CUT-OFF FOR QUANTUM RANDOM WALKS

AMAURY FRESLON

Abstract. We give an introduction to the cut-off phenomenon for random walks on classical and quantum compact groups. We give an example involving random rotations and highlight several open problems.

1. Convergence of quantum random walks

Let G be a compact group and let μ be a probability measure on it. The corresponding random walk is obtained by randomly picking an element of G according to μ at each step and multiplying, say, on the left. After k-steps, the random walk is located in a measurable subset $A \subset G$ with probability

$$\mu^{*k}(A) = \mu^{\otimes k}\{(g_1, \cdots, g_k) \in G^k \mid g_k \cdots g_1 \in A\}.$$

Studying the random walk therefore amounts to studying the sequence of probability measures $(\mu^{*n})_{n\in\mathbb{N}}$. The first natural question is whether this sequence converges. If it does, its limit is a convolution idempotent measure, hence coincides with the Haar measure of a closed subgroup by [Wen54]. In fact, we have the following criterion (see for instance [Str60] for a proof) :

Proposition 1.1. *The sequence of measures $(\mu^{*k})_{k\in\mathbb{N}}$ converges in the weak-∗ topology to the Haar measure of G if and only if the support of μ is not contained in a closed subgroup or in a coset with respect to a closed normal subgroup.*

The convergence then not only holds in the weak-∗ topology, but also for the *total variation distance*, defined as

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{A\subset G} |\mu_1(A) - \mu_2(A)|$$

where the supremum is taken over all Borel subsets A of G. Turning now to the quantum setting, we consider a compact quantum group $\mathbb{G}$ (see for instance [NT13, Ch 1] for an exposition of the basic theory) and a state (i.e. positive linear map) φ on the corresponding Hopf ∗-algebra $\mathcal{O}(\mathbb{G})$. Its convolution powers can be recursively defined by the formula

$$\varphi^{*k+1} = (\varphi \otimes \varphi^{*k}) \circ \Delta = (\varphi^{*k} \otimes \varphi) \circ \Delta.$$

However, in that case there is no criterion ensuring the weak-∗ convergence of $(\varphi^{*k})_{k\in\mathbb{N}}$ to the Haar state of $\mathbb{G}$. One reason for this is that convolution idempotent states on a compact quantum group do not always come from closed quantum subgroups. This is the first problem we want to raise :

Question 1. *What is a necessary and sufficient condition for the sequence $(\varphi^{*k})_{k\in\mathbb{N}}$ to converge in the weak-∗ topology to the Haar state of $\mathbb{G}$?*

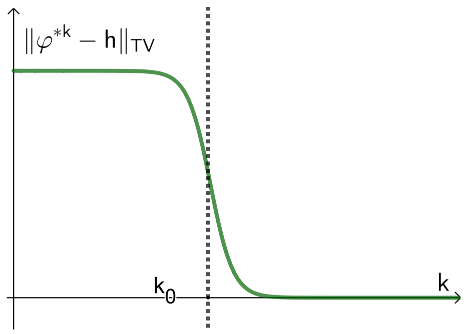
As for the total variation distance, it has a natural analogue in this setting provided the states extend to the von Neumann algebra $L^\infty(\mathbb{G})$. Then,

$$\|\varphi_1 - \varphi_2\|_{TV} = \sup_p |\varphi_1(p) - \varphi_2(p)|$$

where the supremum is taken over all projections of $L^\infty(\mathbb{G})$. Note that it is necessary to use the von Neumann algebra since there are examples of compact quantum groups (for instance O_N^+, by [Voi11, Thm 9.3]) for which the reduced C*-algebra has no non-trivial projection.

2. Cut-off phenomenon and central measures

P. Diaconis and his coauthors discovered in the 1980's that random walks on finite groups sometimes exhibit a surprising behaviour called the *cut-off phenomenon* : for a number of steps the total variation distance to the Haar measure stays close to one and suddenly it drops close to zero. Here is a standard illustration of this :



In the past thirty years lots of examples involving finite groups or Markov chains, as well as a few ones involving infinite compact groups, have been studied. But only last year did the first work on finite quantum groups appear, which is the dissertation of J.P. McCarthy [McC17]. This was followed by two articles by the author, [Fre17] and [Fre18]. The main tool to prove such a result is the so-called *Upper Bound Lemma*. To state it we need to introduce a notation. Given a state φ and an irreducible representation α, we denote by $\widehat{\varphi}(\alpha)$ the matrix with coefficients $\varphi(u^\alpha_{ij})$ for any representative u^α of α.

Lemma 2.1. *Let φ be a state which extends to $L^\infty(\mathbb{G})$, let h be the Haar state of $\mathbb{G}$ and assume that it is tracial. Then,*

$$\|\varphi^{*k} - h\|_{TV}^2 \leqslant \frac{1}{4} \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})\setminus\{\varepsilon\}} d_\alpha \operatorname{Tr}\left(\widehat{\varphi}(\alpha)^{*k}\widehat{\varphi}(\alpha)^k\right). \tag{1}$$

Proof. Here is a sketch of the proof from [Fre17, Lem 2.7]. Because φ extends to $L^\infty(\mathbb{G})$ and h is a faithful trace on this von Neumann algebra, φ is absolutely continuous with respect to h, hence there exists $a_\varphi \in L^1(\mathbb{G})$ such that $\varphi(x) = h(a_\varphi x)$ for all $x \in L^\infty(\mathbb{G})$. Then, by [Fre17, Lem 2.6] the total variation distance equals $\|a_\varphi - 1\|_{L^1(\mathbb{G})}$ up to a factor one half and

$$\|a_\varphi - 1\|^2_{L^1(\mathbb{G})} \leqslant \|a_\varphi - 1\|^2_{L^2(\mathbb{G})} = \left\|(\widehat{\varphi}(\alpha))_{\alpha\in\mathrm{Irr}(\mathbb{G})} - \delta_\varepsilon.1_{\mathbb{C}}\right\|^2_{\ell^2(\widehat{\mathbb{G}})}$$

by the Cauchy-Schwarz inequality and the Plancherel formula. □

Remark 2.2. The link between the total variation distance and the L^1-norm is proved using the traciality of the Haar state. Moreover, the norm in $\ell^2(\mathbb{G})$ has a different expression if h is not assumed to be tracial. It is therefore not clear whether there is a corresponding statement without assuming the Haar state to be tracial.

Formula (1) is not very tractable in general because it requires the computation of traces of large powers of matrices. It becomes much simpler if the matrices are scalar. For classical compact groups, this is easily seen to be equivalent to the fact that the measure μ is conjugation-invariant, i.e. constant on conjugacy classes. In the quantum case, this is equivalent by [CFK14, Prop 6.9] to φ being *central*, i.e. for all $\alpha \in \mathrm{Irr}(\mathbb{G})$, $\varphi(u^\alpha_{ij}) = \varphi(\alpha)\delta_{ij}$ for some constant $\varphi(\alpha)$.

We end this section with a fundamental example of central state : the uniform measure on a conjugacy class. Let $\mathbb{G}$ be a compact quantum group generated by the coefficients of a representation u of dimension N and let $G \subset M_N(\mathbb{C})$ be the classical group of matrices such that $C(G)$ is the abelianization of $C(\mathbb{G})$. We therefore have a surjective homomorphism $u_{ij} \mapsto E^*_{ij|G}$ (where E_{ij} denotes the canonical basis of $M_N(\mathbb{C})$) from $C(\mathbb{G})$ to $C(G)$ which can be composed by the evaluation at a given element $g \in G$. Let us denote by ev_g this composition. Then, the map

$$\varphi_g(u^\alpha_{ij}) = \delta_{ij}\frac{\mathrm{ev}_g(\chi_\alpha)}{d_\alpha}$$

is a central state on $C(\mathbb{G})$, where $\chi_\alpha = u^\alpha_{11} + \cdots + u^\alpha_{d_\alpha d_\alpha}$ is the character of α. Moreover, if $\mathbb{G} = G$ is classical then this is exactly the uniform measure on the conjugacy class of g. We will therefore call φ_g the *uniform measure on the quantum conjugacy class of g*.

3. An example and a problem

We will now give an example of cut-off for compact quantum groups inspired by a classical problem called the *Uniform plane Kac random walk*. Let $\theta \in [0, \pi/2]$ be a fixed angle and consider the following random walk on the unit sphere in $\mathbb{R}^N$: at each step, randomly chose a plane uniformly and then rotate by an angle θ in that plane. This can be obtained by applying to the starting point a random walk on $SO(N)$ obtained by conjugating the rotation matrix of angle θ (in any fixed plane) by a Haar distributed orthogonal matrix. Otherwise said, this is the uniform measure on the conjugacy class of any plane rotation of angle θ. Classically, it follows from works of J. Rosenthal [Ros94] and Y. Jiang and B. Hough [HJ17] that this random walk has a cut-off at $N\ln(N)/2(1-\cos(\theta))$.

Given an orthogonal matrix g, the corresponding state on the quantum orthogonal group O^+_N is completely determined by its value on the fundamental character $\chi_1 = \mathrm{Tr}(u)$. Indeed, if $(P_n)_{n\in\mathbb{N}}$

is the sequence of polynomials defined by $P_0(X) = 1$, $P_1(X) = X$ and

$$XP_n(X) = P_{n+1}(X) + P_{n-1}(X),$$

then it follows from [Ban96] that the irreducible representations of O_N^+ can be indexed by the integers in such a way that $u^0 = \varepsilon$, $u^1 = u$ and the n-th irreducible character is given by $\chi_n = P_n(\chi_1)$. Thus, the uniform measure on the quantum conjugacy class of g only depends on the trace of g. Here appears a surprising fact : two orthogonal matrices are "quantum conjugate" if and only if they have the same trace. This explains that there is no "quantum determinant" on O_N^+, hence no "quantum special orthogonal group". That is the reason why we consider a random walk on O_N^+.

In the case of random rotations, the trace is $N - 2 + 2\cos(\theta)$. This suggests to consider more generally the state $\varphi_{N-\tau}$ sending χ_n to $P_n(N - \tau)$. We then have the following cut-off statement [Fre17, Thm 3.12 and Prop 3.15] :

Theorem 3.1. *There is a function $C(\tau)$ such that for $N \geqslant \tau + C(\tau)$, we have :*

- *for $k = N\ln(N)/\tau - cN$,*
$$\|\varphi_{N-\tau}^{*k} - h\|_{TV} \geqslant 1 - 200e^{-\tau c}$$
- *for any $c > c_0 > 0$ and any $k = N\ln(N)/\tau + cN$,*
$$\|\varphi_{N-\tau}^{*k} - h\|_{TV} \leqslant \frac{1}{2\sqrt{1 - e^{-2\tau c_0}}}e^{-\tau c}.$$

Sketch of proof. The upper bound is proven using Lemma 2.1 together with a fine analysis of the polynomials P_n. For the lower bound, the idea is to evaluate the states at a well-chose projection and use the original definition of the total variation distance. Here, the projection is obtained by applying Borel functional calculus to the fundamental character χ_1. The lower bound then follows from the Chebyshev inequality. □

It would be interesting to try to prove a similar result for some non-central state. These are however not easy to construct or study. One potential family of examples was constructed by U. Franz, A. Kula and S. Skalski in [FKS16, Sec 11]. Let us state this as a question :

Question 2. *Are there non-central states on infinite compact quantum groups exhibiting a cut-off phenomenon ?*

The interest in the cut-off phenomenon originally comes from mixings of decks of cards. One can consider for instance the "random transposition" measure, i.e. the uniform measure φ_{trans} on the conjugacy class of transpositions in S_N or in the quantum permutation group S_N^+. In the quantum as in the classical case, this random walk has a cut-off at $N\ln(N)/2$ steps (see [Fre17, Thm 4.4]). But let us consider a slightly different version. Imagine a deck of N cards spread on a table. Randomly select one of them uniformly, and then another one uniformly. If the same card has been selected twice, nothing is done. Otherwise, the two chosen cards are swapped. The corresponding state is

$$\varphi = \frac{N-1}{N}\varphi_{\text{trans}} + \frac{1}{N}\varphi_e.$$

Classically, that this random walk has a cut-off at $N\ln(N)/2$ steps was the first result in the theory, proved by P. Diaconis and M. Shahshahani in [DS81]. However, in the quantum setting the state φ^{*k} is never bounded on $L^\infty(S_N^+)$. Indeed, it dominates $N^{-k}\varphi_e$ and φ_e is nothing but the *counit*, which is unbounded as soon as the compact quantum group is not *coamenable*. Thus, the total variation distance does not make sense here.

On way round the problem is to consider the transition operator $P_\varphi = (\mathrm{id}\otimes\varphi)\circ\Delta$ which is always bounded on $L^\infty(S_N^+)$. One can then compare $P_{\varphi^{*k}}$ and P_h using various available norms. It seems to us that the most interesting one is the completely bounded norm, leading to the following problem :

Question 3. *Is there a cut-off phenomenon for* $\|P_{\varphi^{*k}} - P_h\|_{cb}$ *?*

References

[Ban96] T. Banica, *Théorie des représentations du groupe quantique compact libre $O(n)$*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 3, 241–244.

[CFK14] F. Cipriani, U. Franz, and A. Kula, *Symmetries of Lévy processes, their Markov semigroups and potential theory on compact quantum groups*, J. Funct. Anal. **266** (2014), no. 5, 2789–2844.

[DS81] P. Diaconis and M. Shahshahani, *Generating a random permutation with random transpositions*, Prob. Theory Related Fields **57** (1981), no. 2, 159–179.

[FKS16] U. Franz, A. Kula, and A. Skalski, *Lévy processes on quantum permutation groups*, Noncommutative analysis, operator theory and applications, 2016, pp. 193–259.

[Fre17] A. Freslon, *Cut-off phenomenon for random walks on free orthogonal quantum groups*, Arxiv preprint arXiv:1711.06555 (2017).

[Fre18] ______, *Quantum reflections, random walks and cut-off*, Arxiv preprint arXiv:1802.09402 (2018).

[HJ17] B. Hough and Y. Jiang, *Cut-off phenomenon in the uniform plane Kac walk*, Ann. Probab. **45** (2017), no. 4, 2248–2308.

[McC17] J.P. McCarthy, *Random walks on finite quantum groups : Diaconis-Shahshahani theory for quantum groups*, Arxiv preprint arXiv:1709.09357 (2017).

[NT13] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*, Cours Spécialisés, SMF, 2013.

[Ros94] J.S. Rosenthal, *Random rotations : characters and random walks on $SO(n)$*, Ann. Probab. **22** (1994), no. 1, 398–423.

[Str60] K. Stromberg, *Probabilities on a compact group*, Trans. Amer. Math. Soc. **94** (1960), no. 2, 295–309.

[Voi11] C. Voigt, *The Baum-Connes conjecture for free orthogonal quantum groups*, Adv. Math. **227** (2011), 1873 – 1913.

[Wen54] J. Wendel, *Haar measure and the semigroup of measures on a compact group*, Proc. Amer. Math. Soc. **5** (1954), no. 6, 923–929.