How to (badly) quantum shuffle cards

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Method :

- Spread the cards on a table ;
- Select one card uniformly at random ;
- Select a second card uniformly at random ;
- Swap the cards if different ;
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Interpretation :

- μ_{tr} = uniform measure on transpositions in S_{52} ;
- $\mu = \dfrac{51}{52}\mu_{\mathrm{tr}} + \dfrac{1}{52}\delta_{\mathrm{id}}$;
- Random walk on S_{52} with driving probability μ.

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Question : Does this acutally mix the cards ?

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Definition

After k iterations, the distribution is given by

$$\mu^{*k} : A \mapsto \mu^{\otimes k}\left(\left\{(\sigma_1, \cdots, \sigma_k) \in S_{52}^k \mid \sigma_1 \cdots \sigma_k \in A\right\}\right).$$

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The deck will eventually be well mixed :

Theorem

The sequence μ^{*k} converges weakly to the Haar measure of S_{52}.

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Question : How fast ?

$$\text{Set } \|\mu - \nu\|_{TV} = \sup_{A \subset S_N} |\mu(A) - \nu(A)| = \frac{1}{2}\|\mu - \nu\|_1$$

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Theorem (Diaconis-Shahshahani)

Set $k_N = N\ln(N)/2$. *For any* $\epsilon > 0$,

$$\left\| \mu^{*\lceil (1-\epsilon)k_N \rceil} - \mathrm{Haar} \right\|_{TV} \xrightarrow[N\to+\infty]{} 1 \quad \& \quad \left\| \mu^{*\lceil (1+\epsilon)k_N \rceil} - \mathrm{Haar} \right\|_{TV} \xrightarrow[N\to+\infty]{} 0$$

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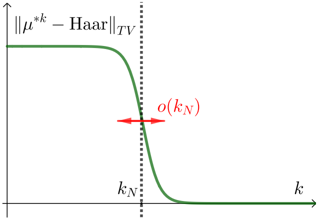
This is the **cut-off phenomenon** :



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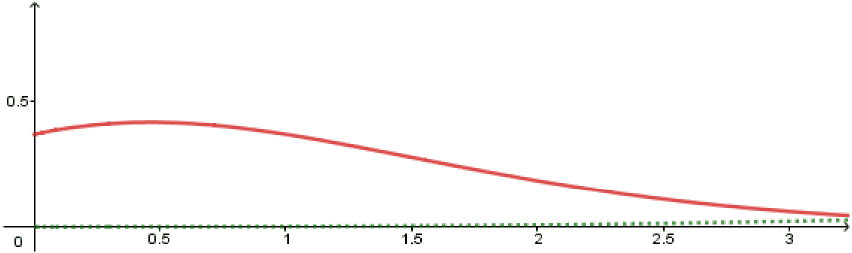
We can **zoom in** on the phase transition :

Theorem (Teyssier)

For any $c \in \mathbb{R}$,

$$\left\| \mu^{*\lceil k_N + cN \rceil} - \text{Haar} \right\|_{TV} \xrightarrow[N \to +\infty]{} \left\| \text{Poiss}(1 + e^{-c}) - \text{Poiss}(1) \right\|_{TV}.$$



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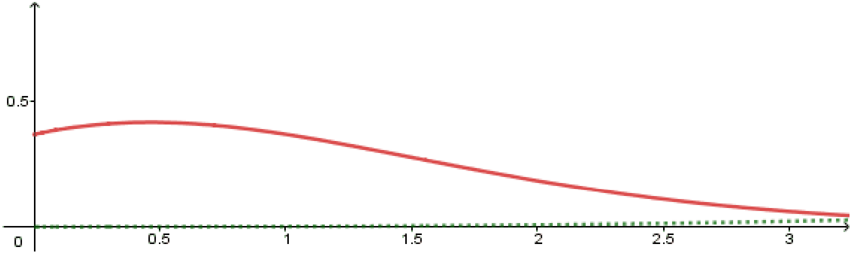

This is the **cutoff profile**.

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Interpretation of the cutoff time :

- Assume all products of k transpositions are different ;
- Then $(N(N-1))^k = N! \rightsquigarrow k \sim \frac{N}{2}$;
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Interpretation of $\left\| \mathrm{Poiss}(1+e^{-c}) - \mathrm{Poiss}(1) \right\|_{TV}$ **:**

- $\mathrm{Poiss}(1)$ = law of fixed points ;
- $\mu^{*\lceil N\ln(N)/2+cN\rceil}$ has too many fixed points.

♡ : Playing cards in the quantum world

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Quantum permutation group : $C(S_N^+)$ universal C*-algebra generated by $(p_{ij})_{1\leqslant i,j\leqslant N}$ such that

1. $p_{ij}^2 = p_{ij} = p_{ij}^*$;

2. $\sum_{k=1}^{N} p_{ik} = 1 = \sum_{k=1}^{N} p_{kj}$;

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Abelianization : $\pi_{\mathrm{ab}} : C(S_N^+) \to C(S_N)$ is given by $p_{ij} \mapsto (\sigma \mapsto \delta_{\sigma(i)j})$.

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Observation : $\mathbb{E}$ = expectation onto central functions.

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$$\begin{aligned}\int_{S_N} f(\sigma)\mathrm{d}\mu_{\mathrm{tr}}(\sigma) &= \int_{S_N}\int_{S_N} f(\tau\sigma\tau^{-1})\mathrm{d}\mu_{\mathrm{tr}}(\sigma)\mathrm{d}\tau \\ &= \int_{S_N}\left(\int_{S_N} f(\tau\sigma\tau^{-1})\mathrm{d}\tau\right)\mathrm{d}\mu_{\mathrm{tr}}(\sigma) \\ &= \int_{S_N}\mathbb{E}(f)\mathrm{d}\mu_{\mathrm{tr}}(\sigma).\end{aligned}$$

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Definition

Let $\mathbb{E}$ be the conditional expectation onto central functions. Then,

$$\varphi_{\mathrm{tr}} : x \mapsto \int_{S_N} \pi_{\mathrm{ab}} \circ \mathbb{E}(x)\mathrm{d}\mu_{\mathrm{tr}}(x).$$

is a state on $C(S_N^+)$, called the *uniform measure on the quantum conjugacy class of transpositions*.

$\diamondsuit$: Quantum transpositions

Set $\varphi_{\mathrm{tr}}^{*k} = \varphi_{\mathrm{tr}}^{\otimes k} \circ \Delta^{(k)}$, then

Theorem (F.-Teyssier-Wang)

Set $k_N = N\ln(N)/2$. *Then, for any* $c > 0$, $\left\| \varphi_{tr}^{*\lceil k_N + cN/2 \rceil} - \mathrm{Haar} \right\|_{C(S_N^+)^*}$ *converges to*

$$\left\| D_{\sqrt{1+e^{-c}}} \left(\mathrm{Meix}^+ \left(\frac{1-e^{-c}}{\sqrt{1+e^{-c}}}, \frac{-e^{-c}}{1+e^{-c}} \right) \right) * \delta_{e^{-c}} - \mathrm{Meix}^+(1,0) \right\|_{TV}$$

where Meix$^+$ *are free Meixner laws.*

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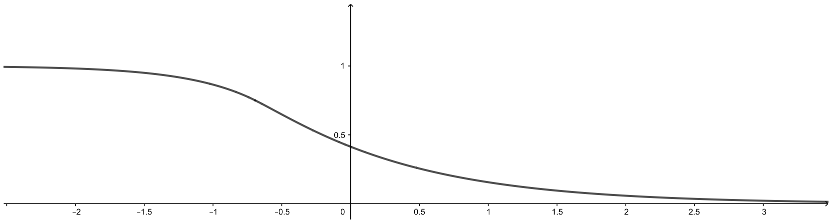

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$$+\,\text{atoms}.$$

- $\mathrm{Meix}^{+}(0,0) = \mathrm{SC}$;
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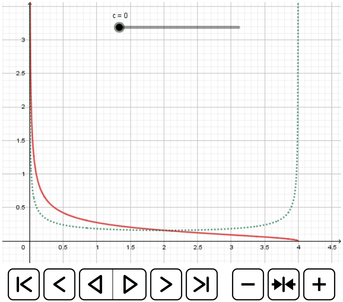
Now that you are here : Orthogonal polynomials for $\mathrm{Poiss}^+(1,1)$: $Q_0(X) = 1$, $Q_1(X) = X$ and

$$XQ_n(X) = Q_{n+1}(X) + Q_n(X) + Q_{n-1}(X).$$

For $c > 0$, φ_{tr} is *absolutely continuous* with respect to the Haar state :

$$\left\|\varphi_{\text{tr}}^{*\lceil k_N + cN/2\rceil} - \text{Haar}\right\|_{C(S_N^+)^*} = \frac{1}{2}\left\|\varphi_{\text{tr}}^{*\lceil k_N + cN/2\rceil} - \text{Haar}\right\|_{L^1(S_N^+)}$$



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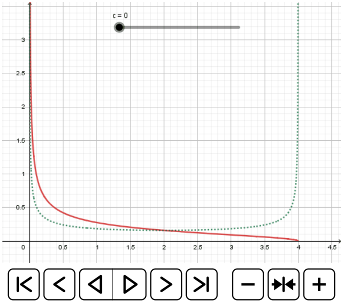

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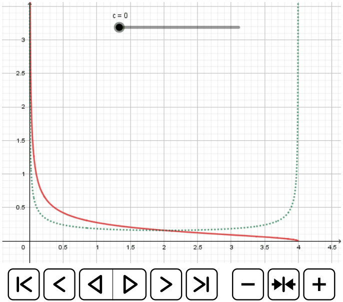

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The BIANE trick :

- Consider the classical process on the central subalgebra ;
- $C(S_N^+)_{\mathrm{central}} = C([0,N])$;
- $\displaystyle\int_0^N f(x)\mu_k^{(N)}(x) = \varphi_{\mathrm{tr}}^{*k}(f).$

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Theorem (F.-Teyssier-Wang)

Setting $k_c = \lceil k_N + cN/2\rceil$, *there exists* $\widetilde{N}(k_c) \in [0,N]$ *and* $\alpha(k_c) > 0$ *such that*

$$\mu_{k_c}^{(N)} = \alpha(k_c)\delta_{\widetilde{N}(k_c)} + \sum_{n=0}^{+\infty}\left[\varphi_{\mathrm{tr}}^{*k_c}(n)Q_n(N) - Q_n(\widetilde{N}(k_c))\right]Q_n \,\mathrm{d}\mathrm{Poiss}^+(1,1).$$

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- $\varphi = \dfrac{N-1}{N}\varphi_{\mathrm{tr}} + \dfrac{1}{N}\varepsilon \geqslant \dfrac{1}{N}\varepsilon$;
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- Flip a biaised coin with probability $1/N$ for heads ;
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- If you get tails, move randomly according to φ_{tr} ;
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For any $c \in \mathbb{R}$,

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Thanks for your attention !