An invitation to quantum stochastic calculus

A. Freslon[†]

These notes are a very elementary introduction to quantum stochastic calculus. It is aimed at people with a probabilistic background and some notions of classical Itô calculus might help, even though it is not mandatory. We have tried to be as careful and precise as possible concerning the operator-theoretic aspects of the construction of the quantum stochastic integral. In that perspective, and to keep the text as simple as possible, we work in a restricted framework which avoids several technicalities while keeping the main features of the theory.

Contents

1 Introduction

The purpose of this text is to give an elemntary yet rigorous introduction to *quantum stochastic calculus*, a noncommutative generalization of the classical Itô calculus within the framework of *quantum probability theory*. This theory extends the tools of stochastic integration to operator-valued processes acting on Fock spaces, providing a natural setting for describing continuous-time evolutions in quantum systems.

Our aim is to make the construction of the quantum stochastic integral accessible to readers with a probabilistic background. Nevertheless, even though familiarity with the basic ideas of classical Itô calculus can be helpful, it is not strictly necessary. We have tried to emphasize the essential analytical and operator-theoretic aspects of the theory while avoiding the most technical functional-analytic difficulties. To this end, we work in a simplified setting, focusing primarily on

[†]A. Freslon, Université Paris-Saclay, CNRS UMR 8628, Laboratoire de Mathématique d'Orsay, 91405 Orsay CEDEX, France.
e-mail : amaury.freslon@universite-paris-saclay.fr
url : https://www.imo.universite-paris-saclay.fr/~freslon/

the fundamental case of creation and annihilation operators and expressing all formulas using inner products with exponential vectors.

The general plan of the exposition is inspired by the survey [Kho91], while the organization and proofs in the final section on quantum stochastic calculus closely follows the classical treatment given in [Par92]. The reader may also see [Mey95] for an alternative introduction to quantum probability theory and a sligthly different exposition of quantum stochastic calculus, including the construction based on so-called *Maasen kernels*.

Let us now outline the structure of the text. In Section 2, we review the basic facts of classical stochastic integration, restricted to the one-dimensional Wiener process. We then develop the *Wiener chaos decomposition*, which expresses the space $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ in terms of iterated Itô integrals. This decomposition serves as a conceptual bridge toward the *symmetric Fock space*, introduced in Section 3 where we reverse the viewpoint: starting from a space of "exponential vectors", we reconstruct operator analogues of the Wiener process.lation processes. This leads naturally to the definition of creation and annihilation processes, which play the role of infinitesimal building blocks of stochastic integration in the quantum setting. We then turn in Section 4 to quantum probability theory. Building on the interpretation of the Wiener process as a combination of creation and annihilation operators on Fock space, we construct a quantum stochastic integral and establish its fundamental properties—linearity, adaptedness, and an analogue of the Itô isometry. This culminates in a quantum Itô formula, which generalizes the classical one to the noncommutative setting. We conclude with a brief discussion of extensions and generalizations, such as the inclusion of number processes and more general operator-valued integrators.

2 The Wiener chaos

Our first goal is to give a description of the Wiener process in the setting of Hilbert spaces through the use of Itô calculus. This will serve as a motivation for the introduction of the fundamental operators of quantum stochastic calculus and the development of an associated integration theory. To start with, it might be helpful to recall some basic facts concerning the standard theory of stochastic integration. We will restrict for simplicity to the case of integration with respect to the one-dimensional Brownian motion and refer the reader to [Øk13] for a comprehensive treatment of the general theory. For the sake of simplicity, we will work with real-valued functions in the sequel, except for one proof where the use of complex numbers is required.

2.1 Preliminaries on Itô calculus

2.1.1 The Itô integral

The aim of classical stochastic integration is to be able to define integrals of functions – or more generally of random variables – with respect to stochastic processes. The prominent example is of course the *Wiener process*, also known as *Brownian motion*. Let us recall what we mean by this.

Definition 2.1 (Wiener process). A family $(W_t)_{t\in\mathbf{R}^+}$ of real-valued random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *(one-dimensional) Wiener process* if it satisfies the following properties:

Independent increments – For any $t_0 < t_1 < \cdots < t_n$, the random variables $W_{t_{i+1}} - W_{t_i}$ are independent for all $0 \leqslant i \leqslant n-1$;

Gaussian distribution – For any Borel subset $A \subset \mathbf{R}$,

$$\mathbb{P}[(W_{s+t} - W_s) \in A] = \frac{1}{\sqrt{2\pi t}} \int_A e^{-x^2/t} \mathrm{d}t$$

Continuity – $\mathbb{P}[t \mapsto W_t \text{ is continuous}] = 1$;

Initialization – $\mathbb{P}[W_0 = 0] = 1$.

It is a fundamental fact of the theory that such processes exist, and we refer for instance to [Øk13, Sec 2.2] for a proof. The core of stochastic calculus is the construction of an integration theory where the usual integration element $\mathrm{d}t$ is replaced by $\mathrm{d}W_t$. We will not explain how one makes rigorous sense of this (see below for the construction in the quantum setting, which uses the same strategy), but simply list some essential properties of that construction. We refer once again to [Øk13] for a detailed treatment.

First, we do not simply want to integrate functions, but more generally random processes. This will require some compatibility with W_t. More precisely, let $\mathscr{F}_t$ be the σ-algebra generated by the random variables $(W_s)_{0\leqslant s\leqslant t}$. These provide a *filtration* of the probability space $(\Omega,\mathscr{F},\mathbb{P})$, and a process $(L(t))_{t\in\mathbf{R}^+}$ is said to be *adapted* if $L(t)$ is measurable with respect to $\mathscr{F}_t$ for all $t\in\mathbf{R}_+$. Given such an adapted process which is moreover *locally square integrable* in the sense that

$$\int_0^t L(s)^2\mathrm{d}s < +\infty$$

for all $t\in\mathbf{R}^+$, we can define a new stochastic process $(X_t)_{t\in\mathbf{R}^+}$ – called its *stochastic integral* – through the formula

$$X(t) = \int_0^t L(s)\mathrm{d}W_s.$$

The process $(X(t))_{\in\mathbf{R}^+}$ satisfies several important properties. The most elementary ones are as follows (see [Øk13, Thm 3.2.1 and Cor 3.2.6] for proofs):

Constant process – If $L(s) = L$ for all almost all $s\in\mathbf{R}^+$, then $\int_0^t L(s)\mathrm{d}W_s = L(W_t - W_0)$ for all $t\in\mathbf{R}_+$;

Linearity – Given two processes $(L(t))_{t\in\mathbf{R}_+}$ and $(L'(t))_{t\in\mathbf{R}_+}$, we have

$$\int_0^t (L(s)+\lambda L'(s))\mathrm{d}W_s = \int_0^t L(s)\mathrm{d}W_s + \lambda\int_0^t L'(s)\mathrm{d}W_s;$$

Adaptedness – $X(t)$ is $\mathscr{F}_t$-measurable for all $t\in\mathbf{R}_+$;

Martingale – $X(t)$ is a *martingale* with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbf{R}_+}$, i.e.

$$\mathbb{E}(X(t)\mid\mathscr{F}_s) = X(s)$$

for all $0\leqslant s<t$.

There are also two more elaborate properties which will be important as motivations in our study of their quantum counterparts. The first one relates stochastic integration to Hilbert spaces.

Proposition 2.2 (ITÔ ISOMETRY)**.** *Let $(L(t))_{t\in\mathbf{R}^+}$ and $(L'(t))_{t\in\mathbf{R}^+}$ be adapted locally square integrable processes and let $(X(t))_{t\in\mathbf{R}^+}$ and $(X'(t))_{t\in\mathbf{R}^+}$ be the associated stochastic integrals. Then, for all $t\in\mathbf{R}^+$,*

$$\mathbb{E}(X(t)X'(t)) = \mathbb{E}\left(\int_0^t L(s)L'(s)\mathrm{d}s\right).$$

Proof. See for instance [Øk13, Lem 3.1.5]. ■

The second property tells us how to perform changes of variables in stochastic integrals, and is the cornerstone of the theory.

THEOREM 2.3 (ITÔ FORMULA) Let $(L(t))_{t\in\mathbf{R}^+}$ be an adapted locally square integrable process and let $(X(t))_{t\in\mathbf{R}^+}$ be its stochastic integral. Let furthermore $g\in\mathscr{C}^2(\mathbf{R}^2)$. Then, the process $(g(t,X(t)))_{t\in\mathbf{R}^+}$ is itself an Itô integral, in the sense that

$$g(t,X(t)) = \int_0^t\left(\frac{\partial g}{\partial t}(s,X(s)) + \frac{L(s)^2}{2}\frac{\partial g}{\partial x^2}(s,X(s))\right)\mathrm{d}s + \int_0^t L(s)\frac{\partial g}{\partial x}(s,X(s))\mathrm{d}W_s$$

||

Proof. See for instance [Øk13, Thm 4.1.2]. ∎

The last statement involves not only a stochastic integral as defined above, but also a classical integral of a stochastic process with respect to the Lebesgue measure ds. Such an integral is straightforward to define, but its appearance calls for an important remark: even if X_t is given by an integral with respect to the Wiener process, performing a change of variables may force the appearance of a deterministic integral. This can be translated into a more common form of the Itô formula once one passes to the language of *stochastic differential equations*.

Instead of expressing $X(t)$ as an integral involving $L(s)$, one may formally write the differential equality

$$\mathrm{d}X(t) = L(t)\mathrm{d}W_t.$$

In that form, Theorem 2.3 reads

$$\mathrm{d}g(t,X(t)) = \frac{\partial g}{\partial t}(t,X(t))\mathrm{d}t + (\mathrm{d}X(t))\frac{\partial g}{\partial x}(t,X(t)) + \frac{1}{2}(\mathrm{d}X(t))^2\frac{\partial^2 g}{\partial x^2}g(t,X(t))\frac{\mathrm{d}t}{\mathrm{d}W_t^2}$$

The first two terms are what is expected from the standard chain rule, but the third one is strange given that there is only one order two term. Moreover, since it corresponds to derivation with respect to x twice, one would expect $(\mathrm{d}W_t)^2$ to appear while we get dt in the end, which is represented by the fraction we used in the formula. This is usually summarized through a modification of the chain rule where products of second order derivatives are given by the following table:

	$\mathrm{d}t$	$\mathrm{d}W_t$
$\mathrm{d}t$	0	0
$\mathrm{d}W_t$	0	$\mathrm{d}t$

An interesting consequence of that fact is the following: consider another quantum stochastic differentiel equation

$$\mathrm{d}Y(t) = K(t)\mathrm{d}W_t.$$

Then, the product $X(t)Y(t)$ again defines a process of the same nature, which is given by the formula

$$\mathrm{d}(X(t)Y(t)) = X(t)K(t)\mathrm{d}W_t + L(t)Y(t)\mathrm{d}W_t + L(t)K(t)\mathrm{d}t.$$

2.1.2 Symmetric functions and iterated integrals

Let us now turn to a more hilbertian point of view. We would like to use stochastic integration to give an alternative description of the Hilbert space $L^2(\Omega,\mathscr{F},\mathbb{P})$. To do this, we need to iterate the construction: starting with a deterministic function f on n variables, we can define

$$\widetilde{f}(t,t_2,\cdots,t_n) = \int_0^t f(t_1,\cdots,t_n)\mathrm{d}W_{t_1}$$

and get a stochastic process, which we can now try to integrate again. However, an issue appears: the new process, seen as depending on t_2, need not be adapted anymore. One way round this problem is to restrict the function f to a smaller space, namely the *simplex*

$$\Delta_n = \left\{(t_1,\cdots,t_n)\in\mathbf{R}_+^n \mid t_1 < t_2 < \cdots < t_n\right\}.$$

From now on, we will work at a fixed time T, meaning that we consider the value of processes at that time, and can therefore work on the simplex

$$\Delta_n(T) = \{(t_1,\cdots,t_n)\in[0,T]^n \mid t_1 < t_2 < \cdots < t_n\}.$$

What we seek is a description of the Hilbert space $L^2(\Omega,\mathscr{F}_T,\mathbb{P})$, and iterated Itô integrals will provide the building blocks.

DEFINITION 2.4 (WIENER CHAOS). The n-th *Wiener chaos* at time T is the closed subspace $K_n(T)$ of $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ spanned by the elements of the form

$$J_n(f)_T = \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n}$$

for $f \in L^2(\Delta_n(T))$.

The point is that these subspaces provide a nice decomposition of the Hilbert space. This is however not a trivial fact, and for the moment we will simply give an elementary property.

Lemma 2.5. *The subspaces $K_n(T)$ are pairwise orthogonal.*

Proof. Let $n > k$ and consider $f \in L^2(\Delta_n(T))$ and $g \in L^2(\Delta_k(T))$. Then, by Proposition 2.2,

$$\begin{aligned}\mathbb{E}(J_n(f)_T J_k(g)_T) &= \mathbb{E}\left(\int_0^T J_{n-1}(f)_s J_{k-1}(g)_s \mathrm{d}s\right) \\ &= \int_0^T \mathbb{E}(J_{n-1}(f)_s J_{k-1}(g)_s) \mathrm{d}s\end{aligned}$$

and by induction it is enough to prove that the inner product vanishes when $k = 0$. But in that case, $J_0(g) = g$ is deterministic, so that

$$\mathbb{E}(J_n(f)_T g(T))) = \mathbb{E}(J_n(f)_T) g(T) = 0$$

since the stochastic integral of a square integrable adapted process, being a martingale, has vanishing expectation. ■

2.2 THE CHAOS DECOMPOSITION

The chaos spaces enable a coordinate free description of the space $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$, and this is what we will prove in this section. In other words we will show that the space

$$K(T) = \bigoplus_{n \in \mathbf{N}} K_n(T)$$

is dense in $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$. In fact we will prove some slightly stronger property. To state it, let us introduce a notation: for a function $f \in L^2(\mathbf{R}^+)$, we define a function $f^{\otimes n} \in L^2(\Delta_n(T))$ through the formula

$$f^{\otimes n}(t_1, \cdots, t_n) = f(t_1) \cdots f(t_n)$$

and we set

$$I_n(f)_T = J_n(f^{\otimes n})_T \in K_n(T).$$

Our building blocks will not be these elements, but rather some series involving them. To describe it, we will need a definition.

DEFINITION 2.6 (EXPONENTIAL PROCESS). For a fixed $\lambda \in \mathbf{R}$ and a deterministic function $f \in L^2(\mathbf{R}_+)$, we define a stochastic process $(M_t(f))_{t \in \mathbf{R}_+}$ through the formula

$$M_f(t) = \exp\left(\lambda \int_0^t f(s) \mathrm{d}W_s - \frac{\lambda^2}{2} \int_0^t f(s)^2 \mathrm{d}s\right).$$

This is called the *exponential process* associated to f with parameter λ.

To see that these are decomposable along the chaos spaces, we will use an elementary computation.

Lemma 2.7 (EXPONENTIAL PROCESS AS A STOCHASTIC INTEGRAL)**.** *The process M_f satisfies the equation*

$$M_f(t) = 1 + \lambda \int_0^t M_s f(s) \mathrm{d}W_s.$$

Proof. Let us write M_f instead of $M_f(t)$ in the sequel to lighten the notations. Setting

$$g(t,x) = \exp\left(\lambda x - \frac{\lambda^2}{2}\int_0^t f(s)^2 \mathrm{d}s\right),$$

we have a deterministic function such that $M_t = g(t, X_t)$, where X_t is the random variable defined as

$$X(t) = \int_0^t f(s)\mathrm{d}W_s,$$

so that $\mathrm{d}X(t) = f(t)\mathrm{d}W_t$. Therefore, the ITÔ FORMULA from Theorem 2.3 reads

$$\begin{aligned}
\mathrm{d}M_f(t) &= \frac{\partial g}{\partial t}(t, X(t))\mathrm{d}t + \frac{\partial g}{\partial x}(t, X(t))\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(\mathrm{d}X(t))^2 \\
&= -\frac{\lambda^2}{2} f(t)^2 g(t, X(t))\mathrm{d}t + \lambda g(t, X(t)) f(t)\mathrm{d}W_t + \frac{\lambda^2}{2} g(t, X(t)) f(t)^2 \mathrm{d}t \\
&= \lambda g(t, X(t)) f(t)\mathrm{d}W_t \\
&= \lambda M_f(t)\mathrm{d}W_t,
\end{aligned}$$

and

$$M_f(t) = M_0 + \int_0^t \mathrm{d}M_s = 1 + \lambda\int_0^t M_s f(s)\mathrm{d}W_s,$$

as claimed. ■

We can now relate stochastic exponentials to the chaos decomposition.

Proposition 2.8. *For any $f \in L^2(\mathbf{R}_+)$ and $\lambda \in \mathbf{R}$, we have for all $t \in \mathbf{R}_+$*

$$M_f(t) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} I_n(f)_t,$$

where $I_0(f)_t = \int_0^t f(s)\mathrm{d}s$.

Proof. Recall first the definition of the *(physicist's) Hermite polynomials* $H_n(x)$: for any $t, x \in \mathbf{R}$, they satisfy the formula

$$e^{tx - \frac{t^2}{2}} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x).$$

Therefore, we have by definition

$$\begin{aligned}
M_f(t) &= \exp\left(\left(\lambda\|f\|_{2,[0,t]}\right)\int_0^t \frac{f(s)}{\|f\|_{2,[0,t]}^2}\mathrm{d}W_s - \frac{(\lambda\|f\|_{2,[0,t]})^2}{2}\right) \\
&= \sum_{n=0}^{+\infty} \frac{\left(\lambda\|f\|_{2,[0,t]}\right)^n}{n!} H_n\left(\int_0^t \frac{f(s)}{\|f\|_{2,[0,t]}^2}\mathrm{d}W_s\right).
\end{aligned}$$

In other words, $M_f(t)$ has a power series expansion in λ.

Observe now that by the formula of Lemma 2.7, a direct iteration yields for any integer k (with the convention that $I_0(f) = 1$)

$$M_f(t) = \sum_{n=0}^{k} \lambda^k I_k(f)_t + \lambda^{k+1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_k} f(t_1)\cdots f(t_{k+1}) M_f(t_{k+1})\mathrm{d}t_{k+1}\cdots\mathrm{d}t_1.$$

This implies that

$$\left.\frac{\mathrm{d}^k M_f(t)}{\mathrm{d}\lambda^k}\right|_{\lambda=0} = k! I_k(f)$$

and the result follows. ■

Remark 2.9. As a byproduct of the previous proof, we have the following formula: for $f \in L^2(\mathbf{R}_+)$ and $t \in \mathbf{R}_+$,

$$I_n(f) = \|f\|_{2,[0,t]}^n H_n\left(\frac{\int_0^t f(s)\mathrm{d}W_s}{\|f\|_{2,[0,t]}}\right).$$

If f has norm one, then this simplifies to

$$I_n(f) = H_n\left(\int_0^t f(s)\mathrm{d}W_s\right).$$

With this in hand, we are ready for the main result of this section.

THEOREM 2.10 (WIENER CHAOS DECOMPOSITION) For any $T \in \mathbf{R}_+$, there is an equality of Hilbert spaces

$$L^2(\Omega, \mathscr{F}_T, \mathbb{P}) = \overline{\bigoplus}_{n\in\mathbf{N}} K_n(T).$$

Proof. We have to prove that if an element in $g \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ is orthogonal to the right-hand side, then it vanishes. By Proposition 2.8, we know that such a g is orthogonal to all stochastic exponentials, and we will show that this forces $g = 0$.

Fix $\lambda_0, \cdots, \lambda_n \in \mathbf{R}$ and $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$. We define a function $f \in L^2([0;T])$ in the following way: for $t_i \leqslant t < t_{i+1}$, $f(t) = \lambda_i$, and $f(T) = \lambda_n$. Then,

$$\begin{aligned}\int_0^T f(t)\mathrm{d}W_t &= \sum_{i=0}^{n-1} \lambda_i (W_{t_{i+1}} - W_{t_i}) \\ &= \sum_{i=1}^{n-1} (\lambda_{i-1} - \lambda_i) W_{t_i} + \lambda_{n-1} W_{t_n}\end{aligned}$$

Moreover,

$$\int_0^T f(t)^2 \mathrm{d}t = \sum_{i=0}^{n-1} \lambda_i^2 (t_{i+1} - t_i)$$

is a constant, so that in that case, $M_f(T)$ is proportional to

$$\exp\left(\sum_{i=1}^{n} \mu_i W_{t_i}\right),$$

where $\mu_i = \lambda_i - \lambda_{i-1}$ for $1 \leqslant i \leqslant n-1$ and $\mu_n = \lambda_{n-1}$. Conversely, any such random variable is of the form $M_f(T)$, so that if a function $g \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ is orthogonal to all the random variables $M_f(T)$, we have in particular for all $\mu_1, \cdots, \mu_n \in \mathbf{R}$ and $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$ that

$$\int_\Omega e^{\mu_1 W_{t_1}(\omega) + \cdots + \mu_n W_{t_n}(\omega)} \overline{g(\omega)} \mathrm{d}\mathbb{P}(\omega) = 0.$$

Note that for the needs of the proof, we have to work here with complex-valued functions, while we were using only real-valued ones before. Consider now the following function of n complex variables:

$$G(z_1, \cdots, z_n) = \int_\Omega e^{z_1 W_{t_1}(\omega) + \cdots + z_n W_{t_n}(\omega)} \overline{g(\omega)} \mathrm{d}\mathbb{P}(\omega).$$

It is analytic and vanishes on $\mathbf{R}^n$, hence it is identically 0. In particular, for any $y_1, \cdots, y_n \in \mathbf{R}$,

$$G(iy_1, \cdots, iy_n) = 0.$$

This enables us to use the Fourier transform. More precisely, if $\phi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ is a smooth function vanishing at infinity and if $\widehat{\phi}$ denotes its Fourier transform, then

$$\begin{aligned}
&\int_\Omega \phi(W_{t_1}(\omega), \cdots, W_{t_n}(\omega))\overline{g(\omega)}\mathrm{d}\mathbb{P}(\omega) \\
&= \int_\Omega \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbf{R}^n} \widehat{\phi}(y_1, \cdots, y_n)e^{iy_1W_{t_1}(\omega)+\cdots+iy_nW_{t_n}(\omega)}\mathrm{d}y_1\cdots\mathrm{d}y_n\right)\overline{g(\omega)}\mathrm{d}\mathbb{P}(\omega) \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n} \widehat{\phi}(y_1, \cdots, y_n)\left(\int_\Omega e^{iy_1W_{t_1}(\omega)+\cdots+iy_nW_{t_n}(\omega)}\overline{g(\omega)}\mathrm{d}\mathbb{P}(\omega)\right)\mathrm{d}y_1\cdots\mathrm{d}y_n \\
&= 0.
\end{aligned}$$

In other words, g is orthogonal to all the random variables of the form $\phi(W_{t_1}, \cdots, W_{t_n})$ for $\phi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$. We are therefore left with proving that the latter condition implies $g = 0$. To do this, let us pick a countable dense subset $(t_n)_{n\in\mathbf{N}}$ of $[0;T]$, and denote by $\mathscr{F}_N$ the σ-algebra generated by $W_{t_1}, \cdots, W_{t_n}$. On the one hand, by the MARTINGALE CONVERGENCE THEOREM (see for instance [Øk13, Cor C.9]), we have

$$g = \mathbb{E}[g \mid \mathscr{F}_T] = \lim_{N\to+\infty} \mathbb{E}[g \mid \mathscr{F}_N]$$

in $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$. On the other hand, by the DOOB-DYNKIN LEMMA (see for instance [Øk13, Lem 2.12]), there exist Borel measurable functions $g_N : \mathbf{R}^N \to \mathbf{R}$ such that

$$\mathbb{E}[g \mid \mathscr{F}_N] = g_N(W_{t_1}, \cdots, W_{t_n}).$$

Approximating g_N by functions in $\mathscr{C}_0^\infty(\mathbf{R}^n)$, we see that

$$\langle g, g_N(W_{t_1}, \cdots, W_{t_n})\rangle = 0,$$

and we conclude that $\|g\|^2 = 0$. ■

Remark 2.11. For simplicity, we have worked at fixed time T in this section. One can use the same strategy to prove a more general isomorphism

$$L^2(W) \cong \overline{\bigoplus_{n\in\mathbf{N}}} K_n,$$

where the left-hand side consists in all square integrable processes in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ adapted to $\mathscr{F}$ and K_n is the closed subspace generated by n-fold iterated Itô integrals.

3 FROM WIENER CHAOS TO FOCK SPACE

Our goal is now to revisit the definition of the Wiener process using the point of view of the chaos decomposition. More precisely, we would like to construct a Hilbert space, together with a family of operators which encode the same information as the process $(W_t)_{t\in\mathbf{R}_+}$. The story starts with very general considerations on Hilbert spaces, for which we assume some basic knowledge on tensor products of vector spaces.

3.1 THE SYMMETRIC FOCK SPACE

We will work for some time in a general abstract framework. For the sake of simplicity, and to stay as close as possible to the setting of real-valued processes on $\mathbf{R}_+$, we will only consider real Hilbert spaces in the sequel. Given two such Hilbert spaces H and K, recall that there is a canonical pre-Hilbert space structure on the vector space tensor product $H \otimes K$, given by

$$\langle x \otimes x', y \otimes y'\rangle = \langle x, x'\rangle\langle y, y'\rangle.$$

Completing if necessary yields a Hilbert space which will again be denoted by $H \otimes K$ (we will never consider non-completed tensor products). In the same way as we could construct the chaos spaces by looking at "exponentials" of functions, we will build a new Hilbert space out of H using some kind of exponential construction.

DEFINITION 3.1 (FOCK SPACE). The *Fock space* over a Hilbert space H is the *Hilbert space direct sum*

$$\mathscr{F}(H) = \overline{\bigoplus}_{n\in\mathbf{N}} H^{\otimes n},$$

where by convention $H^{\otimes 0} = \mathbf{R}$.

If $H = L^2(\mathbf{R}_+)$, then $H^{\otimes n}$ is naturally isomorphic to the space $L^2(\mathbf{R}_+^n)$ through the map sending $f_1 \otimes \cdots \otimes f_n$ to the functions

$$(t_1, \cdots, t_n) \mapsto f_1(t_1)\cdots f_n(t_n).$$

Therefore, $\mathscr{F}(H)$ contains square integrable functions in all number of variables, which is practical for the construction of iterated stochastic integrals. However, as we have already seen, this is too large because of the necessity of adaptedness. Since using the simplex Δ_n makes no sense for an abstract Hilbert space, we have to find a substitute. As it turns out, there is a simple way to do this using symmetrization.

DEFINITION 3.2 (SYMMETRIC FUNCTION). A function $f \in L^2(\mathbf{R}_+^n)$ is said to be *symmetric* if for any $\sigma \in \mathfrak{S}_n$,

$$f\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) = f(t_1, \cdots, t_n).$$

The space of quare summable symmetric functions is denoted by $L_s^2(\mathbf{R}_+^n)$.

Given a symmetric function $f \in L^2(\mathbf{R}_+^n)$, one can of course restrict it to Δ_n to produce a square integrable function. Moreover, f is completely determined by that restriction: for any $(t_1, \cdots, t_n) \in \mathbf{R}_+$, on can always find a permutation $\sigma \in \mathfrak{S}_n$ such that $t_{\sigma(1)} \leqslant \cdots \leqslant t_{\sigma(n)}$ so that f only depends on its values on Δ_n. In other words, we have an isomorphism of Hilbert spaces

$$L^2(\Delta_n) \simeq L_s^2(\mathbf{R}_+^n).$$

To extend the idea of symmetric functions to an arbitrary Hilbert space, simply observe that for two functions, $f, g \in L^2(\mathbf{R}_+)$, we have

$$(f \otimes g)(t_2, t_1) = f(t_2)g(t_1) = g(t_1)f(t_2) = (g \otimes f)(t_1, t_2)$$

so that the function $f \otimes g + g \otimes f$ is symmetric. In other words, to symmetrize the functions, we should simply consider linear combinations of tensors which are symmetric. Concretely, let us define an operator $P_n : H^{\otimes n} \to H^{\otimes n}$ through the formula

$$P_n(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

This extends by linearity to a projection onto the space $H^{\otimes_s n}$ of symmetric tensors, hence we are close to what we want. The only missing piece is to check that the previous operators can be patched together to define a map on the full Fock space.

Lemma 3.3. *The map*

$$P_n : H^{\otimes n} \to H^{\otimes n}$$

is a projection and has norm 1.

Proof. First, observe that for any $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned} \|x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}\| &= \prod_{i=1}^{n} \|x_{\sigma(i)}\| \\ &= \prod_{i=1}^{n} \|x_i\| \\ &= \|x_1 \otimes \cdots \otimes x_n\|, \end{aligned}$$

so that $\|P_n(x_1\otimes\cdots\otimes x_n)\| \leqslant \|x_1\otimes\cdots\otimes x_n\|$ and P_n has norm at most one. Moreover,

$$\begin{aligned} P_n\left(P_n(x_1\otimes\cdots\otimes x_n)\right) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} P_n(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}) \\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} x_{\tau\circ\sigma(1)}\otimes\cdots\otimes x_{\tau\circ\sigma(n)} \end{aligned}$$

For any $\sigma\in\mathfrak{S}_n$, we have $\{\tau\circ\sigma \mid \tau\in\mathfrak{S}_n\} = \{\tau\circ\sigma \mid \tau\in\mathfrak{S}_n\}$, hence the second sum above does not depend on σ and simply equals $P_n(x_1\otimes\cdots\otimes x_n)$. Therefore,

$$\begin{aligned} P_n\left(P_n(x_1\otimes\cdots\otimes x_n)\right) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} P_n(x_1\otimes\cdots\otimes x_n) \\ &= P_n(x_1\otimes\cdots\otimes x_n). \end{aligned}$$

This shows that P_n is a projection and that its norm is exactly one, thereby completing the proof. ■

Assembling these operators, we get a bounded linear map

$$P_s : \mathscr{F}(H)\to\mathscr{F}(H)$$

which coincides with P_n on $H^{\otimes n}$ and whose range is the space that we are looking for.

DEFINITION 3.4 (SYMMETRIC FOCK SPACE). The *symmetric Fock space* over H is the range of P_s, denoted by $\mathscr{F}_s(H)$. Being the range of an orthogonal projection, it is a closed subspace, hence inherits a Hilbert space structure.

Let us illustrate that construction in the case $H = L^2(\mathbf{R}_+)$ which is of prominent interest to us. We claim that the isomorphisms

$$\Phi_n : \left\{ \begin{array}{ccc} L^2(\mathbf{R}_+)^{\otimes n} & \to & L^2(\mathbf{R}_+^n) \\ f_1\otimes\cdots\otimes f_n & \mapsto & \left((t_1,\cdots,t_n)\mapsto \prod_{i=1}^n f_i(t_i)\right) \end{array}\right.$$

map $L^2(\mathbf{R}_+)^{\otimes_s n}$ to $L^2_s(\mathbf{R}_+^n)$. Indeed, for any $\sigma\in\mathfrak{S}_n$,

$$\begin{aligned} \Phi_n\left(f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)}\right)(t_1,\cdots,t_n) &= \prod_{i=1}^n f_{\sigma(i)}(t_i) \\ &= \prod_{i=1}^n f_i(t_{\sigma^{-1}(i)}) \end{aligned}$$

so that

$$\begin{aligned} \Phi_n\circ P_n(f_1\otimes\cdots\otimes f_n) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n f_i(t_{\sigma^{-1}(i)}) \\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n f_i(t_{\sigma(i)}). \end{aligned}$$

The right-hand side is a symmetric function, hence $\Phi_n(L^2(\mathbf{R}_+)^{\otimes_s n})\subset L^2_s(\mathbf{R}_+^n)$. As for the converse inclusion, let $f\in L^2_s(\mathbf{R}_+^n)$ and let $(f_i^\ell)_{1\leqslant i\leqslant n}$ be a family of tuples of functions in $L_2(\mathbf{R}_+)$ such that

$$f = \Phi_n\left(\sum_{\ell\in\mathbf{N}} f_1^\ell\otimes\cdots\otimes f_n^\ell\right),$$

the series being convergent in $L^2(\mathbf{R}_+)^{\otimes n}$. Then, we have

$$\begin{aligned}
f(t_1,\cdots,t_n) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f(t_{\sigma(1)},\cdots,t_{\sigma(n)}) \\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\ell\in\mathbf{N}} f_1^\ell(t_{\sigma(1)})\cdots f_n^\ell(t_{\sigma(n)}) \\
&= \sum_{\ell\in\mathbf{N}}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f_1^\ell(t_{\sigma(1)})\cdots f_n^\ell(t_{\sigma(n)}) \\
&= \sum_{\ell\in\mathbf{N}}\Phi_n\left(P_n(f_1^\ell\otimes\cdots\otimes f_n^\ell)(t_1,\cdots,t_n)\right) \\
&= \Phi_n\left(\sum_{\ell\in\mathbf{N}}P_n(f_1^\ell\otimes\cdots\otimes f_n^\ell)(t_1,\cdots,t_n)\right) \\
&= \Phi_n\left(P_n\left(\sum_{\ell\in\mathbf{N}}f_1^\ell\otimes\cdots\otimes f_n^\ell)(t_1,\cdots,t_n)\right)\right)
\end{aligned}$$

and the proof is complete.

3.2 THE FUNDAMENTAL PROCESSES

The construction of the symmetric Fock space originates in physics. In quantum mechanics the possible states of a system – say a particle – are described by a Hilbert space. If now several systems are coupled, the states are described by the tensor product of the corresponding Hilbert spaces. But what if we couple identical particles that we cannot distinguish ? Then, the correct space describing the system might rather be the symmetric Fock space. There is a subtelty here, since this principle only holds for specific particles called *bosons*, while *fermions* require the use of the antisymmetric version of that construction (that we will not consider in this text).

Anyways, that physical intuition underlying the symmetric Fock space naturally leads to the definition of two fundamental families of operators. Let us right away mention that these operators are *unbounded*, which means that in general, one has to carefully deal with domain issues and all the technicalities of unbounded operators when working with quantum stochastic integration. However, we will take in the sequel an easy path which focuses on a canonical dense subspace of the symmetric Fock space on which all the operators involved are defined.

Let H be a Hilbert space and let $x\in H$. The corresponding *exponential vector* $e(x)$ is defined as

$$e(x) = \sum_{n=0}^{+\infty}\frac{x^{\otimes n}}{\sqrt{n!}} \in \mathscr{F}_s(H),$$

where $x^{\otimes 0} = 1 \in \mathbf{R} = H^{\otimes 0}$ by convention. Observe that the sum converges because $\|x^{\otimes n}\| = \|x\|^n$. The square root in the denominator might be surprising at first sight. However, it will look more natural once we realize that it yields a useful formula for inner products, namely

$$\langle e(x), e(y)\rangle = e^{\langle x,y\rangle}.$$

As we will see, these vectors are very practical for computations, and this is the reason why we will restrict to them in the sequel.

DEFINITION 3.5. The *exponential domain* of H is the subspace $\mathscr{E}(H)\subset\mathscr{F}_s(H)$ of finite linear combinations of exponential vectors.

Before going further, let us check for sanity that restricting to the exponential domain does not “loose information”, in the sense that this subspace is dense in the symmetric Fock space.

Proposition 3.6. *The exponential domain is dense in $\mathscr{F}_s(H)$.*

Proof. Let $x \in H$. We will prove by induction the following assertion, where by convention $x^{\otimes 0} = 1 \in \mathbf{R}$:

$\mathscr{H}_n$: "The vector $x^{\otimes k}$ is in the closure of the exponential domain for all $0 \leqslant k \leqslant n$."

Let us proceed.

Initialization – Observe that by definition, $e(0) \in \mathscr{E}(H)$ is the vector $1 = x^{\otimes 0} \in \mathbf{R} = H^{\otimes 0}$ so that $\mathscr{H}_0$ holds.

Heredity – Assume that $\mathscr{H}_n$ holds for some $n \geqslant 0$, and observe that

$$\frac{1}{\epsilon^{n+1}}\left(e(\epsilon x) - \sum_{k=0}^{n} \frac{\epsilon^k}{\sqrt{k!}} x^{\otimes k}\right) \underset{\epsilon \to 0}{\longrightarrow} \frac{1}{\sqrt{(n+1)!}} x^{\otimes n+1}.$$

Because the right-hand side is in $\overline{\mathscr{E}(H)}$ by $\mathscr{H}_n$, this readily yields $\mathscr{H}_{n+1}$.

We are now left with proving that any symmetric vector in $H^{\otimes_s n}$ is a linear combination of tensor powers. This follows from the following computation: given $x_1, \cdots, x_n \in H$, we have

$$\sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right) (\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n)^{\otimes n} = \sum_{i_1,\cdots,i_n=1}^{n} \left(\sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right) \varepsilon_{i_1} \cdots \varepsilon_{i_n}\right) x_{i_1} \otimes \cdots \otimes x_{i_n}.$$

If all the indices i_k are distinct, then

$$\varepsilon_{i_1} \cdots \varepsilon_{i_n} = \prod_{i=1}^{n} \varepsilon_i,$$

so that

$$\begin{aligned}\sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right) \varepsilon_{i_1} \cdots \varepsilon_{i_n} &= \sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right)^2 \\ &= 2^n.\end{aligned}$$

Otherwise, there are some equal indices, therefore there exists $i_0 \in \{1, \cdots, n\}$ which is different from $i_1, \cdots, i_n$. Let us assume without loss of generality that $i_0 = 1$, so that

$$\begin{aligned}\sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right) \varepsilon_{i_1} \cdots \varepsilon_{i_n} &= \sum_{\varepsilon_2,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=2}^{n} \varepsilon_i\right) \varepsilon_{i_1} \cdots \varepsilon_{i_n} \\ &\quad - \sum_{\varepsilon_2,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=2}^{n} \varepsilon_i\right) \varepsilon_{i_1} \cdots \varepsilon_{i_n} \\ &= 0.\end{aligned}$$

In other words,

$$2^n P_s(x_{i_1} \otimes \cdots \otimes x_{i_n}) = \sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{-1,1\}} \left(\prod_{i=1}^{n} \varepsilon_i\right) (\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n)^{\otimes n}$$

and the proof is complete. ■

We are now ready to make sense of our fundamental operators. Their most general definition is as follows: given a vector $x \in H$, and a tensor $v = x_1 \otimes \cdots \otimes x_n$ in the Fock space $\mathscr{F}(H)$, the operator $A(x)$ "annihilates" the first tensor of v by turning it into

$$A(x)v = \langle x, x_1 \rangle x_2 \otimes \cdots \otimes x_n$$

while the operator $A^\dagger$ "creates" a new tensor in v by turning it into

$$A^\dagger(x)v = x \otimes x_1 \otimes \cdots \otimes x_n.$$

The problem with these descriptions is that they do not preserve symmetric tensors. One therefore has to symetrize the formula: by letting the operator act on all tensors and averaging the results. Instead of doing such computations, we will directly define our operators through their action on exponential vectors, since these are the only ones we are going to work with. This however requires first checking that these vectors form a linearly independent set.

Lemma 3.7. *The exponential vectors form a linearly independent set.*

Proof. Let $x_1, \cdots, x_k \in H$ and $\lambda_1, \cdots, \lambda_k \in \mathbf{R}$ be such that

$$\sum_{i=1}^{k} \lambda_i e(x_i) = 0.$$

For $1 \leqslant i \neq j \leqslant k$, consider the set

$$U_{ij} = \{x \in H \mid \langle x, x_i\rangle \neq \langle x, x_j\rangle\}.$$

It is the complement of a closed hyperplane, hence both open and dense in H. As a consequence, the set

$$U = \bigcap_{1\leqslant i\neq j\leqslant k} U_{ij}$$

is also an open dense subset, and in particular is non-empty. Picking $x \in U$, we then have that the numbers $\theta_i = \langle x, x_i\rangle$ are all distinct. But now we can compute, for any $t \in \mathbf{R}$,

$$\begin{aligned} 0 &= \langle e(tx), \sum_{i=1}^{k} \lambda_i e(x_i)\rangle \\ &= \sum_{i=1}^{k} \lambda_i e^{\langle tx, x_i} \\ &= \sum_{i=1}^{k} \lambda_i e^{t\theta_i}. \end{aligned}$$

The linear independence of the exponential functions then implies $\lambda_i = 0$ for all $1 \leqslant i \leqslant k$ completing the proof. ■

DEFINITION 3.8 (CREATION AND ANNIHILATION OPERATORS). Let $x \in H$. The *annihilation operator* $A(x)$ and the *creation operator* $A^\dagger(x)$ are defined on the exponential domain $\mathscr{E}(H)$ through the formulæ

$$\left\{\begin{array}{lcl} A(x)e(y) &=& \langle x, y\rangle e(y) \\ A^\dagger(x)e(y) &=& \dfrac{\mathrm{d}}{\mathrm{d}t} e(y+tx)\Big|_{t=0} \end{array}\right.$$

Annihilation operators are nice objects because all exponential vectors are proper. Meanwhile, it is essential to note that creation operators do not map the exponential domain to itself ! Notwithstanding, we will ignore the part that is sent "out of exponential vectors" by focusing on *coefficients*, that is, inner products with other exponential vectors. Indeed, one has the following fundamental computation:

$$\begin{aligned} \left\langle e(z), A^\dagger(x)e(y)\right\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle e(z), e(y+tx)\rangle\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} e^{\langle z, y+tx\rangle}\Big|_{t=0} \\ &= \langle z, x\rangle e^{\langle z, y\rangle} \\ &= \langle z, x\rangle\langle e(z), e(y)\rangle \end{aligned}$$

Using this, we can establish some important relations between the creation and annihilation operators. Note that in all the sequel, we will work exclusively with exponential vectors. Some of the results can be extended to other dense subspaces of $\mathscr{F}_s(H)$, sometimes to the price of more involved analysis.

Proposition 3.9. *The creation and annihilation operators are adjoint to one another on the exponential domain, in the sense that for any $x, y, z \in H$,*

$$\langle A(x)e(z), e(y)\rangle = \left\langle e(z), A^\dagger(x)e(y)\right\rangle.$$

Moreover, they satisfy the canonical commutation relations

$$A(x')A^\dagger(x) - A^\dagger(x)A(x') = \langle x', x\rangle \mathrm{Id}$$

for all $x, x' \in H$, in the sense that the equality holds for any coefficient given by exponential vectors.

Proof. To prove the adjunction, we simply compute

$$\begin{aligned}\langle A(x)e(z), e(y)\rangle &= \langle\langle x, z\rangle e(z), e(y)\rangle \\ &= \langle z, x\rangle \langle e(z), e(y)\rangle \\ &= \left\langle e(z), A^\dagger(x)e(y)\right\rangle.\end{aligned}$$

To prove the canonical commutation relations, we proceed similarly using vectors $z, w \in H$. We first observe that

$$\begin{aligned}\langle e(z), A(x)^\dagger A(x')e(y)\rangle &= \langle x', y\rangle \left\langle e(z), A^\dagger(x)e(y)\right\rangle \\ &= \langle z, x\rangle\langle x', y\rangle \langle e(z), e(y)\rangle\end{aligned}$$

while

$$\begin{aligned}\langle e(z), A(x')A^\dagger(x)e(y)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle e(z), A(x')e(y+tx)\right\rangle\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x', y+tx\rangle\langle e(z), e(y+tx)\rangle\Big|_{t=0} \\ &= \langle x', x\rangle\langle e(z), e(y)\rangle + \langle x', y\rangle\langle z, x\rangle\langle e(z), e(y)\rangle.\end{aligned}$$

Substracting the two quantities yields the result. ■

The creation and annihilation operators first appeared in the study of the quantum harmonic oscillator. They were built from the two basic operators of the physical system: the *position* and *momentum* operators. The point was that creation and annihilation give an equivalent description for which the spectral analysis is simpler (see for instance [Hal13, Chap 11]). Let us now reverse the construction, with a little abuse: the definition of the momentum operator P requires the use of complex numbers, hence we will temporarily work with complex Hilbert spaces, but they will not be used afterwards.

DEFINITION 3.10. For $x \in H$, the operators

$$Q(x) = \left(A(x) + A^\dagger(x)\right) \quad \& \quad P(x) = i\left(A(x) - A^\dagger(x)\right)$$

are called respectively the *position* and *momentum* operator.

These operators satisfy an important relation which is one of the corenerstones of quantum mechanics.

Proposition 3.11 (HEISENBERG COMMUTATION RELATIONS)**.** *The position and momentum operators satisfy the following commutation relation on the exponential domain*

$$P(x)Q(x') - Q(x')P(x) = 2i\Re(\langle x, x'\rangle),$$

in the sense that the equality holds for any coefficients given by exponential vectors.

Proof. Recall that the statement is meant to hold for all coefficients given by exponential vectors. Using Proposition 3.9, we compute

$$\begin{aligned}
P(x)Q(x') - Q(x')P(x) &= i\left(A(x)A(x') + A(x)A^\dagger(x') - A^\dagger(x)A(x') - A^\dagger(x)A^\dagger(x')\right) \\
&\quad - i\left(A(x')A(x) + A(x')A^\dagger(x) - A^\dagger(x')A(x) - A^\dagger(x')A^\dagger(x)\right) \\
&= i(\langle x, x'\rangle + \langle x', x\rangle) + i\left(A(x)A(x') - A(x')A(x)\right) - i\left(A^\dagger(x)A^\dagger(x') - A^\dagger(x')A^\dagger(x)\right) \\
&= 2i\Re(\langle x, x'\rangle) + i\left(A(x)A(x') - A(x')A(x)\right) - i\left(A^\dagger(x)A^\dagger(x') - A^\dagger(x')A^\dagger(x)\right)
\end{aligned}$$

We are therefore left with computing the commutator of two creation or annihilation operators on the exponential domain. This follows quite easily from the definitions: for any $y \in H$,

$$\begin{aligned}
A(x')A(x)e(y) &= \langle x, y\rangle A(x')e(y) \\
&= \langle x, y\rangle\langle x', y\rangle e(y) \\
&= \langle x', y\rangle A(x)e(y) \\
&= A(x)A(x')e(y).
\end{aligned}$$

and same for $A^\dagger$,

$$\begin{aligned}
A^\dagger(x')A^\dagger(x)e(y) &= A^\dagger(x')\frac{\mathrm{d}}{\mathrm{d}t}e(y+tx)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}A^\dagger e(y+tx)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}e(y+tx+sx')\Big|_{s=0}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}e(y+tx+sx')\Big|_{t=0}\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}A^\dagger(x)e(y+sx')\Big|_{s=0} \\
&= A^\dagger(x)A^\dagger(x')e(y).
\end{aligned}$$

■

3.3 THE ISOMORPHISM

We will now join the two topics discussed so far – Wiener chaos decomposition and symmetric Fock space – by recovering the Wiener process from the position operator. More precisely, let us fix from now on our Hilbert space to be $H = L^2(\mathbf{R}^+)$. This leads to very natural *operator-valued processes* given by

$$\begin{aligned}
A_t &= A(\mathbf{1}_{[0,t]}); \\
A_t^\dagger &= A^\dagger(\mathbf{1}_{[0,t]}); \\
Q_t &= A_t + A_t^\dagger,
\end{aligned}$$

where $\mathbf{1}_S$ denotes the indicator function of the set S. Our claim is that $(Q_t)_{t\in\mathbf{R}_+}$ is the same in some sense as the Wiener process. To make sense of such an assertion, it is first necessary to recast W_t as an operator. But that is simple to do: simply consider the operator

$$m_{W_t} : L^2(\Omega, \mathscr{F}_t, \mathbb{P}) \to L^2(\Omega, \mathscr{F}_t, \mathbb{P})$$

of multiplication by W_t, i.e. sending g to $W_t g$.

In order to compare Q_t with m_{W_t} we need to map the Hilbert space on which one acts to the Hilbert space on which the other one acts. As it turns out, we have a very natural way of doing this, thanks to the notions of exponentials that we have introduced.

DEFINITION 3.12. For $f \in L^2(\mathbf{R}_+)$, we denote by $\mathscr{E}(f)$ the stochastic exponential associated to f with $\lambda = 1$. We define an operator

$$\Psi : L^2(\Omega, \mathscr{F}_T, \mathbb{P}) \to \mathscr{F}_s(L^2([0,T]))$$

by the formula

$$\Psi(\mathscr{E}(f)_T) = e(f\mathbf{1}_{[0,T]}).$$

We can now conclude this section with a statement describing the Wiener process in the Hilbert space framework.

THEOREM 3.13 (FOCK SPACE REALIZATION OF WIENER PROCESS) For all $T \in \mathbf{R}_+$, the map Ψ is an isomorphism satisfying

$$\Psi^{-1} \circ Q_T \circ \Psi = m_{W_T}$$

on stochastic exponentials.

Proof. We first have to prove that Ψ preserves inner products. Let us start by computing the expectation of a stochastic exponential: because Itô integrals are martingales and therefore have vanishing expectation, the recursion formula from Lemma 2.7 yields for $f \in L^2(\mathbf{R}_+)$ and $t \in \mathbf{R}_+$

$$\mathbb{E}(M_f(t)) = \mathbb{E}\left(1 + \lambda \int_0^t M_f(s) f(s) \mathrm{d}W_s\right) = 1.$$

With this in hand, we can compute the inner product of two stochastic exponentials: for $f, g \in L^2(\mathbf{R}_+)$,

$$\begin{aligned}
\mathbb{E}(\mathscr{E}(f)_T \mathscr{E}(g)_T) &= \mathbb{E}\left(\exp\left(\int_0^T (f(s)+g(s))\mathrm{d}W_s - \frac{1}{2}\left(\|f\|^2_{2,[0,T]} + \|g\|^2_{2,[0,T]}\right)\right)\right) \\
&= e^{-\frac{1}{2}\left(\|f\|^2_{2,[0,T]} + \|g\|^2_{2,[0,T]}\right)} \mathbb{E}\left(\exp\left(\int_0^T (f(s)+g(s))\mathrm{d}W_s\right)\right) \\
&= e^{\langle f\mathbf{1}_{[0,T]}, g\mathbf{1}_{[0,T]}\rangle - \frac{1}{2}\left(\|f+g\|^2_{2,[0,T]}\right)} \mathbb{E}\left(\exp\left(\int_0^T (f(s)+g(s))\mathrm{d}W_s\right)\right) \\
&= e^{\langle f\mathbf{1}_{[0,T]}, g\mathbf{1}_{[0,T]}\rangle} \mathbb{E}\left(\exp\left(\int_0^T (f(s)+g(s))\mathrm{d}W_s - \frac{1}{2}\left(\|f+g\|^2_{2,[0,T]}\right)\right)\right) \\
&= e^{\langle f\mathbf{1}_{[0,T]}, g\mathbf{1}_{[0,T]}\rangle} \mathbb{E}(\mathscr{E}(f+g)) \\
&= e^{\langle f\mathbf{1}_{[0,T]}, g\mathbf{1}_{[0,T]}\rangle} \\
&= \left\langle e(f\mathbf{1}_{[0,T]}), e(g\mathbf{1}_{[0,T]})\right\rangle.
\end{aligned}$$

A first consequence of the previous computation is that Ψ is an isometry on stochastic exponentials. This is implies both that it is injective and that it has dense range. But since exponential vectors are dense in $\mathscr{F}_s(L^2([0,T])$, we conclude that Ψ is an isomorphism of Hilbert spaces.

Let us now prove the equality in the statement. Given $f, g \in L^2([0,T])$,

$$\begin{aligned}
\left\langle \mathscr{E}(g)_T, \left(\Psi^{-1} \circ Q_T \circ \Psi\right)\mathscr{E}(f)_T \right\rangle &= \left\langle \Psi^{-1*}\mathscr{E}(g), Q_T e(f\mathbf{1}_{[0,T]})\right\rangle \\
&= \left\langle e(g\mathbf{1}_{[0,T]}), Q_T e(f\mathbf{1}_{[0,T]})\right\rangle \\
&= \left\langle e(g\mathbf{1}_{[0,T]}), A_T e(f\mathbf{1}_{[0,T]})\right\rangle + \left\langle e(g\mathbf{1}_{[0,T]}), A_T^{\dagger} e(f\mathbf{1}_{[0,T]})\right\rangle \\
&= \left\langle e(\mathbf{1}_{[0,T]}), e(f\mathbf{1}_{[0,T]})\right\rangle \left\langle e(g\mathbf{1}_{[0,T]}), e(f\mathbf{1}_{[0,T]})\right\rangle \\
&+ \left\langle e(g\mathbf{1}_{[0,T]}), e(\mathbf{1}_{[0,T]})\right\rangle \left\langle e(g\mathbf{1}_{[0,T]}), e(f\mathbf{1}_{[0,T]})\right\rangle \\
&= \left(\int_0^T f(s)\mathrm{d}s\right) \left\langle e(g\mathbf{1}_{[0,T]}), e(f\mathbf{1}_{[0,T]})\right\rangle + \left(\int_0^T g(s)\mathrm{d}s\right) \left\langle e(g\mathbf{1}_{[0,T]}), e(f\mathbf{1}_{[0,T]})\right\rangle \\
&= \left(\int_0^T f(s) + g(s)\mathrm{d}s\right) e^{\int_0^T f(s)g(s)\mathrm{d}s}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}\langle \mathscr{E}(g)_T, m_{W_T}\mathscr{E}(f)_T\rangle &= \mathbb{E}(\mathscr{E}(g)_T W_T \mathscr{E}(f)_T)\\ &= \mathbb{E}\left(W_T \exp\left(\int_0^T f(s)+g(s)\mathrm{d}W_s - \frac{1}{2}\int_0^T f(s)^2+g(s)^2\mathrm{d}s\right)\right)\\ &= \mathbb{E}\left(W_T\mathscr{E}(f+g)_T e^{\int_0^T f(s)g(s)\mathrm{d}s}\right)\\ &= \mathbb{E}(W_T\mathscr{E}(f+g)_T)e^{\int_0^T f(s)g(s)\mathrm{d}s}\end{aligned}$$

To complete the computation, observe that $W_T = \int_0^T \mathrm{d}W_s \in K_1(T)$, so that by Lemma 2.5, W_T is orthogonal to $K_n(T)$ for all $n \neq 1$. It then follows from proposition 2.8 that

$$\begin{aligned}\mathbb{E}(W_T\mathscr{E}(f+g)_T) &= \sum_{n=0}^{+\infty}\frac{1}{n!}\mathbb{E}(W_T I_n(f+g)_T)\\ &= \mathbb{E}(W_T I_1(f+g)_T)\\ &= \mathbb{E}\left(I_1(\mathbf{1}_{[0,T)})I_1(f+g)_T\right)\\ &= \mathbb{E}\left(\mathbf{1}_{[0,T]}(f+g)\right)\\ &= \int_0^T f(s)+g(s)\mathrm{d}s\end{aligned}$$

and the proof is complete. ■

Remark 3.14. The same arguments show that the momentum process P_t also encodes the Wiener process on $\mathscr{F}_s(H)$. This is remarkable, since it provides two distinct realization of the Wiener process on the same Hilbert space, which moreover do not commute with one another by Proposition 3.11. This is a typical situation where quantum probability theory exhibits properties which are not classically possible.

Remark 3.15. As in Remark 2.11, one can prove along the same lines the existence of a global isomorphism

$$\Psi : L^2(W) \to \mathscr{F}\left(L^2(\mathbf{R}_+)\right)$$

such that $\Psi^{-1}\circ Q_t \circ \Psi = m_{W_t}$ for all $t \in \mathbf{R}_+$.

4 QUANTUM STOCHASTIC CALCULUS

In this section, we proceed to define integrals of stochastic processes, using the creation and annihilation processes as integrands instead of the Wiener process. As in the classical theory, the strategy is first to construct the integral for processes with a particularly tractable form, and then derive enough information on these to be able to extend the construction to larger families of processes. Our treatment of the theory is a simplified version of [Par92, Part III], which is the standard reference on the subject. We refer to 4.4 for details on the simplifications we made and how the constructions and results can be generalized.

4.1 SIMPLE PROCESSES

We start by defining the class of processes with which we will be working for the first step of the construction. This is very elementary and parallels the classical theory.

DEFINITION 4.1 (SIMPLE PROCESS). A *simple process* is a family $(L(t))_{t\in\mathbf{R}_+}$ of operators on $L^2(\mathbf{R}_+)$ for which there exists $0 = t_0 < t_1 < \cdots < t_n$ such that

- $L(t) = L_{t_j}$ for all $0 \leqslant j \leqslant n-1$ and $t_j \leqslant t < t_{j+1}$;

- $L(t) = L_{t_n}$ for all $t \geqslant t_n$.

The definition of the corresponding integral is straightforward.

DEFINITION 4.2 (STOCHASTIC INTEGRAL OF A SIMPLE PROCESS). Let $M \in \{A, A^\dagger\}$ and let $(L(t))_{t\in\mathbf{R}}$ be a simple stochastic process. The *(quantum) stochastic integral* of $(L(t))_{t\in\mathbf{R}_+}$ with respect to M is the stochastic process $(X(t))_{t\in\mathbf{R}_+}$ defined as

$$X(t) = \begin{cases} \sum_{j=1}^{k} L_{t_{j-1}}(M_{t_j} - M_{t_{j-1}}) + L_{t_k}(M_t - M_{t_k}) & \text{if} \quad t_k \leqslant t < t_{k+1} \\ \sum_{j=1}^{n} L_{t_{j-1}}(M_{t_j} - M_{t_{j-1}}) + L_{t_n}(M_t - M_{t_n}) & \text{if} \quad t_n < t \end{cases}$$

Remark 4.3. It is important to note that this process is of course no longer simple.

Even though this setting is natural, it is too general to produce interesting results. For instance, in classical stochastic calculus, the notion of adaptedness is crucial, and no quantum counterpart of it has appeared yet. Let us work this out now. The definition relies on a specific decomposition of the symmetric Fock space.

Proposition 4.4. *For any $t \in \mathbf{R}_+$, there is an isomorphism*

$$\Phi_t : \mathscr{F}_s\left(L^2(\mathbf{R}_+)\right) \cong \mathscr{F}_s\left(L^2([0,t])\right) \otimes \mathscr{F}_s\left(L^2(]t,+\infty[)\right)$$

given on exponential vectors by

$$\Phi_t(e(f)) = e(f\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[}).$$

Proof. Let us first check that Φ_t preserves inner products. This follows from a direct computation:

$$\begin{aligned} \langle \Phi_t(e(f)), \Phi_t(e(g))\rangle &= \left\langle e(f\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[}), e(f\mathbf{1}_{[0,t]}) \otimes e(g\mathbf{1}_{]t,+\infty[})\right\rangle \\ &= \left\langle e(f\mathbf{1}_{[0,t]}), e(g\mathbf{1}_{[0,t]})\right\rangle \left\langle e(f\mathbf{1}_{]t,+\infty[}), e(g\mathbf{1}_{]t,+\infty[})\right\rangle \\ &= e^{\int_0^t f(s)g(s)\mathrm{d}s} e^{\int_t^{+\infty} f(s)g(s)\mathrm{d}s} \\ &= e^{\int_0^{+\infty} f(s)g(s)\mathrm{d}s} \\ &= \langle e(f), e(g)\rangle. \end{aligned}$$

As a consequence, Φ_t is injective and has dense range. Moreover, let $g \in L^2([0,t])$ and $h \in L^2(]t,+\infty[)$. Then, defining a function f by

$$f(s) = \begin{cases} g(s) & \text{if} \quad s \leqslant t \\ h(s) & \text{if} \quad t < s \end{cases}$$

we have $\Phi_t(e(f)) = e(g) \otimes e(h)$. Since the elements on the right-hand side generate the subspace $\mathscr{E}(L^2[0,T]) \otimes \mathscr{E}(L^2(]T,+\infty[))$ of $\mathscr{F}_s\left(L^2([0,t])\right) \otimes \mathscr{F}_s\left(L^2(]t,+\infty[)\right)$, which is dense by Proposition 3.6, we conclude that Φ_t is surjective, and the proof is complete. ∎

We can now give a natural notion of adaptedness for our processes.

DEFINITION 4.5. A stochastic process $(L(t))_{t\in\mathbf{R}_+}$ is said to be *adapted* if for all $t \in \mathbf{R}_+$, there exists an operator $\widetilde{L}(t)$ acting on $\mathscr{F}_s(L^2([0,t]))$ such that

$$\Phi_t \circ L(t) \circ \Phi_t^{-1} = \widetilde{L}(t) \otimes \mathrm{Id}_{\mathscr{F}_s(L^2(]t,+\infty[))}.$$

To try our hands on the definitions, let us check for sanity that this property is preserved under integration.

Lemma 4.6. *Let $(L(t))_{t\in\mathbf{R}_+}$ be a simple adapted process and let $M \in \{A, A^\dagger\}$. Then, the corresponding stochastic integral $(X(t))_{t\in\mathbf{R}_+}$ is an adapted process.*

Proof. Let us fix $t \in \mathbf{R}_+$ and observe that we can refine the partition in the definition of an adapted process to have $0 = t_0 < t_1 < \cdots < t_n = t$. Then,

$$\begin{aligned}
\Phi_t \circ X(t) \circ \Phi_t^{-1} &= \sum_{j=1}^{n} \Phi_t \circ L_{t_{j-1}} \circ (M_{t_j} - M_{t_{j-1}}) \circ \Phi_t^{-1} \\
&= \sum_{j=1}^{n} \Phi_t \circ L_{t_{j-1}} \circ \Phi_t^{-1} \circ \Phi_t \circ (M_{t_j} - M_{t_{j-1}}) \circ \Phi_t^{-1} \\
&= \sum_{j=1}^{n} \left(\widetilde{L}_{t_{j-1}} \otimes \mathrm{Id}_{\mathscr{F}_s(L^2(]t,+\infty[))}\right) \circ \Phi_t(M_{t_j} - M_{t_{j-1}}) \circ \Phi_t^{-1}.
\end{aligned}$$

Therefore, all that we have to prove is that the fundamental processes are adapted. Let us start with the annihilation process. We have

$$\begin{aligned}
\Phi_t \circ A_t \circ \Phi_t^{-1} e(f\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[}) &= \Phi_t \circ A_t e(f) \\
&= \left(\int_0^t f(s)\mathrm{d}s\right) \Phi_t(e(f)) \\
&= \left(\int_0^t f(s)\mathrm{d}s\right) e(f\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[}) \\
&= \left(A_t e(f\mathbf{1}_{[0,t]})\right) \otimes e(f\mathbf{1}_{]t,+\infty[})
\end{aligned}$$

so that it is adapted. The result for the creation process follows from a similar computation:

$$\begin{aligned}
\Phi_t \circ A_t^\dagger \circ \Phi_t^{-1} e(f\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[}) &= \Phi_t \circ A_t^\dagger e(f) \\
&= \Phi_t \left(\frac{\mathrm{d}}{\mathrm{d}s} e(f + s\mathbf{1}_{[0,t]})\right)\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \Phi_t \left(e(f + s\mathbf{1}_{[0,t]})\right)\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} e(f\mathbf{1}_{[0,t]} + s\mathbf{1}_{[0,t]}) \otimes e(f\mathbf{1}_{]t,+\infty[} + s\mathbf{1}_{[0,t]}\mathbf{1}_{]t,+\infty[})\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} e(f\mathbf{1}_{[0,t]} + s\mathbf{1}_{[0,t]})\Big|_{s=0} \otimes e(f\mathbf{1}_{]t,+\infty[}) \\
&= \left(A_t^\dagger e(f\mathbf{1}_{[0,t]})\right) \otimes e(f\mathbf{1}_{]t,+\infty[})
\end{aligned}$$

∎

As in the classical theory of Itô integration, the idea is now to prove an isometry formula in the spirit of Proposition 2.2 which enables to extend the integration functional from simple processes to more general ones. As a first step, let us try our hands on the definition and compute some simple inner products involving the integral of a simple process. In order to lighten computations in the sequel, we will use the following shorthand notations:

- $\widetilde{L}(t)$ will simply be denoted $L(t)$ as soon as the context is unambiguous;
- $e(f\mathbf{1}_{[0,t]})$ and $e(f\mathbf{1}_{]t,+\infty[})$ will be denoted $e(f_{t]})$ and $e(f_{]t})$ respectively.

Proposition 4.7 (FIRST FUNDAMENTAL LEMMA)**.** *Let L be a simple adapted process, let M in $\{A, A^\dagger\}$ and set*

$$X(t) = \int_0^t L(s)\mathrm{d}M_s.$$

Then, for any $f, g \in L^2(\mathbf{R}_+)$,

$$\langle e(f), X(t)e(g)\rangle = \int_0^t \langle e(f), L(s)e(g)\rangle \,\mathrm{d}\mu(s),$$

where

$$\mathrm{d}\mu(s) = \begin{cases} g(s)\mathrm{d}s & \text{if} \quad M = A \\ f(s)\mathrm{d}s & \text{if} \quad M = A^\dagger \end{cases}$$

Proof. Let $t \in \mathbf{R}_+$ and let us fix a partition $0 = t_0 < \cdots < t_n = t$ of $[0,t]$ with respect to which L is simple. By adaptedness of the process, we have

$$\langle e(f), X(t)e(g)\rangle = \sum_{j=1}^{n} \left\langle e(f_{t_{j-1}]}), L(t_{j-1})e(g_{t_{j-1}]})\right\rangle \left\langle e(f_{[t_{j-1}}), \left(M_{t_j} - M_{t_{j-1}}\right) e(g_{[t_{j-1}})\right\rangle$$

Assume first that $M = A$. Then, we have for any $a < b$ that

$$\begin{aligned} \left\langle e(f_{[a}), (M(b) - M(a))e(g_{[a})\right\rangle &= \left\langle \mathbf{1}_{[0,b]}, g\mathbf{1}_{[a,+\infty[}\right\rangle \left\langle e(f_{[a}), e(g_{[a})\right\rangle - \left\langle \mathbf{1}_{[0,a]}, g\mathbf{1}_{[a,+\infty[}\right\rangle \left\langle e(f_{[a}), e(g_{[a})\right\rangle \\ &= \int_a^b g(s)\mathrm{d}s \left\langle e(f_{[a}), e(g_{[a})\right\rangle \\ &= \left\langle e(f_{[a}), e(g_{[a})\right\rangle \mu([a,b]). \end{aligned}$$

Assume now that $M = A^\dagger$. In that case, by the adjunction relation between A and $A^\dagger$ from Proposition 3.9,

$$\begin{aligned} \left\langle e(f_{[a}), (M(b) - M(a))e(g_{[a})\right\rangle &= \left\langle (A(b) - A(a))e(f_{[a}), e(g_{[a})\right\rangle \\ &= \langle \mathbf{1}_{[0,b]}, f\mathbf{1}_{[a,+\infty[}\rangle \langle e(f_{[a}), e(g_{[a})\rangle - \langle \mathbf{1}_{[0,a]}, f\mathbf{1}_{[a,+\infty[}\rangle \langle e(f_{[a}), e(g_{[a})\rangle \\ &= \int_a^b f(s)\mathrm{d}s \langle e(f_{[a}), e(g_{[a})\rangle \\ &= \langle e(f_{[a}), e(g_{[a})\rangle \mu([a,b]). \end{aligned}$$

To complete the proof, it is enough to observe that by what precedes,

$$\begin{aligned} \langle e(f), X(t)e(g)\rangle &= \sum_{j=1}^{n} \left\langle e(f_{t_{j-1}]}), L(t_{j-1})e(g_{t_{j-1}]})\right\rangle \left\langle e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\right\rangle \mu([t_{j-1}, t_j]) \\ &= \sum_{j=1}^{n} \left\langle e(f), L(t_{j-1})e(g)\right\rangle \mu([t_{j-1}, t_j]) \\ &= \int_0^t \langle e(f), L(s)e(g)\rangle \,\mathrm{d}\mu(s). \end{aligned}$$

■

This is of course not enough for our purpose. Indeed, we would like to compute the norm of a vector of the form $X(t)e(f)$, where X is the integral of a process. This means that we also have to understand the inner product between two such integrals. This is what we will do now. As one may expect, the measures μ corresponding to M_1 and M_2 will appear, but also another term involving both creation and annihilation processes. Before stating and proving that core result, we need a bit more analytical information on quantum stochastic integrals. Namely, we have to make sure that the construction produces processes which are continuous in a reasonable sense.

Lemma 4.8. *Let $(L(s))_{s\in\mathbf{R}_+}$ be a simple adapted process and let $M \in \{A, A^\dagger\}$. Then, the corresponding quantum stochastic integral X is continuous in the sense that for any $f, g \in L^2(\mathbf{R}_+)$, the map*

$$t \longmapsto \langle e(g), X(t)e(f)\rangle$$

is piecewise continuous.

Proof. Let us fix $t \in \mathbf{R}_+ \setminus \{t_1, \cdots, t_n\}$. For any $t' \in \mathbf{R}^+$, we have by the FIRST FUNDAMENTAL LEMMA (Proposition 4.7)

$$\begin{aligned} \left|\langle e(g), X(t)e(f)\rangle - \langle e(g), X(t')e(f)\rangle\right| &\leqslant \left|\int_t^{t'} \langle e(g), L(s)e(f)\rangle \mathrm{d}\mu(s)\right| \\ &= \int_{\min(t,t')}^{\max(t,t')} |\langle e(g), L(s)e(f)\rangle| \,\mathrm{d}\mu(s) \end{aligned}$$

Observe that $\|L(s)e(f)\|$ is bounded for $\min(t,t') \leqslant s \leqslant \max(t,t')$ because it only takes a finite number of values. If C is a bound, then because μ is absolutely continuous with respect to the Lebesgue measure, there exists $\eta > 0$ such that

$$\mu([t,t']) < \frac{\epsilon}{C}$$

as soon as $|t-t'| < \eta$. Therefore, for $|t-t'| < \eta$, we have

$$\left|\langle e(g), X(t)e(f)\rangle - \langle e(g), X(t')e(f)\rangle\right| \leqslant \epsilon,$$

hence the result. ■

We are now ready for the main result of this text.

THEOREM 4.9 (SECOND FUNDAMENTAL LEMMA) Let L, L' be simple adapted processes, let $M_1, M_2 \in \{A, A^\dagger\}$ and set

$$X_i(t) = \int_0^t L_i(s)\mathrm{d}M_i(s).$$

Then, for any $f, g \in L^2(\mathbf{R}_+)$,

$$\begin{aligned}\langle X_1(t)e(f), X_2(t)e(g)\rangle &= \int_0^t \langle L_1(s)e(f), X_2(s)e(g)\rangle\, \mathrm{d}\mu_1(s) \\ &+ \int_0^t \langle X_1(s)e(f), L_2(s)e(g)\rangle\, \mathrm{d}\mu_2(s) \\ &+ \int_0^t \langle L_1(s)e(f), L_2(s)e(g)\rangle\, \mathrm{d}\nu(s),\end{aligned}$$

where

- μ_1 is the measure corresponding to $M_1^\dagger$ in Proposition 4.7;
- μ_2 is the measure corresponding to M_2 in Proposition 4.7;
- $\mathrm{d}\nu = \mathrm{d}s$ if $M_1 = M_2 = A^\dagger$ and $\mathrm{d}\nu = 0$ otherwise.

Note that the definition of μ_1 should not be a surprise, since X_1 is on the right-hand side of the inner product while it was on the left-hand side in Proposition 4.7.

Proof. For the sake of clarity, we will write $M_1(t), M_2(t)$ instead of $M_{1,t}$ and $M_{2,t}$. The computations are similar to those of Proposition 4.7. First of all, let us fix $t \in \mathbf{R}_+$ and choose a partition $t_0 < \cdots < t_n = t$ of $\mathbf{R}_+$ which is adapted to both L_1 and L_2. Then,

$$\langle X_1(t)e(f), X_2(t)e(g)\rangle = \left\langle \sum_{j=1}^n L_1(t_{j-1})\big(M_1(t_j) - M_1(t_{j-1})\big)e(f), \sum_{j=1}^n L_2(t_{j-1})\big(M_2(t_j) - M_2(t_{j-1})\big)e(g)\right\rangle$$

To understand that sum, we are going to decompose it. The basic observation is that by definition,

$$\begin{aligned}&\sum_{j=1}^n \langle L_1(t_{j-1})\big(M_1(t_j) - M_1(t_{j-1})\big)e(f), X_2(t_{j-1})e(g)\rangle \\ &= \sum_{j=1}^n \left\langle L_1(t_{j-1})\big(M_1(t_j) - M_1(t_{j-1})\big)e(f), \sum_{k=1}^{j-1} L_2(t_{k-1})(M_2(t_k) - M_2(t_{k-1}))e(g)\right\rangle\end{aligned}$$

A similar formula holds exchanging the roles of the two processes, and adding both yields the sum we are interested in, except from the diagonal terms. In other words,

$$\begin{aligned}\langle X_1(t)e(f), X_2(t)e(g)\rangle = &\sum_{j=1}^{n} \left\langle L_1(t_{j-1})\left(M_1(t_j) - M_1(t_{j-1})\right)e(f), X_2(t_{j-1})e(g)\right\rangle \\ &+ \sum_{j=1}^{n} \left\langle X_1(t_{j-1})e(f), L_2(t_{j-1})\left(M_2(t_j) - M_2(t_{j-1})\right)e(g)\right\rangle \\ &+ \sum_{j=1}^{n} \left\langle L_1(t_{j-1})\left(M_1(t_j) - M_1(t_{j-1})\right)e(f), L_2(t_{j-1})\left(M_2(t_j) - M_2(t_{j-1})\right)e(g)\right\rangle\end{aligned}$$

We will denote by S_1, S_2 and S_3 the three sums appearing above and study them separately.

We start with S_1. Using the fact that all the processes involved are adapted, we have

$$\begin{aligned}S_1 &= \sum_{j=1}^{n} \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}) \otimes \left(M_1(t_j) - M_1(t_{j-1})\right)e(f_{[t_{j-1}}), X_2(t_{j-1})e(g_{t_{j-1}]}) \otimes e(g_{[t_{j-1}})\right\rangle \\ &= \sum_{j=1}^{n} \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}), X_2(t_{j-1})e(g_{t_{j-1}]})\right\rangle \left\langle \left(M_1(t_j) - M_1(t_{j-1})\right)e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\right\rangle \\ &= \sum_{j=1}^{n} \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}), X_2(t_{j-1})e(f_{t_{j-1}]})\right\rangle \left\langle e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\right\rangle \mu_1([t_{j-1}, t_j]) \\ &= \sum_{j=1}^{n} \left\langle L_1(t_{j-1})e(f), X_2(t_{j-1})e(g)\right\rangle \mu_1([t_{j-1}, t_j]).\end{aligned}$$

A similar computation yields

$$S_2 = \sum_{j=1}^{n} \left\langle X_1(t_{j-1})e(f), L_2(t_{j-1})e(g)\right\rangle \mu_2([t_{j-1}, t_j]).$$

Both S_1 and S_2 are Riemann sums of piecewise continuous functions on the interval $[0, t_n] = [0, t]$ by Lemma 4.8. As a consequence, if we denote by π the partition that we use and set

$$\delta_\pi = \sup_{1 \leqslant j \leqslant n} (t_j - t_{j-1}),$$

then as $\delta_\pi \to 0$, both sums converge, respectively to the first and second terms in the formula of the statement.

We are therefore left with S_3, which can be written as

$$\begin{aligned}S_3 = &\sum_{j=1}^{n} \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}), L_2(t_{j-1})e(g_{t_{j-1}]})\right\rangle \\ &\times \left\langle \left(M_1(t_j) - M_1(t_{j-1})\right)e(f_{[t_{j-1}}), \left(M_2(t_j) - M_2(t_{j-1})\right)e(g_{[t_{j-1}})\right\rangle\end{aligned}$$

The main point is then the computation of the quantity

$$\Gamma_{12}(a, b) = \langle (M_1(b) - M_1(a))e(f), (M_2(b) - M_2(a))e(g)\rangle$$

for $a < b$. Let us split the cases:

- If $M_1 = M_2 = A$, the computation is straightforward and yields

$$\Gamma_{12}(a, b) = \left(\int_a^b f(s)\mathrm{d}s\right)\left(\int_a^b g(s)\mathrm{d}s\right) = \mu_1([a, b])\mu_2([a, b]).$$

- If $M_1 = A$ and $M_2 = A^\dagger$, once first applies M_1, and the uses the adjunction relation to move M_2 to the right-hand side, yielding

$$\Gamma_{12}(a, b) = \left(\int_a^b f(s)\mathrm{d}s\right)\left(\int_a^b f(s)\mathrm{d}s\right) = \mu_1([a, b])\mu_2([a, b]).$$

- If $M_1 = A^\dagger$ and $M_2 = A$, then $\Gamma_{12}(a,b)$ is the same as in the previous situation, hence once again
$$\Gamma_{12}(a,b) = \mu_1([a,b])\mu_2([a,b]).$$
- If $M_1 = M_2 = A^\dagger$, using the adjunction we have to compute a coefficient of the operator.
$$(A(b)-A(a))(A^\dagger(b)-A^\dagger(a)) = A(b)A^\dagger(b) - A(b)A^\dagger(a) - A(a)A^\dagger(b) + A(a)A^\dagger(a).$$
Using the computations in the proof of Proposition 3.9, we have
$$\left\langle A(b)A^\dagger(a)e(f), e(g)\right\rangle = \left\langle \mathbf{1}_{[0,b]}, \mathbf{1}_{[0,a]}\right\rangle \langle e(f), e(g)\rangle + \langle \mathbf{1}_{[0,b]}, g\rangle\langle f, \mathbf{1}_{[0,a]}\rangle\langle e(f), e(g)\rangle.$$
Therefore,
$$\begin{aligned}\Gamma_{12}(a,b) &= (b-a-a+a)\langle e(f), e(g)\rangle + \langle \mathbf{1}_{[0,b]}, g\rangle \left(\langle f, \mathbf{1}_{[0,b]}\rangle - \langle f, \mathbf{1}_{[0,a]}\rangle\right)\langle e(f), e(g)\rangle \\ &\quad - \langle \mathbf{1}_{[0,a]}, g\rangle \left(\langle f, \mathbf{1}_{[0,a]}\rangle - \langle f, \mathbf{1}_{[0,b]}\rangle\right)\langle e(f), e(g)\rangle \\ &= (b-a)\langle e(f), e(g)\rangle + \left(\int_a^b f(s)\mathrm{d}s\right)\langle \mathbf{1}_{[0,b]}, g\rangle\langle e(f), e(g)\rangle \\ &\quad - \left(\int_a^b f(s)\mathrm{d}s\right)\langle \mathbf{1}_{[0,a]}, g\rangle\langle e(f), e(g)\rangle \\ &= (b-a)\langle e(f), e(g)\rangle + \left(\int_a^b f(s)\mathrm{d}s\right)\left(\int_a^b g(s)\mathrm{d}s\right)\langle e(f), e(g)\rangle \\ &= \nu([a,b])\langle e(f), e(g)\rangle + \mu_1([a,b])\mu_2([a,b])\langle e(f), e(g)\rangle.\end{aligned}$$

We have proved the following formula:

$$\begin{aligned}S_3 &= \sum_{j=1}^n \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}), L_2(t_{j-1})e(g_{t_{j-1}]})\right\rangle \langle e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\rangle \nu([t_{j-1}, t_j]) \\ &+ \sum_{j=1}^n \left\langle L_1(t_{j-1})e(f_{t_{j-1}]}), L_2(t_{j-1})e(g_{t_{j-1}]})\right\rangle \langle e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\rangle \mu_1([t_{j-1}, t_j])\mu_2([t_{j-1}, t_j]) \\ &= \int_0^t \langle L_1(s)e(f), L_2(s)e(g)\rangle\, \nu([t_{j-1}, t_j]) + \int_0^t \langle L_1(s)e(f), L_2(s)e(g)\rangle\, \mu_1([t_{j-1}, t_j])\mu_2([t_{j-1}, t_j]) \\ &= \int_0^t \langle L_1(s)e(f), L_2(s)e(g)\rangle\, \nu([t_{j-1}, t_j]) + R.\end{aligned}$$

To conclude the proof, we have to show that $R \to 0$ when $\delta_\pi \to 0$. For that purpose, observe that

$$|R| \leqslant \max_{1\leqslant j\leqslant n} \left|\mu_2([t_{j-1}, t_j])\right| \int_0^t |\langle L_1(s)e(f), L_2(s)e(g)\rangle| \left|\mathrm{d}\mu_1\right|(s).$$

Only the first factor depends on the partition δ_π, and because μ_2 is absolutely continuous with respect to the Lebesgue measure, it has no atom. As a consequence,

$$\max_{1\leqslant j\leqslant n} |\mu_2([t_{j-1}, t_j])| \underset{\delta_\pi\to 0}{\longrightarrow} 0$$

and the proof is complete. ■

4.2 EXTENSION

We will now use the previous results to extend the construction of the quantum Itô integral to more general processes. Let us start with a useful definition.

DEFINITION 4.10 (MEASURABLE PROCESS). An adapted process is said to be *measurable* if for all $f \in L^2(\mathbf{R}_+)$, the map
$$t \mapsto L(t)e(f)$$
is Borel measurable. Two measurable processes L and L' are said to be *equivalent* if for all $f \in L^2(\mathbf{R}_+)$,
$$\{t \mid L(t)e(f) \neq L'(t)e(f)\}$$

has Lebesgue measure 0. We then write $L \sim L'$.

As in the setting of classical Itô calculus, as well as in the theory of Riemann integration, we want to consider processes which are approximable by simple ones.

DEFINITION 4.11 (STOCHASTICALLY INTEGRABLE PROCESS). A process L is said to be *stochastically integrable* if it is adapted, measurable, and for all $t \in \mathbf{R}_+$ there exists a sequence $(L_n)_{n\in\mathbf{N}}$ of simple adapted processes such that for all $f \in L^2(\mathbf{R}_+)$,

$$\int_0^t \|(L_n(s) - L(s))e(f)\|^2 \mathrm{d}s \to 0.$$

Given a sequence of $(L_n)_{n\in\mathbf{N}}$ simple adapted processes approximating a given stochastically integrable process L, one can consider for $M \in \{A, A^\dagger\}$ the new sequence

$$X_n(t) = \int_0^t L_n \mathrm{d}M.$$

What we would like to prove is that the sequence $(X_n)_{n\in\mathbf{N}}$ converges to some adapted process, and that the limit is independent from the choice of the approximating sequence. In that way, we will get a reasonable definition of the quantum stochastic integral of L. To do this, let us start by deriving some bound on the norm of the integral of a simple process.

Lemma 4.12 (BOUNDEDNESS OF THE STOCHASTIC INTEGRAL)**.** *Let L be a simple adapted process, let $M \in \{A, A^\dagger\}$ and let X be the corresponding integral. Then, for any $f \in L^2(\mathbf{R}_+)$,*

$$\|X(t)e(f)\|^2 \leqslant e^{\int_0^t |f(s)|\mathrm{d}s} \int_0^t \|L(s)e(f)\|^2 (|f(s)|+1)\mathrm{d}s$$

Proof. Start by applying the SECOND FUNDAMENTAL LEMMA (Theorem 4.9) with $L_1 = L_2$ and $M_1 = M_2$:

$$\|X(t)e(f)\|^2 = 2\Re\left(\int_0^t \langle X(s)e(f), L(s)e(f)\rangle \mathrm{d}\mu(s)\right) + \int_0^t \|L(s)e(f)\|^2 \mathrm{d}\nu(s)$$

The CAUCHY-SCHWARZ INEQUALITY and the fact that $\mathrm{d}\mu = f\mathrm{d}s$ then yield

$$\|X(t)e(f)\|^2 \leqslant \int_0^t \|X(s)e(f)\|^2 |f(s)|\mathrm{d}s + \int_0^t \|L(s)e(f)\|^2 |f(s)|\mathrm{d}s + \int_0^t \|L(s)e(f)\|^2 \mathrm{d}\nu(s)$$

and because $\mathrm{d}\nu$ is either $\mathrm{d}s$ or 0, we conclude that

$$\|X(t)e(f)\|^2 \leqslant \int_0^t \|X(s)e(f)\|^2 |f(s)|\mathrm{d}s + \int_0^t \|L(s)e(f)\|^2 (|f(s)|+1)\mathrm{d}s.$$

The result now follows from a direct application of GRÖNWALL'S LEMMA. ∎

We are ready to prove the main theorem of this section, which constructs the quantum Itô integral for a large class of processes.

THEOREM 4.13 Let L be a stochastically integrable process and let $M \in \{A, A^\dagger\}$. Then, there exists an adapted process X such that for any $t \in \mathbf{R}_+$ and any sequence $(L_n)_{n\in\mathbf{N}}$ of simple adapted processes which approximate L, we have for all $f \in L^2(\mathbf{R}_+)$ and $t \in \mathbf{R}_+$,

$$\left(\int_0^t L_n(s)\mathrm{d}M\right) e(f) \to X(t)e(f).$$

Proof. For $k,n \in \mathbf{N}$, we have by Lemma 4.12

$$\begin{aligned}\|X_k(t)e(f)-X_n(t)e(f)\|^2 &\leqslant e^{\int_0^t|f(s)|\mathrm{d}s}\int_0^t \|(L_k(s)-L_n(s))e(f)\|^2(|f(s)|+1)\mathrm{d}s\\ &\leqslant e^{\int_0^t|f(s)|\mathrm{d}s}\sup_{0\leqslant s\leqslant t}(|f(s)|+1)\int_0^t \|(L_k(s)-L_n(s))e(f)\|^2\mathrm{d}s\\ &= \alpha(t,f)\int_0^t \|(L_k(s)-L_n(s))e(f)\|^2\mathrm{d}s.\end{aligned}$$

Let now $\epsilon>0$, and let N be such that for all $n\geqslant N$,

$$\int_0^t \|(L_n(s)-L(s))e(f)\|^2 < \frac{\epsilon}{2\alpha(t,f)}.$$

Then, for all $k,n\geqslant N$,

$$\begin{aligned}\int_0^t \|(L_k(s)-L_n(s))e(f)\|^2 &\leqslant \int_0^t \|(L_k(s)-L(s))e(f)\|^2+\int_0^t \|(L(s)-L_n(s))e(f)\|^2\\ &< \frac{\epsilon}{\alpha(t,f)}\end{aligned}$$

and this, together with the first part of the proof yields

$$\|X_k(t)e(f)-X_n(t)e(f)\|^2\leqslant\epsilon$$

for all $k,n\geqslant N$. In other words, the sequence $(X_n(t)e(f))_{n\in\mathbf{N}}$ is Cauchy in $\mathscr{F}_s(H)$ which is complete, hence it converges to a limit which we denote by $X(t)e(f)$.

Let now $(L'_n)_{n\in\mathbf{N}}$ be another sequence of adapted simple processes approximating L and let X' be the corresponding limit process. Then,

$$\begin{aligned}\left\|\left(X'_n(t)-X_n(t)\right)e(f)\right\|^2 &\leqslant \alpha(t,f)\int_0^t \left\|\left(L'_n(s)-L_n(s)\right)\right\|^2\mathrm{d}s\\ &\leqslant \alpha(t,f)\int_0^t \left\|\left(L'_n(s)-L(s)\right)\right\|^2\mathrm{d}s+\alpha(t,f)\int_0^t \|(L(s)-L_n(s))\|^2\mathrm{d}s\end{aligned}$$

Because the right-hand side goes to 0 as n goes to infinity, we conclude that $X'(t)e(f)=X(t)e(f)$, and the proof is complete. ∎

The process given by Theorem 4.13 is called the *(quantum) stochastic integral* of L with respect to M and is denoted by

$$X(t)=\int_0^t L(s)\mathrm{d}M(s).$$

Let us record some elementary properties of that construction.

Proposition 4.14. *Stochastically integrable processes have the following properties:*

1. *If L is stochastically integrable and $L'\sim L$, then L' is stochastically integrable and its integral equals that of L;*
2. *The space $\mathscr{S}$ of stochastically integrable processes up to equivalence is a vector space;*
3. *Stochastic integration is linear on $\mathscr{S}$;*
4. *The stochastic integral of a stochastically integrable process is continuous.*

Proof. We will proceed pointwise.

1. Simply observe that if $L'\sim L$, then any sequence of simple processes approximating L also approximates L'.

2. This is a direct consequence of the definition.

3. This follows from the definition of the stochastic integral of a simple process.

4. Fix $t > 0$ and $f, g \in L^2(\mathbf{R}_+)$. For $t' \in \mathbf{R}_+$, taking limits in the FIRST FUNDAMENTAL LEMMA (Proposition 4.7) shows that

$$\left|\langle e(g), X(t)e(f)\rangle - \langle e(g), X(t')e(f)\rangle\right| \leqslant \left|\int_t^{t'} \langle e(g), L(s)e(f)\rangle \mathrm{d}\mu(s)\right| = \int_{\min(t,t')}^{\max(t,t')} |\langle e(g), L(s)e(f)\rangle| \,\mathrm{d}\mu(s).$$

The result now follows from the fact that $s \mapsto \langle e(g), L(s)e(f)\rangle$ is measurable with respect to the Lebesgue measure by definition of a measurable process, hence with respect to μ, and that μ has no atom since it is absolutely continuous with respect to the Lebesgue measure.

■

Let us show that this class is fairly large, using the following definition:

DEFINITION 4.15 (REGULAR PROCESS). A process L is said to be *regular* if for all $f \in L^2(\mathbf{R}_+)$, the map

$$t \mapsto L(t)e(f)$$

is continuous.

Proposition 4.16. *Regular processes are stochastically integrable.*

Proof. Let us fix a regular process L and $t \in \mathbf{R}_+$. For $n \in \mathbf{N}$, we define L_n by splitting the interval $[0, t]$ into n pieces and fixing the value on each of these small intervals. More precisely, we set, for $t \in \mathbf{R}_+$,

$$L_n(s) = \begin{cases} L\left(\dfrac{tj}{n}\right) & \text{if} \quad \dfrac{tj}{n} \leqslant s < \dfrac{t(j+1)}{n} \\ L(t) & \text{if} \quad t \leqslant s \end{cases}$$

This is by construction a simple process, which is moreover adapted because L is. Moreover, for $s \in [0, t]$ and $f \in L^2(\mathbf{R}_+)$, the regularity of L implies that

$$L_n(s)e(f) \underset{n\to\infty}{\longrightarrow} L(s)e(f).$$

Furthermore, again by regularity, we have that

$$K_t = \sup_{0\leqslant s\leqslant t} \|L(s)e(f)\| < +\infty$$

Therefore, observing that $\|L_n(s)e(f)\| \leqslant K_t$ since L_n takes as values a subset of the values of L, we have

$$\|L_n(s)e(f) - L(s)e(f)\|^2 \leqslant 2K_t^2$$

and a straightforward application of the DOMINATED CONVERGENCE THEOREM yields

$$\int_0^t \|(L_n(s) - L(s))\,e(f)\|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

In other words, L is stochastically integrable. ■

Though Proposition 4.16 provides a large class of stochastically integrable processes, is not as general as the classical construction of the Itô integral which works under the weaker assumption of local square integrability. This is of course not a problem, once we introduce a very natural notion of local square integrability:

DEFINITION 4.17 (SQUARE INTEGRABLE QUANTUM PROCESS). A process L is said to be *locally square integrable* if for all $f \in L^2(\mathbf{R}_+)$,

$$\int_0^t \|L(s)e(f)\|^2 \mathrm{d}s < +\infty.$$

Corollary 4.18. *Locally square integrable processes are stochastically integrable.*

Proof. For fixed t and f, one may define an inner product on the space of locally square integrable processes through the formula

$$\langle L, L' \rangle_{t,f} = \int_0^t \langle L(s)e(f), L'(s)e(f) \rangle \mathrm{d}s.$$

This procudes a Hilbert space $H_{t,f}$ in which regular processes are dense. The result therefore follows from Proposition 4.16. ∎

4.3 A QUANTUM ITÔ FORMULA

So far, we have constructed a quantum Itô integral for regular stochastic processes. But the main tool in Itô calculus is the Itô formula, usually stated in its elementary form

$$\mathrm{d}W_t \mathrm{d}W_t = \mathrm{d}t.$$

So what about the quantum setting ? Here we run into the very issue that we have been trying to avoid all along: because we deal only with unbounded operators, there are analytical subtleties concerning domains which prevent making sense of formulas like $\mathrm{d}A\mathrm{d}A^\dagger$ for instance. Nevertheless, one can get a weak version of the Itô formula by restricting, as we have done all along, to coefficients in the exponential domain. More precisely, taking limits in Theorem 4.9 yields the following:

THEOREM 4.19 (QUANTUM ITÔ FORMULA) Let L_1, L_2 be stochastically integrable processes, and let $M_1, M_2 \in \{A, A^\dagger\}$. If X_1 and X_2 denote the corresponding quantum stochastic integrals, then

$$\begin{aligned}\langle X_1(t)e(f), X_2(t)e(g) \rangle &= \int_0^t \langle L_1(s)e(f), X_2(s)e(g) \rangle \,\mathrm{d}\mu_1(s) \\ &+ \int_0^t \langle X_1(s)e(f), L_2(s)e(g) \rangle \,\mathrm{d}\mu_2(s) \\ &+ \int_0^t \langle L_1(s)e(f), L_2(s)e(g) \rangle \,\mathrm{d}\nu(s)\end{aligned}$$

where

- μ_1 is the measure corresponding to $M_1^\dagger$ in Proposition 4.7;
- μ_2 is the measure corresponding to M_2 in Proposition 4.7;
- $\nu = \mathrm{d}s$ if $M_1 = M_2 = A^\dagger$ and $\nu = 0$ otherwise.

To see why this is an Itô formula, let us get away with domain issues and work as if all compositions and adjoints are well-defined. Consider then the composition $X = X_1^\dagger X_2$. Observe that formally taking adjoints in the FIRST FUNDAMENTAL LEMMA (Proposition 4.7) shows that given a process

$$X(t) = \int_0^t L(s)\mathrm{d}M_s,$$

the process

$$X^\dagger(t) = \int_0^t L^\dagger(s)\mathrm{d}M_s^\dagger,$$

is adjoint to $X(t)$ on exponential vectors. Therefore, shifting all operators to the right-hand sides yields

$$\begin{aligned}\left\langle e(f), X_1^\dagger X_2 e(g)\right\rangle &= \langle X_1(t)e(f), X_2(t)e(g)\rangle \\ &= \int_0^t \left\langle e(f), L_1^\dagger(s) X_2(s) e(g)\right\rangle \mathrm{d}\mu_1(s) \\ &+ \int_0^t \left\langle e(f), X_1^\dagger(s) L_2(s) e(g)\right\rangle \mathrm{d}\mu_2(s) \\ &+ \int_0^t \left\langle e(f), L_1^\dagger(s) L_2(s) e(g)\right\rangle \mathrm{d}\nu(s).\end{aligned}$$

Comparing with Proposition 4.7, we see that the first term above is exactly the one corresponding to the stochastic integral of the process $L_1^\dagger X_2$ with respect to $M_1^\dagger$. Similarly, the second term corresponds to the integral of the process $X_1^\dagger L_2$ with respect to M_2. As for the third term, it vanishes unless $M_1 = M_2 = A^\dagger$, in which case it is just the deterministic integral of the process $L_1^\dagger L_2$ with respect to ds.

In other words, using the differential notations

$$\mathrm{d}X_1 = L_1 \mathrm{d}M_1 \quad \& \quad \mathrm{d}X_2 = L_2 \mathrm{d}M_2,$$

$X_1^\dagger X_2$ is a stochastic integral, and the corresponding process can be written in differential form as

$$\mathrm{d}(X_1^\dagger X_2) = L_1 X_2 \mathrm{d}M_1^\dagger + X_1 L_2 \mathrm{d}M_2 + L_1 L_2 \mathrm{d}M_1^\dagger \mathrm{d}M_2,$$

where $\mathrm{d}M_1^\dagger \mathrm{d}M_2$ is ds if $M_1 = M_2 = A^\dagger$ and 0 otherwise.

Forgetting the adjoints, this can be summarized into the following *Itô table* giving the value of $\mathrm{d}M_1\mathrm{d}M_2$:

M_1 \ M_2	A	$A^\dagger$
A	0	ds
$A^\dagger$	0	0

Remembering now that $Q_t = A_t + A_t^\dagger$ is the Wiener process, we immediatly recover the celebrated formula:

$$\mathrm{d}Q_t \mathrm{d}Q_t = \mathrm{d}t.$$

4.4 Further perspectives

As mentionned in Section 1, we have been working in the simplest possible framework: we focused on the symmetric Fock space over $L^2(\mathbf{R}_+)$, considered only the creation and annihilation operators as integrands, and only worked with coefficients given by exponential vectors. This shows that there are at least three directions in which one may enlarge our setting. Let us dicuss them one by one.

More Hilbert spaces

First, we have been working in $\mathscr{F}_s(L^2(\mathbf{R}_+))$ all along. This is natural, since $\mathbf{R}_+$ is needed in the framework to provide the time-dependence of the processes and make sense of adaptedness. Nevertheless, one may want the operators to act on a larger space. This can be done by fixing another Hilbert space K – called the *source space* – and considering operators on $K \otimes \mathscr{F}_s(L^2(\mathbf{R}_+))$ instead. One may then define similarly adapted and simple processes, and proceed to prove a bound on the norm of the corresponding stochastic integrals enabling the extension of the framework. However, one does not in general want to impose the operators defining the processes to be bounded on K. Therefore, we will have to assume that for a given process, there is a common dense subspace on which the operators act at all time. One then has to deal with this all along the computations, and to be careful about what happens when trying to compare two processes.

Another way of extending the framework could be to replace $L^2(\mathbf{R}_+)$ by another Hilbert space H. The main question then is how to make sense of time evolution and adaptedness. The solution is to fix a "time measure" of H by means of a map

$$\varphi : \mathscr{F}(\mathbf{R}_+) \to \mathscr{P}(H),$$

where $\mathscr{F}(\mathbf{R}_+)$ is the Borel σ-algebra of $\mathbf{R}_+$ and $\mathscr{P}(H)$ is the set of all orthogonal projections in H. The idea is then that the range of $\varphi([0,t])$ will then be the set of vectors available up to time t, so that we can make sense of adapted processes. However, this can only work if φ statisfies some natural conditions (for instance $\varphi([0,t]) \leqslant \varphi([O,t'])$ for $t \leqslant t'$) making it a *projection-valued measure*. The usual conditions are as follows:

- $\varphi(\emptyset) = 0$ and $\varphi(\mathbf{R}_+) = \mathrm{Id}$;
- Given pairwise disjoint subsets $(E_i)_{i\in\mathbf{N}}$ in $\mathscr{F}(\mathbf{R}_+)$, for any $v \in H$,

$$\left(\sum_{i=1}^{n} \varphi(E_i)\right) v \to \varphi\left(\bigcup_{i=1}^{\infty} E_i\right) v$$

Assuming that this is done, we still have to replace the elements $\mathbf{1}_{[0,t]}$ in the definition of the creation and annihilation operators. This will be done through maps $x : \mathbf{R}_+ \to H$ which are compatible with the time evolution φ in the sense that for any $s < t$,

$$\varphi([0,s])x(t) = x(s).$$

By an obvious anology, such a one parameter family of vectors is called a *φ-martingale*. Note that in that setting, the measures μ and ν appearing in Theorem 4.9 have to be changed since they will depend on φ and x.

Before turning to the next point, let us mention another constructions based on the Fock space: the *antisymmetric Fock space* $\mathscr{F}_a(H)$. It is given by a projection $P_a : \mathscr{F}(H) \to \mathscr{F}(H)$ which takes thje signature of the permutations in account when averaging:

$$P_a(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

A simple computation shows that if one of the tensors is a linear combination of the others, then the result is 0. This implies for instance that if H is finite-dimensional, then $\mathscr{F}_a(H)$ also is, in contrast with the symmetric case. This also implies that there is no exponential vectors in $\mathscr{F}_a(H)$ so that one has to find other dense subspaces to work. Despite these differences, a similar theory of stochastic integration can be developped in that setting. Since the use of the antisymmetric Fock space is used in quantum physics to describe the statistics of particles called fermions, this is usually called *fermionic stochastic calculus* and details can be found for instance in [AH84].

More processes

Even though the addition of a source space is important to enlarge the setting and allow for a richer theory, it is more of a technical extension of the framework. The second point brings in a new concept. Indeed, quantum mechanics uses a third type of operators besides creation and annihilation: the *conservation* (or *gauge*) operators. These are not associated to a vector in H but to a bounded linear map $T \in \mathscr{B}(H)$. The precise definition involves the notion of second differential quantization and we will therefore skip, but we give the expression of coefficients between exponential vectors:

$$\langle e(y), \Lambda(T)e(x)\rangle = \langle y, Tx\rangle\langle e(y), e(x)\rangle.$$

Observe that formally, $\Lambda(T)^* = \Lambda(T^*)$, so that $\Lambda(T)$ is essentially self-adjoint when T is self-adjoint. Using the multiplication operators $T_t = m_{\mathbf{1}_{[0,t]}}$ then provides an adapted process $(\Lambda_t = \Lambda(T_t))_{t\in\mathbf{R}_+}$ with respect to which one may perform stochastic integration. One important feature of that new integrator is that it enables the realization of the standard *Poisson* process on the symmetric Fock space, in a way similar to the Wiener process in Theorem 2.10. The Poisson process with intensity $\alpha > 0$ will then correspond to the operators

$$\Lambda_t + \sqrt{\alpha} Q_t + \alpha t.$$

Of course, the introduction of conservation operators leads to a larger Itô table for computing products of quantum stochastic integrals, which is the following one:

M_1 \ M_2	A	$A^\dagger$	Λ
A	0	$\mathrm{d}s$	$\mathrm{d}A$
$A^\dagger$	0	0	0
Λ	0	$\mathrm{d}A^\dagger$	$\mathrm{d}\Lambda$

If one uses the more general framework explained in the previous point involving a time evolution φ in a general Hilbert space H, then an operator $T \in \mathscr{B}(H)$ is compatible with φ is $\varphi([0,t])T = T\varphi([0,t])$ for all $t \in \mathbf{R}_+$. Then, denoting by T_t that operator, one gets a conservation process $\Lambda(T_t)$.

As a finel comment, let us mention that the main issue when dealing with conservation operators is that they are not directly defined on exponential vectors. This raises the question of a better dense subspace to work with. A natural choice would be the subspace of finite tensors, which is simply the algebraic direct sum of vector spaces

$$\mathscr{F}(H)_{\mathrm{fin}} = \bigoplus_{n \in \mathbf{N}} H^{\otimes_s n}.$$

The drawback of that space is that it does not contain exponential vectors, so that for instance one cannot directly realize the Wiener process on it. This motivates the search for better subspaces, but we will not go into it here.

REFERENCES

[AH84] D. APPLEBAUM & R. HUDSON – "Fermion Itô's formula and stochastic evolutions", *Comm. Math. Phys.* **96** (1984), no. 4, p. 473–496.

[Hal13] B. HALL – *Quantum theory for mathematicians*, Graduate Texts in Mathematics, vol. 267, Springer, 2013.

[Kho91] A. KHOLEVO – "Quantum stochastic calculus", *J. Math. Sci.* **56** (1991), p. 2609–2624.

[Mey95] P.-A. MEYER – *Quantum probability for probabilists*, Lecture Notes in Mathematics, vol. 1538, Springer, 1995.

[Øk13] B. ØKSENDAL – *Stochastic differential equations: an introduction with applications*, Universitext, Springer, 2013.

[Par92] K. PARTHASARATHY – *An introduction to quantum stochastic calculus*, Monographs in Mathematics, vol. 85, Birkhaüser, 1992.