Classical spaces and quantum symmetries

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These are notes originally written as a support for a talk on results obtained in collaboration with Simeng Wang and Frank Taipe in [FTW21]. The talk was intended for an audience with a background in operator algebras, hence all basics concerning C*-algebras are assumed to be known to the reader.

1 The quantum symmetry problem

Non-commutative geometry emerged from the idea of considering non-commutative algebras as substitutes for classical spaces, for instance when their topological or geometric properties become ill-behaved (see for instance [Con94] for numerous illustrations of that principle). As soon as such spaces where considered, the following question arose :

Question. *What are the symmetries of a non-commutative space ?*

Stated in such generality, the question does not make much sense, but it suggests interesting problems. To illustrate this, let us focus on the case of C*-algebras, seen as generalizations of locally compact Hausdorff spaces. Of course, they have a group of $*$-homomorphisms, but could there be more ? At the same time as the appearance of non-commutative geometry, the concept of *quantum group* emerged to unify several situations where Hopf algebraic structures play an important rôle as generalized symmetries. In particular, *compact quantum groups* were developed at that time at the intersection of both worlds and it is therefore natural to wonder whether they have a systematic interpretation in terms of symmetries of non-commutative spaces.

1.1 The setting

Compact quantum groups are a generalization of the theory of compact groups to the setting of non-commutative geometry introduced by S.L. Woronowicz in [Wor87] and [Wor98]. This means that the basic object is a "non-commutative compact space", i.e. a unital C*-algebra. Since it is intended to play the rôle of the algebra of continuous functions on a fictious object $\mathbb{G}$, we usually denote it by $C(\mathbb{G})$. We will however simplify the setting here, because we are mainly interested in the quantum permutation group S_N^+ (see Definition 1.6 below) which belongs to the specific class of *orthogonal compact matrix quantum groups*.

Definition 1.1. An orthogonal compact matrix quantum group[1] is given by a unital C*-algebra $C(\mathbb{G})$ together with N^2 generators $(u_{ij})_{1\leqslant i,j\leqslant N}$ such that

(1) $U = [u_{ij}]_{1\leqslant i,j\leqslant N}$ is an orthogonal matrix, i.e. for all $1 \leqslant i,j \leqslant N$, $u_{ij}^* = u_{ij}$ and

$$\sum_{k=1}^{N} u_{ik}u_{jk} = \delta_{ij}.1_{C(\mathbb{G})} = \sum_{k=1}^{N} u_{ki}u_{kj}.$$

(2) There exists a (necessarily unique) $*$-homomorphism[2] $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ such that for all $1 \leqslant i,j \leqslant N$,

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

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For reasons which will become clearer later on (see Section 2.1), U is called the *fundamental representation* of $\mathbb{G}$.

The basic example is of course that of "usual" (we say "classical") groups :

Example 1.2. Let $G \subset O_N$ be a closed subgroup and let $c_{ij} \in C(G)$ be the function sending a matrix $g \in G$ to its (i,j)-th coefficient. Then, the elements $(c_{ij})_{1\leqslant i,j\leqslant N}$ satisfy Condition (1) because all elements of G are orthogonal matrices, and Condition (2) holds by using the map Δ induced by the matrix product : identifying $C(G)\otimes C(G)$ with $C(G\times G)$, we have

$$\Delta(f) : (g,h) \mapsto f(gh).$$

Eventually, by the Stone-Weierstrass theorem, the coefficient functions generate a dense subalgebra of the algebra $C(G)$ of continuous complex-valued functions on G. In conclusion any compact group of orthogonal matrices is an orthogonal compact matrix quantum group, and such kind of quantum groups will be called *classical*[3].

Because of the analogy between Δ and the matrix product, the former is called the *coproduct* of the quantum group $\mathbb{G}$. Just like classical groups, quantum groups can act on commutative and non-commutative spaces. For the sake of simplicity, we will focus here on actions on unital C*-algebras.

DEFINITION 1.3. A *continuous right action*[4] of $\mathbb{G}$ on a unital C*-algebra A is a $*$-homomorphism

$$\alpha : A \to A \otimes C(\mathbb{G})$$

satisfying the equations

$$(\alpha\otimes \mathrm{id})\circ\alpha = (\mathrm{id}\otimes\Delta)\circ\alpha \quad \& \quad (\mathrm{id}\otimes\varepsilon)\circ\alpha = \mathrm{id}_A,$$

where $\varepsilon : C(\mathbb{G}) \to \mathbf{C}$ is the unique[5] $*$-homomorphism such that $\varepsilon(u_{ij}) = \delta_{i,j}$.

Let us briefly explain the meaning of these two equations. The first one translates the fact that we have a "quantum semigroup" action, that is to say compatibility of the action map with the group operation. As for the second one, it is analogous to the fact that the neutral element acts trivially.

Remark 1.4. The continuity of the action is not given by a specific axiom since it is contained in the existence of the $*$-homomorphism α. The reader is invited to check that if X is a compact space and $A = C(X)$, then the notion above coincides for classical groups with that of a continuous right action. Moreover, there is nothing specific about the action being on the right, the definition of left actions being similar.

Observe that any compact (quantum) group can act on any (not necessarily commutative) space through the *trivial action*

$$\alpha_{\mathrm{triv}} : x \mapsto x\otimes 1.$$

Therefore, to consider that a quantum group embodies genuine quantum symmetries of a space, one should rule out this kind of example. There are several notions for that, and we will now define two of them.

DEFINITION 1.5. An action α of a compact quantum group $\mathbb{G}$ on a C*-algebra A is said to be

- *Faithful* if there is no C*-subalgebra $C(\mathbb{H}) \subset C(\mathbb{G})$ such that

$$\Delta(C(\mathbb{H})) \subset C(\mathbb{H})\otimes C(\mathbb{H})$$

  and

$$\alpha(A) \subset A\otimes C(\mathbb{H});$$

- *Ergodic* if

$$\mathrm{Fix}(\alpha) := \{x\in A \mid \alpha(x) = x\otimes 1\} = \mathbf{C}.1_A.$$

Assuming G to be classical in the definition of faithfulness, it follows that $C(\mathbb{H})$ is the algebra of functions on a quotient group of G. The condition therefore means that the action does not factor through any quotient. As for the second one, it can be shown to be equivalent to the existence of a unique state in A which is invariant under the action. Classically, the state corresponds to a measure on the space which is therefore ergodic with respect to the action, hence the name.

1.2 THE QUANTUM PERMUTATION GROUPS

The simplest of all examples is certainly the finite-dimensional commutative C*-algebras, i.e. $\mathbf{C}^N$. It turns out that besides its classical $*$-automorphism group $\mathfrak{S}_N$, this space has quantum symmetries. This was first shown by SH. WANG in a seminal paper [Wan98] by constructing the following quantum group :

DEFINITION 1.6. We denote by $C(S_N^+)$ the universal C*-algebra with generators $(u_{ij})_{1\leqslant i,j\leqslant N}$ satisfying the following relations :

(1) $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $1 \leqslant i,j \leqslant N$;

(2) $\displaystyle\sum_{k=1}^{N} u_{ik} = 1 = \sum_{k=1}^{N} u_{kj}$ for all $1 \leqslant i,j \leqslant N$.

Because the elements u_{ij} are orthogonal projections, they are pairwise orthogonal on each row and column since their sum is a projection. This shows that $U = [u_{ij}]_{1\leqslant i,j\leqslant N}$ is orthogonal. As for the existence of the map Δ, it follows from the universal property[6] of $C(S_N^+)$. Therefore, this defines an orthogonal compact matrix quantum group called the *quantum permutation group* and denoted by S_N^+.

The quantum permutation group naturally acts on $\mathbf{C}^N$. Indeed, denoting by $(e_i)_{1\leqslant i\leqslant N}$ the canonical basis of $\mathbf{C}^N$, we can set

$$\alpha_U(e_i) = \sum_{i=1}^{N} e_j \otimes u_{ij}.$$

It is very easy to check that this action is ergodic[7], and even more is true.

THEOREM 1.7 (WANG) Let $\mathbb{G}$ be a compact quantum group and let α be an action of $\mathbb{G}$ on $\mathbf{C}^N$. Then, there is a surjective $*$-homomorphism $\pi : C(S_N^+) \to C(\mathbb{G})$ such that

$$\alpha = (\mathrm{id} \otimes \pi) \circ \alpha_U.$$

In other words, any action on $\mathbf{C}^N$ factors through S_N^+, so that the latter can rightfully be termed the *quantum automorphism group* of $\mathbf{C}^N$. Despite that remarkable result, it turns out that the general situation is very difficult to understand. For instance, SH. WANG showed in [Wan98] that if A is a non-commutative finite-dimensional C*-algebra, then there it has no quantum automorphism group. The reason is that any action of a quantum group automatically fixes a state on the C*-algebra, but no non-trivial action can fix all of them.

This suggests that one considers an extra structure on the space and restricts attention to actions preserving that structure. But then, exploring commutative spaces leads to a series of negative results. Let us briefly summarize some of them,

- If M is a compact connected smooth manifold, and if $\mathbb{G}$ is a quantum group acting faithfully and *smoothly*[8] on M, then $\mathbb{G}$ is a subgroup of the classical diffeomorphism group $\mathrm{Diff}(M)$ (D. GOSWAMI [Gos20]) ;

- If (M,g) is a compact connected Riemaniann manifold with strictly negative curvature and $\mathbb{G}$ is a quantum group acting faithfully *isometrically*[9] on the underlying geodesic metric space, then $\mathbb{G}$ is a subgroup of the classical isometry group $\mathrm{Iso}(X,d)$ (A. CHIRVASITU [Chi16]) ;

- Almost all compact metric spaces (in the Baire sense) have trivial quantum automorphism group, i.e. any action of a compact quantum group on them must be trivial (A. CHIRVASITU [Chi21]).

The first two results critically use connectedness, and combined with the fact that the only known ergodic compact quantum group action on a classical space is an action on a discrete space, this led to the conjecture that connected compact spaces should not have any quantum symmetry in general. However, it is extremely difficult to say something about actions of arbitrary compact quantum groups on arbitrary compact spaces. One way around can be to consider actions of a *given* compact quantum group on arbitrary spaces. This is the line of thought triggered by the following question of D. GOSWAMI :

Question. *Can S_N^+ act faithfully on a connected compact space ?*

It turns out that any action of S_N^+ is either trivial or faithful, and that there does exist non-trivial action on classical connected spaces. This was shown by H. HUANG in [Hua13] through an explicit construction.

Example 1.8 (HUANG)**.** Let Y be a compact connected space. If Y_N denotes the disjoint unions of N copies of Y, then there is an isomorphism

$$C(Y_N) \simeq C(Y) \otimes \mathbf{C}^N.$$

Letting S_N^+ act trivially on the first tensorand and through α_U on the second one then provides an action on Y_N. Of course, Y_N is not connected, but we can make it so by gluing the various copies together. To do this, let $Z \subset Y$ be a closed subset and consider an element

$$a = \sum_{i=1}^{N} f_i \otimes e_i \in C(Y) \otimes \mathbf{C}^N$$

such that for any $z \in Z$ and any $1 \leqslant i,j \leqslant N$, $f_i(z) = f_j(z)$. Then, if $\mathrm{ev}_z : C(Y) \to \mathbf{C}$ denotes the evaluation map at some $z \in Z$, we have

$$\begin{aligned}
(\mathrm{ev}_z \otimes e_k^* \otimes \mathrm{id}) \circ \alpha(a) &= (\mathrm{ev}_z \otimes \mathrm{id} \otimes \mathrm{id}) \left(\sum_{i=1}^{N} \sum_{j=1}^{N} f_i \otimes e_j \otimes u_{ij} \right) \\
&= \sum_{i=1}^{N} f_i(z) u_{ik} \\
&= f_1(z) \sum_{i=1}^{N} u_{ik} \\
&= f_1(z).
\end{aligned}$$

This means that if $A \subset C(Y_N)$ denotes the subalgebra of functions which all coincide on Z, then $\alpha(A) \subset A \otimes C(S_N^+)$. In other words, we have produced an action on A. Gelfand duality then shows that $A = C(X)$, where X is the quotient of Y_N identifying all copies of Z pointwise. This is compact and connected, and not a point if $Z \neq Y$.

There remains to check that the action is non-trivial. To see this, let $x \notin Z$ and let $f \in C(Y)$ be such that $f(x) = 1$ and $f(z) = 0$ for all $z \in Z$ (such a function exists by Urysohn's Lemma). Setting $a = f \otimes e_1$, we see that

$$(\mathrm{ev}_x \otimes e_2^* \otimes \mathrm{id}) \circ \alpha(a) = u_{12} \neq 0,$$

so that $\alpha(a) \neq a \otimes 1$.

Observing that the action is nevertheless not ergodic (consider constant tuples of non-constant functions), H. HUANG asked the following strengthened question :

Question. *Can S_N^+ act ergodically on a connected compact space ?*

We will explain hereafter why the answer is no, but this first requires taking a different look at actions.

2 THE CATEGORICAL SIDE OF LIFE

2.1 TANNAKA-KREIN DUALITY

Quantum groups, like their classical analogues, can be considered under several points of view. In particular, the *Tannaka-Krein duality* established by S.L. Woronowicz in [Wor88] shows that they can be seen as a C*-tensor category[10] equipped with a specific functor. Let us explain how this works.

Let $\mathbb{G}$ be a compact quantum group with fundamental representation U. It defines a map $\rho_U : \mathbf{C}^N \to \mathbf{C}^N \otimes C(\mathbb{G})$ through the formula

$$\rho_U(e_i) = \sum_{j=1}^{k} e_j \otimes u_{ij}$$

which is the same, if $\mathbb{G}$ is classical, as a unitary representation of the underlying group. It is straightforward to check that

$$(\rho_U \otimes \mathrm{id}) \circ \rho_U = (\mathrm{id} \otimes \Delta) \circ \rho_U \quad \& \quad (\mathrm{id} \otimes \varepsilon) \circ \rho_U = \mathrm{id}_{\mathbf{C}^N}.$$

These are the defining equations of an action, except that ρ_U is only linear and not multiplicative in general. Such a map is therefore called a *representation* of $\mathbb{G}$, the terminology being justified by the classical case.

Example 2.1. Let G be a compact group of matrices. Then, a linear map $\rho : V \to V \otimes C(G)$ for some finite-dimensional vector space V. For $g \in G$, let ev_g be the corresponding evaluation map and set

$$\phi_g : v \in V \mapsto (\mathrm{id} \otimes \mathrm{ev}_g) \circ \rho(v) \in V.$$

This is a linear map and if we assume that $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho$, then

$$\begin{aligned}
\phi_g \circ \phi_h &= (\mathrm{id} \otimes \mathrm{ev}_g) \circ \rho \circ (\mathrm{id} \otimes \mathrm{ev}_h) \circ \rho \\
&= (\mathrm{id} \otimes \mathrm{ev}_g \otimes \mathrm{ev}_h) \circ (\rho \otimes \mathrm{id}) \circ \rho \\
&= (\mathrm{id} \otimes \mathrm{ev}_g \otimes \mathrm{ev}_h) \circ (\mathrm{id} \otimes \Delta) \circ \rho \\
&= (\mathrm{id} \otimes (\mathrm{ev}_g \otimes \mathrm{ev}_h) \circ \Delta) \circ \rho
\end{aligned}$$

But for a function $f \in C(G)$,

$$(\mathrm{ev}_g \otimes \mathrm{ev}_h) \circ \Delta(f) = f(gh) = \mathrm{ev}_{gh}(f)$$

so that in the end, $\phi_g \circ \phi_h = \phi_{gh}$. Moreover, because $\mathrm{ev}_{\mathrm{Id}} = \varepsilon$, we have $\phi_{\mathrm{Id}} = \mathrm{Id}_V$. This means that $\phi : G \times V \to V$ is a linear action of G i.e. a representation, which is moreover continuous.

For a classical group of orthogonal matrices, ρ_U is just the inclusion $G \hookrightarrow M_N(\mathbf{C})$ and this map "contains" in a sense all the information about the group G. And it does indeed, even if $\mathbb{G}$ is a quantum group. To explain this, let us first consider the tensor powers of ρ_U, that is to say, for $n \in \mathbf{N}$, the representations

$$\rho_U^{\otimes n} : \left(\mathbf{C}^N\right)^{\otimes n} \to \left(\mathbf{C}^N\right)^{\otimes n} \otimes C(\mathbb{G})$$

given by

$$\rho_U^{\otimes n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \sum_{j_1, \cdots, j_n = 1}^{N} e_{j_1} \otimes \cdots \otimes e_{j_n} \otimes u_{i_1 j_1} \cdots u_{i_n j_n}.$$

We want to build a category with $(\rho_U^{\otimes n})_{n \in \mathbf{N}}$ as objects, but we lack morphisms for the moment. The natural ones to consider are the following :

DEFINITION 2.2. A linear map $T : \left(\mathbf{C}^N\right)^{\otimes n} \to \left(\mathbf{C}^N\right)^{\otimes m}$ is an *intertwiner* if

$$(T \otimes \mathrm{id}) \circ \rho_U^{\otimes n} = \rho_U^{\otimes m} \circ T.$$

We can now define the category $\mathrm{Rep}(\mathbb{G}, U)$ whose objects are tensor powers of ρ_U and morphisms are intertwiners. This is a C*-tensor category and there is moreover a unitary tensor functor[11] to the category of finite-dimensional Hilbert spaces

$$\mathscr{F}_U : \mathrm{Rep}(\mathbb{G}, U) \to \mathrm{Hilb}_f$$

sending $\rho_U^{\otimes n}$ to $(\mathbf{C}^N)^{\otimes n}$ and being the identity on morphisms. We now have everything in hand to state the celebrated duality theorem proven by S.L. WORONOWICZ in [Wor88].

> **THEOREM 2.3** (WORONOWICZ) Let $\mathfrak{C}$ be a C*-tensor category with object set $\mathbf{N}$ and let $\mathscr{F} : \mathfrak{C} \to \mathrm{Hilb}_f$ be a unitary tensor functor. Then, there exists a quantum group $\mathbb{G}$ with fundamental representation U such that $\mathfrak{C}$ is isomorphic to $\mathrm{Rep}(\mathbb{G}, U)$ and $\mathscr{F}$ is isomorphic to $\mathscr{F}_U$. Moreover, the pair $(\mathbb{G}, U)$ is unique up to isomorphism.

2.2 A DUALITY THEOREM FOR ERGODIC ACTIONS

Theorem 2.3 tells us that $\mathrm{Rep}(\mathbb{G}, U)$ is in a sense "the same thing" as the quantum group $\mathbb{G}$. As a consequence, any notion concerning quantum groups should have an equivalent categorical description, usually in terms of a specific functor on $\mathrm{Rep}(\mathbb{G}, U)$. Let us apply this principle to ergodic actions.

Given an ergodic action α of $\mathbb{G}$ on a unital C*-algebra A, we would like to build a functor $\mathscr{F}_\alpha$ encoding its properties. We first have to find the images of the objects $\rho_U^{\otimes n}$. To do this, observe that α is in particular a representation of $\mathbb{G}$ on the vector space underlying A. We can therefore try to "compare" it with $\rho_U^{\otimes n}$ by considering the subspaces

$$A_n = \mathrm{Fix}\left(A \otimes (\mathbf{C}^N)^{\otimes n}\right),$$

where the action $\rho_n : A \otimes (\mathbf{C}^N)^{\otimes n} \to A \otimes (\mathbf{C}^N)^{\otimes n} \otimes C(\mathbb{G})$ is given by[12]

$$\rho_n(x \otimes v) = \alpha(x)_{13} \rho^{\otimes n}(v)_{23}.$$

It then follows from the general theory developped by F. BOCA in [Boc95] that

1. A_n is finite-dimensional for all $n \in \mathbf{N}$;
2. $\bigcup_{n\in\mathbf{N}} A_n$ is dense[13] in A ;
3. If $T : (\mathbf{C}^N)^{\otimes n} \to (\mathbf{C}^N)^{\otimes m}$ is an intertwiner, then $(\mathrm{id}_A \otimes T)$ commutes with the corresponding representations of $\mathbb{G}$, hence restricts to a map from A_n to A_m.

This gives a unitary functor

$$\mathscr{F}_\alpha : \mathrm{Rep}(\mathbb{G}, U) \to \mathrm{Hilb}_f$$

sending $\rho_U^{\otimes n}$ to A_n. The only issue is with the tensor structure. Of course, there needs to be an issue since we are consider actions and we know that unitary tensor functors recover quantum groups. And indeed, it turns out that even though there always exist embeddings

$$\imath_{n,m} : A_n \otimes A_m \hookrightarrow A_{n+m}$$

given by $\imath_{n,m}(x \otimes y) = x_{12} y_{13}$, these fail to be surjective in general. We therefore need to weaken our definition of a tensor functor.

DEFINITION 2.4. Let $\mathfrak{C}$ be a C*-tensor category with object set $\mathbf{N}$. A unitary functor

$$\mathscr{F} : \mathfrak{C} \to \mathrm{Hilb}_f$$

is said to be a *weak unitary tensor functor* if there exist embeddings

$$\imath_{n,m} : \mathscr{F}(n) \otimes \mathscr{F}(m) \hookrightarrow \mathscr{F}(n+m)$$

for all $n, m \in \mathbf{N}$, satisfying some natural compatibility conditions[14].

We can now state the first important theorem of our work [FTW21, Thm 3.3].

> **THEOREM 2.5** (F.-TAIPE-WANG) Let $\mathbb{G}$ be a quantum group and let $\mathscr{F}$ be a weak unitary tensor functor on $\mathrm{Rep}(\mathbb{G}, U)$. Then, there exists an ergodic action α of $\mathbb{G}$ on a unital C*-algebra A such that $\mathscr{F}$ is isomorphic to $\mathscr{F}_\alpha$.

Proof. We will not explain the proof here, but simply mention that the result is a variant of a similar reconstruction theorem by C. PINZARI and J. ROBERTS in [PR08]. The difference is that they use a functor defined on the category of all finite-dimensional representations while our functor is defined on a smaller full subcategory. This entails differences in the proofs, even though the spirit is the same. Let us also mention that similar results were obtained by N. NESHVEYEV for abritrary (not necessarily ergodic) actions, see [Nes14]. ■

One can go futher in that direction and try to characterize properties of the action in terms of $\mathscr{F}_\alpha$. This is done for one of the most important properties, that of being *(braided commutative) Yetter-Drinfeld*. We refer to [FTW21, Sec 3.2 and Appendix A] for more details.

3 APPLICATION TO QUANTUM SYMMETRIES

The work [FTW21] gives two applications of Theorem 2.5. The first one is the construction of ergodic actions of a given quantum group. Indeed, there is a family of quantum groups called *easy quantum groups* (introduced by T. BANICA and R. SPEICHER in [BS09]) for which the category $\mathrm{Rep}(\mathbb{G}, U)$ has a nice combinatorial description based on partitions of finite sets. One may therefore take advantage of the underlying combinatorics to build weak unitary tensor functors, and then investigate the corresponding actions. This is the subject of [FTW21, Sec 4] but we will not say anything more about it here.

3.1 THE MAIN RESULT

The other application is the "inverse quantum rigidity problem", that is to say investigating the classical spaces on which a given quantum group *cannot* act ergodically. In that direction, we obtained in [FTW21, Thm 6.5] the following result :

THEOREM 3.1 (F.-TAIPE-WANG) Let X be compact connected topological space. Then, S_N^+ cannot act ergodically on X unless it is a point.

We will now explain the proof of this result. But before, let us mention that it also works for other important examples of quantum groups, and in particular for all easy quantum groups corresponding to non-crossing partitions in the sens of [BS09] (see [FTW21, Thm 6.7]).

The proof requires a bit more detail on the representation theory of S_N^+. More precisely, a representation is said to be *irreducible* if its self-intertwiners form a one-dimensional space. The irreducible representations of S_N^+ were classified by T. BANICA in [Ban99].

THEOREM 3.2 (BANICA) The irreducible representations of S_N^+ can be indexed by the integers in such a way that ρ_0 is the trivial representation (the one underlying the trivial action on $\mathbf{C}$), $\rho_U = \rho_0 \oplus \rho_1$ and for any $n \in \mathbf{N}$,

$$\rho_1 \otimes \rho_n = \rho_{n-1} \oplus \rho_n \oplus \rho_{n+1}.$$

A straightforward induction yields the more general formula

$$\rho_k \otimes \rho_n = \bigoplus_{i=|k-n|}^{k+n} \rho_i. \tag{1}$$

Now if X is a compact space and α an ergodic action of S_N^+ on $C(X)$, we have a corresponding weak unitary tensor functor $\mathscr{F}_\alpha$. This is only defined on tensor powers of ρ_U, but standard arguments enable to extend it to all finite-dimensional representations and in particular to all irreducible ones. The idea is then that the commutativity of X forces some relations on the possible images of the representations ρ_n.

To be more precise, we need to introduce some notations. For $k \in \mathbf{N}$, it follows from Equation (1) that $(\mathbf{C}^N)^{\otimes k}$ contains a unique subrepresentation isomorphic to ρ_k. Denoting by H_k the subspace on which it acts and using once again Equation (1), there exists for each $0 \leqslant n \leqslant 2k$ a unique subspace of $H_k \otimes H_k$ such that the corresponding subrepresentation of $\rho_k \otimes \rho_k$ is ρ_n. We will denote by $P_n^{k,k}$ the orthogonal projection onto that subspace. By definition, this is an intertwiner, hence once the functor $\mathscr{F}_\alpha$ is extended to all finite-dimensional representations it makes sense to consider $\mathscr{F}_\alpha(P_n^{k,k})$.

Lemma 3.3. *With the previous notations, assume that $\mathscr{F}_\alpha(P_n^{k,k}) \neq 0$ for some $n \geqslant 2$ and $k \geqslant n/2$. Then, the range of $P_n^{k,k}$ is contained in an eigenspace of the flip map σ_k on $H_k \otimes H_k$.*

Proof. Theorem 2.5 comes with an explicit description of A by generators and relations expressed in terms of irreducible representations. Then, commutativity of A implies that some sum involving all subrepresentations of $\rho_k \otimes \rho_k$ is invariant under the flip map. But since ρ_n appears with multiplicity one in $\rho_k \otimes \rho_k$, each summand of the previous sum must itself be invariant under the flip map and the result follows. We refer to [FTW21, Lem 6.1] for details. ■

The main virtue of the necessary condition appearing in Lemma 3.3 is that it can be directly checked by exhibiting vectors which are neither symmetric nor anti-symmetric.

Proposition 3.4. *Let $n \geqslant 2$. Then, there is a vector in the range of $P_n^{k,k}$ which is not symmetric nor anti-symmetric. Therefore, $\mathscr{F}_\alpha(P_n^{k,k}) = 0$ for any action α on a classical space.*

Proof. The proof is done by building an explicit vector and checking that it is not an eigenvector for the flip map σ. The crucial point here is that we have a good combinatorial description of the projection $P_n^{k,k}$ given in [FW16]. We refer to [FTW21, Prop 6.4] for details. ■

Bringing together the two previous facts, one can eventually prove Theorem 3.1.

Proof of Theorem 3.1. In the same way as we can decompose A using the representations $\rho_U^{\otimes n}$, we can decompose it using the irreducible representations. By the general theory of [Boc95], this yields finite-dimensional subspaces $A^{(n)}$ for all $n \in \mathbf{N}$ such that

$$\mathscr{A} = \bigoplus_{n \in \mathbf{N}} A^{(n)}$$

is dense in A and $A^{(n)}$ is equivalent as a representation to a direct sum of copies of ρ_n. Let now k be the smallest strictly positive integer such that $A^{(k)} \neq 0$. Using once again the explicit description of A in terms of $\mathscr{F}_\alpha$, we see that given $x, y \in A^{(n)}$, the product xy must lie in $A^{(0)} \oplus A^{(1)} \subset A^{(0)} \oplus A^{(k)}$. As a consequence,

$$A' = A^{(0)} \oplus A^{(k)}$$

is a finite-dimensional C*-subalgebra[15] of A of dimension at least 2. In particular, it contains a non-trivial projection, contradicting connectedness of X. ■

3.2 FURTHER QUESTIONS

This result, though satisfying, raises other questions. For instance, we still do not know of any classical space, apart from finite ones, on which S_N^+ can act ergodically. It is therefore tempting to conjecture that such a space must be very far from connected, and the next question pushes that idea to the extreme :

Question. *If S_N^+ acts ergodically on a compact space X, then does it follow that X is finite ?*

There are weaker notions of being "far from connected" than discreteness. For instance, one could soften the previous question by asking whether X needs to be totally disconnected. Since the Cantor set is a particularly nice example of totally disconnected but not dicrete space, investigating its quantum symmetries could be a first step in that direction.

If we drop the commutativity assumption, very little is known. Of course S_N^+ can act non-trivially on many non-commutative C*-algebras (starting with $C(S_N^+)$ itself), but as we saw in the proof of Theorem 3.1 the key property of connectedness that we used is the absence of non-trivial projection. The next question is therefore a natural generalization.

Question. *Can S_N^+ act ergodically on a projectionless C*-algebra ?*

Eventually, going back to H. HUANG's example at the beginning, we may even wonder whether it is possible to classify all actions of S_N^+ on a compact topological space. Indeed, given an action on X, we have by Gelfand duality that $C(X)^\alpha = C(Y)$ for some quotient space $X \twoheadrightarrow Y$. The fibres are the orbits in the sense of [Hua16], on which the action is ergodic by [Hua16, Thm 4.15]. Therefore, a positive answer to the first question above would give us more information on X. It is quite unclear however how far this can be pushed. Let us just summarize this as a general question :

Question. *What can be said of a general non-trivial action of S_N^+ on a compact space X ?*

REFERENCES

[Ban99] T. BANICA – "Symmetries of a generic coaction", *Math. Ann.* **314** (1999), no. 4, p. 763–780.

[Boc95] F. BOCA – "Ergodic actions of compact matrix pseudogroups on C*-algebras", *Astérisque* **232** (1995), p. 93–109.

[BS09] T. BANICA & R. SPEICHER – "Liberation of orthogonal Lie groups", *Adv. Math.* **222** (2009), no. 4, p. 1461–1501.

[Chi16] A. CHIRVASITU – "Quantum rigidity of negatively curved manifolds", *Comm. Math. Phys.* **344** (2016), no. 1, p. 193–221.

[Chi21] — , "Generic quantum metric rigidity", *Int. Math. Res. Not.* **2021** (2021), no. 18, p. 14379–14397.

[Con94] A. CONNES – *Noncommutative geometry*, Academic Press, 1994.

[FTW21] A. FRESLON, F. TAIPE & S. WANG – "Tannaka-Krein reconstruction and ergodic actions of easy quantum groups", *arXiv preprint* (2021).

[FW16] A. FRESLON & M. WEBER – "On the representation theory of partition (easy) quantum groups", *J. Reine Angew. Math.* **720** (2016), p. 155–197.

[Gos20] D. GOSWAMI – "Non-existence of genuine (compact) quantum symmetries of compact, connected smooth manifolds", *Adv. Math.* **369** (2020), no. 1, p. 107181.

[Hua13] H. HUANG – "Faithful compact quantum group actions on connected compact metrizable spaces", *J. Geom. Phys.* **70** (2013), p. 232–236.

[Hua16] H. HUANG – "Invariant subsets under compact quantum group actions", *J. Noncommut. Geom.* **10** (2016), no. 2, p. 447–469.

[Nes14] S. NESHVEYEV – "Duality theory for nonergodic actions", *Münster J. Math.* **7** (2014), p. 413 – 437.

[NT13] S. NESHVEYEV & L. TUSET – *Compact quantum groups and their representation categories*, Cours Spécialisés, vol. 20, Société Mathématique de France, 2013.

[PR08] C. PINZARI & J. E. ROBERTS – "A duality theorem for ergodic actions of compact quantum groups on c*-algebras", *Comm. Math. Phys.* **277** (2008), no. 2, p. 385–421.

[Wan98] S. WANG – "Quantum symmetry groups of finite spaces", *Comm. Math. Phys.* **195** (1998), no. 1, p. 195–211.

[Wor87] S. WORONOWICZ – "Compact matrix pseudogroups", *Comm. Math. Phys.* **111** (1987), no. 4, p. 613–665.

[Wor88] — , "Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups", *Invent. Math.* **93** (1988), no. 1, p. 35–76.

[Wor98] — , "Compact quantum groups", *Symétries quantiques (Les Houches, 1995)* (1998), p. 845–884.

NOTES

1. From now on we will simply write "quantum group" since this is the only kind of quantum groups we will consider. Note that even the terminology "orthogonal compact matrix quantum group" is not completely accurate since we are furthermore assuming the quantum group to be of so-called *Kac type*. We will however not enter these details here.

2. All tensor products of C*-algebras in this document are maximal.

3. One may prove, using Gelfand duality, that conversely any quantum group with commutative C*-algebra is classical, see for instance [NT13, 1.1.2].

4. Abbreviated to "action" in the sequel.

5. Consider the abelianization map $\pi_{\mathrm{ab}} : C(\mathbb{G}) \to C(\mathbb{G})_{\mathrm{ab}}$. As stated earlier, $C(\mathbb{G})_{\mathrm{ab}}$ is isomorphic to $C(G)$ for a compact group of orthogonal matrices G. Composing π_{ab} with the evaluation at the identity matrix then yields the existence of ε. Uniqueness follows from the definition.

6. Simply check that the elements $v_{ij} = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$ satisfy all the defining relations.

7. If $x = \sum_{i=1}^{N} \lambda_i e_i$ is fixed, then $(e_k^* \otimes \mathrm{id}) \circ \alpha(x) = \lambda_k.1$ for all $1 \leqslant k \leqslant N$. However, direct computation yields
$$(e_k^* \otimes \mathrm{id}) \circ \alpha(x) = \sum_{i=1}^{N} \lambda_i u_{ik}$$
so that $\lambda_k = \sum_{i=1}^{N} \lambda_i u_{ik}$. Multiplying that equation by u_{jk} for some $1 \leqslant j \leqslant k$ leads to $\lambda_k u_{jk} = \lambda_j u_{jk}$ so that all the coefficients are equal, hence $x = \lambda.1$.

8. This means that $\alpha(\mathscr{C}^\infty(M)) \subset \mathscr{C}^\infty(M)$ and that the span of $\alpha(\mathscr{C}^\infty(M))(1 \otimes C(\mathbb{G}))$ is dense in $\mathscr{C}^\infty(M) \otimes C(\mathbb{G})$ in the Fréchet topology.

9. An action on a metric space (X, d) is said to be isometric if $\alpha(d_x)(y) = S \circ \alpha(d_y)(x)$ for all $x, y \in X$, where $d_x : y \mapsto d(x, y)$ and $S : C(\mathbb{G}) \to C(\mathbb{G})$ is the unique anti-homomorphism such that $S(u_{ij}) = u_{ji}$. This is called the *antipode* of $\mathbb{G}$.

10. See for instance [NT13, Chap 2] for details on that notion.

11. Being "unitary' means that it commutes with taking adjoints on morphisms, while being "tensor" means that there are isomorphisms
$$\mathscr{F}_U(\rho_U^{\otimes n}) \otimes \mathscr{F}_U(\rho_U^{\otimes n}) \simeq \mathscr{F}_U\left(\rho_U^{\otimes(n+m)}\right)$$
which satisfy a set of natural compatibility conditions (see for instance [NT13, Def 2.1.3]).

12. We use here the *leg-numbering notations* : if T is an operator acting on a tensor product of two vector spaces, then T_{ij} is its extension to a larger tensor product acting only on the i-th and j-th tensorands.

13. Note that the union is not increasing in general, but the inclusion $A_n.A_m \subset A_{n+m}$ always holds.

14. These are straightforward adaptations of the compatibility conditions between the isomorphisms giving a usual tensor structure on a functor.

15. We have not proven that A' is stable under the involution, but this directly follows from the definition of the involution in the reconstruction theorem together with the fact that all the irreducible representations of S_N^+ are self-conjugate.