On The classification Of Partition quantum groups

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Contents

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1 INTRODUCTION

These notes were originally written as a support for a series of two lectures given in Oberwolfach on the occasion of the workshop *"Operator algebraic quantum groups"* [2]. The aim of these lectures was to give a snapshot of some of the current developments around classification of families of compact quantum groups.

One of the main focus at the moment is the classification of structures linked to partitions of finite sets. This is an active research topic evolving rapidly, so that an exhaustive review is unfortunately out of our scope. We have therefore made the choice of devoting the first lecture to an overview of several recent results in the area by other authors. We would like both to isolate some key ideas which seem important to us, as well as convince the reader that this is a broad and lively topic. As for the second lecture, it will be devoted to our own contribution to the subject.

Before delving into partition quantum groups, let us mention some other classification problems for compact quantum groups which will not be treated hereafter. The first examples of compact quantum groups were the deformations of the group $SU(2)$ and more generally $SU(N)$, due to S.L. Woronowicz in [Wor87]. The construction was soon generalized by M. Rosso to arbitrary compact Lie groups in [Ros90] and the reader may find a detailed description in [NT13, Sec 2.4]. However, S.L. Woronowicz noticed already in [Wor87] that his procedure may yield more compact quantum groups. This requires, through Tannaka-Krein duality, the explicit construction of some intertwiners between the fundamental representation u and $u^{\otimes N}$. Even for $N = 3$, there is no complete classification of these potential intertwiners to this day, despite results by A. Kula in [Kul15].

This is connected to the broader question of classifying fibre functors on given tensor categories, or even more generally classifying quantum groups having given fusion rules. The problem has been solved for $SU(2)$ (see [Ban96]) and $SO(3)$ (see [Mro15]) but this is a difficult problem in general, with recent advances in particular due to S. Neshveyev and M. Yamashita in [NY16]. There, the authors classify all dimension-preserving fibre functors on the representation category of $SU(3)$. Nothing is known however for dimension increasing fibre functors [3].

PREREQUISITES

These lectures were aimed at experts, we therefore assume a firm knowledge in compact quantum group theory and in particular Tannaka-Krein duality. The interested reader may refer to the books [Tim08] and [NT13] for detailed treatments of the subject. For another reader-friendly introduction, including the combinatorial approach to compact quantum groups, see [Web17].

2 FIRST LECTURE : TAKE IT EASY

2.1 THE SETTING

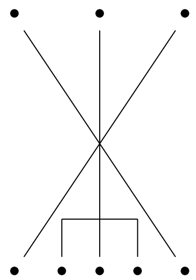
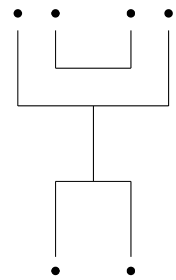
After a period of maturation in the work of T. Banica and several co-authors, compact quantum groups related to partitions were properly formalized by T. Banica and R. Speicher in the influential paper [BS09] under the name of *easy quantum groups*. This was restricted to the orthogonal case but it was clear that it could be extended to the unitary case, and this was done by M. Weber and P. Tarrago in [TW17]. Around the same time, we introduced a more general setting, both because the generalization was natural and because it was suited to the investigation of compact quantum groups with free fusion semi-ring (see [Fre17, Sec 5] for details). We will therefore use this general setting. For the sake of completeness and in order to fix notations, we will recall the basic definitions.

A *partition* is given by two integers k and ℓ and a partition p of the set $\{1,\dots,k+\ell\}$. It is very useful to represent such partitions as diagrams, in particular for computational purposes.

2. Oberwolfach Miniworkshop 1941a, October 7 – 11, 2019.

3. Note that building a non-amenable Kac-type compact quantum group monoidally equivalent to $SU_q(3)$ (necessarily for $q+q^{-1}\in\mathbf{N}$) has been an open problem for several years now. It is interesting because this would yield a nice example of a property (T) discrete quantum group.

A diagram consists in an upper row of k points, a lower row of ℓ points and some strings connecting these points if and only if they belong to the same set of the partition. Let us consider for instance the partitions $p_1 = \{\{1,8\},\{2,6\},\{3,4\},\{5,7\}\}$ and $p_2 = \{\{1,4,5,6\},\{2,3\}\}$. Their diagram representation is :

$p_1 =$  $p_2 =$ 

When manipulating partitions, the crucial notion is that of a block.

DEFINITION 2.1. Let p be a partition.

- A maximal set of points which are all connected (i.e. one of the subsets defining the partition) is called a *block* of p,
- If moreover this block consists only of neighbouring points, then it is called an *interval*,
- If b contains both upper and lower points (i.e. the subset contains an element of $\{1,\dots,k\}$ and an element of $\{k+1,\dots,k+\ell\}$), then it is called a *through-block*,
- Otherwise, it is called a *non-through-block*.

The total number of through-blocks of the partition p is denoted by $t(p)$.

Even though we will mention some aspects of the general case, these lectures mainly focus on the specific family of *non-crossing partitions* in the following sense :

DEFINITION 2.2. Let p be a partition. A *crossing* in p is a tuple $k_1 < k_2 < k_3 < k_4$ of integers such that :

- k_1 and k_3 are in the same block,
- k_2 and k_4 are in the same block,
- the four points are *not* in the same block.

If there is no crossing in p, then it is said to be a *non-crossing* partition. The set of non-crossing partitions will be denoted by NC.

To generalize this setting, the idea is to further colour the points of the partitions with elements of a fixed set.

DEFINITION 2.3. A *colour set* is a set $\mathscr{A}$ together with an involution denoted by $x \mapsto x^{-1}$. An $\mathscr{A}$-coloured partition is a partition together with an element of $\mathscr{A}$ attached to each point. A coloured partition is said to be non-crossing if the underlying uncoloured partition is non-crossing. The set of $\mathscr{A}$-coloured non-crossing partitions will be denoted by $NC^{\mathscr{A}}$.

Let p be an $\mathscr{A}$-coloured partition. Reading from left to right, we can associate to the upper row of p a word w on $\mathscr{A}$ and to its lower row (again reading from left to right) a word w' on $\mathscr{A}$. For a set of partitions $\mathscr{C}$, we will denote by $\mathscr{C}(w,w')$ the subset of all partitions in $\mathscr{C}$ such that the upper row is coloured by w and the lower row is coloured by w' and we will denote by $|w|$ the length of a word w. There are several fundamental operations available on partitions called the *category operations* :

- If $p \in \mathscr{C}(w,w')$ and $q \in \mathscr{C}(z,z')$, then $p \otimes q \in \mathscr{C}(w.z,w'.z')$ is their *horizontal concatenation*, i.e. the first $|w|$ of the $|w|+|z|$ upper points are connected by p to the first $|w'|$ of the $|w'|+|z'|$ lower points, whereas q connects the remaining $|z|$ upper points with the remaining $|z'|$ lower points.

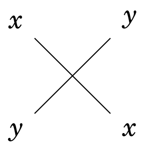
- If $p \in \mathscr{C}(w,w')$ and $q \in \mathscr{C}(w',w'')$, then $qp \in \mathscr{C}(w,w'')$ is their *vertical concatenation*, i.e. $|w|$ upper points are connected by p to $|w'|$ middle points and the lines are then continued by q to $|w''|$ lower points. This process may produce loops in the partition. More precisely, consider the set L of elements in $\{1,\dots,|w'|\}$ which are not connected to an upper point of p nor to a lower point of q. The lower row of p and the upper row of q both induce a partition of the set L. For $x,y \in L$, let us set $x \sim y$ if x and y belong either to the same block of the partition induced by p or to the one induced by q. The transitive closure of $\sim$ is an equivalence relation on L and the corresponding partition is called the *loop partition* of L, its blocks are called *loops* and their number is denoted by $\mathrm{rl}(q,p)$. To complete the operation, we remove all the loops. Note that we can only perform this vertical concatenation if the words associated to the lower row of p and the upper row of q match.

- If $p \in \mathscr{C}(w,w')$, then $p^* \in \mathscr{C}(w',w)$ is the partition obtained by reflecting p with respect to an horizontal axis between the two rows (without changing the colours).

- If $w = w_1\dots w_n$, $w' = w'_1\dots w'_k$ and $p \in \mathscr{C}(w,w')$, then rotating the extreme left point of the lower row of p to the extreme left of the upper row and changing its colour to its inverse yields a partition $q \in \mathscr{C}((w'_1)^{-1}w_1\dots w_n, w'_2\dots w'_k)$. The partition q is called a *rotated version* of p. One can also perform rotations on the right and from the upper to the lower row.

Let us say that for an element $x \in \mathscr{A}$, the *x-identity partition* is the partition $| \in \mathscr{C}(x,x)$ coloured with x on both ends. We are now ready for the definition of a category of coloured partitions, the fundamental object of these lectures.

DEFINITION 2.4. A *category of $\mathscr{A}$-coloured partitions* $\mathscr{C}$ is the data of a set of $\mathscr{A}$-coloured partitions $\mathscr{C}(w,w')$ for all words w and w' on $\mathscr{A}$, which is stable under all the category operations and contains the x-identity partition for all $x \in \mathscr{A}$. If $\mathscr{C}$ moreover contains the partition



for all $x,y \in \mathscr{A}$, then it is said to be *symmetric*.

Such a data gives rise to a compact quantum group in the same way as for easy quantum groups. In order to give a precise statement, let us fix some notations. Let N be a given integer, for each colour $x \in \mathscr{A}$, we consider a copy V^x of $\mathbf{C}^N$ and for a word $w = w_1\cdots w_n$ on $\mathscr{A}$, we set

$$V^w = V^{w_1} \otimes \cdots \otimes V^{w_n}.$$

Given representation $(u^x)_{x\in\mathscr{A}}$ and a word w on $\mathscr{A}$, one defines in the same way the tensor product representation u^w. Moreover, if p is an $\mathscr{A}$-coloured partition, we can associate to it two words on $\mathscr{A}$ by reading its upper and lower row from left to right. For an $\mathscr{A}$-coloured category of partitions $\mathscr{C}$, we then denote by $\mathscr{C}(w,w')$ the set of its partitions with upper word w and lower word w'. With these notations, for $p \in \mathscr{C}(w,w')$ we can define a linear map $T_p : V^w \to V^{w'}$ by the same formula as for easy quantum groups, namely

$$T_p : e_{i_1}^{w_1} \otimes \cdots \otimes e_{i_n}^{w_n} \mapsto \sum_{j_1,\cdots,j_k} \delta_p(\mathbf{i},\mathbf{j}) e_{j_1}^{w'_1} \otimes \cdots \otimes e_{j_k}^{w'_k},$$

where $\delta_p(\mathbf{i},\mathbf{j}) = 1$ if and only if all strings of the partition p connect equal indices of the multi-index $\mathbf{i} = (i_1,\dots,i_n)$ in the upper row with equal indices of the multi-index $\mathbf{j} = (j_1,\dots,j_k)$ in the lower row. Here is now the precise existence statement, proven in [Fre17, Thm 3.2.3] :

> **THEOREM 2.5** Let $\mathscr{A}$ be a colour set, let $\mathscr{C}$ be a category of $\mathscr{A}$-coloured partitions and let N be an integer. Then, there exists a compact quantum group $\mathbb{G}$ together with unitary representations $(u^x)_{x\in\mathscr{A}}$ of dimension N such that
>
> - Any finite-dimensional representation of $\mathbb{G}$ is equivalent to a subrepresentation of the tensor product u^w for some word w on $\mathscr{A}$,

- For any two words w, w' on $\mathscr{A}$,

$$\mathrm{Mor}_{\mathbb{G}}(u^{w}, u^{w'}) = \mathrm{Vect}\{T_p \mid p \in \mathscr{C}(w, w')\}.$$

The compact quantum group $\mathbb{G}$ is called the *partition quantum group* associated to $\mathscr{C}$ and N and is denoted by $\mathbb{G}_N(\mathscr{C})$. Moreover, $\mathbb{G}_N(\mathscr{C})$ is classical if and only if $\mathscr{C}$ is symmetric.

Remark 2.6. One may wonder about the assumption that all the representations u^x should have the same dimension. Removing it may yield to trouble depending on the partitions in $\mathscr{C}$, but in some cases it is possible. The first appearance of this phenomenon is in the work of D. Gromada and M. Weber [GW19b] which we will not discuss here.

Remark 2.7. One important source of examples in quantum group theory is twisting. It is indeed possible to twist the maps T_p to produce new examples. Some particular twistings have been studied in detail by T. Banica, and the question of finding all possible ways of twisting this construction is still open, see for instance [Ban19].

2.2 Orthogonal easy quantum groups

The first case to consider is obviously that of a colour set reduced to one point, which must be its own inverse. This means that we are working with quantum subgroups of O_N^+ (see [Wan95] for the definition), which is the reason why they are called *orthogonal*.

2.2.1 The non-crossing case

Let us consider the non-crossing case, which should be the simplest. Because our aim is to discuss the general classification problem, let us start by stating a classification theorem, and then discuss the key features of the proof. The result we will consider is due to the joint efforts of T. Banica and R. Speicher in [BS09] and M. Weber [Web13], to which we refer for the definitions of the various compact quantum groups involved [4].

Theorem 2.8 (Banica–Speicher, Weber) There exist exactly seven easy orthogonal non-crossing quantum groups : O_N^+, B_N^+, H_N^+, S_N^+, $B_N^+ * \mathbf{Z}_2$, $B_N^+ \times \mathbf{Z}_2$ and $S_N^+ \times \mathbf{Z}_2$.

Proof. The first thing one may try when classifying structures, is to build invariants, and hope that they will completely capture the structure in question. In our case, there two natural invariants which are easy to define for categories of partitions. The first one is the *Block Size*

$$BS(\mathscr{C}) = \{n \in \mathbf{N} \mid \exists p \in \mathscr{C} \text{ with a block of size } n\}.$$

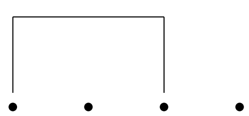
This invariant can take four different values : $\{2\}$, $\{1,2\}$, $2\mathbf{N}$ and $\mathbf{N}$ giving the four main families of compact quantum groups. Since it only depends on the blocks of $\mathscr{C}$, we may call this a *local invariant*.

One must now find a way of distinguishing, say, $S_N^+ \times \mathbf{Z}_2$ from S_N^+. This can be done using the *Odd Block Number*

$$BN(\mathscr{C}) = \{n \mid \exists p \in \mathscr{C} \text{ with } n \text{ blocks of odd size }\}$$

which can be either $\mathbf{N}$ or $2\mathbf{N}$. By contrast with the block size, we will call this a *global invariant*.

These two invariants together are not sufficient to obtain a full classification, since they cannot distinguish $B_N^+ * \mathbf{Z}_2$ from $B_N^+ \times \mathbf{Z}_2$. The final step is however not done by introducing a new invariant, but by considering the presence or absence of a peculiar partition, called the *positioner partition*



4. We will use, for simplicity, the notation $*$ instead of $\hat{*}$ to denote the usual free product of compact quantum groups. This means that $B_N^+ * \mathbf{Z}_2$ is the compact quantum group with C*-algebra $C(B_N^+) * C^*(\mathbf{Z}_2)$.

The presence of this partition does not translate into a numerical invariant, but rather into a *commutation property* : using it and standard manipulation on partitions, we can move singletons around without leaving the category of partitions. ■

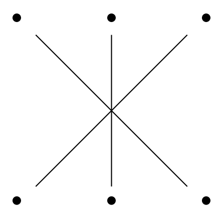
Remark 2.9. From this one may also obtain a classification of the classical easy orthogonal compact groups. Indeed, given a symmetric category of partitions $\mathscr{C}$, it follows from the definitions that $\mathscr{C} \cap NC$ is a category of non-crossing partitions. Conversely, given a category of non-crossing partitions, one can add the crossing partition $\{\{1,3\},\{2,4\}\}$ and this generates a symmetric category of partitions. These operations are not inverse to one another, since $B_N^+ * \mathbf{Z}_2$ and $B_N^+ \times \mathbf{Z}_2$ both collapse to $B_N \times \mathbf{Z}_2$, but at least it yields all possible classical groups (see for instance [TW18, Lem 8.2]).

2.2.2 The full classification

It was already known at the time of the aforementioned works that there exist orthogonal easy quantum groups which are neither classical nor non-crossing, the first examples being the so-called *half-liberations*. These are obtained by adding the relation

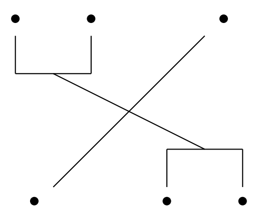
$$abc = cba$$

for all triples (a,b,c) of generators, which is easily seen to be given by the following partition :



M. Weber was able to show that any *non-hyperoctaedral* (see [Web13, Thm 3.12] for the definition) easy quantum group is either classical, non-crossing, or obtained from those through half-liberation, making a total of 13 easy quantum groups. It therefore remains to classify hyperoctaedral quantum groups, i.e. those whose category of partitions contain the four-block but not the double singleton.

The hyperocteadral case was then completely classified by S. Raum and M. Weber in a series of papers (we refer the reader to [RW16] and references therein for more details). Again, the strategy relies on the definition of some specific partition which help distinguish fundamental properties of the associated quantum groups. This is the so-called *pair positioner partition*



It leads to the following dichotomy :

- If $\mathscr{C}$ contains the pair positioner partition then it can be decomposed as a kind of semi-direct product of a discrete group acting on $C(S_N)$. Once again, this can be seen as a commutation relation : any four-block partition can be moved along the partitions in $\mathscr{C}$.

- If $\mathscr{C}$ does not contain the pair positioner partition, then $\mathscr{C}$ is called an *interpolating category*. One then defines a numerical invariant, the maximum of the *wdepth* of $\mathscr{C}$, and shows that it completely classifies the interpolating categories. In other words, there is just one extra series $H_N^{[s]}$ besides non-hyperoctaedral and group-theoretical quantum groups. Note that the maximum of the wdepth is a local invariant which can be encoded by a partition π_k.

Remark 2.10. There is a precise characterization in [RW15] of those compact quantum groups of semi-direct product type which are easy. The non-easy ones exhibit an interesting "skew-easy" structure investigated by L. Maaßen in [Maa19]. To understand their structure, L. Maaßen introduces a variant of categories of partitions called *skew categories of partitions*. Using a suitable adaptation of the definition of the operators T_p, she is then able to completely classify all compact quantum groups of semi-direct product type.

2.3 UNITARY EASY QUANTUM GROUPS

It is tempting, in view of the full classification of orthogonal easy quantum groups achieved by S. Raum and M. Weber, to hope for a similar classification in the unitary case, i.e. with the colour set $\mathscr{A} = \{\circ, \bullet\}$ with $\circ^{-1} = \bullet$. Things turn out, however, to be more complicated.

2.3.1 The non-crossing case

A starting idea is to try to refine the invariants $BS(\mathscr{C})$ and $BN(\mathscr{C})$ from the orthogonal case. P. Tarrago and M. Weber did this in [TW18] by defining three new numerical invariants called the *colouring parameters* [5]. Let us start with a convenient definition.

DEFINITION 2.11. Let p be a partition of $\{1, \cdots, k\}$. A sub-partition q (i.e. a union of blocks of p) is said to be *full* if, up to a rotation of p, it is a partition of $\{a, \cdots, a+b\}$ for some $1 \leqslant a \leqslant a+b \leqslant k$.

DEFINITION 2.12. Let $\mathscr{C}$ be a category of non-crossing partitions coloured with $\{\circ, \bullet\}$. For a partition $p \in \mathscr{C}$ lying on one line, let us denote by $c(p)$ the difference between the number of white points and the number of black points, called the *colour sum* of p. Then,

- The *global colouring parameter* of $\mathscr{C}$ is the minimum $k(\mathscr{C})$ of the numbers $|c(p)|$ for all partitions in $p \in \mathscr{C}$ lying on one line,
- The *first local colouring parameter* of $\mathscr{C}$ is the minimum $d_{\circ\bullet}(\mathscr{C})$ of the numbers $c(p)$ for all full sub-partitions p appearing between two connected points with different colours of a partition in $\mathscr{C}$ lying on one line,
- The *second local colouring parameter* of $\mathscr{C}$ is the minimum $d_{\bullet\bullet}(\mathscr{C})$ of the numbers $c(p)$ for all full sub-partitions p appearing between two connected points with the same colour of a partition in $\mathscr{C}$ lying on one line.

Using these, they were able to classify categories of non-crossing unitary easy quantum groups

THEOREM 2.13 (Tarrago-Weber) Let us say that a category of coloured partitions is *globally colourised* if it is stable under inversion of colours, and *locally colourised* otherwise. Then, the quadruple

$$(BS(\mathscr{C}), k(\mathscr{C}), d_{\circ\bullet}(\mathscr{C}), d_{\bullet\bullet}(\mathscr{C}))$$

is a complete invariant for globally colourised categories of non-crossing partitions, as well as for locally colourised ones.

Proof. The proof is of course extremely involved and out of our scope. Let us simply mention the rough strategy. Given the invariants, one builds partitions "encoding" them and proves that they must belong to $\mathscr{C}$. Then, one shows that $\mathscr{C}$ is in fact generated by these partitions. ∎

Interestingly, P. Tarrago and M. Weber showed in [TW17] that the previous combinatorial invariant translate into operations generalizing the free complexification introduced by T. Banica in [BB07].

DEFINITION 2.14. Let $\mathbb{G}$ be a compact quantum group with a fundamental representation u and let d be an integer. The d-free complexification of $\mathbb{G}$ is the compact quantum group given by the C*-subalgebra of $C(\mathbb{G}) * C(\mathbf{Z}_d)$ generated by the coefficients of uz (where z denotes the fundamental

5. The names and notations here are ours and differ from those given in [TW18].

representation of $\mathbf{Z}_d$) together with the restriction of the coproduct. The d-tensor complexification is obtained similarly using $C(\mathbb{G})\otimes C(\mathbf{Z}_d)$. Eventually, the image of the d-free complexification in the quotient of $C(\mathbb{G})*C(\mathbf{Z}_d)$ by the relations

$$\left(u_{ij}z^r\right)^* = u_{ij}z^r$$

for $r \mid d$, is called the *r-self-adjoint d-free complexification*.

The classification of [TW18] can then be restated in the following way :

THEOREM 2.15 (Tarrago-Weber) All non-crossing partition unitary easy quantum groups can be obtained from orthogonal easy quantum groups and free wreath products by applying (r-self-adjoint) d-free and d-tensor complexifications.

2.3.2 Crossing pair partitions

Allowing crossings the situation becomes much more involved, even in the simplest instance $BS(\mathscr{C}) = \{2\}$. Indeed, in the orthogonal case there are only three corresponding quantum groups, namely O_N, O_N^+ and the half-liberation O_N^* while in the unitary case the classification of all easy quantum groups sitting in between U_N and U_N^+ was only recently obtained by A. Mang and M. Weber in [MW19a] and [MW19b].

To see the difficulty, simply wonder at the following question : what is the analogue of half-liberation for non-self-adjoint generators ? Should one consider the relation

$$abc = cba$$

just for the generators u_{ij} or also for their adjoints ? And what about the relations

$$ab^*c = cb^*a$$

or other variants ? The solution is to think about the problem from another angle. Let us get back to numerical invariants and define the following :

DEFINITION 2.16. Let us say that a *sector* in a partition $p \in \mathscr{P}_2^{\circ,\bullet}$ lying on one line is a full sub-partition whose endpoints are connected. We define the *sector colour number* $\sigma(\mathscr{C})$ of $\mathscr{C}$ to be the minimum of the numbers $|c(p)|$ for all sectors p of partitions in $\mathscr{C}$ such that $c(p) \neq 0$. If there is no such partition, we set $\sigma(\mathscr{C}) = 0$.

A. Mang and M. Weber showed in [MW19a, Prop 8.1] that $\sigma(\mathscr{C})$ in a sense classifies "half" of the easy quantum groups between U_N and U_N^+. To classify the other half, one needs an additional invariant :

DEFINITION 2.17. The *colour semi-group* of $\mathscr{C} \subset \mathscr{P}_2^{\circ,\bullet}$ is the set $D(\mathscr{C})$ of all values of $c(p)$, where p is a full sub-partition of a partition in $\mathscr{C}$ whose endpoints belong to two blocks which cross.

As the name indicates, $D(\mathscr{C})$ is a sub-semi-group of $(\mathbf{N}, +)$ as proven in [MW19b, Prop 7.14]. Moreover, by [MW19a, Prop 8.1] and [MW19b, Thm 8.3], this is enough to complete the classification :

THEOREM 2.18 (Mang-Weber) The pair $(\sigma(\mathscr{C}), D(\mathscr{C}))$ is a complete invariant for categories of partitions $NC_2^{\circ,\bullet} \subset \mathscr{C} \subset \mathscr{P}_2^{\circ,\bullet}$.

Proof. Let us simply mention that, once again, the spirit of the proof is to translate the invariants into partitions. Here, this gives rise to a large family of so-called *bracket partitions* with a rich combinatorial structure. One very nice feature of these partitions is that they are rotations of partitions giving half-liberation-type relations. In a sense, this quarter turn rotation was the key to the classification. ■

2.4 BEFORE THE BREAK

2.4.1 A brief summary

We have encountered a number of tools so far for proving classification theorems, and we would like to pause a moment to look at their main features. We have somehow three types of tools : local invariants, global invariants and specific partitions implementing "commutation relations". Let us work one-by-one using the translation of these invariant into relations in the compact quantum group as explained in [TW18].

In the unitary case, global invariants consist in the global colouring property, the uncoloured invariant BN and the global colouring number $k(\mathscr{C})$. The latter is intimately linked to the group of one-dimensional representations of the compact quantum group. It is usually $\mathbf{Z}_{k(\mathscr{C})}$ or a quotient of it.

As for local invariants, they first give through BS a splitting into four subclasses in which we can work separately. Then, $d_{\circ\bullet}(\mathscr{C})$ gives commutation relations : the subgroup of the group of one-dimensional representations commuting with the fundamental representation is exactly $\mathbf{Z}_{d_{\circ\bullet}(\mathscr{C})}$. And the invariant $d_{\bullet\bullet}(\mathscr{C})$ gives r-self-adjointness, which is a kind of twisted commutation relation.

We are left with the positioner partition but the previous point strongly suggests that it can be exchanged for a local invariant. However, we do not know how to do this at the moment.

2.4.2 Hard quantum groups ?

As the term *easy* suggests, one reason for the definition of easy quantum groups is that they should be more amenable to classification because we can resort to the rich combinatorics of partitions. One may nevertheless wonder for other classes of quantum groups to classify. It turns out that even building non-easy quantum groups is not a simple task in general and only recently were new families of quantum groups defined and classified.

More generally, any compact quantum group $S_N \subset \mathbb{G} \subset O_N^+$ is determined by a family of linear combination of partitions. It is extremely difficult to find explicit linear combinations which do not yield a genuine category of partitions in the end, as the following open problem shows :

Question. *Is there a compact quantum group $S_N \subset \mathbb{G} \subset S_N^+$ for $N \geqslant 6$? In other words, given all non-crossing partitions and a linear combination of crossing ones, can one always build all partitions ?*

The first progress in this direction is a very recent work of D. Gromada and M. Weber [GW19a], which is doubly interesting. First, it describes quantum groups which are not easy but with explicit linear combinations of partitions generating their intertwiners, paving the way for a deeper study of these objects. Second, the examples were obtained through computer assisted computations, and this suggests to further investigate the potential of computers in the study of such combinatorial quantum groups.

3 SECOND LECTURE : THE MORE THE MERRIER

In this lecture, we will explain our personal contribution to the subject, based on the works [Fre19] and [Fre18]. It is concerned with the classification of partition quantum groups associated to categories of non-crossing partitions coloured by a set $\mathscr{A} = \{x, y\}$ with $x^{-1} = x$ and $y^{-1} = y$ (and some general results for an arbitrary colour set). This means that these quantum groups are naturally quotients of $O_N^+ * O_N^+$, and we will see that they form a large family with many interesting new examples. More importantly, our method is quite different from the ones explained in the first lecture, and we believe that it may complement them in the general classification program.

3.1 FROM PARTITIONS TO REPRESENTATION THEORY

Our strategy heavily relies upon a joint work with M. Weber [FW16] linking non-crossing partitions to the representation theory of the corresponding compact quantum group. We therefore start by briefly reviewing these results. The fundamental object is the following :

DEFINITION 3.1. A partition p is said to be *projective* if $pp = p = p^*$. The map T_p is then a scalar multiple of a projection.

An important fact, proved in [FW16, Prop 2.18], is that for any partition r, r^*r (hence also rr^*) is always projective. Based on this fact and the analogy with projections on Hilbert spaces, we will say that a projective partition p is *equivalent* to another projective partition q if there exists a third partition r such that

$$p = r^*r \text{ and } q = rr^*.$$

In [FW16, Sec 4], given a category of coloured partitions[6], we associate to any projective partition $p \in \mathscr{C}$ and integer N a unitary representation u_p of $\mathbb{G}_N(\mathscr{C})$ in such a way that the following properties are satisfied (see [FW16, Prop 4.15, Thm 4.18 and Prop 4.22] and [Fre14, Lem 5.1] for proofs):

1. u_p is irreducible for all p,
2. Any irreducible representation of $\mathbb{G}_N(\mathscr{C})$ is equivalent to u_p for some p,
3. u_p is one-dimensional if and only if $t(p) = 0$, where $t(p)$ denotes the number of *through-blocks* of p,
4. $u_p \sim u_q$ if and only if $p \sim q$.

Remark 3.2. There is also an explicit formula for the fusion rules, given in [FW16, Thm 4.27], but we will not use it in the sequel.

3.2 THE CLASSIFICATION

3.2.1 Three constructions

To explain our strategy, let us go back to the easy orthogonal case. The local invariant BS gives us a set of four base cases, namely

$$\mathscr{S} = \{O_N^+, B_N^+ * \mathbf{Z}_2, H_N^+, S_N^+ \times \mathbf{Z}_2\}.$$

Then, we refine with the global invariant BN which distinguishes whether the non-trivial one-dimensional representation given by $\mathbf{Z}_2$ is made trivial or not. This suggests to consider more generally relations at the level of one-dimensional representations. Let us give it a name for convenience :

DEFINITION 3.3. A compact quantum group $\mathbb{H}$ is said to be a quotient of a compact quantum group $\mathbb{G}$ *by group-like relations* if $C(\mathbb{H})$ is a quotient of $C(\mathbb{G})$ by a Hopf $*$-ideal generated by elements of the form

$$x - 1,$$

where $x \in C(\mathbb{G})$ is a group-like element.

The key observation is that this operation leaves the class of non-crossing partition quantum groups invariant, as shown in [Fre19, Prop 3.8] :

Proposition 3.4. *If $\mathbb{G}$ is a non-crossing partition quantum group, then any quotient of $\mathbb{G}$ by group-like relations is again a non-crossing partition quantum group.*

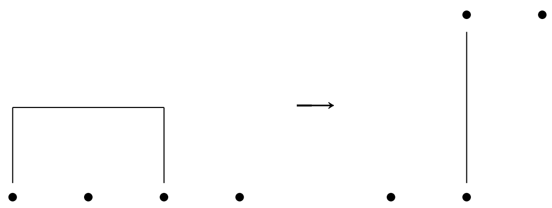
Proof. If x is a group-like element by which we want to quotient, let us take a partition p such that $u_p = x$. Because $t(p) = 0$, we can find a partition b lying on one line such that $p = b^*b$. Then simply consider the category of partitions generated by that of $\mathbb{G}$ and b. ■

6. The original work was only done in the uncoloured case, but carries on straightforwardly to the general setting.

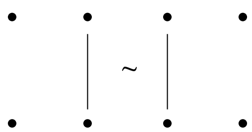
This yields the first step of our strategy. Let $\mathscr{C}$ be a category of non-crossing coloured partitions, and let $\mathscr{C}' \subset \mathscr{C}$ be a subcategory containing all the projective partitions of $\mathscr{C}$. Then, if $p \in \mathscr{C} \setminus \mathscr{C}'$, we can rotate it on one line to obtain a new partition p', and consider $q = p'^* p'$. By assumption $q \in \mathscr{C}'$, and adding p to $\mathscr{C}'$ is the same as adding the relation $u_q = 1$ to $\mathbb{G}_N(\mathscr{C}')$. Thus, we can recover $\mathbb{G}_N(\mathscr{C})$ from $\mathbb{G}_N(\mathscr{C}')$ by adding group-like relations.

This reduces the classification to categories of partitions which are generated by their projective partitions. Instead of trying to list them, we will push the previous idea further and try to find other constructions which preserve the class of partition quantum groups so that we could reduce the classification to a *generating set* like $\mathscr{S}$.

Considering again the orthogonal easy case to get some inspiration, the last ingredient in the classification was the positioner partition and we already mentioned that it corresponds to a kind of commutation property of the category of partitions. To make this rigorous, let us rotate it



This rotated version implements an equivalence



exactly meaning that the fundamental representation $u_{|}$ of B_N^+ commutes with the non-trivial one-dimensional representation $u_{\cdot}$ corresponding to $\mathbf{Z}_2$. Abstracting the idea yields to the following definition :

DEFINITION 3.5. A compact quantum group $\mathbb{H}$ is said to be a quotient of a compact quantum group $\mathbb{G}$ *by commutation relations* if $C(\mathbb{H})$ is a quotient of $C(\mathbb{G})$ by a Hopf $*$-ideal generated by elements of the form

$$xv_{ij} - v_{ij}x$$

for all $1 \leqslant i, j \leqslant \dim(v)$, where $x \in C(\mathbb{G})$ is a group-like element and v is a representation of $\mathbb{G}$.

Let us highlight the difference with group-like relations. If $\mathscr{C}$ denotes the category of representations of $B_N^+ * \mathbf{Z}_2$ and p is the positioner partition, then

$$\langle \mathscr{C}, p \rangle = \langle \mathscr{C}, p^* p \rangle$$

so that this new category of partitions is still generated by its projective partitions, even though the partition we added to it is not projective. The reason for this is that any category of partitions containing $p^* p$ must contain p, hence $u_{p^* p}$ must be the trivial representation there. As a consequence, adding p does not produce a quotient by group-like relations. The important fact is of course that, like for group-like relations, commutation relations preserve the partition structure.

Proposition 3.6. *If $\mathbb{G}$ is a non-crossing partition compact quantum group, then any quotient of $\mathbb{G}$ by commutation relations is again a non-crossing partition compact quantum group.*

Proof. Given v and x, we take partitions p and q such that $u_p \sim v$ and $u_q = x$. Writing $q = b^* b$ with b lying on one line, we just have to add $b \otimes p \otimes b^*$ to obtain $\mathbb{H}$. ∎

We have formalized and generalized now both the global invariants (group-like relations) and the commutation partitions (commutation relations), but this is not enough. Indeed, these constructions do not increase the number of colours in the colour set, so that starting with $\mathscr{S}$ we will remain in the class of orthogonal easy quantum groups. To be able to build new objects, we must combine elements of $\mathscr{S}$. The obvious way to do this is through free product constructions, and it turns out that we can even throw in amalgamation in a broad sense.

Assume that we are given two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ together with a third compact quantum group $\mathbb{H}$ and embeddings

$$i_k : C(\mathbb{H}) \hookrightarrow C(\mathbb{G}_k)$$

intertwining the coproducts. One can then construct the amalgamated free product by quotienting $C(\mathbb{G}_1) * C(\mathbb{G}_2)$ by the $*$-ideal generated by $i_1(C(\mathbb{H})) - i_2(C(\mathbb{H}))$ (see [Wan95] for details). But if the (non-amalgamated) free product $\mathbb{G}_1 * \mathbb{G}_2$ has one-dimensional representations, then one may twist, say, $i_1(C(\mathbb{H}))$ before identifying it with $i_2(C(\mathbb{H}))$, leading to the following notion :

DEFINITION 3.7. With the previous notations, a compact quantum group $\mathbb{G}$ is said to be a *twisted amalgamated free product* if it is the quotient of $C(\mathbb{G}_1) * C(\mathbb{G}_2)$ by the Hopf $*$-ideal generated by

$$x i_1(C(\mathbb{H})) x^{-1} - i_2(C(\mathbb{H}))$$

for some group-like element $x \in C(\mathbb{G}_1) * C(\mathbb{G}_2)$.

An example (without twisting) will be treated in detail in Section 3.3.1. Once again, this operation can be encoded with partitions as soon as the embedding satisfy some kind of compatibility.

Proposition 3.8. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ are non-crossing partition compact quantum groups and let $\mathbb{H}$ be a common quantum subgroup. Assume that there is a generating set $X = \{v_1, \cdots, v_n\}$ of irreducible representations of $\mathbb{H}$, and partitions $p_1, \cdots p_n, q_1, \cdots q_n$ such that for all $1 \leqslant i \leqslant n$,*

$$i_1(v_i) = u_{p_i} \text{ and } i_2(v_i) = u_{q_i}.$$

Then, any twisted amalgamated free product (regardless of $\mathbb{H}$) is a non-crossing partition compact quantum group.

Proof. Let us first mention that to build the free product, one takes disjoint copies $\mathscr{A}_1$ and $\mathscr{A}_2$ of the colour sets of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively, and then considers the category of partitions generated by $\mathscr{C}_1^{\mathscr{A}_1}$ and $\mathscr{C}_2^{\mathscr{A}_2}$ in $P^{\mathscr{A}_1 \sqcup \mathscr{A}_2}$. Consider then an irreducible representation u of $\mathbb{H}$, and partitions $p_1 \in \mathscr{C}_1^{\mathscr{A}_1}$ and $p_2 \in \mathscr{C}_2^{\mathscr{A}_2}$ representing it. They must have the same number of through-blocks, hence we can "merge" them by gluing the upper row of p_1 to the lower row of p_2. If p denotes the resulting partition, and if $q = b^*b$ represents x, then we set $p_{u,x} = b \otimes p \otimes b^*$. Adding these for all irreducible representations in X yields the desired category of partitions. ∎

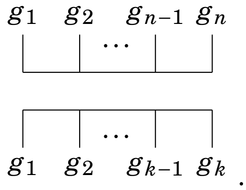
The surprising and, at least to us, satisfying feature of our method is that the three operations above are somehow enough to describe all non-crossing partition quantum groups on the colour set $\mathscr{A}$.

3.2.2 Free wreath products of pairs and the classification

Our claim is that all non-crossing partition quantum groups on two self-inverse colours can be built from the three operations. Of course, such a statement is vacuous as long as the generating set is not given. The best one may hope is certainly to start with $\mathscr{S}$. This is in fact almost true, except that we miss one family which is a generalization of the free wreath product construction. Let us introduce it first.

Given a discrete group Γ and a symmetric generating set $S \subset \Gamma$ containing the neutral element, one may consider the category $\mathscr{C}_{\Gamma,S}$ of all non-crossing partitions coloured by Γ (with the involution given by inversion in the group) such that in each block, the product of the elements in the upper row equals the product of the elements in the lower row. It was proven by F. Lemeux in [Lem15] that the corresponding quantum group is the *free wreath product* $\widehat{\Gamma} \wr_* S_N^+$ introduced by J. Bichon in [Bic04]. The representation theory of these objects is well-known, and in particular they have no non-trivial one-dimensional representation. There is however a rather natural way of adding one-dimensional representations.

Let $\lambda \in \Gamma$, which can be written as $\lambda = g_1 \cdots g_n$, on the generators in S and consider the partition

$\beta_\lambda =$ 

If we apply Theorem 2.5 to the category of partitions generated by β_λ and $\mathscr{C}$, then we get a quotient of the free wreath product with a non-trivial one-dimensional representation, namely u_{β_λ}. Doing this for all the elements of a fixed subgroup $\Lambda \subset \Gamma$ produces the *free wreath product of the pair* (Γ, Λ), introduced in [Fre19] and denoted by $H_N^{++}(\Gamma, \Lambda)$. By construction, these are partition quantum groups and cannot be obtained by the previous operations. For instance, consider the free wreath product

$$(\mathbf{Z}_2 * \mathbf{Z}_2) \wr_* S_N^+ \simeq H_N^+ \underset{S_N^+}{*} H_N^+.$$

Any subgroup Λ of the infinite dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$ gives rise to a free wreath product of pairs which is a non-crossing partition quantum group on two self-inverse colours and whose C*-algebra is a quotient of that of a free product. However, since there is no one-dimensional representation in the free product to quotient by, and amalgamation would just yield another free wreath product, this object cannot be obtained by the construction of Section 3.2.1.

In fact, free wreath products of pairs form a closed family of compact quantum groups in a strong sense, as the following result proven in [Fre19, Prop 3.18] shows :

Proposition 3.9. *Let $\mathbb{G}$ be a quotient of $H_N^{++}(\Gamma, \Lambda)$, then there exists a group $\Lambda \subset \widetilde{\Lambda} \subset \Gamma$ and a normal subgroup $\Lambda_0 \subset \widetilde{\Lambda}$ such that*

$$\mathbb{G} \simeq H_N^{++}(\Gamma/\Lambda_0, \widetilde{\Lambda}/\Lambda_0).$$

The previous result leads to the second best statement one can make, and this one happily holds and is the content of the article [Fre19].

THEOREM 3.10 (F.) Any non-crossing partition quantum group on two self-inverse colours is either

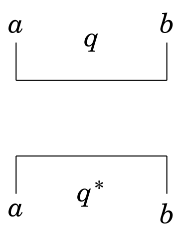
- Obtained from the set $\mathscr{S} = \{O_N^+, B_N^+ * \mathbf{Z}_2, H_N^+, S_N^+ \times \mathbf{Z}_2\}$ using twisted amalgamation, commutation relations and group-like relations,
- O r a free wreath product of a pair.

Sketch of proof for a special case. The proof is extremely involved and covers the entire article [Fre19], we will therefore not explain it here. Let us nevertheless illustrate how it works in a simple case. Consider a category of non-crossing partitions $\mathscr{C}$ with $BS(\mathscr{C}) = \{1, 2\}$ and containing double singletons coloured both by x and y. The corresponding compact quantum group is a quotient of $(B_N^+ * \mathbf{Z}_2) * (B_N^+ * \mathbf{Z}_2)$ and we will denote by $\mathscr{C}_0 \subset \mathscr{C}$ the category of partitions of this free product.

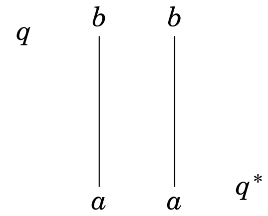
Let $p \in \mathscr{C} \setminus \mathscr{C}_0$ be a projective partition. A straightforward induction shows that p is a horizontal concatenation of the form

$$p = (b_0^* b_0) \otimes | \otimes (b_1^* b_1) \otimes | \otimes \cdots \otimes (b_{n-1}^* b_{n-1}) \otimes | \otimes (b_n^* b_n)$$

An easy lemma (see [Fre19, Prop 3.5]) shows that $b_i^* b_i \in \mathscr{C}$ for all $0 \leqslant i \leqslant n$. Moreover, up to decomposing again into horizontal concatenation, we may assume that b_i is a sector, so that $b_i^* b_i$ has the following form :



for some partition q lying on one line. Rotating then yields



As a consequence, adding $b_i^*b_i$ to $\mathscr{C}_0$ is the same as either quotienting $C(\mathbb{G}_N(\mathscr{C}_0))$ by commutation relations (if $a = b$) or performing a twisted amalgamation (if $a \neq b$). One then only has to prove that p can be reconstructed from the partitions $b_i^*b_i$ (see for instance the proof of [Fre19, Thm 5.10]) to conclude that there exists a category of partitions $\mathscr{C}_0 \subset \mathscr{C}_1 \subset \mathscr{C}$ containing p such that it is obtained from the initial free product by our operations. Iterating this construction, we end up with a category of partitions

$$\mathscr{C}_0 \subset \widetilde{\mathscr{C}} \subset \mathscr{C}$$

satisfying the following properties :

- Any projective partition of $\mathscr{C}$ lies in $\widetilde{\mathscr{C}}$,
- $\mathbb{G}_N(\widetilde{\mathscr{C}})$ is obtained from $\mathbb{G}_N(\mathscr{C}_0)$ by twisted amalgamation and quotienting by commutation relations.

Now, any additional partition in $\mathscr{C} \setminus \widetilde{\mathscr{C}}$ gives a group-like relation, and the proof is complete. ■

The previous result can be turned, with some extra work (see Section 3.3 for a glimpse), into a list of all possible categories of partitions. This is based on the fact that we know all quantum subgroups of elements of $\mathscr{S}$ and that the groups of one-dimensional representations are always dihedral groups, of which all the subgroups are known. That list, given in [Fre19, Thm 8.1], is roughly one page long and not very enlightening. On the contrary, our statement suggests the following question :

Question. *Can any non-crossing partition quantum group on an arbitrary number of colours which are all their own inverses be obtained from the set $\mathscr{S}$ using twisted amalgamation, commutation relations and group-like relations as soon as it is not a free wreath product of a pair ?*

There is a way of getting rid of the dichotomy between free wreath products of pairs and the rest, thanks to the following definition :

DEFINITION 3.11. Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H}_1$, $\mathbb{H}_2$ be compact quantum groups such that

- There are injective $*$-homomorphisms $i_k : C(\mathbb{H}_k) \to C(\mathbb{G})$ intertwining the coproducts for $k = 1, 2$,
- There is an isomorphism of compact quantum groups $\varphi : C(\mathbb{H}_1) \to C(\mathbb{H}_2)$.

Then, the quotient of $C(\mathbb{G})$ by the Hopf $*$-ideal generated by

$$i_1(C(\mathbb{H}_1)) - i_2 \circ \varphi(C(\mathbb{H}_1)).$$

is a compact quantum group called a *collapsing* of $\mathbb{G}$.

Using this, we can give a more concise classification, which has not appeared in this form yet.

Corollary 3.12. *All non-crossing partition quantum groups coloured by $\mathscr{A}$ are obtained from $\mathscr{S} * \mathscr{S}$ by collapsings and group-like relations.*

Proof. It is clear that both quotient by commutation relations and twisted amalgamation are collapsings. As for free wreath products of pairs, we start with $(\mathbf{Z}_2 * \mathbf{Z}_2)\wr_* S_N^+$, which is an amalgamated free product, hence a collapsing of $H_N^+ * H_N^+$. Let $\lambda \in \mathbf{Z}_2 * \mathbf{Z}_2$ and assume that it can be written as $(xy)^k$, where x and y are the canonical generators of the free product. Rotating β_λ so that the upper row has colouring xx, we see that it implements an equivalence between the representations u^{xx} and $u^{\gamma\gamma}$, where $\gamma = (y(xy)^{k-1})$. These two representations generate isomorphic quantum subgroups of the free wreath product, an explicit isomorphism sending u^{xx} to $u^{\gamma\gamma}$ coefficient-wise. As a consequence, the corresponding collapsing is isomorphic to $H_N^{++}(\langle\lambda\rangle,\Gamma)$. If now $\lambda = (xy)^k x$, then we get an equivalence between u^{xx} and $u^{\gamma\gamma^{-1}}$, with $\gamma = (xy)^k$. Once again this leads to a collapsing, hence the result. ■

Note that this is less precise than our previous statement, which gave the explicit form of the possible collapsings in all cases.

3.2.3 To unitarity and beyond

What if we now tried this approach for the general case ? If we allow different colours to be inverse to one another, we need at least an additional ingredient : the free complexification operations of P. Tarrago and M. Weber explained in Section 2.3.1. This leads to the following question :

Question. *Can any non-crossing partition quantum group be obtained from the set $\mathcal{S}$ and the free wreath products of pairs using twisted amalgamation, commutation relations, group-like relations and complexifications ?*

Let us see what we can precisely say in the unitary case. Considering the categories of partitions given by P. Tarrago and M. Weber in [TW18] we first see that in the globally colourised case, everything is obtained by taking the tensor complexification and adding group-like relations. Of course, one cannot add commutation relations since the group of one-dimensional representations is already central. As for the locally colourised case, here is a series of observations :

1. In the case $BS(\mathscr{C}) = \{2\}$, the only possibility is U_N^+, which is the free complexification of O_N^+.
2. The case $BS(\mathscr{C}) = \{1,2\}$ involves the compact quantum group C_N^+ whose category of partitions is generated by the singletons. Let us first consider the category generated by the double white singleton. The corresponding compact quantum group is easily seen to be isomorphic to $U_{N-1}^+ * \mathbf{Z}$ and we obtain C_N^+ by adding the group-like relation making the $\mathbf{Z}$ factor trivial. Everything is then obtained by adding commutation relations, group-like relations and a twisted commutation relation corresponding to the r-self-adjointness.
3. In the case $BS(\mathscr{C}) = 2\mathbf{N}$, we already know that we have the quantum reflection groups $H_N^{s+} = \mathbf{Z}_s \wr_* S_N^+$ which are collapsings of
$$H_N^{\infty+} = \mathbf{Z} \wr_* S_N^+.$$
Indeed, simply consider the two copies of $H_N^{\infty+}$ generated respectively by the fundamental representation u^1 and the representation $u^{-(s-1)}$. One can then further take the tensor complexification and quotient by group-like relations as before. Note that free complexification is also possible but yields the same compact quantum group except for $s = 2$.
4. Eventually, in the case $BS(\mathscr{C}) = \mathbf{N}$, one simply starts with $S_N^+ \times \mathbf{Z}_2$ and after freely complexifying it, adds commutation relations and group-like relations.

Summing up everything, we obtain the following statement :

Corollary 3.13. *The set of non-crossing partition quantum groups on at most two colours coincides with the set of compact quantum groups obtained from $\mathcal{S} * \mathcal{S}$ using collapsing, free and tensor complexifications by $\mathbf{Z}$ and group-like relations.*

3.3 WHAT ABOUT INVARIANTS ?

In this exposition, we have purposely emphasised the difference between the methods of [Fre19] and the ones of previous similar works, based on invariants. This does not mean however that invariants disappeared. For instance, the classification in [Fre19] starts by splitting into four cases according to the local invariant BS, even though one may have to split again according to the sizes of blocks with only one colour.

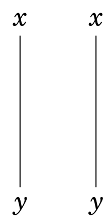
Furthermore, if one wants a finer understanding of specific examples, then it is necessary to go back to the language of invariants. This is the strategy used in [Fre18] to compute the representation theory of some non-crossing partition quantum groups and we will now explain it to illustrate our point.

3.3.1 A global invariant

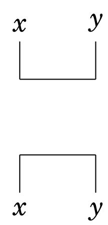
Let us start with the free product $O_N^+ * O_N^+$ and consider the common quantum subgroup PO_N^+ generated by the tensor square of the fundamental representation. Performing amalgamation yields the compact quantum group

$$O_N^{++} = O_N^+ \underset{PO_N^+}{*} O_N^+.$$

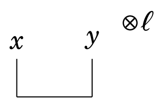
Note that the amalgamation cannot be twisted since O_N^+ has no non-trivial one-dimensional representations. However, O_N^{++} does have such representations. To see this, let us first denote by x the colour corresponding to the first copy of O_N^+ and by y the colour corresponding to the second one. Following the proof of Proposition 3.8, we see that the category of partitions of O_N^{++} is generated by



which after rotation yields



The latter partition implements a non-trivial one-dimensional representation which is easily proven to have infinite order (see [Fre18, Lem 3.2]). By the classification theorem, all we can build from this are the quotients by the group-like relations $s^\ell = 1$ for $\ell \in \mathbf{N}$, yielding compact quantum groups denoted by $O_N^{++}(\ell)$. This is of course equivalent to adding the partition



If one wants to compute the representation theory of $O_N^{++}(\ell)$, the description of the category of partitions with generators is not practical, since we need to classify projective partitions up to equivalence. It would be better to describe this category through a global invariant.

This is doable once one realizes that global invariants should not be numbers but rather groups or semi-groups. Indeed, the block numbers form a sub-semi-group of $\mathbf{N}$. Even better, the global colouring of partitions in the unitary case, that is to say the number of white points minus the number of black points, form a subgroup of $\mathbf{Z}$ and the global colouring parameter is just a generator of this subgroup. Indeed, given a word w on $\{\circ,\bullet\}$, we can define an element $\varphi(w)$ of $\mathbf{Z}$ by sending $\circ$ to 1 and $\bullet$ to -1. If a partition p has upper colouring w and lower colouring w', we then set

$$\varphi(p) = \varphi(w)\varphi(w')^{-1}.$$

The same idea works here, except that because $x^{-1} = x$ and $y^{-1} = y$, the invariant will be a subgroup of $\mathbf{Z}_2 * \mathbf{Z}_2$.

DEFINITION 3.14. To a word w on $\{x, y\}$ we associate an element $\varphi(w) \in \mathbf{Z}_2 * \mathbf{Z}_2$ by sending x to the first generator and y to the second one. If a partition p has upper colouring w and lower colouring w', we then set

$$\varphi(p) = \varphi(w)\varphi(w')^{-1}.$$

Eventually, we define $\mathscr{D}_\ell$ to be the category of all non-crossing pair partitions coloured with $\{x, y\}$ such that

$$\varphi(p) \in \left\langle (xy)^\ell \right\rangle \subset \mathbf{Z}_2 * \mathbf{Z}_2.$$

One easily checks that $\mathscr{D}_\ell$ is indeed a category of partitions and, by construction, $\mathscr{C}_\ell \subset \mathscr{D}_\ell$. In view of the classification given in Theorem 3.10, they should be equal, and this is indeed the case, see for instance [Fre18, Cor 3.7]. The point, however, is that we *do not need* this global invariant at any point in the classification.

3.3.2 Local invariants

As an example of a local invariant, we now consider free wreath products of pairs. Once again, the numerical invariants can be seen as subgroups. This is also true for the local colouring parameters of P. Tarrago and M. Weber, as explained in [TW17, Lem 2.14]. Here is the local invariant we need.

DEFINITION 3.15. Let $\mathscr{C}$ be a category of coloured non-crossing partitions. Its *local subgroup invariant* is the subgroup generated by $\varphi(p)$ for all full sub-partitions p of a partition in $\mathscr{C}$.

Here we mean that this invariant is a subgroup of the free product of one copy of $\mathbf{Z}_2$ for each self-inverse colour, and one copy of $\mathbf{Z}$ for each other pair of mutually inverse colours, with the obvious extension of the map φ. The key idea is that if p_λ denotes the upper block of β_λ, then $\varphi(p_\lambda) = \lambda$, so that one may hope to recover Λ as the local subgroup invariant. This is the case and the proof uses the following definition :

DEFINITION 3.16. Let Γ be a discrete group with a symmetric generating set S and let $\Lambda \subset \Gamma$ be a subgroup. We define $\mathscr{D}_{\Gamma,\Lambda,S}$ to be the set of all partitions $p \in NC^S(w, w')$ such that

- $\varphi(w) = \varphi(w')$ as elements of Γ,
- For any full sub-partition of p with upper and lower colourings v and v' respectively,

$$\varphi(v)^{-1}\varphi(v') \in \Lambda.$$

Note that it is not obvious that $\mathscr{D}_{\Gamma,\Lambda,S}$ is a category of partitions, because one has to prove that the local condition is preserved under vertical concatenation. This was proven in [Fre18, Lem 4.2] and one deduces from this the expected result (see [Fre18, Cor 4.5]).

Proposition 3.17. *The compact quantum group associated to $\mathscr{D}_{\Gamma,\Lambda,S}$ is the free wreath product of the pair* (Γ, Λ).

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