

Property (T) is relevant to our problem because it survives the passage to the von Neumann algebra : A. CONNES introduced in [6] a notion of property (T) for II_1 -factors and observed that, for an i.c.c. group Γ , the factor $L(\Gamma)$ has property (T) if and only if Γ has it, the theory being then developed by A. CONNES and V. JONES in [7]. As a consequence, we get the following result.

THEOREM 2.7 (CONNES, CONNES–JONES) Let Γ and Γ' be i.c.c. groups such that $L(\Gamma) \simeq L(\Gamma')$. If Γ has property (T), then Γ' also has property (T).

This is our first really non-trivial invariant. For instance, it implies that $L(\mathrm{PSL}_3(\mathbb{Z}))$ is neither isomorphic to $L(\Gamma)$ for an infinite i.c.c. amenable group Γ , nor to a free group factor $L(\mathbb{F}_n)$.

2.3 CONNES' RIGIDITY PROBLEM

Property (T) factors turn out to be rigid in several respects. In the introduction of [6], right after introducing property (T) for factors, A. CONNES lists among the open questions internal to the theory the study of the von Neumann algebras of “rigid” discrete groups, “say Γ of finite covolume in a simple Lie group of rank ≥ 2 ”, and writes⁵ [6, p. 45] (with our notations) :

⁴Let $\mathbb{H}$ be the skew-field of quaternions and consider on $\mathbb{H}^{n+1}$ the quaternionic Hermitian form $\langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n - \bar{x}_{n+1} y_{n+1}$ of signature $(n, 1)$. Then, $\mathrm{Sp}(n, 1)$ is the group of $(n+1) \times (n+1)$ quaternionic matrices preserving this form. It is a simple Lie group of real rank one, which acts by isometries on the quaternionic hyperbolic space of (quaternionic) dimension n , just as $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ act on the real and complex hyperbolic spaces. Recall also that a *lattice* in a Lie group G is a discrete subgroup Γ such that G/Γ has finite volume, the fundamental example being $\mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$.

⁵Our translation. The original reads : “*Problème*. Montrer que si Γ_1, Γ_2 sont non isomorphes $R(\Gamma_1)$ et $R(\Gamma_2)$ sont non isomorphes. Bien sûr ici Γ_1 et Γ_2 sont rigides au sens ci-dessus”.

Problem. Show that if Γ_1, Γ_2 are non-isomorphic, then $L(\Gamma_1)$ and $L(\Gamma_2)$ are non-isomorphic. Of course here Γ_1 and Γ_2 are rigid in the above sense.

This is exactly our guiding question, restricted to rigid groups. The word “rigid” can be understood either in the sense of Connes’ example (lattices in higher rank simple Lie groups), or as “having property (T)”, which is the property discussed in the same paragraph. For the most natural examples of higher rank lattices, the problem specialises to the following question.

Question. Can we have $L(\mathrm{PSL}_n(\mathbb{Z})) \simeq L(\mathrm{PSL}_m(\mathbb{Z}))$ for $n \neq m$?

Remark 2.8. The use of projective groups is necessary because for n even, $\{-\mathrm{Id}\}$ is a non-trivial conjugacy class. Let us add that $\mathrm{PSL}_n(\mathbb{Z})$ is known to be i.c.c for all $n \geq 2$ and have property (T) for all $n \geq 3$.

A claimed negative answer was released in [24]. The broadest reading of Connes’ problem is the form under which it is usually quoted in the literature, as *Connes’ rigidity conjecture* (see for instance [4]). It states that in the property (T) world, the von Neumann algebra should remember *everything*.

Question. Let Γ and Γ' be i.c.c. groups with property (T). If $L(\Gamma) \simeq L(\Gamma')$, do we have $\Gamma \simeq \Gamma'$?

Groups whose von Neumann algebra remembers them in this strong sense are called *W*-superrigid*. It took a long time to find even a single example of such a group : the first ones were obtained by A. IOANA, S. POPA and S. VAES in [15], using Popa’s deformation/rigidity theory, and the first examples with property (T) are due to I. CHIFAN, A. IOANA, D. OSIN and B. SUN [4]. However, counterexamples to this general formulation of the conjecture have been announced : one was obtained by an AI model [23], and another one, independently, by S. ZHOU [30], also with the help of AI tools. In the construction of [30], the two groups are semi-direct products $\Gamma_i = D \rtimes_{\theta_i} H$ of an abelian group D by the same group H . Using the Fourier transform on D , one gets

$$L(D \rtimes_{\theta_i} H) \simeq L^\infty(\widehat{D}) \rtimes H,$$

and the phenomenon that we observed for abelian groups can then be exploited to identify the two von Neumann algebras, while the groups remain non-isomorphic.

3 A BANACH SPACE APPROACH

In view of what precedes, we have to find invariants of group von Neumann algebras which can be computed and compared. The first result distinguishing infinitely many property (T) factors follows this strategy and is due to M. COWLING and U. HAAGERUP [8].

THEOREM 3.1 (COWLING–HAAGERUP) Let Γ be a lattice in $\mathrm{Sp}(n, 1)$ and Γ' a lattice in $\mathrm{Sp}(m, 1)$. If $L(\Gamma) \simeq L(\Gamma')$, then $n = m$.

Since these lattices have property (T) for $n, m \geq 2$, this yields infinitely many pairwise non-isomorphic property (T) factors. Let us explain the idea of the proof. It relies on the approximation of the identity map of $L(\Gamma)$ by finite rank maps of a specific form,

borrowed from classical harmonic analysis. Given a function $\varphi : \Gamma \rightarrow \mathbb{C}$, we can try to define a *Fourier multiplier* on $L(\Gamma)$ by the formula

$$m_\varphi \left(\sum_{g \in \Gamma} \hat{x}(g) \lambda(g) \right) = \sum_{g \in \Gamma} \varphi(g) \hat{x}(g) \lambda(g),$$

exactly as one does for $\Gamma = \mathbb{Z}$. This makes sense at least on the group algebra $\mathbb{C}[\Gamma]$, and the question is whether m_φ extends to a bounded map on $L(\Gamma)$. If φ is finitely supported, then m_φ is well-defined on all of $L(\Gamma)$ and has finite rank, since its range is contained in $\text{Span}\{\lambda(g) \mid \varphi(g) \neq 0\}$, and if φ is close to 1 on a large set, then m_φ should be close to the identity. To measure how good such an approximation is, M. COWLING and U. HAAGERUP used a norm which is slightly stronger than the operator norm. For every $k \geq 1$, the space $M_k(L(\Gamma))$ of $k \times k$ matrices with entries in $L(\Gamma)$ acts on $\ell^2(\Gamma)^k$, hence carries a natural operator norm, and any linear map $T : L(\Gamma) \rightarrow L(\Gamma)$ can be *amplified* into a map

$$\text{id}_k \otimes T : M_k(L(\Gamma)) \rightarrow M_k(L(\Gamma)).$$

The map T is said to be *completely bounded* if

$$\|T\|_{cb} = \sup_{k \geq 1} \|\text{id}_k \otimes T\| < +\infty.$$

DEFINITION 3.2. The *Cowling–Haagerup constant* $\Lambda_{cb}(\Gamma) \in [1, +\infty]$ is the infimum of the constants C such that there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of finitely supported functions on Γ converging pointwise to 1 with $\|m_{\varphi_n}\|_{cb} \leq C$ for all $n \in \mathbb{N}$ (with $\Lambda_{cb}(\Gamma) = +\infty$ if no such sequence exists).

Let us stress a crucial feature of this definition : even though $L(\Gamma)$ is an algebra with an involution, neither the product nor the adjoint plays any role in the definition of Λ_{cb} . All that is used is the vector space $L(\Gamma)$, together with the norms on its matrix amplifications $M_k(L(\Gamma))$. Such a collection of matrix norms is called an *operator space structure*, and the Cowling–Haagerup constant is really an invariant of the operator space $L(\Gamma)$. This observation will be the key to the passage to other spaces in the sequel.

The other fundamental fact is that, although the definition involves the group through the multipliers m_φ , the constant $\Lambda_{cb}(\Gamma)$ only depends on the von Neumann algebra $L(\Gamma)$: it is also the best constant C for which the identity map of $L(\Gamma)$ can be approximated pointwise (in the weak operator topology) by weak*-continuous finite rank maps T with $\|T\|_{cb} \leq C$, see [3, Sec 12.3]. One inequality is clear, and the other one is obtained by averaging an arbitrary finite rank map into a multiplier. As a consequence, if $L(\Gamma) \simeq L(\Gamma')$, then $\Lambda_{cb}(\Gamma) = \Lambda_{cb}(\Gamma')$. For instance, $\Lambda_{cb}(\Gamma) = 1$ if Γ is amenable (see Proposition 3.6 below), and U. HAAGERUP proved in [13] that $\Lambda_{cb}(\mathbb{F}_N) = 1$ as well, using the functions φ_t of Section 2.2. The theorem then follows from the computation, which is the heart of [8], that

$$\Lambda_{cb}(\Gamma) = 2n - 1$$

for any lattice Γ in $\text{Sp}(n, 1)$.

Unfortunately, this invariant is useless for Connes’ problem : for $n \geq 3$, one has $\Lambda_{cb}(\text{PSL}_n(\mathbb{Z})) = +\infty$, meaning that no controlled approximation by finite rank maps exists, and in fact these groups fail even weaker approximation properties by results of V. LAFFORGUE and M. DE LA SALLE [19] and U. HAAGERUP and T. DE LAAT [14]. The idea of looking at approximation properties might nevertheless be a good one, but it has to be applied to other objects built from $L(\Gamma)$. This is what we now explain.

3.1 THE TRACE AND NON-COMMUTATIVE L^p -SPACES

The new idea is to exploit less structure on $L(\Gamma)$ – as with the Cowling–Haagerup constant – but on more spaces. To build these additional spaces, we will take inspiration from measure theory. Namely, let us define

$$\tau(x) = \langle x\delta_e, \delta_e \rangle = \widehat{x}(e).$$

One checks that $\tau(1) = 1$, that $\tau(x^*x) > 0$ for $x \neq 0$, and that

$$\tau(xy) = \tau(yx)$$

for all $x, y \in L(\Gamma)$ ⁶. In technical terms, τ is a *faithful normal tracial state*. When $\Gamma = \mathbb{Z}$, it corresponds under the Fourier transform to the integral $f \mapsto \int_{\mathbb{T}} f$ with respect to the Lebesgue measure. The existence of such a trace is precisely what it means for the factor to be of type II_1 , and a fundamental result of F. J. MURRAY and J. VON NEUMANN is that it is then unique (see for instance [1]). This yields :

Proposition 3.3. *Let Γ and Γ' be i.c.c. groups with traces τ and τ' respectively. If $\pi : L(\Gamma) \rightarrow L(\Gamma')$ is an isomorphism, then $\tau' \circ \pi = \tau$.*

Now that we have an analogue of an integral, we can do analysis. Just as one defines L^p -spaces of functions from the Lebesgue integral, one can define L^p -spaces for the trace τ . This goes back to I. SEGAL and J. DIXMIER, and we refer to [25] for a modern account.

DEFINITION 3.4. For $1 \leq p < +\infty$, the *non-commutative L^p -space* $L^p(L(\Gamma), \tau)$ is the completion of $L(\Gamma)$ for the norm⁷

$$\|x\|_p = \tau(|x|^p)^{1/p},$$

where $|x| = (x^*x)^{1/2}$. We also set $L^\infty(L(\Gamma), \tau) = L(\Gamma)$ with the operator norm.

Here are some sanity checks : for Γ abelian we recover the usual spaces $L^p(\widehat{\Gamma})$, for $p = 2$ we recover the Hilbert space $\ell^2(\Gamma)$, and the familiar properties (Hölder's inequality, duality between L^p and $L^{p'}$ with $1/p + 1/p' = 1$, interpolation) all hold. The point of this construction for us is the following straightforward consequence of the previous proposition.

Corollary 3.5. *If Γ and Γ' are i.c.c. and $L(\Gamma) \simeq L(\Gamma')$, then $L^p(L(\Gamma), \tau) \simeq L^p(L(\Gamma'), \tau')$ isometrically (in fact completely isometrically) for all $1 \leq p \leq +\infty$.*

In other words, the whole family of Banach spaces $(L^p(L(\Gamma), \tau))_{1 \leq p \leq \infty}$ is an invariant of the von Neumann algebra. The case $p = \infty$ is the von Neumann algebra itself, where the Cowling–Haagerup constant lives, while the case $p = 2$ is a Hilbert space, hence carries no information. The idea is that there should be interesting information hiding in between.

3.2 APPROXIMATION PROPERTIES OF NON-COMMUTATIVE L^p -SPACES

The strategy is now to transfer the ideas of M. COWLING and U. HAAGERUP to non-commutative L^p -spaces : we ask whether the identity map of $L^p(L(\Gamma), \tau)$ can be approximated pointwise by finite rank maps in a controlled way. As we stressed above, the Cowling–Haagerup constant does not use the algebra structure of $L(\Gamma)$, but only the possibility of amplifying it to matrices. This is precisely what still makes sense for

⁶It is enough to check it on $\lambda(g)$ and $\lambda(h)$, for which both sides are equal to 1 if $gh = e$ and 0 otherwise.

⁷It is not completely trivial to check that the formula below indeed defines a norm. This requires proving first a non-commutative version of Young and Hölder inequalities.

$L^p(L(\Gamma), \tau)$, which is no longer an algebra : for every $k \geq 1$, the space $M_k(L^p(L(\Gamma), \tau))$ can be identified with the non-commutative L^p -space $L^p(M_k(L(\Gamma)), \text{Tr} \otimes \tau)$, hence carries a natural norm. This yields an operator space structure on $L^p(L(\Gamma), \tau)$, due to G. PISIER (see [25]), and a linear map T on $L^p(L(\Gamma), \tau)$ is then said to be *completely bounded* if the amplifications $\text{id}_k \otimes T$ are uniformly bounded, with $\|T\|_{cb}$ the supremum of their norms.

Fourier multipliers also make sense on $L^p(L(\Gamma), \tau)$: the formula defining m_φ makes sense on the group algebra $\mathbb{C}[\Gamma]$, which is dense in $L^p(L(\Gamma), \tau)$ for $p < +\infty$, and the question is whether it extends to a bounded (or completely bounded) map. As before, positive definite functions provide the best possible situation.

Proposition 3.6. *If φ is positive definite and $\varphi(e) = 1$, then m_φ extends to a complete contraction (i.e. $\|\text{id}_k \otimes m_\varphi\| \leq 1$ for all $k \geq 1$) on $L^p(L(\Gamma), \tau)$ for all $1 \leq p \leq +\infty$.*

Proof. Let us start with $p = \infty$. Write $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ with $\|\xi\| = 1$ and let

$$V : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma) \otimes H$$

be the isometry defined by $V\delta_h = \delta_h \otimes \pi(h)^*\xi$. A direct computation gives, for all $g, h \in \Gamma$,

$$V^*(\lambda(g) \otimes 1)V\delta_h = \langle \pi(h)^*\xi, \pi(gh)^*\xi \rangle \delta_{gh} = \varphi(g)\lambda(g)\delta_h.$$

Hence, $m_\varphi(x) = V^*(x \otimes 1)V$ for all $x \in \mathbb{C}[\Gamma]$, and the right-hand side makes sense and takes values in $L(\Gamma)$ for every $x \in L(\Gamma)$. Maps of this form are completely positive and have cb-norm at most $\|V\|^2 = 1$. Next, one checks on the elements $\lambda(g)$ that

$$\tau(m_\varphi(x)y^*) = \tau(xm_{\bar{\varphi}}(y)^*),$$

and $\bar{\varphi}$ is also positive definite. Hence, by duality, m_φ is a complete contraction on $L^1(L(\Gamma), \tau)$, and the case of general p follows by interpolation. ■

As a consequence, if Γ is amenable, the multipliers m_{φ_n} associated to the functions from the definition of amenability are finite rank complete contractions on $L^p(L(\Gamma), \tau)$ which converge to the identity pointwise (because they obviously do so on the dense subspace $\mathbb{C}[\Gamma]$ and are uniformly bounded) for finite p . For $\Gamma = \mathbb{Z}$ and the functions of Example 2.3, this is nothing but FEJÉR'S THEOREM : under the isomorphism $L(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$, m_{φ_n} becomes the convolution $f \mapsto K_n * f$, i.e. the map sending a function to the Cesàro means of its Fourier series, and these converge to f in $L^p(\mathbb{T})$ for $1 \leq p < \infty$. In other words, for amenable groups all the non-commutative L^p -spaces have the best possible approximation properties (and in the case $p = \infty$, this shows that $\Lambda_{cb}(\Gamma) = 1$).

For non-amenable groups, there is no hope of using finitely supported positive definite functions, and we need a notion of approximation which does not require the approximating maps to be contractions, nor even uniformly bounded. An interesting candidate was introduced by E. EFFROS and Z.-J. RUAN (see [12]) in the general setting of operator spaces. Let $S^p[L^p(L(\Gamma))]$ denote the non-commutative L^p -space of the weak operator closure $\mathcal{B}(\ell^2(\mathbb{N})) \bar{\otimes} L(\Gamma)$ of the algebraic tensor product

$$\mathcal{B}(\ell^2(\mathbb{N})) \otimes L(\Gamma) \subset \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^2(\Gamma)$$

with respect to $\text{Tr} \otimes \tau$. Its elements should be thought of as infinite matrices with entries in $L^p(L(\Gamma), \tau)$, so that this is simply the infinite-dimensional version of the amplifications considered above. Every finite rank map T on $L^p(L(\Gamma), \tau)$ then extends to a bounded map $\text{id}_{\ell^2(\mathbb{N})} \otimes T$ on $S^p[L^p(L(\Gamma))]$, acting entrywise.

DEFINITION 3.7. Let $1 \leq p < +\infty$. The space $L^p(L(\Gamma), \tau)$ has the *operator approximation property* (OAP for short) if there exists a net $(T_i)_i$ of finite rank maps on $L^p(L(\Gamma), \tau)$ such that for every $x \in S^p[L^p(L(\Gamma))]$,

$$\|(\text{id}_{\ell^2(\mathbb{N})} \otimes T_i)(x) - x\|_p \longrightarrow 0.$$

In other words, the identity map can be approximated pointwise by finite rank maps, not only on $L^p(L(\Gamma), \tau)$ but also on its matrix-valued version. Contrary to the definition of the Cowling–Haagerup constant, no uniform bound on the maps T_i is required, but just as there, only the operator space structure of $L^p(L(\Gamma), \tau)$ is involved. Here are some basic facts :

- For $p = 2$, the OAP always holds : $S^2[L^2(L(\Gamma))]$ is simply the Hilbert space $\ell^2(\mathbb{N} \times \Gamma)$, and one can take for T_i the orthogonal projections onto $\text{Span}\{\delta_g \mid g \in F_i\}$ for an increasing sequence of finite sets F_i exhausting Γ (these are the multipliers associated to the indicator functions of the sets F_i).
- If Γ is amenable, the OAP holds for all p : by Proposition 3.6, the maps $\text{id}_{\ell^2(\mathbb{N})} \otimes m_{\varphi_n}$ are contractions on $S^p[L^p(L(\Gamma))]$, and they converge to the identity on a dense subspace, hence everywhere.
- For group von Neumann algebras, an averaging argument shows that if the OAP holds, then it can be witnessed by Fourier multipliers $T_i = m_{\varphi_i}$ associated to finitely supported functions φ_i (see [3, Thm 12.4.9]).

The last point shows that deciding whether $L^p(L(\Gamma), \tau)$ has the OAP is a genuinely hard problem of harmonic analysis on groups, analogous to the classical problem of determining which functions define bounded Fourier multipliers on $L^p(\mathbb{R}^d)$. Moreover, the definition only involves the Banach space $L^p(L(\Gamma), \tau)$ together with its matrix-valued versions, which are also preserved by isomorphisms of von Neumann algebras (they are the non-commutative L^p -spaces of $\mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} L(\Gamma)$). Therefore, the set

$$\mathcal{P}(\Gamma) = \{p \in [1, +\infty) \mid L^p(L(\Gamma), \tau) \text{ has the OAP}\}$$

only depends on $L(\Gamma)$. It always contains $p = 2$, and one can show using duality that for $1 < p < +\infty$, it contains p if and only if it contains the conjugate exponent p' . If one could compute it for $\text{PSL}_n(\mathbb{Z})$ and find different answers for different values of n , this would answer Connes' problem for these groups. Here is what is known so far.

THEOREM 3.8 (LAFFORGUE–DE LA SALLE, DE LAAT–DE LA SALLE)

- For $p > 4$ or $p < 4/3$, $L^p(L(\text{SL}_3(\mathbb{Z})), \tau)$ does not have the OAP.
- Let $d \geq 2$. For $p > 2 + \frac{2}{d-1}$ or $p < \frac{2d}{d+1}$, the space $L^p(L(\text{SL}_{2d-1}(\mathbb{Z})), \tau)$ does not have the OAP.

The first statement is from [19], where it provided the first examples of non-commutative L^p -spaces without good approximation properties for $p \neq 2$, and the second one is from [9]. Recall that $\text{SL}_n(\mathbb{Z}) = \text{PSL}_n(\mathbb{Z})$ for n odd. The striking feature of this result is that the “forbidden” region grows with d and in the limit approaches every $p \neq 2$. This suggests the following strategy : if one could prove that $L^p(L(\text{SL}_3(\mathbb{Z})), \tau)$ has the OAP for *some* $p \neq 2$, say $p = 2 + \varepsilon$, then this would imply that $L(\text{SL}_3(\mathbb{Z})) \neq L(\text{SL}_{2d-1}(\mathbb{Z}))$ as soon as $2 + 2/(d-1) < 2 + \varepsilon$. More ambitiously, if the critical exponent were exactly $2 + 2/(d-1)$ for $\text{SL}_{2d-1}(\mathbb{Z})$, one would answer Connes' problem for all odd values of n .

Unfortunately, proving that the OAP holds for some $p \neq 2$ seems to be very difficult, since it requires the construction of good Fourier multipliers on $L^p(L(\Gamma), \tau)$ for groups which are very far from being amenable. This is the subject of recent work of M. DE LA SALLE, which shows that this question is deeply connected to hard problems of Euclidean harmonic analysis [10].

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